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Non-Gaussian effects on quantum entropies

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ABSTRACT

A deduction of generalized quantum entropies within the non-Gaussian frameworks, Tsallis and Kaniadakis, is derived using a generalized combinatorial method and the so-called q and κ calculus. In agreement with previous results, we also show that for the Tsallis formulation the q-quantum entropy is well-defined for values of the nonextensive parameter q lying in the interval [0,2].

1. Introduction

The notion of entropy is directly connected to information associated with the degrees of freedom of physical system [1,2]. Recently, a considerable effort has been made toward the development of a mathematical generalization of this concept aiming at better understanding a number of physical systems [3–5]. Some generalizations are the so-called non-Gaussian (nG) statistics which are based on the entropic measure [3,4]

$$S_q = -\sum_{i=1}^{w} p_i^q \ln_q p_i \quad \text{and} \quad S_\kappa = -\sum_{i=1}^{w} p_i \ln_\kappa p_i, \tag{1}$$

where p_i is the microstate probability, w stands for the number of states and the nG parameters (q, κ) take the values $q = 1$ and $\kappa = 0$ for Gaussian and $q \neq 1$ and $\kappa \neq 0$ for nG statistics. From the mathematical point of view, the above statistics are based on the deformed functions given, respectively, by

$$\exp_q(p) = (1 + (1-q)p)^{1/1-q}, \qquad \ln_q = \frac{p^{1-q} - 1}{1-q}, \tag{2a}$$

$$\exp_\kappa(p) = (\sqrt{1+\kappa^2 p^2} + \kappa p)^{1/\kappa}, \qquad \ln_\kappa = \frac{p^\kappa - p^{-\kappa}}{2\kappa}. \tag{2b}$$

The effect of nonextensivity and nonadditivity has been largely studied in the context of quantum mechanics. In this particular context, the generalized Bose–Einstein and Fermi–Dirac distributions in nonextensive systems have been investigated by the at least three different methods, namely: (i) the asymptotic approximation proposed by Tsallis et al. [6], which derived the expression for the canonical partition function valid for $|q-1|/k_B T \rightarrow 0$; (ii) The factorization approximation considered by Büyükiliç et al. [7] to evaluate the grand-canonical partition function and (iii) the exact approach developed by Rajagopal et al. [8] which derived the exact integral representation for the grand-canonical partition

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function of nonextensive systems. The connection between the Kaniadakis framework and quantum statistics has been investigated using the maximal entropy principle [4,9], as well as through applications using the relativistic nuclear equation of state in the context of the Walecka quantum hadrodynamics theory [10].

Recently, a proof of the nG quantum H-theorem in the context of both Tsallis and Kaniadakis formalisms was derived by considering statistical correlations under a collisional term from the quantum Boltzmann equation [11,12]. In the Tsallis case, the positiveness of time variation of the quantum entropy S_q^Q combined with a duality transformation discussed in Refs. [13,14] implied that the nonextensive parameter q must lie in the interval [0, 2] (the same is not true for the Kaniadakis case). Additionally, the stationary states are described by quantum q- and κ-power law extensions of the Fermi–Dirac and Bose–Einstein distributions.

However, in dealing with quantum Tsallis and Kaniadakis frameworks, a particular attention must be paid to the generalization of entropy which plays a fundamental role within the domain of nG quantum H-theorem [11,12]. Specifically, the Bose–Einstein and Fermi–Dirac distributions are calculated through a very concrete mathematical basis of combinatorial nature. Our goal in this Paper is to show that the Tsallis and Kaniadakis quantum entropies can be deduced within the context of a generalized combinatorial method, similarly to ordinary Gaussian statistics. In agreement with previous results, we show that by introducing the so-called q-Stirling's formula, q-multinomial coefficients and the duality transformation, the q-quantum entropy can be determined with the nonextensive parameter q in the interval [0, 2].

This Paper is organized as follows. In Section 2, we give a brief description of the main considerations on the combinatorial structure of the standard quantum entropy. A deduction of the Tsallis and Kaniadakis quantum entropies based on the generalized combinatorial structure is made in Sections 3 and 4, respectively. We summarize the main conclusions in Section 5. In the Appendix we show a κ-generalization of Stirling approximation used in Section 4 to derive Kaniadakis quantum entropy.

2. Quantum entropy in standard statistics

Let us start by representing the main considerations of the combinatorial structure of the standard quantum entropy.[2] The quantum entropy is calculated through mathematical structure, e.g.: first, we describe the energy eigenstates W of a quantum gas by considering the arrangement of the g_α and n_α quantities. In the case of a Bose–Einstein and Fermi–Dirac gas, respectively, we have [1]

$$W = \prod_\alpha W_\alpha^{\{+\}} = \prod_\alpha \frac{(n_\alpha + g_\alpha - 1)!}{n_\alpha!(g_\alpha - 1)!} \tag{3}$$

and

$$W = \prod_\alpha W_\alpha^{\{-\}} = \prod_\alpha \frac{g_\alpha!}{n_\alpha!(g_\alpha - n_\alpha)!}. \tag{4}$$

Second, by applying the logarithms functions in the above quantities and using the Stirling approximation for large n

$$\ln(n!) = n[\ln(n) - 1], \quad n \gg 1 \tag{5}$$

we can calculate, by using the functional $S^Q = -\ln W$, the explicit expressions for standard quantum entropy for Bose–Einstein and Fermi–Dirac gas, in the following form

$$S^Q = -\sum_\alpha [n_\alpha \ln(n_\alpha) - (n_\alpha \pm g_\alpha)\ln(n_\alpha \pm g_\alpha) \pm g_\alpha \ln(g_\alpha)], \tag{6}$$

where the upper signs refer to the Bose–Einstein and the lower signs to the Fermi–Dirac gases.

3. Tsallis quantum entropy

In order to obtain the generalized quantum entropy in Tsallis framework [11], let us first introduce the so-called q-Stirling approximation proposed in Ref. [15]. Next, we present the generalized multinomial coefficients for the Bose–Einstein and Fermi–Dirac gas.

3.1. q-Stirling's approximation

In Refs. [15,16], it was shown that the q-factorial $n!_q$ for $n \in N$ and $q > 0$ is defined by

$$n!_q := 1 \otimes_q 2 \otimes_q 3 \otimes_q \cdots \otimes_q n. \tag{7}$$

[2] We consider a spatially homogeneous gas of N particles (bosons or fermions) enclosed in a volume V. We also assume that this gas is appropriately specified by regarding the states of energy for a single particle in the container as divided up into groups of g_α neighboring states, and by stating the number of particles n_α assigned to each such group g_α [1].

