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MSSM-like from $SU_5 \times D_4$ models

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Abstract

Using finite discrete group characters and symmetry breaking by hyperflux as well as constraints on top-quark family, we study minimal low energy effective theory following from $SU_5 \times D_4$ models embedded in F-theory with non-abelian flux. Matter curves spectrum of the models is obtained from $SU_5 \times S_5$ theory with monodromy S_5 by performing two breakings: first from symmetric group S_5 to S_4 subsymmetry, and next to dihedral D_4 subgroup. As a consequence, and depending on the ways of decomposing triplets of S_4, we end with three types of D_4-models. Explicit constructions of these theories are given and a MSSM-like spectrum is derived.

1. Introduction

Recently, there has been an increasing interest in building $SU_5 \times \Gamma$ GUT models, with discrete symmetries Γ, embedded in Calabi–Yau compactification of F-theory down to 4D space time [1–11]; and in looking for low energy minimal prototypes with broken monodromies [12–19]. This class of supersymmetric GUTs with discrete groups leads to quasi-realistic field spectrum having quark and lepton mass matrices with properties fitting with MSSM requirements. In the geometric engineering of these F-GUTs, splitting spectral cover method together with Galois

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theory tools is used to generate appropriate matter curves spectrum [20–25]; and a *geometric* Z_2 parity has been also introduced to suppress unwanted effects such as exotic couplings and undesired proton decay operators [26–29].

In this paper, we develop another manner to deal with monodromy of F-GUT that is different from the one proposed first in [18], and further explored in [27,30,31], where matter curves of the same orbit of monodromy are identified. In our approach, we use the non-abelian flux conjecture of [15,16] to think of monodromy group of F-theory SU_5 models as a non-abelian flavor symmetry Γ. Non-trivial irreducible representations of the non-abelian discrete group Γ are used to host the three generations of fundamental matter; a feature that opens a window to build semi-realistic models with matter curves distinguished from each other in accord with mass hierarchy and mixing neutrino physics [32–34].

In this work, we study the family of supersymmetric $SU_5 \times \Gamma_p \times U(1)^{5-p}$ models in the framework of F-theory GUT; with *non-abelian* monodromies Γ_p contained in the permutation group $\mathbb{S}_5$ [30–42]; and analyse the realisation of low energy constraints under which one can generate an effective field spectrum that resembles to MSSM. A list of main constraints leading to a good low energy spectrum is described in section 5; it requires amongst others a tree-level Yukawa coupling for top-quark family. To realise this condition with non-abelian Γ_p, we consider the case where Γ_p is given by the order *8* dihedral group $\mathbb{D}_4$; this particular non-abelian discrete symmetry has representations which allow more flexibility in accommodating matter generations. Recall that the non-abelian alternating $\mathbb{A}_4$ group has no irreducible doublet as shown in the character relation $12 = 3^2 + 1^2 + 1^2 + 1^2$; and the irreducible representation of non-abelian $\mathbb{S}_4$ and $\mathbb{S}_3$, which can be respectively read from $24 = 3^2 + 3^2 + 2^2 + 1^2 + 1^2$ and $6 = 2^2 + 1^2 + 1^2$, has a doublet and two singlets. The non-abelian dihedral group $\mathbb{D}_4$ however has representations $\boldsymbol{R}_i$ with dimensions, that can be read from $8 = 2^2 + 1^2 + 1^2 + 1^2 + 1^2$, seemingly more attractable phenomenologically; it has *5* irreducible $\boldsymbol{R}_i$'s, four singlets, indexed by their basis characters as $\mathbf{1}_{++}$, $\mathbf{1}_{+-}$, $\mathbf{1}_{-+}$, $\mathbf{1}_{--}$, and an irreducible doublet $\mathbf{2}_{00}$, offering therefore several pictures to accommodate the three generations of matter of the electroweak theory; in particular more freedom in accommodating top quark family.

To deal with the engineering of $SU_5 \times \mathbb{D}_4$-models, we develop a *new method* based on finite discrete group characters $\chi_{\boldsymbol{R}_i}$, avoiding as a consequence the complexity of Galois theory approach. The latter is useful to study F-theory models with the dihedral $\mathbb{D}_4$ and the alternating $\mathbb{A}_4$ subgroups of $\mathbb{S}_4$ as they are not directly reached by the standard splitting spectral cover method; they are obtained in Galois theory by putting constraints on the discriminant of underlying spectral covers, and introducing other monodromy invariant of the covers such a resolvent [14,15,29].

To derive the $\mathbb{D}_4$-matter curves spectrum in $SU_5 \times \mathbb{D}_4$-models, we think of it in terms of a two steps descent from $\mathbb{S}_5$-theory: a first descent down to $\mathbb{S}_4$, and a second one to $\mathbb{D}_4$ by turning on appropriate flux that will be explicitly described in this work, see also appendix C. By studying all scenarios of breaking the triplets $\mathbb{S}_4$-theory in terms of irreducible $\mathbb{D}_4$-representations, we end with three kinds of $\mathbb{D}_4$-models: one having a field spectrum involving all $\mathbb{D}_4$-representations including doublet $\mathbf{2}_{00}$ (*model I*), the second theory (*model II*) has no doublet $\mathbf{2}_{00}$ nor the singlet $\mathbf{1}_{--}$, and the third model has no $\mathbf{2}_{00}$, but does have $\mathbf{1}_{--}$. We have studied the curves spectrum of the three $\mathbb{D}_4$-models; and we have found that only model III allows a tree level 3-couplings and exhibits phenomenologically interesting features.

The presentation is as follows: In section 2, we study the $SU_5 \times \mathbb{S}_5$ model, and describe the picture of the two steps breaking $\mathbb{S}_5 \to \mathbb{S}_4 \to \mathbb{S}_3$ by using standard methods. In section 3, we introduce our method; and we revisit the construction of the $\mathbb{S}_4$- and $\mathbb{S}_3$-models from the view of discrete group characters. In section 4, we use character group method to build three

$SU_5 \times \mathbb{D}_4 \times U_1^{\perp}$ models. In section 5, we solve basic conditions for deriving MSSM-like spectrum from $SU_5 \times \mathbb{D}_4 \times U_1^{\perp}$ models. In section 6, we conclude and make discussions. Last section is devoted to three appendices: In appendix A, we give relations regarding group characters. In appendix B, we report details on other results obtained in this study; and in appendix C we exhibit the link between non-abelian monodromies and flavor symmetry.

2. Spectral covers in $SU_5 \times \Gamma$ models

In F-GUT models with SU_5 gauge symmetry, matter curves carry quantum numbers in $SU_5 \times SU_5^{\perp}$ bi-representations following from the breaking of E_8 as given below

$$\begin{aligned} \mathbf{248} \quad \rightarrow \quad & (\mathbf{24}, \mathbf{1}_{\perp}) \oplus (\mathbf{1}, \mathbf{24}_{\perp}) \oplus \\ & (\mathbf{10}, \mathbf{5}_{\perp}) \oplus \left(\overline{\mathbf{10}}, \bar{\mathbf{5}}_{\perp}\right) \oplus \\ & \left(\bar{\mathbf{5}}, \mathbf{10}_{\perp}\right) \oplus \left(\mathbf{5}, \overline{\mathbf{10}}_{\perp}\right) \end{aligned} \tag{2.1}$$

In this SU_5 theory, the perpendicular $SU_5^{\perp}$ is restricted to its Cartan–Weyl subsymmetry $\left(U_1^{\perp}\right)^4$, see appendix C for some explicit details; and the matter content of the model is labelled by five weights t_i like

$$\mathbf{10}_{t_i}, \quad \overline{\mathbf{10}}_{-t_i}, \quad \bar{\mathbf{5}}_{t_i+t_j}, \quad \mathbf{5}_{-t_i-t_j}, \quad \mathbf{1}_{t_i-t_j} \tag{2.2}$$

with traceless condition

$$t_1 + t_2 + t_3 + t_4 + t_5 = 0 \tag{2.3}$$

The components of the five 10-plets $\mathbf{10}_{t_i}$ and those of the ten 5-plets $\bar{\mathbf{5}}_{t_i+t_j}$ are related to each other by monodromy symmetries Γ; offering a framework of approaching GUT-models with discrete symmetries originating from geometric properties of the elliptic Calabi–Yau fourfold $CY4$ which, naively, can be thought of as given by the 4-dim complex space

$$CY4 \sim E \times \mathcal{B}_3 \tag{2.4}$$

In this fibration, the complex 3-dim base $\mathcal{B}_3$ contains the complex GUT surface $\mathcal{S}_{GUT}$ wrapped by 7-brane; and the complex elliptic curve E fiber is as follows

$$y^2 = x^3 + b_5 xy + b_4 x^2 z + b_3 y z^2 + b_2 x z^3 + b_0 z^5 \tag{2.5}$$

where the homology classes $[x]$, $[y]$, $[z]$ and $[b_k]$; associated with the holomorphic sections x, y, z and b_k, are expressed in terms of the Chern class $c_1 = c_1(\mathcal{S}_{GUT})$ of the tangent bundle of the $\mathcal{S}_{GUT}$ surface; and the Chern class $-t$ of the normal bundle $\mathcal{N}_{\mathcal{S}_{GUT}|\mathcal{B}_3}$ like

$$\begin{aligned} \left[y\right] &= 3\left(c_1 - t\right), \qquad & [z] &= -t \\ [x] &= 2\left(c_1 - t\right), \qquad & [b_k] &= (6c_1 - t) - kc_1 \end{aligned} \tag{2.6}$$

2.1. Matter curves in $SU_5 \times \mathbb{S}_5$ model

Matter curves of $SU_5 \times U(1)^{5-k} \times \Gamma_k$ models live on GUT surface $\mathcal{S}_{GUT}$ with monodromy symmetries Γ_k contained in $\mathbb{S}_5$, the Weyl group of $SU_5^{\perp}$, see eq. (C.8) of appendix C. In the case of $\Gamma_5 = \mathbb{S}_5$; these curves organise into reducible multiplets[1] of $\mathbb{S}_5$ with the following characteristic properties

Matters curves	Weights	$\mathbb{S}_5$ repres	Homology classes	Holomorphic sections
10_{t_i}	t_i	5	$\eta - 5c_1$	$b_5 = b_0 \prod_{i=1}^{5} t_i$
$\bar{5}_{t_i+t_j}$	$t_i + t_j$	10	$\eta' - 10c_1$	$d_{10} = d_0 \prod_{j>i=1}^{5} T_{ij}$
$1_{t_i-t_j}$	$t_i - t_j$	20	$\eta'' - 20c_1$	$g_{20} = g_0 \prod_{i\neq j=1}^{5} S_{ij}$

(2.7)

where the t_is as above, $T_{ij} = t_i + t_j$ with $i < j$, and $S_{ij} = t_i - t_j$ with $i \neq j$. These t_is, T_{ij}s, and S_{ij}s are respectively interpreted as the simple zeros of the spectral covers $\mathcal{C}_5 = 0$ describing ten-plets, $\mathcal{C}_{10} = 0$ describing five-pelts and $\mathcal{C}_{20} = 0$ for flavon singlets [45–50]

$$\begin{aligned}
\mathcal{C}_5 &= b_0 \prod_{i=1}^{5} (s - t_i) \equiv b_0 \prod_{i=1}^{5} s_i \\
\mathcal{C}_{10} &= d_0 \prod_{j>i=1}^{5} \left(s - T_{ij}\right) \equiv d_0 \prod_{j>i=1}^{5} s_{ij} \\
\mathcal{C}_{20} &= g_0 \prod_{i\neq j}^{5} \left(s - S_{ij}\right) \equiv g_0 \prod_{i\neq j}^{5} s'_{ij}
\end{aligned} \tag{2.8}$$

The homology classes of the complex curves in (2.7) are nicely obtained by defining the spectral covers in terms of the usual holomorphic sections; for the 5-sheeted covering of $\mathcal{S}_{GUT}$, we have

$$\mathcal{C}_5 = b_0 s^5 + b_1 s^4 + b_2 s^3 + b_3 s^2 + b_4 s + b_5 = 0 \tag{2.9}$$

with $b_1 = 0$ due to traceless condition; and homology classes of the complex holomorphic sections b_k as follows

Holomorphic sections	Homology classes
s	$-c_1$
b_k	$\eta - kc_1$

(2.10)

with canonical homology class η given by

$$\eta = 6c_1 - t \tag{2.11}$$

with c_1 and $-t$ as in eqs. (2.6). From these relations, the homology class $\left[10_{t_i}\right] = \left[\mathcal{C}_5|_{s=0}\right]$ is given by $[b_5]$; by using $b_5 = b_0 \prod_{i=1}^{5} t_i$, we have $[b_5] = \eta - 5c_1$ in agreement with (2.6). For the

[1] An equivalent spectrum can be also given by using irreducible representations of $\mathbb{S}_5$ and their characters; to fix ideas see the analogous $\mathbb{S}_4$- and $\mathbb{S}_3$-models studied in section 3.