By considering the q-product [17,18]

$$x \otimes_q y := \begin{cases} [x^{1-q} + y^{1-q} - 1]^{\frac{1}{1-q}} & \text{if } x, y > 0 \text{ and } x^{1-q} + y^{1-q} - 1 > 0 \\ 0 & \text{otherwise,} \end{cases} \tag{8}$$

the following expression for q-Stirling formula is obtained

$$\ln_q(n!_q) \simeq \begin{cases} \dfrac{n}{2-q}(\ln_q n - 1) & \text{if } q > 0 \text{ and } q \neq 1, 2 \\ n - \ln n & \text{if } q = 2, \end{cases} \tag{9}$$

where in the limit $q \to 1$, Eq. (5) is fully recovered.

3.2. Bose–Einstein gas

Let us now consider the Bose–Einstein gas in which the q-multinomial coefficient can be defined using the q-product and the q-division as follows

$$W_q = \prod_\alpha {}_{\otimes^q} W_\alpha^{\{+,q\}} = \prod_\alpha {}_{\otimes^q} \{(n_\alpha + g_\alpha - 1)!_q \oslash^q [n_\alpha!_q \otimes^q (g_\alpha - 1)!_q]\}, \tag{10}$$

where the above product reads [17,18]

$$\prod_{\alpha=1}^{l} {}_{\otimes^q} x_\alpha := x_1 \otimes^q x_2 \otimes^q \cdots \otimes^q x_l. \tag{11}$$

Here, by taking the q-logarithm in the expression of the q-number of states W_q, we obtain

$$\ln_q W_q = \sum_\alpha \{\ln_q[(n_\alpha + g_\alpha - 1)!_q] - \ln_q(n_\alpha!_q) - \ln_q[(g_\alpha - 1)!_q]\}. \tag{12}$$

Now, by using the q-Stirling formula for large n, we arrive to the following cases:

For $q = 2$

$$\begin{aligned} \ln_q W_q &\simeq \sum_\alpha [(n_\alpha + g_\alpha) - \ln(n_\alpha + g_\alpha) - n_\alpha + \ln(n_\alpha) - g_\alpha + \ln(g_\alpha)] \\ &= \sum_\alpha [\ln(n_\alpha) + \ln(g_\alpha) - \ln(g_\alpha - n_\alpha)]. \end{aligned} \tag{13}$$

For $q > 0, q \neq 2$

$$\begin{aligned} \ln_q W_q &\simeq \sum_\alpha \left\{ \frac{(n_\alpha + g_\alpha)}{2-q} \left[\ln_q(n_\alpha + g_\alpha) - 1\right] - \frac{n_\alpha}{2-q}\left[\ln_q(n_\alpha) - 1\right] - \frac{g_\alpha}{2-q}\left[\ln_q(g_\alpha) - 1\right] \right\} \\ &= \sum_\alpha \left\{ \frac{(n_\alpha + g_\alpha)}{2-q} \left[\frac{(n_\alpha + g_\alpha)^{1-q} - 1}{1-q}\right] - \frac{n_\alpha}{2-q}\left[\frac{(n_\alpha^{1-q} - 1)}{1-q}\right] - \frac{g_\alpha}{2-q}\left[\frac{(g_\alpha^{1-q})}{1-q}\right] \right\} \\ &= \sum_\alpha \frac{1}{2-q} \left\{ (n_\alpha + g_\alpha)^{2-q} \ln_{2-q}(n_\alpha + g_\alpha) - n_\alpha^{2-q} \ln_{2-q}(n_\alpha) - g_\alpha^{2-q} \ln_{2-q}(g_\alpha) \right\}. \end{aligned} \tag{14}$$

By introducing the duality transformation proposed in Refs. [13,14], i.e, $q^* \to (2-q)$ in the expression above, we obtain

$$q^* \ln_{2-q^*} W_{2-q^*} = \sum_\alpha \left\{ (n_\alpha + g_\alpha)^{q^*} \ln_{q^*}(n_\alpha + g_\alpha) - n_\alpha^{q^*} \ln_{q^*}(n_\alpha) - g_\alpha^{q^*} \ln_{q^*}(g_\alpha) \right\}, \tag{15}$$

which is the nonadditive quantum entropy for Bose–Einstein gas.

3.3. Fermi–Dirac gas

Analogously to the previous result, for the Fermi–Dirac gas, the q-multinomial coefficient is defined as follows

$$W_q = \prod_\alpha {}_{\otimes^q} W_\alpha^{\{-,q\}} = \prod_\alpha {}_{\otimes^q} \{g_\alpha!_q \oslash^q [n_\alpha!_q \otimes^q (g_\alpha - n_\alpha)!_q]\}. \tag{16}$$

By making similar calculations, we obtain:

For $q = 2$

$$\begin{aligned}\ln_q(W_q) &\simeq \sum_\alpha [g_\alpha - \ln(g_\alpha) - n_\alpha + \ln(n_\alpha) - (g_\alpha - n_\alpha) + \ln(g_\alpha - n_\alpha)] \\ &= \sum_\alpha [\ln(n_\alpha) - \ln(g_\alpha) + \ln(g_\alpha - n_\alpha)]. \end{aligned} \tag{17}$$

For $q > 0, q \neq 2$

$$\begin{aligned}\ln_q(W_q) &\simeq \sum_\alpha \left\{ \frac{g_\alpha}{2-q}\left[\ln_q(g_\alpha)\right] - \frac{n_\alpha}{2-q}\left[\ln_q(n_\alpha)\right] - \frac{(g_\alpha - n_\alpha)}{2-q}\left[\ln_q(g_\alpha - n_\alpha)\right]\right\} \\ &= \sum_\alpha \left\{ \frac{g_\alpha}{2-q}\left[\frac{g_\alpha^{1-q}-1}{1-q}\right] - \frac{n_\alpha}{2-q}\left[\frac{n_\alpha^{1-q}-1}{1-q}\right] - \frac{(g_\alpha - n_\alpha)}{2-q}\left[\frac{(g_\alpha - n_\alpha)^{1-q}-1}{1-q}\right]\right\} \\ &= \sum_\alpha \frac{1}{2-q}\left\{-(g_\alpha - n_\alpha)^{2-q}\ln_{2-q}(g_\alpha - n_\alpha) - n_\alpha^{2-q}\ln_{2-q}(n_\alpha) + g_\alpha^{2-q}\ln_{2-q}(g_\alpha)\right\}. \end{aligned} \tag{18}$$

Again, if we consider the duality transformation, we find

$$q^* \ln_{2-q^*}(W_{2-q^*}) = \sum_\alpha \left\{-(g_\alpha - n_\alpha)^{q^*}\ln_{q^*}(g_\alpha - n_\alpha) - n_\alpha^{q^*}\ln_{q^*}(n_\alpha) + g_\alpha^{q^*}\ln_{q^*}(g_\alpha)\right\}, \tag{19}$$

which is the nonadditive quantum entropy for Fermi–Dirac gas.

Finally, the nonadditive quantum entropy $S_{q^*}^Q$, given by expressions (15) and (19), can be written in a more general form, i.e.,

$$\begin{aligned} S_{q^*}^Q &= q^* \ln_{2-q^*}\left(W_{2-q^*}^{(\pm)}\right) \\ &= -\sum_\alpha [n_\alpha^{q^*}\ln_{q^*}(n_\alpha) \mp (g_\alpha \pm n_\alpha)^{q^*}\ln_{q^*}(g_\alpha \pm n_\alpha) \pm g_\alpha^{q^*}\ln_{q^*}(g_\alpha)], \end{aligned} \tag{20}$$

where the upper sign refers to bosons and the lower one to fermions. Note that, when we take the limit $q^* \to 1$, the above expression reduces to the standard case of Eq. (6). Note also that the nonextensive quantum entropy for Bose–Einstein and Fermi–Dirac gases, arbitrarily introduced in the Refs. [11,19], are exactly equivalent to expression (20), which has been obtained through the generalized combinatorial method. In particular, the q-Stirling's formula, q-multinomial coefficients and the duality transformation provide the quantum entropy with the nonextensive parameter q^* in the interval [0, 2]. This fully corroborate the results obtained through the quantum H-theorem [11] and the second law of thermodynamics in quantum regime [20].

4. Kaniadakis quantum entropy

The derivation of the generalized quantum entropy in the context of the Kaniadakis statistics is based on the results obtained in Ref. [21]. The main expressions, i.e., the κ-Stirling formula and new κ-product are presented in the Appendix. As we will demonstrate, the generalized multinomial coefficients for the Bose–Einstein and Fermi–Dirac gas provide the Kaniadakis quantum entropy.

4.1. Bose–Einstein gas

For the Bose–Einstein gas, the so-called κ-multinomial coefficients will be defined through the "standard" and new κ-product and κ-division, i.e.:

4.1.1. The "standard" κ-product

The structure of the κ-multinomial coefficient is given by

$$W_\kappa = \prod_\alpha {}_{\otimes^\kappa} W_\alpha^{\{+,\kappa\}} = \prod_\alpha {}_{\otimes^\kappa}\{(n_\alpha + g_\alpha - 1)!_\kappa \oslash^\kappa [n_\alpha !_\kappa \otimes^\kappa (g_\alpha - 1)!_\kappa]\}, \tag{21}$$

where

$$\prod_{\alpha=1}^{l} {}_{\otimes^\kappa} x_\alpha := x_1 \otimes^\kappa x_2 \otimes^\kappa \cdots \otimes^\kappa x_l \tag{22}$$

is the κ-product.

Now, by using the Eq. (21) we obtain the expressions

$$\prod_{\alpha}{}_{\otimes^{\kappa}}\left(W_{\alpha}^{\{+,\kappa\}}\right)^{\otimes^{\kappa}[(g_{\alpha}+n_{\alpha})^{-\kappa}]}=\prod_{\alpha}{}_{\otimes^{\kappa}}\{(n_{\alpha}+g_{\alpha}-1)!_{\kappa}\oslash^{\kappa}[n_{\alpha}!_{\kappa}\otimes^{\kappa}(g_{\alpha}-1)!_{\kappa}]\}^{\otimes^{\kappa}(g_{\alpha}+n_{\alpha})^{-\kappa}}, \tag{23}$$

and

$$\prod_{\alpha}{}_{\otimes^{\kappa}}\left(W_{\alpha}^{\{+,\kappa\}}\right)^{\otimes^{\kappa}[(g_{\alpha}+n_{\alpha})^{\kappa}]}=\prod_{\alpha}{}_{\otimes^{\kappa}}\{(n_{\alpha}+g_{\alpha}-1)!_{\kappa}\oslash^{\kappa}[n_{\alpha}!_{\kappa}\otimes^{\kappa}(g_{\alpha}-1)!_{\kappa}]\}^{\otimes^{\kappa}(g_{\alpha}+n_{\alpha})^{\kappa}}. \tag{24}$$

Here, by taking the $\ln_{\kappa}$ from the expressions (23) and (24), using the κ-power property

$$x^{\otimes^{\kappa} a}:=\exp_{\kappa}[a\ln_{\kappa}(x)] \tag{25}$$

and the approximation (57) (see the Appendix), we obtain

$$\begin{aligned}\ln_{\kappa}\left[\prod_{\alpha}{}_{\otimes^{\kappa}}\left(W_{\alpha}^{\{+,\kappa\}}\right)^{\otimes^{\kappa}[(g_{\alpha}+n_{\alpha})^{-\kappa}]}\right]&=\sum_{\alpha}\ln_{\kappa}\left\{\exp_{\kappa}\left[(g_{\alpha}+n_{\alpha})^{-\kappa}\ln_{\kappa}\left((n_{\alpha}+g_{\alpha}-1)!_{\kappa}\oslash^{\kappa}(n_{\alpha}!_{\kappa}\otimes^{\kappa}(g_{\alpha}-1)!_{\kappa})\right)\right]\right\}\\&=\sum_{\alpha}(g_{\alpha}+n_{\alpha})^{-\kappa}\{\ln_{\kappa}(g_{\alpha}+n_{\alpha}-1)!_{\kappa}-\ln_{\kappa}(n_{\alpha}!_{\kappa})-\ln_{\kappa}[(g_{\alpha}-1)!_{\kappa}]\}\\&=\sum_{\alpha}(g_{\alpha}+n_{\alpha})^{-\kappa}\left[\frac{(g_{\alpha}+n_{\alpha})^{1+\kappa}}{2\kappa(1+\kappa)}-\frac{(g_{\alpha}+n_{\alpha})^{1-\kappa}}{2\kappa(1-\kappa)}-\frac{n_{\alpha}^{1+\kappa}}{2\kappa(1+\kappa)}+\frac{n_{\alpha}^{1-\kappa}}{2\kappa(1-\kappa)}\right.\\&\qquad\left.-\frac{g_{\alpha}^{1+\kappa}}{2\kappa(1+\kappa)}+\frac{g_{\alpha}^{1-\kappa}}{2\kappa(1-\kappa)}\right]\end{aligned} \tag{26}$$