10-sheeted covering, we have

$$\mathcal{C}_{10} = \sum_{k=0}^{10} d_k s^{10-k} \tag{2.12}$$

and leads to the homology class $[d_{10}] = \eta' - 10c_1$ where, due to $d_0 = b_0^3$, the class η' can be related to the canonical η of the 5-sheeted cover like 3η. Similar relation can be written down for singlets

$$\mathcal{C}_{20} = \sum_{k=0}^{20} g_k s^{20-k} \tag{2.13}$$

leading to $\left[g_{20}\right] = \eta'' - 20c_1$ with the property $\eta'' = 9\eta$.

For later use, we consider together with (2.7) the so called geometric Z_2 parity of [19], but as approached in [14,15] in dealing with local models. For simplicity, we use a short way to introduce this parity by requiring, up to an overall phase, invariance of $\mathcal{C}_5 = 0$, $\mathcal{C}_{10} = 0$, $\mathcal{C}_{20} = 0$ under the following transformations along the spectral fiber; see [14–16] for explicit details,

$$\begin{aligned} s_i' &= e^{-i\phi} s_i \\ b_k' &= e^{i\left[\beta+(5-k)\phi\right]} b_k \\ d_k' &= e^{i\left[\gamma+(10-k)\phi\right]} d_k \\ g_k' &= e^{i\left[\delta+(20-k)\phi\right]} g_k \end{aligned} \tag{2.14}$$

Under this phase change, the spectral covers eqns transform like

$$\begin{aligned} \mathcal{C}_5' &= e^{i\beta} \mathcal{C}_5 \\ \mathcal{C}_{10}' &= e^{i\gamma} \mathcal{C}_{10} \\ \mathcal{C}_{20}' &= e^{i\delta} \mathcal{C}_{20} \end{aligned} \tag{2.15}$$

Focusing on 10-plets, and equating above $\mathcal{C}_5'$ with the one deduced from construction of [16] namely $\mathcal{C}_5' = e^{i(\zeta-\phi)}\mathcal{C}_5$, we learn that we should have $\beta = \zeta - \phi$, and therefore $b_k' = e^{i[\zeta+(k-6)\phi]} b_k$. For the particular choice $\phi = \pi$, we have $s_i' = -s_i$ and

$$b_k' = (-)^k e^{i\zeta} b_k \tag{2.16}$$

If we put $\zeta = 0$, we get $\left(b_0', b_5'\right) = (+b_0, -b_5)$; while by taking $\zeta = \pi$, we have $\left(b_0', b_5'\right) = (-b_0, +b_5)$; below we set $\zeta = \pi$. To get the parity of the holomorphic sections d_k and g_k of eqs. (2.8), we use their relationships with the b_k coefficients. By help of the relations $d_{10} = b_3^2 b_4 - b_2 b_3 b_5 + b_0 b_5^2$ and $g_{20} = 256 b_4^5 b_0^4 + \ldots$, it follows that $Z_2(d_{10}) \sim Z_2(b_3^2 b_4)$ and $Z_2(g_{20}) = Z_2(b_4^5 b_0^4)$, so we have [27,30,31]

$$\begin{aligned} &Z_2(d_{10}) = -1, \qquad Z_2(g_{20}) = -1, \qquad Z_2(b_5) = +1 \\ &Z_2(d_0) = -1, \qquad Z_2(g_0) = -1, \qquad Z_2(b_0) = -1 \end{aligned} \tag{2.17}$$

in agreement with the homology class properties $\eta' = 3\eta$ and $\eta'' = 9\eta$.

2.2. Models with broken $\mathbb{S}_5$

To engineer matter curves with monodromy $\Gamma_k \subset \mathbb{S}_5$, we generally use spectral cover splitting method combined with constraints inspired from Galois theory [14–16,26,27]. In this study, we develop a new method without need of the involved tools of Galois group theory, our approach uses characters $\chi_{\mathbf{R}}(g)$ of discrete group representations, and relies directly the roots of the spectral covers. To illustrate the method, but also for later use, we first study the two interesting cases by using the standard method:

- $\Gamma_4 = \mathbb{S}_4 \subset \mathbb{S}_5$,
- $\Gamma_3 = \mathbb{S}_3 \subset \mathbb{S}_5$.

The case $\Gamma_4 = \mathbb{D}_4$ requires more tools: it will be studied later after revisiting $\mathbb{S}_4$- and $\mathbb{S}_3$-models from the view of characters of their representations.

2.2.1. $\mathbb{S}_4$-model in standard approach

To engineer the breaking of $\mathbb{S}_5$ down to $\mathbb{S}_4$, we proceed as follows: First, we use $\mathbb{S}_5$-invariance to rewrite the holomorphic polynomial $\mathcal{C}_5$ like

$$\mathcal{C}_5 = \frac{b_0}{5!} \sum_{\sigma \in \Gamma} \prod_{i=1}^{5} \left(s - t_{\sigma(i)}\right) \tag{2.18}$$

and similarly for $\mathcal{C}_{10}$ and $\mathcal{C}_{20}$. To break $\mathbb{S}_5$ down to $\mathbb{S}_4$, we impose a condition fixing one of the weight [51], for example

$$\sigma\left(t_5\right) = t_5 \qquad \Leftrightarrow \qquad \sigma\left(5\right) = 5 \tag{2.19}$$

This requirement breaks $\mathbb{S}_5$ down to one of the five possible $\mathbb{S}_4$ subgroups living inside $\mathbb{S}_5$, and leads to the following features:

(**a**) the traceless condition (2.3) of the orthogonal $SU_5^{\perp}$ is solved as $t_5 = -\left(t_1 + t_2 + t_3 + t_4\right)$; it is manifestly $\mathbb{S}_4$-invariant. To deal with this t_5 weight, we shall think about the breaking of $\mathbb{S}_5$ down to $\mathbb{S}_4$ in terms of the descent of the symmetry $SU_5 \times U\left(1\right)^{5-k} \times \Gamma_k$ from $k=5$ to $k=4$ as follows [58,59]

$$\begin{aligned} SU_5 \times U\left(1\right)^{5-5} \times \mathbb{S}_5 \quad &\rightarrow \quad SU_5 \times U\left(1\right)^{5-4} \times \mathbb{S}_4 \\ &\sim \quad SU_5 \times \mathbb{S}_4 \times U\left(1\right) \end{aligned} \tag{2.20}$$

(**b**) the spectral covers $\mathcal{C}_5$ and $\mathcal{C}_{10}$ split as the product of two factors: ($\boldsymbol{\alpha}$) the spectral cover $\mathcal{C}_5$ factorises like $\mathcal{C}_4 \times \mathcal{C}_1$ with

$$\mathcal{C}_4 = A_0 \prod_{i=1}^{4} \left(s - t_i\right), \qquad \mathcal{C}_1 = a_0 \left(s - t_5\right) \tag{2.21}$$

and[2]

[2] The holomorphic sections A_l and a_m eqs. (2.21) are directly derived by expanding the factorised forms of the spectral covers $\mathcal{C}_4$ and $\mathcal{C}_1$; we will not give these details here; for example the relevant A_4 and a_1 are given by $A_4 = A_0 \prod_{i=1}^{4} t_i$ and $a_1 = -a_0 t_5$.

$$\begin{aligned} b_0 &= A_0 \times a_0 \\ b_5 &= A_4 \times a_1 \end{aligned} \tag{2.22}$$

together with the transformations following from (2.14)–(2.15). Notice that the above factorisations put conditions on the field $\mathcal{K}$ where live the holomorphic sections; a feature that is also predicted by Galois theory [28,29]. As a naive illustration, we use the comparison with arithmetics in the set of integers $\mathcal{Z}$; an integer number like 6 can be factorised in $\mathcal{Z}$ as $6 = 2 \times 3$, while a prime integer like 5 has no factorisation.

By using $\mathcal{C}_5' = \mathcal{C}_4' \times \mathcal{C}_1'$ and equating $e^{i(\zeta-\phi)}(\mathcal{C}_4 \times \mathcal{C}_1)$ with $\left(e^{i\xi}\mathcal{C}_4\right) \times \left(e^{i\psi}\mathcal{C}_1\right)$, it follows that $\zeta - \phi = \xi + \psi$, and

$$\begin{aligned} A_4' &= e^{i\xi} A_4 \\ a_1' &= e^{i(\zeta-\xi-\phi)} a_1 \end{aligned} \tag{2.23}$$

from which we learn that A_4 and a_1 sections transform differently, and then $Z_2(b_4) = Z_2(A_4) \times Z_2(a_1)$. ($\beta$) the $\mathcal{C}_{10}$ splits in turns like $\tilde{\mathcal{C}}_6 \times \tilde{\mathcal{C}}_4$ with

$$\mathcal{C}_6 = \tilde{A}_0 \prod_{j>i=1}^{4} \left(s - T_{ij}\right), \qquad \mathcal{C}_4 = \tilde{a}_0 \prod_{i=1}^{4} (s - T_{i5}) \tag{2.24}$$

and

$$\begin{aligned} d_0 &= \tilde{A}_0 \times \tilde{a}_0 \\ d_{10} &= \tilde{A}_6 \times \tilde{a}_4 \end{aligned} \tag{2.25}$$

as well as $\tilde{\mathcal{C}}_6 = e^{2i\tilde{\xi}}\tilde{\mathcal{C}}_6$ and $\tilde{\mathcal{C}}_4 = e^{2i\tilde{\psi}}\tilde{\mathcal{C}}_4$ with $\tilde{\xi} + \tilde{\psi} = \tilde{\zeta} - \phi$.

Under the above splitting, the spectrum (2.7) decomposes in terms of reducible $\mathbb{S}_4$ multiplets as follows

Curves	Weights	S_4	$U_1^{\perp}$	Homology	Sections	Z_2	$U(1)_Y$ flux
10_{t_i}	t_i	4	0	$\eta - 4c_1 + \chi$	A_4	$\varkappa_4$	N
10_{t_5}	t_5	1	1	$-\chi - c_1$	a_1	$\varkappa_1$	$-N$
$5_{t_i+t_j}$	t_i+t_j	6	0	$\eta' - 6c_1 + \tilde{\chi}$	$\tilde{A}_6$	$\tilde{\varkappa}_6$	N
$5_{t_i+t_5}$	t_i+t_5	4	1	$-\tilde{\chi} - 4c_1$	$\tilde{a}_4$	$\tilde{\varkappa}_4$	$-N$

(2.26)

with

$$\begin{aligned} A_4 &= A_0 \prod_{i=1}^{4} t_i, \qquad \tilde{A}_6 = \tilde{A}_0 \prod_{j>i=1}^{4} T_{ij} \\ a_1 &= a_0 t_5, \qquad \tilde{a}_4 = \tilde{a}_0 \prod_{i=1}^{4} T_{i5} \end{aligned} \tag{2.27}$$

and where $\varkappa_i$ and $\tilde{\varkappa}_k$ refer to Z_2 parities; for instance

$$\begin{aligned} \varkappa_4 &= Z_2(A_4), \qquad & \tilde{\varkappa}_6 &= Z_2\left(\tilde{A}_6\right) \\ \varkappa_1 &= Z_2(a_1), \qquad & \tilde{\varkappa}_4 &= Z_2(\tilde{a}_4) \\ \varkappa_4\varkappa_1 &= Z_2(b_5), \qquad & \tilde{\varkappa}_4\tilde{\varkappa}_6 &= Z_2(d_{10}) \end{aligned} \tag{2.28}$$

The last column of eq. (2.26) refers to the hyperflux of the $U(1)_Y$ gauge field strength; it breaks SU_5 gauge symmetry down to standard model gauge invariance, and also pierces the matter curves of the model as shown in table.