and

$$\begin{aligned}\ln_{\kappa}\left[\prod_{\alpha}{}_{\otimes^{\kappa}}\left(W_{\alpha}^{\{+,\kappa\}}\right)^{\otimes^{\kappa}[(g_{\alpha}+n_{\alpha})^{\kappa}]}\right]&=\sum_{\alpha}\ln_{\kappa}\left\{\exp_{\kappa}\left[(g_{\alpha}+n_{\alpha})^{\kappa}\ln_{\kappa}\left((n_{\alpha}+g_{\alpha}-1)!_{\kappa}\oslash^{\kappa}(n_{\alpha}!_{\kappa}\otimes^{\kappa}(g_{\alpha}-1)!_{\kappa})\right)\right]\right\}\\&=\sum_{\alpha}(g_{\alpha}+n_{\alpha})^{\kappa}\left[\frac{(g_{\alpha}+n_{\alpha})^{1+\kappa}}{2\kappa(1+\kappa)}-\frac{(g_{\alpha}+n_{\alpha})^{1-\kappa}}{2\kappa(1-\kappa)}-\frac{n_{\alpha}^{1+\kappa}}{2\kappa(1+\kappa)}+\frac{n_{\alpha}^{1-\kappa}}{2\kappa(1-\kappa)}+\right.\\&\qquad\left.-\frac{g_{\alpha}^{1+\kappa}}{2\kappa(1+\kappa)}+\frac{g_{\alpha}^{1-\kappa}}{2\kappa(1-\kappa)}\right].\end{aligned} \tag{27}$$

As one may check, the Gaussian limit $\kappa\to 0$ in Eqs. (21), (23) and (24) leads to the standard coefficient presented in Eq. (3).

4.1.2. The new κ-product

The κ-multinomial coefficient is given by

$$W_{\kappa}=\prod_{\alpha}{}_{\odot_{\kappa}}W_{\alpha}^{\{+,\kappa\}}=\prod_{\alpha}{}_{\odot_{\kappa}}\{(n_{\alpha}+g_{\alpha}-1)!^{\kappa}\oslash_{\kappa}[n_{\alpha}!^{\kappa}\odot_{\kappa}(g_{\alpha}-1)!^{\kappa}]\} \tag{28}$$

where

$$\prod_{\alpha=1}^{l}{}_{\odot_{\kappa}}x_{\alpha}:=x_1\odot_{\kappa}x_2\odot_{\kappa}\cdots\odot_{\kappa}x_l \tag{29}$$

is the so-called new κ-product.

By using Eq. (28), we obtain the expressions

$$\prod_{\alpha}{}_{\odot_{\kappa}}\left(W_{\alpha}^{\{+,\kappa\}}\right)^{\odot_{\kappa}[(g_{\alpha}+n_{\alpha})^{-\kappa}]}=\prod_{\alpha}{}_{\odot_{\kappa}}\{(n_{\alpha}+g_{\alpha}-1)!^{\kappa}\oslash_{\kappa}[n_{\alpha}!^{\kappa}\odot_{\kappa}(g_{\alpha}-1)!^{\kappa}]\}^{\odot_{\kappa}(g_{\alpha}+n_{\alpha})^{-\kappa}} \tag{30}$$

and

$$\prod_{\alpha}{}_{\odot_{\kappa}}\left(W_{\alpha}^{\{+,\kappa\}}\right)^{\odot_{\kappa}[(g_{\alpha}+n_{\alpha})^{+\kappa}]}=\prod_{\alpha}{}_{\odot_{\kappa}}\{(n_{\alpha}+g_{\alpha}-1)!^{\kappa}\oslash_{\kappa}[n_{\alpha}!^{\kappa}\odot_{\kappa}(g_{\alpha}-1)!^{\kappa}]\}^{\odot_{\kappa}(g_{\alpha}+n_{\alpha})^{\kappa}}. \tag{31}$$

By taking u_{κ} in Eqs. (30) and (31), using the κ-power property

$$x^{\odot_{\kappa} a}:=u_{\kappa}^{-1}[au_{\kappa}(x)] \tag{32}$$

and the approximation (64) (see the Appendix), we obtain

$$u_\kappa\Big[\prod_\alpha \odot_\kappa \big(W_\alpha^{\{+,\kappa\}}\big)^{\odot_\kappa[(g_\alpha+n_\alpha)^{-\kappa}]}\Big] = \sum_\alpha u_\kappa \left\{u_\kappa^{-1}\Big[(g_\alpha+n_\alpha)^{-\kappa} u_\kappa\big((n_\alpha+g_\alpha-1)!^\kappa \oslash_\kappa [n_\alpha!^\kappa \odot_\kappa (g_\alpha-1)!^\kappa]\big)\Big]\right\}$$
$$= \sum_\alpha (g_\alpha+n_\alpha)^{-\kappa}\{u_\kappa[(g_\alpha+n_\alpha-1)!^\kappa] - u_\kappa(n_\alpha!^\kappa) - u_\kappa[(g_\alpha-1)]!^\kappa\}$$
$$= \sum_\alpha (g_\alpha+n_\alpha)^{-\kappa}\left[\frac{(g_\alpha+n_\alpha)^{1+\kappa}}{2(1+\kappa)} + \frac{(g_\alpha+n_\alpha)^{1-\kappa}}{2(1-\kappa)} - \frac{n_\alpha^{1+\kappa}}{2(1+\kappa)} - \frac{n_\alpha^{1-\kappa}}{2(1-\kappa)} - \frac{g_\alpha^{1+\kappa}}{2(1+\kappa)} - \frac{g_\alpha^{1-\kappa}}{2(1-\kappa)}\right] \tag{33}$$

and

$$u_\kappa\Big[\prod_\alpha \odot_\kappa \big(W_\alpha^{\{+,\kappa\}}\big)^{\odot_\kappa[(g_\alpha+n_\alpha)^{+\kappa}]}\Big] = \sum_\alpha u_\kappa \left\{u_\kappa^{-1}\Big[(g_\alpha+n_\alpha)^{\kappa} u_\kappa\big((n_\alpha+g_\alpha-1)!^\kappa \oslash_\kappa [n_\alpha!^\kappa \odot_\kappa (g_\alpha-1)!^\kappa]\big)\Big]\right\}$$
$$= \sum_\alpha (g_\alpha+n_\alpha)^{\kappa}\left[\frac{(g_\alpha+n_\alpha)^{1+\kappa}}{2(1+\kappa)} + \frac{(g_\alpha+n_\alpha)^{1-\kappa}}{2(1-\kappa)} - \frac{n_\alpha^{1+\kappa}}{2(1+\kappa)} - \frac{n_\alpha^{1-\kappa}}{2(1-\kappa)} - \frac{g_\alpha^{1+\kappa}}{2(1+\kappa)} - \frac{g_\alpha^{1-\kappa}}{2(1-\kappa)}\right]. \tag{34}$$

In the Gaussian limit $\kappa \to 0$, Eqs. (28), (30) and (31) reduce to standard coefficient (Eq. (3)).