2.2.2. $\mathbb{S}_3$-model in standard approach

The breaking of $\mathbb{S}_5$ down to $\mathbb{S}_3$ may be obtained from above $\mathbb{S}_4$ model by further breaking $\mathbb{S}_4$ down to $\mathbb{S}_3$; this corresponds to $SU_5 \times U(1)^{5-5} \times \mathbb{S}_5 \rightarrow SU_5 \times U(1)^{5-3} \times \mathbb{S}_3$. This can be realised by fixing one of the four t_i roots, say t_4, so that the breaking pattern is given by

$$\begin{aligned} SU_5 \times U(1)^{5-5} \times \mathbb{S}_5 \quad &\rightarrow \quad SU_5 \times U(1)^{5-3} \times \mathbb{S}_3 \\ &\sim \quad SU_5 \times \mathbb{S}_3 \times U(1)^2 \end{aligned} \tag{2.29}$$

Setting $U(1)^2 = U_1^{\perp} \times U_1^{\perp}$, the previous $\mathbb{S}_4$ spectrum decomposes into reducible $\mathbb{S}_3$ multiplets as follows,

Curves	S_3	$U_1^{\perp} \times U_1^{\perp}$	Homology	Section	$U(1)_Y$ flux
10_{t_i}	3	$(0,0)$	$\eta - 3c_1 - \chi - \chi'$	A'_3	$-N-P$
10_{t_4}	1	$(1,0)$	$\chi' - c_1$	A'_1	P
10_{t_5}	1	$(0,1)$	$\chi - c_1$	a_1	N
$5_{t_i+t_j}$	3	$(0,0)$	$\eta' - 3c_1 - \tilde{\chi} - \tilde{\chi}'$	$\tilde{A}'_3$	$-N-P$
$5_{t_i+t_4}$	3	$(1,0)$	$\tilde{\chi}' - 3c_1$	$\tilde{A}''_3$	P
$5_{t_i+t_5}$	3	$(0,1)$	$\tilde{\chi} - 3c_1 - \tilde{\chi}'$	$\tilde{a}'_3$	$N-P$
$5_{t_4+t_5}$	1	$(1,1)$	$\tilde{\chi}' - c_1$	$\tilde{a}''_1$	P

(2.30)

with

$$b_5 = \left(A'_3 A'_1\right) \times a_1, \qquad d_{10} = \left(\tilde{A}'_3 \tilde{A}''_3\right) \times \left(\tilde{a}'_3 \tilde{a}''_1\right) \tag{2.31}$$

where A'_3, A'_1, a_1 and $\tilde{A}'_3$, $\tilde{A}''_3$, $\tilde{a}'_3$, $\tilde{a}''_1$ are given by relations of form as in (2.27). An extra column for Z_2-parity can be also added as in (2.26) with the property

$$\begin{aligned} Z_2(b_5) &= Z_2(A'_3) \times Z_2(A'_1) \times Z_2(a_1) \\ Z_2(d_{10}) &= Z_2(\tilde{A}'_3) \times Z_2(\tilde{A}''_3) \times Z_2(\tilde{a}'_3) \times Z_2(\tilde{a}''_1) \end{aligned} \tag{2.32}$$

Observe also that here we have two new homology class cycles χ and χ' with

$$\int_{\chi} \mathcal{F}_X = N, \qquad \int_{\chi'} \mathcal{F}_X = P \tag{2.33}$$

The non-zero P is responsible for the second splitting; this is because the breaking of $\mathbb{S}_5$ down to $\mathbb{S}_3$ has been undertaken into two stages: first $\mathbb{S}_5 \rightarrow \mathbb{S}_4$; and second $\mathbb{S}_4 \rightarrow \mathbb{S}_3$. In what follows we extend this idea to the breaking pattern of $\mathbb{S}_5$ down to $\mathbb{D}_4$.

3. Revisiting $\mathbb{S}_4$ and $\mathbb{S}_3$-models

In this section, we develop tools towards the study of the breaking of $\mathbb{S}_5$ monodromy down to its $\mathbb{D}_4$ sub-symmetry. To our knowledge these tools, have not been used before, even for $\mathbb{S}_n$ permutation groups, so we begin by revisiting the $\mathbb{S}_4$- and $\mathbb{S}_3$-models from the view of characters of their irreducible representations, and turn in next section to develop the $\mathbb{D}_4$ theory.

3.1. $SU_5 \times \mathbb{S}_4 \times U_1^{\perp}$ model

In the canonical t_i-weight basis, the matter spectrum of $\mathbb{S}_4$-model is given by (2.26); there matter curves are organised into *reducible* multiplets of $\mathbb{S}_4 \times U_1^{\perp}$. Below, we give another manner to approach the spectrum of $\mathbb{S}_4$-model.

By help of the standard relation $24 = 1^2 + 1^2 + 2^2 + 3^2 + 3^2$ showing that $\mathbb{S}_4$ has *5 irreducible* representations $\boldsymbol{R}_i$ and *5 conjugacy* classes $\mathfrak{C}_i$ [39–42], and by using properties of the irreducible $\boldsymbol{R}_i$ representations of $\mathbb{S}_4$ given in appendix, eq. (2.26) may be expressed in terms of the $\boldsymbol{R}_i$s and their $\chi_R^{(a,b,c)}$ characters as follows

Curves	Weights	Irrep S_4	$\chi_R^{(a,b,c)}$	$U_1^{\perp}$	Homology	$U(1)_Y$ flux
10_{x_i}	x_i	$\mathbf{3}$	$(1, 0, -1)$	0	$\eta - 3c_1$	0
10_{x_4}	x_4	$\mathbf{1}$	$(1, 1, 1)$	0	$\chi - c_1$	N
10_{t_5}	t_5	$\mathbf{1}$	$(1, 1, 1)$	1	$-\chi - c_1$	$-N$
$5_{X_{ij}}$	X_{ij}	$\mathbf{3}'$	$(-1, 0, 1)$	0	$\eta' - 3c_1$	0
$5_{X_{i4}}$	X_{i4}	$\mathbf{3}$	$(1, 0, -1)$	0	$-3c_1 + \chi'$	N
$5_{X_{i5}}$	X_{i5}	$\mathbf{3}$	$(1, 0, -1)$	0	$-3c_1 - \chi'$	$-N$
$5_{X_{45}}$	X_{45}	$\mathbf{1}$	$(1, 1, 1)$	1	$-c_1$	0

(3.1)

Notice that $\mathbb{S}_4$ has three generators denoted here by (a, b, c) and chosen as given by 2-, 3- and 4-cycles; they obey amongst others the cyclic properties $a^2 = b^3 = c^4 = I_{id}$; these three generators are non-commuting permutation operators making extraction of full information from them a difficult task, but part of these information is given their $\chi_R^{(a,b,c)}$'s; these characters are real numbers as collected in following table [39–42],

χ_{ij}	$\chi_{\boldsymbol{I}}$	$\chi_{\mathbf{3}'}$	$\chi_{\mathbf{2}}$	$\chi_{\mathbf{3}}$	χ_{ϵ}
a	1	-1	0	1	-1
b	1	0	-1	0	1
c	1	1	0	-1	-1

(3.2)

Notice also that the **4**- and **6**-representations of $\mathbb{S}_4$, which have been used in the canonical formulation of section 2, are decomposed in (3.1) as direct sums of irreducible components as follows:

$$\begin{aligned} \mathbf{4}_{(2,1,0)} &= \mathbf{1}_{(1,1,1)} \oplus \mathbf{3}_{(1,0,-1)} \\ \mathbf{6}_{(0,0,0)} &= \mathbf{3}_{(1,0,-1)} \oplus \mathbf{3}'_{(-1,0,1)} \end{aligned} \tag{3.3}$$

Notice moreover that the previous t_i-weights are now replaced by new quantities x_i given by some linear combinations of the t_i's fixed by representation theory of $\mathbb{S}_4$. One of these weights, say x_4, is given by the usual completely $\mathbb{S}_4$-symmetric term

$$x_4 \sim (t_1 + t_2 + t_3 + t_4) \tag{3.4}$$

transforming in the trivial representation of $\mathbb{S}_4$; the three other x_i are given by some orthogonal linear combinations of the four t_i's that we express as follows

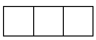
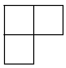
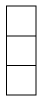
$$x_i = \alpha_i t_1 + \beta_i t_2 + \gamma_i t_3 + \delta_i t_4 \tag{3.5}$$

These three weights transform as an irreducible triplet of $\mathbb{S}_4$; but seen that we have two kinds of 3-dim representations in $\mathbb{S}_4$ namely $\mathbf{3}$ and $\mathbf{3}'$, the explicit expressions of (3.5) depend in which of the two representations the x_is are sitting; details are reported in appendix where one also finds the relationships $t_\mu = U_{\mu\rho} x_\rho$ and $t_\mu \pm t_\nu = \left(U_{\mu\rho} \pm U_{\nu\rho}\right) x_\rho$. Notice finally that the explicit expressions of $X_{\mu\nu}$ weights in (3.1) are not needed in our approach; their role will be played by the characters of the representations.

3.2. $SU_5 \times \mathbb{S}_3 \times \left(U_1^\perp\right)^2$ model

The spectrum of GUT-curves of the $SU_5 \times \mathbb{S}_3 \times \left(U_1^\perp\right)^2$ model follows from the spectrum of the $SU_5 \times \mathbb{S}_5$ theory by using splitting spectral method. By working in the canonical basis for t_i-weights, this spectrum, expressed in terms of reducible multiplets, is given by (2.30). Here, we revisit the $SU_5 \times \mathbb{S}_3 \times \left(U_1^\perp\right)^2$ curves spectrum by using irreducible representations of $\mathbb{S}_3$ and their characters.

We start by recalling that $\mathbb{S}_3$ has three irreducible representations as shown of the usual character relation $6 = 1^2 + 1^{2\prime} + 2^2$ linking the order of $\mathbb{S}_3$ to the squared dimensions of its irreducible representations; these irreducible representations are nicely described in terms of Young diagrams [42]

$\mathbf{1}$: , $\mathbf{2}$: , $\mathbf{1}'$:  (3.6)

The group $\mathbb{S}_3$ is a non-abelian discrete group; it has two non-commuting generators (a, b) satisfying $a^2 = b^3 = 1$ with characters as follows

χ_R	$\chi_{\mathbf{1}}$	$\chi_{\mathbf{2}}$	χ_ϵ
a	1	0	-1
b	1	-1	1

(3.7)

The spectrum of matter curves in the $\mathbb{S}_3$-model is obtained here by starting from the $\mathbb{S}_4$ spectrum (t_1, t_2, t_3) (2.30); and then breaking $\mathbb{S}_4$ monodromy to $\mathbb{S}_3 \times \mathbb{S}_1$. We find

Curves	Weights	Irrep S_3	$\chi_R^{(a,b)}$	$U_1^\perp$	Homology	$U(1)_Y$ flux
10_{x_i}	x_i	$\mathbf{2}$	$(0, -1)$	0	$\eta - 2c_1 - \chi'$	$-P$
10_{x_3}	x_3	$\mathbf{1}$	$(1, 1)$	0	$-\chi - c_1$	$-N$
10_{x_4}	x_4	$\mathbf{1}$	$(1, 1)$	0	$\chi' - c_1$	P
10_{t_5}	t_5	$\mathbf{1}$	$(1, 1)$	1	$\chi - c_1$	N
$5_{X_{ij}}$	X_{ij}	$\mathbf{2}$	$(0, -1)$	0	$\eta' - 2c_1$	0
$5_{X_{i3}}$	X_{i3}	$\mathbf{1}$	$(-1, 1)$	0	$-c_1 - \chi' - \chi$	$-P - N$
$5_{X_{i4}}$	X_{i4}	$\mathbf{2}$	$(0, -1)$	0	$-2c_1$	0
$5_{X_{34}}$	X_{34}	$\mathbf{1}$	$(1, 1)$	0	$\chi' - c_1$	P
$5_{X_{i5}}$	X_{i5}	$\mathbf{2}$	$(0, -1)$	0	$-2c_1$	0
$5_{X_{35}}$	X_{35}	$\mathbf{1}$	$(1, 1)$	0	$-c_1 - \chi' + \chi$	$N - P$
$5_{X_{45}}$	X_{45}	$\mathbf{1}$	$(1, 1)$	1	$\chi' - c_1$	P

(3.8)

where the integers P and N are as in eq. (2.33).

4. $SU_5 \times \mathbb{D}_4$ models

First notice that the engineering of the $SU_5 \times \mathbb{D}_4 \times U_1^{\perp}$ theory has been recently studied in [16] by using Galois theory, but here we use a method based on characters of the irreducible representations of $\mathbb{D}_4$, and find at the end that there are in fact three kinds of $SU_5 \times \mathbb{D}_4 \times U_1^{\perp}$ models, they are explicitly constructed in this section. To that purpose, we first review useful aspects on characters of the dihedral group, then we turn to construct the three $\mathbb{D}_4 \times U_1^{\perp}$ models.

4.1. Characters in $\mathbb{D}_4$ models

The dihedral $\mathbb{D}_4$ is an order *8* subgroup of $\mathbb{S}_4$ with no 3-cycles; there are three kinds of such subgroups inside $\mathbb{S}_4$; an example of $\mathbb{D}_4$ subgroup is the one having the following elements

$$I_{id}, \qquad \begin{matrix} (24) \\ (13) \end{matrix}, \qquad \begin{matrix} (13)\,(24) \\ (12)\,(34) \\ (14)\,(23) \end{matrix}, \qquad \begin{matrix} (1234) \\ (1432) \end{matrix} \tag{4.1}$$

with non-commuting generators $a = \langle(24)\rangle$ and $b = \langle(1234)\rangle$ satisfying $a^2 = b^4 = I$ and $aba = b^3$. The two other $\mathbb{D}_4'$ and $\mathbb{D}_4''$ have similar contents; but with other transpositions and 4-cycles. In terms of (a, b) generators, the eight elements (4.1) of the dihedral $\mathbb{D}_4$ reads as

$$I_{id}, \qquad \begin{matrix} a \\ b^2 a \end{matrix}, \qquad \begin{matrix} b^2 \\ ab \\ ba \end{matrix}, \qquad \begin{matrix} b \\ b^3 \end{matrix} \tag{4.2}$$

they form *5* conjugacy classes as follows

$$\begin{aligned} &\mathfrak{C}_1 = \{I_{id}\}, \qquad \mathfrak{C}_2 = \left\{b^2\right\}, \qquad \mathfrak{C}_3 = \left\{b, b^3\right\} \\ &\mathfrak{C}_4 = \{a\}, \qquad \mathfrak{C}_5 = \{ab\} \end{aligned} \tag{4.3}$$

The dihedral group $\mathbb{D}_4$ has also *5* irreducible representations $\boldsymbol{R}_i$; this can be directly learnt on the character formula $8 = 1_1^2 + 1_2^2 + 1_3^2 + 1_4^2 + 2^2$, linking the order of $\mathbb{D}_4$ with the sum of d_i^2, the squares of the dimensions d_i of the irreducible $\boldsymbol{R}_i$ representations of $\mathbb{D}_4$. So, the order *8* dihedral group has four irreducible representations with 1-dim; and a fifth irreducible $\mathbb{D}_4$-representation with 2-dim [42]. The character table of $\mathbb{D}_4$ representations is given by

$\mathfrak{C}_i \backslash \chi_{\boldsymbol{R}_j}$	$\chi_{\mathbf{1}_1}$	$\chi_{\mathbf{1}_2}$	$\chi_{\mathbf{1}_3}$	$\chi_{\mathbf{1}_4}$	$\chi_{\mathbf{2}}$	Number
$\mathfrak{C}_1$	1	1	1	1	2	1
$\mathfrak{C}_2$	1	1	1	1	−2	1
$\mathfrak{C}_3$	1	1	−1	−1	0	2
$\mathfrak{C}_4$	1	−1	1	−1	0	2
$\mathfrak{C}_5$	1	−1	−1	1	0	2

(4.4)

from which we learn the following characters of the (a, b) generators

$\chi_{ij}^{(g)}$	$\chi_{\mathbf{1}_1}$	$\chi_{\mathbf{1}_2}$	$\chi_{\mathbf{1}_3}$	$\chi_{\mathbf{1}_4}$	$\chi_{\mathbf{2}}$
a	1	−1	1	−1	0
b	1	1	−1	−1	0

(4.5)

For other features see [41]. With these tools at hand, we turn to engineer the $SU_5 \times \mathbb{D}_4 \times U_1^{\perp}$ models with dihedral monodromy symmetry.

4.2. *Three $\mathbb{D}_4$-models*

As in the case of $\mathbb{S}_3$ monodromy, the breaking of $\mathbb{S}_4$ down to $\mathbb{D}_4$ is induced by non-zero flux piercing the curves of the $SU_5 \times \mathbb{S}_4 \times U_1^{\perp}$ model. Using properties from the character table of $\mathbb{D}_4$, we distinguish three kinds of models depending on the way the $\mathbb{S}_4$-irreducible triplets have been pierced; there are three possibilities and are as described in what follows:

4.2.1. *First case:* $\mathbf{3} = \mathbf{1}_{+,-} \oplus \mathbf{2}_{0,0}$

In this model, the various irreducible triplets of $\mathbb{S}_4$; in particular those involved in:

(i) the five 10-plets namely $\mathbf{5} = \mathbf{1} \oplus \mathbf{3} \oplus \mathbf{1}_{t_5}$, and
(ii) the ten 5-plets which includes the four 10-plets charged under $U_1^{\perp}$ namely $\mathbf{4}_{t_5} = \mathbf{1}_{t_5} \oplus \mathbf{3}_{t_5}$, and the six uncharged 10-plets given by $\mathbf{6} = \mathbf{3} \oplus \mathbf{3}'$,

are decomposed as sums of two singlets $\mathbf{1}_{p,q} + \mathbf{1}_{p',q'}$ and a doublet $\mathbf{2}_{0,0}$. The character properties of the $\mathbb{D}_4$-representations indicate that the decompositions of the triplets should be as

$$\begin{array}{lll} \mathbf{3}|_{\mathbb{D}_4} & = & \mathbf{1}_{+,-} \oplus \mathbf{2}_{0,0} \\ \mathbf{3}'|_{\mathbb{D}_4} & = & \mathbf{1}_{-,+} \oplus \mathbf{2}_{0,0} \end{array} \tag{4.6}$$

By substituting these relations back into the restricted spectrum resulting from (3.1), we end with the following $SU_5 \times \mathbb{D}_4 \times U_1^{\perp}$ spectrum

- *five 10-plets*

Curves	Weights	$\mathbb{D}_4$	$\chi_{\mathbf{R}}^{(a,b)}$	$U_1^{\perp}$	Homology	$U(1)_Y$ flux
10_{y_i}	y_i	$\mathbf{2}$	$(0,0)$	0	$\eta - 2c_1 - \varphi$	$-N-P$
10_{y_3}	y_3	$\mathbf{1}$	$(1,-1)$	0	$-c_1$	0
10_{y_4}	y_4	$\mathbf{1}$	$(1,1)$	0	$\chi' - c_1$	P
10_{t_5}	t_5	$\mathbf{1}$	$(1,1)$	1	$\chi - c_1$	N

(4.7)

where $\chi_{\mathbf{R}}^{(a,b)}$ stands for the character of the generators in the $\mathbf{R}$ representation; $\varphi = \chi + \chi'$, and the integers N and P as in eqs. (2.33). Notice that the multiplets 10_{y_4} and 10_{t_5} transform in the same trivial $\mathbb{D}_4$-representation; but having different t_5-charges; the 10_{y_3} transforms also as a singlet; but with character $(1,-1)$; it is a good candidate for accommodating the top-quark family.

- *ten 5-plets*

Curves	Weight	$\mathbb{D}_4$	$\chi_{\mathbf{R}}^{(a,b)}$	$U_1^{\perp}$	Homology	$U(1)_Y$ flux
$5_{Y_{i3}}$	Y_{i3}	$\mathbf{2}$	$(0,0)$	0	$\eta'-2c_1+\varphi$	$N+P$
$5_{Y_{12}}$	Y_{12}	$\mathbf{1}$	$(-1,1)$	0	$-\chi-c_1$	$-N$
$5_{Y_{i4}}$	Y_{i4}	$\mathbf{2}$	$(0,0)$	0	$-\chi'-c_1$	$-P$
$5_{Y_{34}}$	Y_{34}	$\mathbf{1}$	$(1,-1)$	0	$-2c_1$	0
$5_{Y_{i5}}$	Y_{i5}	$\mathbf{2}$	$(0,0)$	1	$-\chi'-c_1$	$-P$
$5_{Y_{35}}$	Y_{35}	$\mathbf{1}$	$(1,-1)$	1	$-\chi-c_1$	$-N$
$5_{Y_{45}}$	Y_{45}	$\mathbf{1}$	$(1,1)$	1	$\varphi-2c_1$	$N+P$

(4.8)

where we have set $\varphi=\chi+\chi'$. From this table, we learn that among the ten 5-plets, two sit in the $1_{+,-}$ representation with character $(1,-1)$, but with different t_5 charges; one in $1_{-,+}$ with character $(-1,1)$ with no t_5 charge; and a fourth in the trivial representation of $\mathbb{D}_4$ with a unit t_5 charge.

- *flavons*
  Among the *24* flavons of the $SU_5\times\mathbb{S}_4\times U_1^{\perp}$ model, there are *20* ones charged under $\mathbb{D}_4$ monodromy symmetry; but because of hermitic feature, they can be organised into $\mathbf{10}\oplus\mathbf{10}'$ subsets with opposite $\mathbb{D}_4$ characters and opposite t_5 charges. Moreover due to reducibility of the 10-dim multiplet as $\mathbf{10}=\mathbf{4}_{t_5}\oplus\mathbf{6}$, which is also equal to $(\mathbf{1}_{t_5}\oplus\mathbf{3}_{t_5})\oplus(\mathbf{3}\oplus\mathbf{3}')$; and therefore to the direct sum $\mathbf{1}_{+,+}^{t_5}\oplus\left(\mathbf{1}_{+,-}^{t_5}\oplus\mathbf{2}_{0,0}^{t_5}\right)$ plus $(\mathbf{1}_{+,-}\oplus\mathbf{2}_{0,0})+(\mathbf{1}_{-,+}\oplus\mathbf{2}_{0,0})$; one ends with (α) flavons doublets ϑ^i, $\vartheta_{t_5}^i$ having character $(0,0)$ with and without t_5 charges, and (β) flavon singlets having characters $(\pm1,\pm1)$ with and without t_5 charges; they are as collected below:

Curves	Weights	$\mathbb{D}_4$ irrep	$\chi_{\mathbf{R}}^{(a,b)}$ character	t_5 charge
$1_{\pm Z_{i3}}$	$\pm Z_{i3}$	$\mathbf{2}$	$(0,0)$	0
$1_{\pm Z_{12}}$	$\pm Z_{12}$	$\mathbf{1}$	$\pm(-1,1)$	0
$1_{\pm Z_{i4}}$	$\pm Z_{i4}$	$\mathbf{2}$	$(0,0)$	0
$1_{\pm Z_{34}}$	$\pm Z_{34}$	$\mathbf{1}$	$\pm(1,-1)$	0
$1_{\pm Z_{i5}}$	$\pm Z_{i5}$	$\mathbf{2}$	$(0,0)$	∓1
$1_{\pm Z_{35}}$	$\pm Z_{35}$	$\mathbf{1}$	$\pm(1,-1)$	∓1
$1_{\pm Z_{45}}$	$\pm Z_{45}$	$\mathbf{1}$	$\pm(1,1)$	∓1

(4.9)

4.2.2. Second case: $\mathbf{3}=\mathbf{1}_{+,-}\oplus\mathbf{1}_{+,-}\oplus\mathbf{1}_{-,+}$

This is a completely reducible model; under restriction to dihedral subsymmetry, the $\mathbf{3}$ and $\mathbf{3}'$ triplets of $\mathbb{S}_4$ are decomposed as follows

$$\begin{aligned}\mathbf{3}|_{\mathbb{D}_4} &= \mathbf{1}_{+,-}\oplus\mathbf{1}_{+,-}\oplus\mathbf{1}_{-,+}\\ \mathbf{3}'\big|_{\mathbb{D}_4} &= \mathbf{1}_{-,+}\oplus\mathbf{1}_{-,+}\oplus\mathbf{1}_{+,-}\end{aligned}\tag{4.10}$$

by substituting these decompositions back into the spectrum of $SU_5\times\mathbb{S}_4\times U_1^{\perp}$-theory given by (3.1), we obtain the curves spectrum of the second $SU_5\times\mathbb{D}_4\times U_1^{\perp}$-model:

- *five 10-plets*
  The spectrum of the 10-plets in the $\mathbb{D}_4$-model II can be also deduced from (4.7) by splitting the $\mathbf{2}_{0,0}$ doublet as $\mathbf{1}_{+,-}\oplus\mathbf{1}_{-,+}$; we have

Curves	$\mathbb{D}_4$ irrep	Character	$U_1^{\perp}$	Homology	$U(1)_Y$ flux
$10_{+,-}$	$\mathbf{1}_{+,-}$	$(1,-1)$	0	$\eta - c_1 - \chi - \chi'$	$-N-P$
$10_{-,+}$	$\mathbf{1}_{-,+}$	$(-1,1)$	0	$-c_1$	0
$10_{+,-}$	$\mathbf{1}_{+,-}$	$(1,-1)$	0	$-c_1$	0
$10_{+,+}$	$\mathbf{1}_{+,+}$	$(1,1)$	0	$\chi' - c_1$	P
$10^{t_5}_{+,+}$	$\mathbf{1}_{+,+}$	$(1,1)$	1	$\chi - c_1$	N

(4.11)

Here we have two matter multiplets namely $10_{+,+}$ and $10^{t_5}_{+,+}$; they transform in the same trivial $\mathbb{D}_4$-representation with character $(1,1)$; but having different t_5-charges. We also have two $10_{+,-}$ multiplets transforming in $1_{+,-}$ with character $(1,-1)$; but with different fluxes; and one multiplet $10_{-,+}$ with character $(-1,1)$; it will be interpreted in appendix B as the one accommodating the top-quark family.

- *ten 5-plets*

Curves	$\mathbb{D}_4$ irrep	$\chi_{\mathbf{R}}^{(a,b)}$	$U_1^{\perp}$	Homology	$U(1)_Y$ flux
$5_{+,-}$	$\mathbf{1}_{+,-}$	$(1,-1)$	0	$\eta' - c_1 + \chi + \chi'$	$N+P$
$5_{-,+}$	$\mathbf{1}_{-,+}$	$(-1,1)$	0	$-c_1$	0
$5_{-,+}$	$\mathbf{1}_{-,+}$	$(-1,1)$	0	$-\chi - c_1$	$-N$
$5_{+,-}$	$\mathbf{1}_{+,-}$	$(1,-1)$	0	$-\chi' - c_1$	$-P$
$5_{-,+}$	$\mathbf{1}_{-,+}$	$(-1,1)$	0	$-c_1$	0
$5_{+,-}$	$\mathbf{1}_{+,-}$	$(1,-1)$	0	$-c_1$	0
$5^{t_5}_{+,-}$	$\mathbf{1}_{+,-}$	$(1,-1)$	1	$-\chi' - c_1$	$-P$
$5^{t_5}_{-,+}$	$\mathbf{1}_{-,+}$	$(-1,1)$	1	$-c_1$	0
$5^{t_5}_{+,-}$	$\mathbf{1}_{+,-}$	$(1,-1)$	1	$-\chi - c_1$	$-N$
$5^{t_5}_{+,+}$	$\mathbf{1}_{+,+}$	$(1,1)$	1	$-2c_1 + \chi + \chi'$	$N+P$

(4.12)

where N and P as in eqs. (2.33).