Now, using the relations (26)–(27) and (33)–(34), we find

$$\ln_\kappa\Big[\prod_\alpha \otimes^\kappa \big(W_\alpha^{\{+,\kappa\}}\big)^{\otimes^\kappa[(g_\alpha+n_\alpha)^{-\kappa}]}\Big] + \frac{1}{\kappa} u_\kappa\Big[\prod_\alpha \odot_\kappa \big(W_\alpha^{\{+,\kappa\}}\big)^{\odot_\kappa[(g_\alpha+n_\alpha)^{-\kappa}]}\Big]$$
$$= \sum_\alpha (g_\alpha+n_\alpha)^{-\kappa}\left[\frac{(g_\alpha+n_\alpha)^{1+\kappa}}{\kappa(1+\kappa)} - \frac{n_\alpha^{1+\kappa}}{\kappa(1+\kappa)} - \frac{g_\alpha^{1+\kappa}}{\kappa(1+\kappa)}\right]$$
$$= \frac{1}{\kappa(\kappa+1)}\sum_\alpha \left[(g_\alpha+n_\alpha) - g_\alpha\left(\frac{g_\alpha}{g_\alpha+n_\alpha}\right)^\kappa - n_\alpha\left(\frac{n_\alpha}{g_\alpha+n_\alpha}\right)^\kappa\right] \tag{35}$$

and

$$\ln_\kappa\Big[\prod_\alpha \otimes^\kappa \big(W_\alpha^{\{+,\kappa\}}\big)^{\otimes^\kappa[(g_\alpha+n_\alpha)^{+\kappa}]}\Big] - \frac{1}{\kappa} u_\kappa\Big[\prod_\alpha \odot_\kappa \big(W_\alpha^{\{+,\kappa\}}\big)^{\odot_\kappa[(g_\alpha+n_\alpha)^{+\kappa}]}\Big]$$
$$= \sum_\alpha (g_\alpha+n_\alpha)^{\kappa}\left[-\frac{(g_\alpha+n_\alpha)^{1-\kappa}}{\kappa(1-\kappa)} + \frac{n_\alpha^{1-\kappa}}{\kappa(1-\kappa)} + \frac{g_\alpha^{1-\kappa}}{\kappa(1-\kappa)}\right]$$
$$= \frac{1}{\kappa(1-\kappa)}\sum_\alpha \left[-(g_\alpha+n_\alpha) + g_\alpha\left(\frac{g_\alpha}{g_\alpha+n_\alpha}\right)^{-\kappa} + n_\alpha\left(\frac{n_\alpha}{g_\alpha+n_\alpha}\right)^{-\kappa}\right]. \tag{36}$$

From the above equations, it is possible to obtain the κ-quantum entropy for a Boson–Einstein gas, i.e.,

$$S_\kappa^Q = \frac{(1+\kappa)}{2}\left\{\ln_\kappa\Big[\prod_\alpha \otimes^\kappa \big(W_\alpha^{\{+,\kappa\}}\big)^{\otimes^\kappa[(g_\alpha+n_\alpha)^{-\kappa}]}\Big] + \frac{1}{\kappa} u_\kappa\Big[\prod_\alpha \odot_\kappa \big(W_\alpha^{\{+,\kappa\}}\big)^{\odot_\kappa[(g_\alpha+n_\alpha)^{-\kappa}]}\Big]\right\}$$
$$+ \frac{(1-\kappa)}{2}\left\{\ln_\kappa\Big[\prod_\alpha \otimes^\kappa \big(W_\alpha^{\{+,\kappa\}}\big)^{\otimes^\kappa[(g_\alpha+n_\alpha)^{\kappa}]}\Big] - \frac{1}{\kappa} u_\kappa\Big[\prod_\alpha \odot_\kappa \big(W_\alpha^{\{+,\kappa\}}\big)^{\odot_\kappa[(g_\alpha+n_\alpha)^{\kappa}]}\Big]\right\} \tag{37}$$

or, equivalently,

$$S_\kappa^Q = -\sum_\alpha \left[n_\alpha \ln_\kappa\left(\frac{n_\alpha}{g_\alpha+n_\alpha}\right) + g_\alpha \ln_\kappa\left(\frac{g_\alpha}{g_\alpha+n_\alpha}\right)\right]. \tag{38}$$

4.2. Fermi–Dirac gas

Similarly to the previous results, we have:

4.2.1. The "standard" κ-product

In this case, the κ-multinomial coefficient is given by

$$W_\kappa = \prod_\alpha {}_{\otimes^\kappa} W_\alpha^{\{-,\kappa\}} = \prod_\alpha {}_{\otimes^\kappa} \{g_\alpha!_\kappa \oslash^\kappa [n_\alpha!_\kappa \otimes^\kappa (g_\alpha - n_\alpha)!_\kappa]\} \tag{39}$$

where the product was defined in Eq. (22).

The above expression leads to

$$\prod_\alpha {}_{\otimes^\kappa} \left(W_\alpha^{\{-,\kappa\}}\right)^{\otimes^\kappa [(g_\alpha+n_\alpha)^{-\kappa}]} = \prod_\alpha {}_{\otimes^\kappa} \{g_\alpha!_\kappa \oslash^\kappa [n_\alpha!_\kappa \otimes^\kappa (g_\alpha - n_\alpha)!_\kappa]\}^{\otimes^\kappa (g_\alpha - n_\alpha)^{-\kappa}} \tag{40}$$

and

$$\prod_\alpha {}_{\otimes^\kappa} \left(W_\alpha^{\{-,\kappa\}}\right)^{\otimes^\kappa [(g_\alpha+n_\alpha)^{+\kappa}]} = \prod_\alpha {}_{\otimes^\kappa} \{g_\alpha!_\kappa \oslash^\kappa [n_\alpha!_\kappa \otimes^\kappa (g_\alpha - n_\alpha)!_\kappa]\}^{\otimes^\kappa (g_\alpha - n_\alpha)^{\kappa}}. \tag{41}$$