In this model, there is no flavon doublets; there are only singlet flavons transforming in the representations $\mathbf{1}_{+,+}$, $\mathbf{1}_{-,-}$, $\mathbf{1}_{+,-}$, $\mathbf{1}_{-,+}$ with and without t_5 charges; they are denoted in what follows as $\vartheta_{p,q}$ and $\vartheta^{\pm t_5}_{p,q}$ with $p, q = \pm 1$.

4.2.3. *Third case:* $\mathbf{3} = \mathbf{1}_{+,+} \oplus \mathbf{1}_{-,-} \oplus \mathbf{1}_{+,-}$

This $\mathbb{D}_4$-model differs from the previous one by the characters of the singlets; since in this case the $\mathbb{S}_4$-triplets $\mathbf{3}|_{\mathbb{S}_4}$ and $\mathbf{3}'|_{\mathbb{S}_4}$ are decomposed in terms of irreducible representations of $\mathbb{D}_4$ like

$$\begin{aligned} \mathbf{3}|_{\mathbb{D}_4} &= \mathbf{1}_{+,+} \oplus \mathbf{1}_{-,-} \oplus \mathbf{1}_{+,-} \\ \mathbf{3}'|_{\mathbb{D}_4} &= \mathbf{1}_{+,+} \oplus \mathbf{1}_{-,-} \oplus \mathbf{1}_{-,+} \end{aligned} \tag{4.13}$$

Substituting these relationships back into (3.1), we get the curve spectrum of the third model namely:

- *five 10-plets*

Curves	$\mathbb{D}_4$ irrep	Character	$U_1^{\perp}$	Homology	$U(1)_Y$ flux
$10_{+,+}$	$\mathbf{1}_{+,+}$	$(1,1)$	0	$\eta - c_1 - \varphi$	$-N-P$
$10_{-,-}$	$\mathbf{1}_{-,-}$	$(-1,-1)$	0	$-c_1$	0
$10_{+,-}$	$\mathbf{1}_{+,-}$	$(1,-1)$	0	$-c_1$	0
$10_{+,+}$	$\mathbf{1}_{+,+}$	$(1,1)$	0	$\chi' - c_1$	P
$10_{+,+}^{t_5}$	$\mathbf{1}_{+,+}$	$(1,1)$	1	$\chi - c_1$	N

(4.14)

Here we have three $10_{p,q}$ matter multiplets in the trivial $\mathbb{D}_4$-representation with character $(p,q)=(1,1)$; one of them namely $10_{+,+}^{t_5}$ having a t_5 charge and the two others not. A fourth curve $10_{+,-}$ in $1_{+,-}$ without t_5 charge nor a flux; and a fifth $10_{-,-}$ in $1_{-,-}$ with no t_5 but carrying a flux.

- *ten 5-plets*

Curves	$\mathbb{D}_4$ irrep	$\chi_{\mathbf{R}}^{(a,b)}$	$U_1^{\perp}$	Homology	$U(1)_Y$ flux
$5_{+,+}$	$\mathbf{1}_{+,+}$	$(1,1)$	0	$\eta' - c_1 + \chi + \chi'$	$N+P$
$5_{-,-}$	$\mathbf{1}_{-,-}$	$(-1,-1)$	0	$-\kappa_1\chi' - c_1$	$-\kappa_1 P$
$5_{-,+}$	$\mathbf{1}_{-,+}$	$(-1,1)$	0	$-\chi - c_1$	$-N$
$5_{+,+}$	$\mathbf{1}_{+,+}$	$(1,1)$	0	$-\kappa_2\chi' - c_1$	$-\kappa_2 P$
$5_{-,-}$	$\mathbf{1}_{-,-}$	$(-1,-1)$	0	$-c_1$	0
$5_{+,-}$	$\mathbf{1}_{+,-}$	$(1,-1)$	0	$-c_1$	0
$5_{+,+}^{t_5}$	$\mathbf{1}_{+,+}$	$(1,1)$	1	$-\kappa_1\chi' - c_1$	$-\kappa_1 P$
$5_{-,-}^{t_5}$	$\mathbf{1}_{-,-}$	$(-1,-1)$	1	$-\kappa_2\chi' - c_1$	$-\kappa_2 P$
$5_{+,-}^{t_5}$	$\mathbf{1}_{+,-}$	$(1,-1)$	1	$-\chi - c_1$	$-N$
$5_{+,+}^{t_5}$	$\mathbf{1}_{+,+}$	$(1,1)$	1	$-2c_1 + \chi + \chi'$	$N+P$

(4.15)

with $\kappa_1 + \kappa_2 = 1$ whose values will be fixed by the derivation of MSSM. The ten 5-plets $5_{p,q}$ splits as follows: *4* with $p=q=1$; the two $5_{+,+}^{t_5}$ having a t_5 charge and the two others $5_{+,+}$ chargeless; the $U_1^{\perp}$ charges and the (N,P) fluxes allow to distinguish the four. There are also *3* types of $5_{-,-}$-plets; two $5_{+,-}$ and one $5_{-,+}$. This model has no flavon doublets: there are only singlet flavons $\vartheta_{p,q}$ and $\vartheta_{p,q}^{\pm t_5}$ with $p,q=\pm 1$.

5. MSSM like spectrum

First, we describe the breaking of the $SU_5 \times \mathbb{D}_4 \times U_1^{\perp}$ theory down to supersymmetric standard model, then we study the derivation of the spectrum of MSSM like model with $\mathbb{D}_4$ monodromy and where the heaviest top-quark family is singled out.

5.1. Breaking gauge symmetry

Gauge symmetry is broken by $U(1)_Y$ hyperflux; by assuming doublet–triplet splitting produced by N units of $U(1)_Y$, but still preserving $\mathbb{D}_4 \times U_1^{\perp}$, the 10-plets and 5-plets get decomposed into irreducible representations of standard model symmetry. The 5-plets of the $SU_5 \times \mathbb{D}_4 \times U_1^{\perp}$ models with multiplicity M_5 split as [60,61]

$$\begin{aligned} n_{(3,1)_{-1/3}} - n_{(\bar{3},1)_{+1/3}} &= M_5 \\ n_{(1,2)_{+1/2}} - n_{(1,2)_{-1/2}} &= M_5 + N \end{aligned} \tag{5.1}$$

leading to a difference between number of triplets and doublets in the low energy MSSM effective theory. These two relations are important since for $N \neq 0$ the correlation is some how relaxed; by choosing

$$M_5^{(Higgs)} = 0 \tag{5.2}$$

the coloured triplet–antitriplet fields $(3,1)_{-1/3}$ and $\left(\bar{3},1\right)_{+1/3}$ in the Higgs matter curve come in pair that form heavy massive states; which decouple at low energy. Moreover, by making particular choices of the $M_5^{(matter)}$ multiplicities, we can also have the desired matter curve properties for accommodating fermion families; in particular the chirality property $n_{(1,2)_{+1/2}} \neq n_{(1,2)_{-1/2}}$ which is induced by hyperflux. Furthermore, due to the flux, we also have different numbers of down quarks d_L^c and lepton doublets L.

For the 10-plets of the GUT-model with multiplicity M_{10}, we have the following decompositions [27,62,63]

$$\begin{array}{lll} n_{(3,2)_{+1/6}} - n_{\left(\bar{3},2\right)_{-1/6}} & = & M_{10} \\ n_{\left(\bar{3},1\right)_{-2/3}} - n_{(3,1)_{+2/3}} & = & M_{10} - N \\ n_{(1,1)_{+1}} - n_{(1,1)_{-1}} & = & M_{10} + N \end{array} \tag{5.3}$$

The first relation with $M_{10} \neq 0$ generates up-quark chirality since the number $n_{(3,2)_{+1/6}}$ of $Q_L = (3,2)_{+1/6}$ representations differs from the number $n_{\left(\bar{3},2\right)_{+1/6}}$ of $\bar{Q}_L = \left(\bar{3},2\right)_{-1/6}$. With non-zero units of hyperflux, the two extra relations leads to the other desired splitting; the second relation leads for $N \neq 0$ to lifting the multiplicities between $Q = (3,2)_{+1/6}$ and $u^c = \left(\bar{3},1\right)_{-2/3}$ while the third relation ensures the chirality property of e_L^c.

In what follows, we study the derivation of an effective matter curve spectrum that resembles to the field content of MSSM. In addition to three families and

$$\sum M_{10} + \sum M_5 = 0 \tag{5.4}$$

as well as total hyperflux conservation

$$\sum_{fluxes} N_i = 0 \tag{5.5}$$

we demand the following:

- only a tree-level Yukawa coupling is allowed; and is given by the top-quark family,
- the heaviest third generation is the least family affected by hyperflux,
- MSSM matter generations are in $\mathbb{D}_4 \times U_1^{\perp}$ representations,
- no dimension 4 and 5 proton decay operators are allowed,
- no μ-term at a tree level,
- two Higgs doublets H_u and H_d as required by MSSM.

5.2. Building the spectrum

Seen that there are three possible $SU_5 \times \mathbb{D}_4 \times U_1^{\perp}$ models, we focus on the first model with curve spectrum given by eqs. (4.7)–(4.8); and consider first the 10-plets; then turn after to 5-plets. Results regarding the two other models II and III are reported in appendix B.

5.2.1. Ten-plets sector in $\mathbb{D}_4$-model I

The five 10-plets of the $\mathbb{D}_4$ model carry different quantum numbers with respect to $\mathbb{D}_4 \times U_1^{\perp}$ representations, different hyperflux units (N, P); and different $M_{10}^{(n)}$ multiplicities satisfying the properties (5.3). By thinking about $\sum M_{10}$ as given by the number of MSSM generations

$$\sum M_{10} = 3 \tag{5.6}$$

and taking into account that the two components of the 10_i-doublet are monodromy equivalent; it follows that one of the five 10-plets should be disregarded; at least at a tree level analysis. Moreover, using the property that top-quark 10-plet should be a $\mathbb{D}_4$-singlet; one may choose the $M_{10}^{(n)}$'s as in following table,

Curves	$\mathbb{D}_4$ irrep	$U_1^{\perp}$	$U(1)_Y$ flux	Multiplicity	
10_i	$\mathbf{2}_{0,0}$	0	$-N-P$	$M_{10}^{(a)}$	2
10_3	$\mathbf{1}_{+,-}$	0	0	$M_{10}^{(3)}$	1
10_4	$\mathbf{1}_{++}$	0	P	$M_{10}^{(4)}$	0
10_5	$\mathbf{1}_{++}$	1	N	$M_{10}^{(5)}$	0

(5.7)

where chiral modes of 10_4 have been ejected ($M_{10}^{(4)} = 0$). Notice that the top-quark generation can a priori be taken in any one of the three $\mathbb{D}_4$-singlets; that is either 10_3 or 10_4; or 10_5; the basic difference between these $\mathbb{D}_4$-singlets is given by t_5 charge and hyperflux. But the choice of the 10_3-multiplet looks be the natural one as it is unaffected by hyperflux, a desired property for MSSM and beyond; and has no t_5 charge

$$\mathbf{10}_3 = \left(Q_L, U_L^c, e_L^c\right) \equiv \mathbf{10}_{+,-} \tag{5.8}$$

This multiplet captures also an interesting signature of $\mathbb{D}_4$ monodromy in the sense it behaves as a $\mathbb{D}_4$-singlet $1_{+,-}$ with non-trivial character $(+1, -1)$. The importance of this feature at modelling level is twice: (**i**) first it fixes the quantum numbers of the 5_{H_u} Higgs representation as a $\mathbb{D}_4$-singlet $5_{p,q}$ as shown on the tree level top-quark Yukawa coupling

$$10_{+,-} \otimes 10_{+,-} \otimes 5_{H_u} \tag{5.9}$$

Monodromy invariance of (5.9) under $\mathbb{D}_4 \times U_1^{\perp}$ requires 5_{H_u} in the trivial representation with no t_5 charge; i.e: $5_{H_u} \sim 1_{+,+}$. However, an inspection of the characters of the $U_1^{\perp}$ chargeless 5-plets revels that there is no $\left(5_{+,+}\right)_{t_5=0}$ in the spectrum of the $\mathbb{D}_4 \times U_1^{\perp}$-models I and II constructed above. To bypass this constraint, we realise the role of the Higgs 5_{H_u} by allowing VEVs to come from flavons as well; in other words by thinking of 5_{H_u} as follows

$$5_{H_u} \rightarrow 5_{p,q} \otimes \vartheta_{p',q'} \qquad \text{with} \qquad pp' = 1,\ qq' = 1 \tag{5.10}$$

where $\vartheta_{p',q'}$ stands for a flavon in the representation $1_{p',q'}$.