Again, by taking $\ln_\kappa$ in the expressions (40) and (41), using (25) and the approximation (57), is possible to show that

$$\begin{aligned}
\ln_\kappa \left[\prod_\alpha {}_{\otimes^\kappa} \left(W_\alpha^{\{-,\kappa\}}\right)^{\otimes^\kappa [(g_\alpha+n_\alpha)^{-\kappa}]}\right] &= \sum_\alpha \ln_\kappa \left\{\exp_\kappa\left[(g_\alpha + n_\alpha)^{-\kappa} \ln_\kappa \left(g_\alpha!_\kappa \oslash^\kappa [n_\alpha!_\kappa \otimes^\kappa (g_\alpha - n_\alpha)!_\kappa]\right)\right]\right\} \\
&= \sum_\alpha (g_\alpha - n_\alpha)^{-\kappa} \{\ln_\kappa (g_\alpha!_\kappa) - \ln_\kappa (n_\alpha!_\kappa) - \ln_\kappa [(g_\alpha - n_\alpha)!_\kappa]\} \\
&= \sum_\alpha (g_\alpha - n_\alpha)^{-\kappa} \left[\frac{g_\alpha^{1+\kappa}}{2\kappa(1+\kappa)} - \frac{g_\alpha^{1-\kappa}}{2\kappa(1-\kappa)} - \frac{n_\alpha^{1+\kappa}}{2\kappa(1+\kappa)} + \frac{n_\alpha^{1-\kappa}}{2\kappa(1-\kappa)} \right. \\
&\quad \left. - \frac{(g_\alpha - n_\alpha)^{1+\kappa}}{2\kappa(1+\kappa)} + \frac{(g_\alpha - n_\alpha)^{1-\kappa}}{2\kappa(1-\kappa)}\right]
\end{aligned} \tag{42}$$

and

$$\begin{aligned}
\ln_\kappa \left[\prod_\alpha {}_{\otimes^\kappa} \left(W_\alpha^{\{-,\kappa\}}\right)^{\otimes^\kappa [(g_\alpha+n_\alpha)^{+\kappa}]}\right] &= \sum_\alpha \ln_\kappa \left\{\exp_\kappa\left[(g_\alpha + n_\alpha)^{\kappa} \ln_\kappa \left(g_\alpha!_\kappa \oslash^\kappa [n_\alpha!_\kappa \otimes^\kappa (g_\alpha - n_\alpha)!_\kappa]\right)\right]\right\} \\
&= \sum_\alpha (g_\alpha - n_\alpha)^{\kappa} \left[\frac{g_\alpha^{1+\kappa}}{2\kappa(1+\kappa)} - \frac{g_\alpha^{1-\kappa}}{2\kappa(1-\kappa)} - \frac{n_\alpha^{1+\kappa}}{2\kappa(1+\kappa)} + \frac{n_\alpha^{1-\kappa}}{2\kappa(1-\kappa)} \right. \\
&\quad \left. - \frac{(g_\alpha - n_\alpha)^{1+\kappa}}{2\kappa(1+\kappa)} + \frac{(g_\alpha - n_\alpha)^{1-\kappa}}{2\kappa(1-\kappa)}\right],
\end{aligned} \tag{43}$$

whose the Gaussian limit $\kappa \to 0$ furnishes the standard coefficient given in Eq. (3).

4.2.2. The new κ-product

Now, the κ-multinomial coefficient is given by

$$W_\kappa = \prod_\alpha {}_{\odot_\kappa} W_\alpha^{\{-,\kappa\}} = \prod_\alpha {}_{\odot_\kappa} \{g_\alpha!^\kappa \oslash_\kappa [n_\alpha!^\kappa \odot_\kappa (g_\alpha - n_\alpha)!^\kappa]\} \tag{44}$$

and the above product was defined in (29).

The expression above leads to

$$\prod_\alpha {}_{\odot_\kappa} \left(W_\alpha^{\{-,\kappa\}}\right)^{\odot_\kappa [(g_\alpha-n_\alpha)^{-\kappa}]} = \prod_\alpha {}_{\odot_\kappa} \{g_\alpha!^\kappa \oslash_\kappa [n_\alpha!^\kappa \odot_\kappa (g_\alpha - n_\alpha)!^\kappa]\}^{\odot_\kappa (g_\alpha - n_\alpha)^{-\kappa}} \tag{45}$$

and

$$\prod_\alpha {}_{\odot_\kappa} \left(W_\alpha^{\{-,\kappa\}}\right)^{\odot_\kappa [(g_\alpha-n_\alpha)^{\kappa}]} = \prod_\alpha {}_{\odot_\kappa} \{g_\alpha!^\kappa \oslash_\kappa [n_\alpha!^\kappa \odot_\kappa (g_\alpha - n_\alpha)!^\kappa]\}^{\odot_\kappa (g_\alpha - n_\alpha)^{\kappa}}. \tag{46}$$

Again, taking u_κ in (45)–(46) and considering Eqs. (32) and (64), we obtain

$$u_\kappa \left[\prod_\alpha {}_{\odot_\kappa} \left(W_\alpha^{\{-,\kappa\}}\right)^{\odot_\kappa [(g_\alpha+n_\alpha)^{-\kappa}]}\right] = \sum_\alpha u_\kappa \left\{u_\kappa^{-1}\left[(g_\alpha - n_\alpha)^{-\kappa} u_\kappa \left(g_\alpha!^\kappa \oslash_\kappa [n_\alpha!^\kappa \odot_\kappa (g_\alpha - n_\alpha)!^\kappa]\right)\right]\right\}$$

$$= \sum_\alpha (g_\alpha - n_\alpha)^{-\kappa} \{u_\kappa(g_\alpha!^\kappa) - u_\kappa(n_\alpha!^\kappa) - u_\kappa[(g_\alpha - n_\alpha)!^\kappa]\}$$

$$= \sum_\alpha (g_\alpha - n_\alpha)^{-\kappa} \left[\frac{g_\alpha^{1+\kappa}}{2(1+\kappa)} + \frac{g_\alpha^{1-\kappa}}{2(1-\kappa)} - \frac{n_\alpha^{1+\kappa}}{2(1+\kappa)} - \frac{n_\alpha^{1-\kappa}}{2(1-\kappa)} - \frac{(g_\alpha - n_\alpha)^{1+\kappa}}{2(1+\kappa)} - \frac{(g_\alpha - n_\alpha)^{1-\kappa}}{2(1-\kappa)}\right] \tag{47}$$