(**ii**) second it gives an important tool to distinguish between matter and Higgs in the 5-plets sector as manifestly exhibited by the tri-coupling $10_{+,-} \otimes \bar{5}^M \otimes \bar{5}_{H_d}$. This interaction requires matter $\bar{5}_3^M$ and Higgs $\bar{5}_{H_d}$ to be in different $\mathbb{D}_4$-singlets $1_{p,q}$ and $1_{p',q'}$ with $pp' = 1$ and $qq' = -1$; see discussion given later on.

By choosing the hyperflux units as $N = P = 1$; and using (5.3) we obtain the matter content

Curves	$\mathbb{D}_4$	$U_1^{\perp}$	Flux	Matter content	Z_2 parity
10_i	$\mathbf{2}_{0,0}$	0	-2	$2Q_L \oplus 4e_L^c$	$\varkappa_{43} = -$
10_3	$\mathbf{1}_{+,-}$	0	0	$Q_L \oplus U_L^c \oplus e_L^c$	$\varkappa_{42} = -$
10_4	$\mathbf{1}_{+,+}$	0	1	$U_L^c \ominus e_L^c$	$\varkappa_{41} = +$
10_5	$\mathbf{1}_{+,+}$	1	1	$U_L^c \ominus e_L^c$	$\varkappa_1 = +$

(5.11)

Notice that by following [16] using Galois theory, the 10-plets have been attributed $\mathbb{Z}_2$ parity charges as reported by the last column of above table. In our formulation these parities correspond to $s_i \to -s_i$ and $\varkappa_1$ and $\varkappa_4 = \varkappa_{41}\varkappa_{42}\varkappa_{43}$ as in eq. (2.28); by help of (2.14) and (2.22) we obtain

$$\begin{aligned} &\mathbb{Z}_2(b_5) = +1, \qquad \mathbb{Z}_2(b_0) = -1 \\ &\mathbb{Z}_2(d_{10}) = -1, \qquad \mathbb{Z}_2(d_0) = -1 \end{aligned} \tag{5.12}$$

in agreement with (2.17).

5.2.2. Five-plets sector

Like for 10-plets, the ten 5-plets carry different quantum numbers of $\mathbb{D}_4 \times U_1^{\perp}$ representations, hyperflux units (N, P) and $M_5^{(n)}$ multiplicities as in (5.1). To have a matter curve spectrum that resembles to MSSM, we choose the $M_5^{(n)}$'s and the hyperflux as

Curves	$\mathbb{D}_4$ irrep	$U_1^{\perp}$	Homology	Flux	Multiplicity
$5_{Y_{i3}}$	$\mathbf{2}_{0,0}$	0	$\eta' - 2c_1 - \chi' + \xi'$	$-N-P$	$M_5^{(1)}$
$5_{Y_{12}}$	$\mathbf{1}_{-,+}$	0	$\chi' - c_1$	N	$M_5^{(2)}$
$5_{Y_{i4}}$	$\mathbf{2}_{0,0}$	0	$\xi' - c_1$	P	$M_5^{(3)}$
$5_{Y_{34}}$	$\mathbf{1}_{+,-}$	0	$-2c_1$	0	$M_5^{(4)}$
$5_{Y_{i5}}$	$\mathbf{2}_{0,0}$	-1	$\xi' - c_1$	P	$M_5^{(5)}$
$5_{Y_{35}}$	$\mathbf{1}_{+,-}$	-1	$\chi' - c_1$	N	$M_5^{(6)}$
$5_{Y_{45}}$	$\mathbf{1}_{+,+}$	-1	$-2c_1 - \chi' - \xi'$	$-N-P$	$M_5^{(7)}$

(5.13)

where χ' and ξ' are two classes playing similar role as in the case of breaking $\mathbb{S}_5$ monodromy down to $\mathbb{S}_3$. By using (5.4)–(5.6), we have

$$\sum M_5 = -\sum M_{10} = -3 \tag{5.14}$$

and thinking of this number as $\sum M_5 = 3 - 6$, a possible configuration for a MSSM like spectrum is given by

$$\begin{aligned} M_5^{(1)} &= 2 \\ M_5^{(2)} &= 0 \\ M_5^{(3)} &= -4 \\ M_5^{(4)} &= 0 \\ M_5^{(6)} &= 1 \\ M_5^{(7)} &= -2 \end{aligned} \tag{5.15}$$

By choosing the hyperflux as $N = P = 1$, and putting back into above table, we obtain, after relabelling, the 5-plets

Curves	$\mathbb{D}_4$	$U_1^{\perp}$	Flux	$M_5^{(n)}$	Matter	Parity
$(5_i^M)_0$	$\mathbf{2}_{0,0}$	0	-2	2	$2\bar{d}_L^c$	$\tilde{\varkappa}_{61} = +$
$(5_{-,+}^{H_u})_0$	$\mathbf{1}_{-,+}$	0	1	0	H_u	$\tilde{\varkappa}_{62} = +$
$(5_{+,-}^M)_0$	$\mathbf{1}_{+,-}$	0	1	-4	$-4\bar{d}_L^c - 3\bar{L}$	$\tilde{\varkappa}_{63} = -$
$(5_{+,+}^{H_d})_{-t_5}$	$\mathbf{1}_{+,+}$	-1	1	0	$-H_d$	$\tilde{\varkappa}_{41} = +$
$(5_{+,-}^M)_{-t_5}$	$\mathbf{1}_{+,-}$	-1	1	1	$\bar{d}_L^c$	$\tilde{\varkappa}_{42} = +$
$(5_i^M)_{-t_5}$	$\mathbf{2}_{0,0}$	-1	-2	-2	$-2\bar{d}_L^c$	$\tilde{\varkappa}_{43} = +$
$(5_i^M)_0$	$\mathbf{2}_{0,0}$	0	0	0	0	$\tilde{\varkappa}_{64} = +$

(5.16)

From this table we learn that the up-Higgs 5-plet $\left(5_{-,+}^{H_u}\right)_0$ has a character equal to $(-1, +1)$ and no t_5 charge; by substituting in (5.10), we obtain $5_{H_u} \sim \left(5_{-,+}^{H_u}\right)_0 \otimes \left(\vartheta_{-,+}\right)_0$.

We also learn that the 5-plet $(5_{+,-}^M)_0$ is the least multiplet affected by hyperflux; and because of our assumptions, it is the candidate for matter $\bar{5}_3^M$; the partner of 10_3 in the underlying SO_{10} GUT-model. With this choice, the down-type quarks tri-coupling for the third family namely $10_3 \otimes \bar{5}_3^M \otimes \bar{5}^{H_d}$; and which we rewrite like

$$10_{+,-} \otimes \bar{5}_{p,q}^M \otimes \bar{5}_{p',q'}^{H_d} \qquad \text{with} \qquad pp' = 1, qq' = -1 \tag{5.17}$$

This coupling requires the matter $\bar{5}_3^M$ and the down-Higgs $\bar{5}^{H_d}$ multiplets to belong to different $\mathbb{D}_4$ singlets seen that 10_3 is in $\mathbf{1}_{+,-}$ representation. However, the candidates $(\bar{5}_{-,-}^{H_d})_{t_5}$ and $\bar{5}_3^M \equiv (\bar{5}_{-,+}^M)_0$ are ruled out because of the non conservation of t_5 charge. Nevertheless, a typical diagonal mass term of third family may be generated by using a flavon $\vartheta_{-t_5}^{+}$ carrying -1 unit charge under $U_1^{\perp}$ and transforming as a trivial $\mathbb{D}_4$ singlet. This leads to the realisation $\bar{5}^{H_d} \sim (\bar{5}_{-,-}^{H_d})_{t_5}(\vartheta_{++})_{-t_5}$; and then to

$$\left(10_{+,-}\right)_0 \otimes \left(\bar{5}_{-,+}^M\right)_0 \otimes \left(\bar{5}_{-,-}^{H_d}\right)_{t_5} \otimes \left(\vartheta_{++}\right)_{-t_5} \tag{5.18}$$

Non-diagonal 4-order coupling superpotentials with one $\left(10_{+,-}\right)_0$ are as follows[3]

$$\begin{aligned}
&\left(10_{+,-}\right)_0 \otimes \left(10_{+,+}\right)_0 \otimes \left(5_{-,+}^{H_u}\right)_0 \otimes \left(\vartheta_{-,-}\right)_0 \\
&\left(10_{+,-}\right)_0 \otimes \left(10_{+,+}\right)_{t_5} \otimes \left(5_{-,+}^{H_u}\right)_0 \otimes \left(\vartheta_{-,-}\right)_{-t_5} \\
&\left(10_{+,-}\right)_0 \otimes \left(5_{-,+}^{H_u}\right)_0 \otimes \left(10_{0,0}^i\right)_0 \otimes \left(\vartheta_{0,0}^i\right)_0 \\
&\left(10_{+,-}\right)_0 \otimes \left(10_{+,-}\right)_0 \otimes \left(5_{-,+}^{H_u}\right)_0 \otimes \left(\vartheta_{-,+}\right)_0 \\
&\left(10_{+,-}\right)_0 \otimes \left(5_{-,+}^{H_u}\right)_0 \otimes \left(10_{0,0}^i\right)_0 \left(\vartheta_{0,0}^i\right)_0
\end{aligned}$$

[3] A complete classification requires also use Z_2 partity; see [16].

$$\begin{aligned}&\left(10_{+,+}\right)_{0}\otimes\left(5_{-,+}^{H_u}\right)_{0}\otimes\left(10_{0,0}^{i}\right)_{0}\otimes\left(\vartheta_{0,0}^{i}\right)_{0}\\&\left(10_{+,+}\right)_{t_5}\otimes\left(5_{-,+}^{H_u}\right)_{0}\otimes\left(10_{0,0}^{i}\right)_{0}\otimes\left(\vartheta_{0,0}^{i}\right)_{-t_5}\end{aligned}\tag{5.19}$$

Below, we discuss some properties of these couplings.

5.3. *More on couplings in $\mathbb{D}_4$ model I*

First, we study the quark sector; and turn after to the case of leptons.

5.3.1. *Quark sector*

From the view of supersymmetric standard model with $SU(3)\times SU_L(2)\times U_Y(1)$ gauge symmetry; and denoting the triplet and doublet components of the Higss 5-plets $5^{H_x}=3^{H_x}\oplus 2^{H_x}$ respectively like $D_x\oplus H_x$, the usual tree level up/down-type Yukawa couplings in SU_5 model split like

$$\begin{aligned}10^M.10^M.5^{H_u}\quad&\rightarrow\quad Qu^cH_u+u^ce^cD_u^c+QQD_u^c\\10^M.\overline{5}^M.\overline{5}^{H_d}\quad&\rightarrow\quad Qd^cH_d+e^cLH_d+QD_d^cL\end{aligned}\tag{5.20}$$

They involve up/down Higgs triplets D_u^c and D_d^c, which are exotic to MSSM; but with the hyperflux $U_Y(1)$ choice we have made in $SU_5\times\mathbb{D}_4\times U_1^{\perp}$ model (5.11), (5.16), they are removed; therefore we have

$$\begin{aligned}10^M.10^M.5^{H_u}\quad&\rightarrow\quad Qu^cH_u\\10^M.\overline{5}^M.\overline{5}^{H_d}\quad&\rightarrow\quad Qd^cH_d+e^cLH_d\end{aligned}\tag{5.21}$$

with right hand sides capturing same monodromy representations as left hand sides; that is Q, u^c same $\mathbb{D}_4\times U_1^{\perp}$ representations as 10^M, and so on. In what follows, we study each of these terms separately by taking into account $\vartheta_{p,q}$ flavon contributions up to order four couplings; some of these flavons are interpreted as right neutrinos; they will be discussed at proper time.