and

$$u_\kappa\left[\prod_\alpha \odot_\kappa \left(W_\alpha^{\{-,\kappa\}}\right)^{\odot_\kappa[(g_\alpha+n_\alpha)^\kappa]}\right] = \sum_\alpha u_\kappa \left\{u_\kappa^{-1}\left[(g_\alpha - n_\alpha)^\kappa u_\kappa\left(g_\alpha!^\kappa \oslash_\kappa [n_\alpha!^\kappa \odot_\kappa (g_\alpha - n_\alpha)!^\kappa]\right)\right]\right\}$$

$$= \sum_\alpha (g_\alpha - n_\alpha)^{\kappa} \left[\frac{g_\alpha^{1+\kappa}}{2(1+\kappa)} + \frac{g_\alpha^{1-\kappa}}{2(1-\kappa)} - \frac{n_\alpha^{1+\kappa}}{2(1+\kappa)} - \frac{n_\alpha^{1-\kappa}}{2(1-\kappa)} - \frac{(g_\alpha - n_\alpha)^{1+\kappa}}{2(1+\kappa)} - \frac{(g_\alpha - n_\alpha)^{1-\kappa}}{2(1-\kappa)}\right]. \tag{48}$$

Whose the Gaussian limit $\kappa \to 0$ furnishes the coefficient given in Eq. (3). Now, by using (47)–(48), we obtain

$$\ln_\kappa\left[\prod_\alpha \otimes^\kappa \left(W_\alpha^{\{-,\kappa\}}\right)^{\otimes^\kappa[(g_\alpha+n_\alpha)^{-\kappa}]}\right] + \frac{1}{\kappa} u_\kappa\left[\prod_\alpha \odot_\kappa \left(W_\alpha^{\{-,\kappa\}}\right)^{\odot_\kappa[(g_\alpha+n_\alpha)^{-\kappa}]}\right]$$

$$= \sum_\alpha (g_\alpha - n_\alpha)^{-\kappa}\left[\frac{g_\alpha^{1+\kappa}}{\kappa(1+\kappa)} - \frac{n_\alpha^{1+\kappa}}{\kappa(1+\kappa)} - \frac{(g_\alpha - n_\alpha)^{1+\kappa}}{\kappa(1+\kappa)}\right]$$

$$= \frac{1}{\kappa(\kappa+1)} \sum_\alpha \left[g_\alpha\left(\frac{g_\alpha}{g_\alpha - n_\alpha}\right)^\kappa - n_\alpha\left(\frac{n_\alpha}{g_\alpha - n_\alpha}\right)^\kappa - (g_\alpha - n_\alpha)\right] \tag{49}$$

and

$$\ln_\kappa\left[\prod_\alpha \otimes^\kappa \left(W_\alpha^{\{-,\kappa\}}\right)^{\otimes^\kappa[(g_\alpha+n_\alpha)^{\kappa}]}\right] - \frac{1}{\kappa} u_\kappa\left[\prod_\alpha \odot_\kappa \left(W_\alpha^{\{-,\kappa\}}\right)^{\odot_\kappa[(g_\alpha+n_\alpha)^{\kappa}]}\right]$$

$$= \sum_\alpha (g_\alpha - n_\alpha)^{\kappa}\left[-\frac{g_\alpha^{1-\kappa}}{\kappa(1-\kappa)} + \frac{n_\alpha^{1-\kappa}}{\kappa(1-\kappa)} + \frac{(g_\alpha - n_\alpha)^{1-\kappa}}{\kappa(1-\kappa)}\right]$$

$$= \frac{1}{\kappa(\kappa+1)} \sum_\alpha \left[-g_\alpha\left(\frac{g_\alpha}{g_\alpha - n_\alpha}\right)^\kappa + n_\alpha\left(\frac{n_\alpha}{g_\alpha - n_\alpha}\right)^\kappa + (g_\alpha - n_\alpha)\right]. \tag{50}$$

By combining the above expressions, the κ-quantum entropy for Fermi–Dirac gas can be written as

$$S_\kappa^Q = \frac{(1+\kappa)}{2}\left\{\ln_\kappa\left[\prod_\alpha \otimes^\kappa \left(W_\alpha^{\{-,\kappa\}}\right)^{\otimes^\kappa[(g_\alpha+n_\alpha)^{-\kappa}]}\right] + \frac{1}{\kappa} u_\kappa\left[\prod_\alpha \odot_\kappa \left(W_\alpha^{\{-,\kappa\}}\right)^{\odot_\kappa[(g_\alpha+n_\alpha)^{-\kappa}]}\right]\right\}$$
$$+ \frac{(1-\kappa)}{2}\left\{\ln_\kappa\left[\prod_\alpha \otimes^\kappa \left(W_\alpha^{\{-,\kappa\}}\right)^{\otimes^\kappa[(g_\alpha+n_\alpha)^{\kappa}]}\right] - \frac{1}{\kappa} u_\kappa\left[\prod_\alpha \odot_\kappa \left(W_\alpha^{\{-,\kappa\}}\right)^{\odot_\kappa[(g_\alpha+n_\alpha)^{\kappa}]}\right]\right\} \tag{51}$$

or, equivalently,

$$S_\kappa^Q = -\sum_\alpha \left[n_\alpha \ln_\kappa\left(\frac{n_\alpha}{g_\alpha - n_\alpha}\right) - g_\alpha \ln_\kappa\left(\frac{g_\alpha}{g_\alpha - n_\alpha}\right)\right]. \tag{52}$$

Finally, the κ-quantum entropy S_κ^Q given by expressions (38) and (52) can write in a general form, i.e.,

$$S_\kappa^Q = -\sum_\alpha \left[n_\alpha \ln_\kappa\left(\frac{n_\alpha}{g_\alpha \pm n_\alpha}\right) \pm g_\alpha \ln_\kappa\left(\frac{g_\alpha}{g_\alpha \pm n_\alpha}\right)\right], \tag{53}$$

where the upper and lower signs refer to bosons and fermions, respectively. In the limit $\kappa \to 0$, the above expression reduces to the quantum entropy showed in Eq. (6). Note also that the κ-quantum entropy assumed in Ref. [12] to derive the quantum H-theorem in Kaniadakis framework is the same expression above which was derived through the generalized combinatorial method.

5. Final remarks

In several recent analyses, an expression to the Tsallis and Kaniadakis quantum entropies were arbitrarily assumed [19,11,12]. In this paper, we have not only shown that these assumptions are valid but also have provided a consistent mathematical derivation of S_q^Q and S_κ^Q using the generalized combinatorial method which is based on the generalization of the factorial and Stirling formulas [15,16,21]. In the Tsallis framework, the calculation follows the so-called q-algebra [17,18], and the generalization of Stirling formula is valid for $q > 0$ and $q \neq 2$. In the Kaniadakis derivation, we have considered the κ-generalization of Stirling approximation and a new κ-product of Ref. [21].