- *Up-type Yukawa couplings*

Because of the $\mathbb{D}_4\times U_1^{\perp}$ monodromy charge of the up-Higgs 5-plet like $(5_{-,+}^{H_u})_0$, there is no monodromy invariant 3-coupling type $10^M.10^M.5^{H_u}$. As shown by eq. (5.10), one needs to go to higher orders by implementing flavons with quantum numbers depending on the monodromy representation of the 10-plets. Indeed, by focusing on the third generation $10_3^M\equiv(10_{+,-})_0$; we can distinguish diagonal and non-diagonal interactions; an inspection of $\mathbb{D}_4$ quantum numbers of matter and Higgs multiplets reveals that we need $\mathbb{D}_4$-charged flavons to have monodromy invariant superpotentials as shown below

$$W_{top}^{(4)}=\alpha_3Tr[(10_{+,-})_0\otimes(10_{+,-})_0\otimes(5_{-,+}^{H_u})_0\otimes(\vartheta_{-,+})_0]\tag{5.22}$$

By restricting to VEVs $\left\langle\vartheta_{-,+}\right\rangle=\rho_0$ and $\langle H_u\rangle=\upsilon_u$; this non-renormalisable coupling leads to the top quark mass term $m_tQ_3u_3^c$ with m_t equal to $\alpha_3\upsilon_u\rho_0$. Such a term should be thought of as a particular contribution to a general up-quark mass terms $u_i^cM^{ij}u_j$ with 3×3 mass matrix as follows

$$M_{u,c,t}=\upsilon_u\begin{pmatrix}* & * & *\\ * & * & *\\ * & * & \alpha_3\rho_0\end{pmatrix}\tag{5.23}$$

where the $(*)$s refer to contributions coming from other terms including non-diagonal couplings; one of them is

$$Tr[(10_{+,-})_0 \otimes (5^{H_u}_{-,+})_0 \otimes (10_{0,0})_0 \otimes (\vartheta_{0,0})'_0] \tag{5.24}$$

it involves a 10-plet doublet $(10_{0,0})_0 \equiv (10_i)_0$ and a flavon doublet $(\vartheta_{0,0})'_0 \equiv (\vartheta_i)'_0$ with VEVs (ρ_1, ρ_2); the latter $(\vartheta_{0,0})'_0$ will be combined the 10_i-plet doublet like $(10_{0,0})_0 \otimes (\vartheta_{0,0})'_0$ to make a scalar. Indeed, the tensor product can be reduced as direct sum over irreducible representations of $\mathbb{D}_4$ having amongst others the $\mathbb{D}_4$-component

$$S_{-,-} = (10_{0,0})_0 \otimes (\vartheta_{0,0})'_0\big|_{-,-} \tag{5.25}$$

with $(-,-)$ charge character. This negative charge is needed to compensate the $(-,-)$ charge coming from $(10_{+,-})_0 \otimes (5^{H_u}_{-,+})_0$. Restricting to quarks, this reduction corresponds to $(10_{0,0})_0 \otimes (\vartheta_{0,0})'_0 \to Q_i \otimes \rho_i$ with

$$Q_i \otimes \rho_i|_{(-,-)} = Q_1\rho_2 - Q_2\rho_1 \tag{5.26}$$

Putting back into (5.24), and thinking of $S_{-,-}$ in terms of the linear combination $\alpha_2(Q_1\rho_2 - Q_2\rho_1)$ of quarks, we obtain $\alpha_2 v_u(Q_1\rho_2 - Q_2\rho_1)u_3^c$; which can be put into the form $u_i^c M^{ij} u_j$ with mass matrix as

$$M_{u,c,t} = v_u \begin{pmatrix} * & * & \alpha_2\rho_2 \\ * & * & -\alpha_2\rho_1 \\ * & * & \alpha_3\rho_0 \end{pmatrix} \tag{5.27}$$

One can continue to fill this mass matrix by using the VEV's of other flavons; however to do that, one needs to rule out couplings with those flavons describing right neutrinos ν_i^c. Extending ideas from [16], the 3 generations of the right handed neutrinos ν_i^c in $SU_5 \times \mathbb{D}_4 \times U_1^\perp$ model should be as

$$\begin{array}{lcl} \nu_3^c & \to & (\vartheta_{+,-})_0 \\ (\nu_1^c, \nu_2^c)^\top & \to & (\vartheta_{0,0})_0 \end{array} \tag{5.28}$$

with the following features among the set of 15 flavons of the model

Flavons	SU_5	$\mathbb{D}_4$ irrep	$U_1^\perp$	Z_2 Parity	VEV
$(\vartheta_{0,0})'_0$	1	2	0	$+$	$(\rho_1, \rho_2)^\top$
$(\vartheta_{-,+})_0$	1	1	0	$+$	ρ_0
$(\vartheta_{0,0})_{\pm t_5}$	1	2	± 1	$+$	$(\sigma_1, \sigma_2)^\top$
$(\vartheta_{+,-})_{\pm t_5}$	1	1	± 1	$+$	$-$
$(\vartheta_{+,+})_{\pm t_5}$	1	1	± 1	$\mp$	ω
$(\vartheta_{0,0})_0 = (\nu_1^c, \nu_2^c)^\top$	1	2	0	$-$	$-$
$(\vartheta_{+,-})_0 = \nu_3^c$	1	1	0	$-$	$-$

(5.29)

Therefore, the contribution to (5.27) coming from the diagonal couplings of the doublets $(10_{0,0})_0$ follows from

$$W^{(4)} = Tr[(5^{H_u}_{-,+})_0 \otimes [(10_{0,0})_0 \otimes (10_{0,0})_0]\big|_{p,q} \otimes (\vartheta_{-,+})_0] \tag{5.30}$$

However, though monodromy invariant, this couplings cannot generate the mass term $mQ_{1,2}u^c_{1,2}$ since the matter curve $(10_{0,0})_0$ don't contain the quark $u^c_{1,2}$; so the mass matrix (5.27) for the up-type quarks is

$$M_{u,c,t} = v_u \begin{pmatrix} 0 & 0 & \alpha_2\rho_2 \\ 0 & 0 & -\alpha_2\rho_1 \\ 0 & 0 & \alpha_3\rho_0 \end{pmatrix} \tag{5.31}$$

it is a rank one matrix; it gives mass to the third generation (top-quark), while the two first generations are massless.

Masses for lighter families

The rank one property of above mass matrix (5.31) is a known feature in GUT models building including F-Theory constructions; see for instance [36,43,44,64]. To generate masses for the up-quarks in the first two generations, different approaches have been used in literature: (i) approach based on flux corrections using non-perturbative effects [20] or non-commutative geometry [21]; and (ii) method using δW deformations of the GUT superpotential W by higher order chiral operators [14,43,44,64–66]. Following the second way of doing, masses to the two lighter families are generated by higher dimensional operators corrections that are invariant under $\mathbb{D}_4$ symmetry and Z_2 parity. This invariance requirement leads to involve 6- and 7-dimensional chiral operators which contribute to the up-quark mass matrix as follows

$$\delta W = \sum_{i=1}^{5} x_i \delta W_i \tag{5.32}$$

with

$$\begin{aligned} \delta W_1 &= \left(10^i_{0,0}\right)_0 \otimes (10_{+,+})_0 \otimes \left(5^{H_u}_{-,+}\right)_0 \otimes \left(\vartheta^i_{0,0}\right)_{-t_5} \otimes (\vartheta_{+,+})_{t_5} \\ \delta W_2 &= \left(10^i_{0,0}\right)_0 \otimes (10_{+,+})_{t_5} \otimes \left(5^{H_u}_{-,+}\right)_0 \otimes \left(\vartheta^i_{0,0}\right)_{-t_5} \otimes \left(\vartheta^i_{0,0}\right)_{-t_5} \otimes (\vartheta_{+,+})_{t_5} \\ \delta W_3 &= \left(10^i_{0,0}\right)_0 \otimes (10_{+,+})_{t_5} \otimes \left(5^{H_u}_{-,+}\right)_0 \otimes \left(\vartheta^i_{0,0}\right)_{-t_5} \otimes (\vartheta_{+,-})_{-t_5} \otimes (\vartheta_{+,+})_{t_5} \end{aligned} \tag{5.33}$$

and

$$\begin{aligned} \delta W_4 &= (10_{+,-})_0 \otimes (10_{+,+})_0 \otimes \left(5^{H_u}_{-,+}\right)_0 \otimes (\vartheta_{-,+})_0 \otimes (\vartheta_{+,-})_{-t_5} \otimes (\vartheta_{+,+})_{t_5} \\ \delta W_5 &= (10_{+,-})_0 \otimes (10_{+,+})_{t_5} \otimes \left(5^{H_u}_{-,+}\right)_0 \otimes \left(\vartheta^i_{0,0}\right)^2_{-t_5} \otimes (\vartheta_{+,+})_{t_5} \end{aligned} \tag{5.34}$$

Notice that the adjunction of $(\vartheta_{+,+})_{t_5}$ chiral superfield is required by invariance under Z_2 parity. Using this deformation, a higher rank up-quark mass matrix is obtained as usual by giving VEVs to flavons as in (5.29) and $\left\langle(\vartheta_{+,-})_{-t_5}\right\rangle = \varphi$. By calculating the product of the operators in eqs. (5.33)–(5.34) using $\mathbb{D}_4$ fusion rules, we obtain

$$\begin{aligned} x_1\delta W_1 &= \left(10^i_{0,0}\right)_0 \otimes (10_{+,+})_0 \otimes \left(5^{H_u}_{-,+}\right)_0 \otimes \left(\vartheta^i_{0,0}\right)_{-t_5} \otimes (\vartheta_{+,+})_{t_5} \\ &= x_1 v_u (Q_1\sigma_1 - Q_2\sigma_2)u^c_2\omega \end{aligned}$$

and

$$\begin{aligned} x_2\delta W_2 &= \left(10^i_{0,0}\right)_0 \otimes (10_{+,+})_{t_5} \otimes \left(5^{H_u}_{-,+}\right)_0 \otimes \left(\vartheta^i_{0,0}\right)_{-t_5} \otimes (\vartheta_{+,-})_{-t_5} \otimes (\vartheta_{+,+})_{t_5} \\ &= x_2 v_u (Q_1\sigma_2 - Q_2\sigma_1)u^c_1\omega\varphi \end{aligned} \tag{5.35}$$

The operator

$$\left(10^{i}_{0,0}\right)_0 \otimes \left(10_{+,+}\right)_{t_5} \otimes \left(5^{H_u}_{-,+}\right)_0 \otimes \left(\vartheta^{i}_{0,0}\right)_{-t_5} \otimes \left(\vartheta^{i}_{0,0}\right)_{-t_5} \otimes \left(\vartheta_{+,+}\right)_{t_5}$$

contributes in the up-quark mass matrix (5.31) as a correction to the matrix elements $m_{1,1}$ and $m_{1,2}$; it has the same role as the higher operator (5.35); so we will not take it into account in the quark mass matrix. Expanding the remaining operators by help of the $\mathbb{D}_4$ rules, we have

$$\begin{aligned} x_4\delta W_4 &= \left(10_{+,-}\right)_0 \otimes \left(10_{+,+}\right)_0 \otimes \left(5^{H_u}_{-,+}\right)_0 \otimes \left(\vartheta_{-,+}\right)_0 \otimes \left(\vartheta_{+,-}\right)_{-t_5} \otimes \left(\vartheta_{+,+}\right)_{t_5} \\ &= x_4 v_u \rho_0 \varphi \omega Q_3 u^c_2 \end{aligned}$$

and

$$\begin{aligned} x_5\delta W_5 &= \left(10_{+,-}\right)_0 \otimes \left(10_{+,+}\right)_{t_5} \otimes \left(5^{H_u}_{-,+}\right)_0 \otimes \left(\vartheta^{i}_{0,0}\right)^2_{-t_5} \otimes \left(\vartheta_{+,+}\right)_{t_5} \\ &= x_5 v_u \omega Q_3 u^c_1 \left(\sigma_1\sigma_2 - \sigma_2\sigma_1\right) = 0 \end{aligned}$$

Summing up all contributions, we end with the following up-quark matrix

$$M_{u,c,t} = v_u \begin{pmatrix} x_2\sigma_1\omega\varphi & x_1\omega & \alpha_2\rho_2 \\ -x_2\sigma_1\omega\varphi & -x_1\omega & -\alpha_2\rho_1 \\ 0 & x_4\rho_0\varphi\omega & \alpha_4\rho_0 \end{pmatrix} \tag{5.36}$$

• *Down-type Yukawa*

Following the same procedure as in up-Higgs type coupling, we can build invariant operators for the down-type Yukawa

$$\begin{aligned} &(10_{+,-})_0 \otimes (\overline{5}^{M}_{+,-})_0 \otimes (\overline{5}^{H_d}_{+,+})_{t_5} \otimes \left(\vartheta_{+,+}\right)_{-t_5} \\ &(\overline{5}^{M}_{+,-})_0 \otimes (\overline{5}^{H_d}_{+,+})_{t_5} \otimes (10_{0,0})_0 \otimes \left(\vartheta_{0,0}\right)_{-t_5} \end{aligned} \tag{5.37}$$

Restricting VEV of down Higgs $\langle H_d\rangle = v_d$, and using the flavons VEVs as in (5.29) as well as taking into account multiplicities, the first coupling gives a mass term of the form $m_i d^c_i Q_3$ with $m_i = \omega v_d y_{3,i}$ where $y_{3,i}$ are coupling constants. For the second term, we need to reduce $(10_{0,0})_0 \otimes \left(\vartheta_{0,0}\right)_{-t_5}$ into irreducible $\mathbb{D}_4$ representations; and restricts to the component $S_{(+,-)} = Q_i \otimes \sigma_i|_{(+,-)}$ with

$$S_{(+,-)} = Q_1\sigma_1 + Q_2\sigma_2 \tag{5.38}$$

So the couplings in eqs. (5.37) may expressed like

$$y_{3,i} Q_3 d^c_i \omega v_d + y_{1,i}(Q_1\sigma_1 + Q_2\sigma_2) d^c_i v_d \tag{5.39}$$

leading to the mass matrix

$$m_{d,s,b} = v_d \begin{pmatrix} y_{1,1}\sigma_1 & y_{1,2}\sigma_1 & y_{1,3}\sigma_1 \\ y_{1,1}\sigma_2 & y_{1,2}\sigma_2 & y_{1,3}\sigma_2 \\ y_{3,1}\omega & y_{3,2}\omega & y_{3,3}\omega \end{pmatrix} \tag{5.40}$$

5.3.2. Lepton sector

First we consider the charged leptons; and then turn to neutrinos.