Finally, it should be emphasized that for the Tsallis case, the combination of the q-Stirling's formula, q-multinomial coefficients and duality transformation [13,14] has constrained the nonextensive parameter to interval of validity $q \in [0, 2]$, which is fully consistent with the results of Refs. [11,20,22] and also with the bounds obtained from several independent studies involving the Tsallis nonextensive framework [23].

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Appendix. κ-generalization of Stirling approximation

In Section 4, we have calculated the quantum entropy in the context of Kaniadakis framework. Based on the results of Ref. [21], the definition of the standard and new κ-products and of the so-called Stirling approximation are presented below. The functions $\exp_\kappa$ and $\ln_\kappa$ of the variables x and y are defined in (2b).

A.1. The "standard" κ-product

The standard κ-product is defined by Kaniadakis [4]

$$x \otimes^\kappa y \equiv \exp_\kappa[\ln_{\{\kappa\}}(x) + \ln_{\{\kappa\}}(y)]$$
$$= \left[\left(\frac{x^\kappa - x^{-\kappa}}{2}\right) + \left(\frac{y^\kappa - y^{-\kappa}}{2}\right) + \sqrt{1 + \left[\left(\frac{x^\kappa - x^{-\kappa}}{2}\right) + \left(\frac{y^\kappa - y^{-\kappa}}{2}\right)\right]^2}\,\right]^{\frac{1}{\kappa}}, \tag{54}$$

and κ-division by

$$x \oslash^\kappa y \equiv \exp_\kappa[\ln_{\{\kappa\}}(x) - \ln_{\{\kappa\}}(y)]$$
$$= \left[\left(\frac{x^\kappa - x^{-\kappa}}{2}\right) - \left(\frac{y^\kappa - y^{-\kappa}}{2}\right) + \sqrt{1 + \left[\left(\frac{x^\kappa - x^{-\kappa}}{2}\right) - \left(\frac{y^\kappa - y^{-\kappa}}{2}\right)\right]^2}\,\right]^{\frac{1}{\kappa}}. \tag{55}$$

By using this κ-product, the κ-factorial $n!_\kappa$ with $n \in N$ is given by

$$n!_\kappa \equiv 1 \otimes^\kappa 2 \otimes^\kappa \cdots \otimes^\kappa n$$
$$= \left[\sum_{\alpha}^{n}\left(\frac{\alpha^\kappa - \alpha^{-\kappa}}{2}\right) + \sqrt{\left[\sum_{\alpha=1}^{n}\left(\frac{\alpha^\kappa - \alpha^{-\kappa}}{2}\right)\right]^2 + 1}\,\right]^{\frac{1}{\kappa}} = \exp_\kappa\left[\sum_{\alpha}^{n} \ln_{\{\kappa\}}(\alpha)\right]. \tag{56}$$

In this approach, the Stirling approximation for large n is well approximated by the integral, i.e.,

$$\ln_\kappa(n!_\kappa) = \sum_{\alpha=1}^{n} \ln_\kappa(\alpha) \approx \int_0^n \mathrm{d}x \ln_\kappa(x) = \frac{n^{1+\kappa}}{2\kappa(1+\kappa)} - \frac{n^{1-\kappa}}{2\kappa(1-\kappa)}. \tag{57}$$

A.2. The new κ-product

Here, we introduce another κ-generalization of the products based on the following function

$$u_\kappa(x) \equiv \frac{x^\kappa + x^{-\kappa}}{2} = \cosh[\kappa \ln(x)]. \tag{58}$$

The basic properties of $u_\kappa(x)$ are given by

$$\lim_{\kappa\to 0} u_\kappa(x) = 1, \quad u_\kappa(x) = u_\kappa\left(\frac{1}{x}\right) u_\kappa(x) \geq 1, \tag{59}$$

and its inverse function is defined as

$$u_\kappa^{-1}(x) \equiv [\sqrt{x^2-1}+x]^{\frac{1}{\kappa}}, \quad (x \geq 1). \tag{60}$$

Now, by using the above definitions, the new κ-product and the new κ-division can be written as

$$\begin{aligned} x \odot_\kappa y &\equiv u_\kappa^{-1}[u_\kappa(x)+u_\kappa(y)] \\ &= \left[\left(\frac{x^\kappa+x^{-\kappa}}{2}\right)+\left(\frac{y^\kappa+y^{-\kappa}}{2}\right)+\sqrt{\left[\left(\frac{x^\kappa+x^{-\kappa}}{2}\right)+\left(\frac{y^\kappa+y^{-\kappa}}{2}\right)\right]^2-1}\right]^{\frac{1}{\kappa}} \end{aligned} \tag{61}$$

and

$$\begin{aligned} x \oslash_\kappa y &\equiv u_\kappa^{-1}[u_\kappa(x)-u_\kappa(y)] \\ &= \left[\left(\frac{x^\kappa+x^{-\kappa}}{2}\right)+\left(\frac{y^\kappa+y^{-\kappa}}{2}\right)+\sqrt{\left[\left(\frac{x^\kappa+x^{-\kappa}}{2}\right)-\left(\frac{y^\kappa+y^{-\kappa}}{2}\right)\right]^2-1}\right]^{\frac{1}{\kappa}} \end{aligned} \tag{62}$$

where $\left(\frac{x^\kappa+x^{-\kappa}}{2}\right)-\left(\frac{y^\kappa+y^{-\kappa}}{2}\right) \geq 1$.

Using the new κ-product, the associated κ-factorial can be introduced as

$$\begin{aligned} n!^\kappa &\equiv 1 \odot_\kappa 2 \odot_\kappa \cdots \odot_\kappa n \\ &= \left[\sum_\alpha^n \left(\frac{\alpha^\kappa+\alpha^{-\kappa}}{2}\right)+\sqrt{\left[\sum_{\alpha=1}^n \left(\frac{\alpha^\kappa+\alpha^{-\kappa}}{2}\right)\right]^2-1}\right]^{\frac{1}{\kappa}}. \end{aligned} \tag{63}$$

Finally, the κ-Stirling approximation can be written as

$$u_\kappa(n_\alpha!^\kappa) = \sum_{\alpha=1}^n u_\kappa(\alpha) \approx \int_0^n \mathrm{d}x u_\kappa(x) = \frac{n_\alpha^{1+\kappa}}{2(1+\kappa)} + \frac{n_\alpha^{1-\kappa}}{2(1-\kappa)}. \tag{64}$$

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