• *Charged leptons*

Charged leptons masses are determined by the same operators used in the case of the down quark sector $10^M \otimes \overline{5}^M \otimes \overline{5}^{H_d}$; using spectrum eqs. (5.11), (5.16), the appropriate operators which provide mass to charged leptons are

$$\begin{aligned}&(10_{+,-})_0 \otimes (\overline{5}^M_{+,-})_0 \otimes (\overline{5}^{H_d}_{+,+})_{t_5} \otimes \left(\vartheta_{+,+}\right)_{-t_5}\\&(10_{0,0})_0 \otimes (\overline{5}^M_{+,-})_0 \otimes (\overline{5}^{H_d}_{+,+})_{t_5} \otimes \left(\vartheta_{0,0}\right)_{-t_5}\end{aligned} \tag{5.41}$$

giving the lepton mass term $m^{ij} e^c_i L_j$ with mass matrix

$$m_{e,\mu,\tau} = v_d \begin{pmatrix} z_{1,1}\sigma_1 & z_{1,2}\sigma_1 & z_{1,3}\sigma_1 \\ z_{1,1}\sigma_2 & z_{1,2}\sigma_2 & z_{1,3}\sigma_2 \\ z_{3,1}\omega & z_{3,2}\omega & z_{3,3}\omega \end{pmatrix} \tag{5.42}$$

• *Neutrinos*

Right handed neutrinos are as in eq. (5.28), they have negative R-parity. Dirac neutrino term is embedded in the coupling $\left(\nu^c_i \otimes \overline{5}^M\right) \otimes 5^{H_u}$ where the right neutrino ν^c_i is an SU_5 singlet; it allows a total neutrino mass matrix using see-saw I mechanism [18]. The invariant operators that give the Dirac neutrino in $SU_5 \times \mathbb{D}_4 \times U_1^{\perp}$ model are

$$\begin{aligned}&x_{1,i}\left(\vartheta_{+,-}\right)_0 \otimes (\overline{5}^M_{+,-})_0 \otimes (5^{H_u}_{-,+})_0 \otimes \left(\vartheta_{-,+}\right)_0\\&x_{2,i}\left(\vartheta_{0,0}\right)_0 \otimes (\overline{5}^M_{+,-})_0 \otimes (5^{H_u}_{-,+})_0 \otimes \left(\vartheta_{0,0}\right)'_0\end{aligned} \tag{5.43}$$

Using the $\mathbb{D}_4$ algebra rules and flavon VEV's, these couplings lead to

$$\begin{aligned}&x_{1,i} v_u \rho_0 L_i \nu^c_3\\&x_{2,i} v_u \rho_2 L_i \nu^c_1 - x_{2,i} v_u \rho_1 L_i \nu^c_2\end{aligned} \tag{5.44}$$

and then to a Dirac neutrino mass matrix as

$$m_D = v_u \begin{pmatrix} x_{2,1}\rho_2 & -x_{2,1}\rho_1 & x_{1,1}\rho_0 \\ x_{2,2}\rho_2 & -x_{2,2}\rho_1 & x_{1,2}\rho_0 \\ x_{2,3}\rho_2 & -x_{2,3}\rho_1 & x_{1,3}\rho_0 \end{pmatrix} \tag{5.45}$$

The Majorana neutrino term is given by $M\nu^c_i \otimes \nu^c_j$; by using eqs. (5.11), (5.16), the Majorana neutrino couplings in $SU_5 \times \mathbb{D}_4 \times U_1^{\perp}$ model are as follows

$$\begin{aligned}&\left(\vartheta_{+,-}\right)_0 \otimes \left(\vartheta_{+,-}\right)_0\\&\left(\vartheta_{0,0}\right)_0 \otimes \left(\vartheta_{0,0}\right)_0\\&\left(\vartheta_{+,-}\right)_0 \otimes \left(\vartheta_{0,0}\right)_0 \otimes \left(\vartheta_{0,0}\right)'_0\end{aligned} \tag{5.46}$$

we can also add the singlet $\left(\vartheta_{-,+}\right)_0$ as a correction of the last two operators. The operators in above (5.46) lead to

$$m\nu^c_3\nu^c_3, \qquad M\nu^c_1\nu^c_2, \qquad \lambda\nu^c_3(\nu^c_1\rho_1 + \nu^c_2\rho_2) \tag{5.47}$$

and ends with a Majorana neutrino mass matrix like

$$m_M = \begin{pmatrix} 0 & M & \lambda\rho_1 \\ M & 0 & \lambda\rho_2 \\ \lambda\rho_1 & \lambda\rho_2 & m \end{pmatrix} \tag{5.48}$$

The general neutrino mass matrix is calculated using see-saw I mechanism; it reads as $\boldsymbol{M}_\nu = -m_D m_M^{-1} m_D^\top$; and leads to the following effective neutrino mass matrix

$$\boldsymbol{M}_\nu \simeq \xi_0 \begin{pmatrix} m_{1,1} & m_{1,2} & m_{1,3} \\ m_{1,2} & m_{2,2} & m_{2,3} \\ m_{1,3} & m_{2,3} & m_{3,3} \end{pmatrix} \tag{5.49}$$

with

$$\begin{aligned} m_{1,1} &= \lambda^2 x_{2,1}^2 \rho_2^4 - 2x_{2,1}^2 \rho_1\rho_2 mM + 2\lambda x_{2,1}\rho_2^2(\lambda x_{2,1}\rho_1^2 - x_{1,1}M\rho_0) \\ &\quad + (\lambda x_{2,1}\rho_1^2 + x_{1,1}M\rho_0)^2 \\ m_{2,2} &= \lambda^2 x_{2,2}^2 \rho_2^4 - 2x_{2,2}^2 \rho_1\rho_2 mM + 2\lambda x_{2,2}\rho_2^2(\lambda x_{2,2}\rho_1^2 - x_{1,2}M\rho_0) \\ &\quad + (\lambda x_{2,2}\rho_1^2 + x_{1,2}M\rho_0)^2 \\ m_{3,3} &= \lambda^2 x_{2,3}^2 \rho_2^4 - 2x_{2,3}^2 \rho_1\rho_2 mM + 2\lambda x_{2,3}\rho_2^2(\lambda x_{2,3}\rho_1^2 - x_{1,3}M\rho_0) \\ &\quad + (\lambda x_{2,3}\rho_1^2 + x_{1,3}M\rho_0)^2 \end{aligned} \tag{5.50}$$

and

$$\begin{aligned} m_{1,2} &= \lambda^2 x_{2,1}x_{2,2}\rho_2^4 - 2x_{2,1}x_{2,2}\rho_1\rho_2 mM + (\lambda x_{2,1}\rho_1^2 + x_{1,1}\rho_0 M)(\lambda x_{2,2}\rho_1^2 + x_{1,2}\rho_0 M) \\ &\quad + \lambda\rho_2^2[2\lambda x_{2,1}x_{2,2}\rho_1^2 - \rho_0 M(x_{1,1}x_{2,2} + x_{2,1}x_{1,2})] \\ m_{1,3} &= \lambda^2 x_{2,1}x_{2,3}\rho_2^4 - 2x_{2,1}x_{2,3}\rho_1\rho_2 mM + (\lambda x_{2,1}\rho_1^2 + x_{1,1}\rho_0 M)(\lambda x_{2,3}\rho_1^2 + x_{1,3}\rho_0 M) \\ &\quad + \lambda\rho_2^2[2\lambda x_{2,1}x_{2,3}\rho_1^2 - \rho_0 M(x_{1,1}x_{2,3} + x_{2,1}x_{1,3})] \\ m_{2,3} &= \lambda^2 x_{2,2}x_{2,3}\rho_2^4 - 2x_{2,2}x_{2,3}\rho_1\rho_2 mM + (\lambda x_{2,2}\rho_1^2 + x_{1,2}\rho_0 M)(\lambda x_{2,3}\rho_1^2 + x_{1,3}\rho_0 M) \\ &\quad + \lambda\rho_2^2[2\lambda x_{2,2}x_{2,3}\rho_1^2 - \rho_0 M(x_{1,3}x_{2,2} + x_{2,3}x_{1,2})] \end{aligned} \tag{5.51}$$

and where we have set

$$\xi_0 = \frac{v_u^2}{M(mM - 2\lambda^2\rho_1\rho_2)} \tag{5.52}$$

To obtain neutrino mixing compatible with experiments we need a particular parametrisation and some approximations on $\boldsymbol{M}_\nu$. To that purpose, recall that there are three approaches to mixing using: (i) the well know *Tribimaximal* (TBM) mixing matrix, (ii) *Bimaximal* (BM) and (iii) *Democratic* (DC); all of the TBM, BM and DC mixing matrices predict a zero value for the angle θ_{13}. However recent results reported by MINOS [24], Double Chooz [25], T2K [54], Daya Bay [55], and RENO [56] collaborations revealed a non-zero θ_{13}; such non-zero θ_{13} has been recently subject of great interest; in particular by perturbation of the TBM mixing matrix [57].

To estimate the proper masses of the $\boldsymbol{M}_\nu$ matrix; we diagonalise it by using the unitary U_{TBM} TBM mixing matrix; we use the μ–τ symmetry requiring $m_{2,2} = m_{3,3},\ m_{1,2} = m_{1,3}$; as well as the condition $m_{2,3} = m_{1,1} + m_{1,2} - m_{2,2}$. So we have $\boldsymbol{M}_\nu^{diag} = U_{TBM}^\top \boldsymbol{M}_\nu U_{TBM}$ with

$$U_{TBM} = \begin{pmatrix} \sqrt{\frac{2}{3}} & \frac{1}{\sqrt{3}} & 0 \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \end{pmatrix} \tag{5.53}$$

and therefore

$$\boldsymbol{M}_{\nu}^{diag} \simeq \xi_0 \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix} \tag{5.54}$$

with eigenvalues as

$$\begin{aligned} \lambda_1 &= \xi_0(m_{1,1} - m_{1,2}) \\ \lambda_2 &= \xi_0\left(m_{1,1} + 2m_{1,2}\right) \\ \lambda_3 &= \xi_0\left(2m_{3,2} - m_{1,1} - m_{1,2}\right) \end{aligned} \tag{5.55}$$

6. Conclusion and discussions

In this paper, we have developed a method based on characters of discrete group representations to study $SU_5 \times \mathbb{D}_4 \times U_1^{\perp}$-GUT models with dihedral monodromy symmetry. After having revisited the construction of $SU_5 \times \mathbb{S}_4 \times U_1^{\perp}$ and $SU_5 \times \mathbb{S}_3 \times \left(U_1^{\perp}\right)^2$ models from the character representation view, we have derived three $SU_5 \times \mathbb{D}_4 \times U_1^{\perp}$ models (referred here to as I, II and III) with curves spectrum respectively given by eqs. (4.7)–(4.8), (4.11)–(4.12) and (4.14)–(4.15). These models follow from the three different ways of decomposing the irreducible $\mathbb{S}_4$-triplets in terms of irreducible representations of $\mathbb{D}_4$; see eqs (4.6), (4.10), (4.13); such richness may be interpreted as due to the fact that $\mathbb{D}_4$ has four kinds of singlets with generator group characters given by the (p, q) pairs with $p, q = \pm 1$.

Then we have focused on the curve spectrum (4.7)–(4.8) of the first $SU_5 \times \mathbb{D}_4 \times U_1^{\perp}$ model; and studied the derivation of a MSSM-like spectrum by using particular multiplicity values and turning on adequate fluxes. We have found that with the choice of: (i) top-quark family $\mathbf{10}_3$ as $(\mathbf{10}_{+-})_0$, transforming into a $\mathbb{D}_4$-singlet with $\chi^{(a,b)}$ character equal to $(1, -1)$; and (ii) a 5^{H_u} up-Higgs as $\left(5_{-,+}\right)_0$, transforming into a different $\mathbb{D}_4$-singlet with character equal to $(-1, 1)$; there is no tri-Yukawa couplings of the form

$$\left(10_{+,-}\right)_0 \otimes \left(10_{+,-}\right)_0 \otimes \left(5^{H_u}\right)_{++}$$

as far as $\mathbb{D}_4 \times U_1^{\perp}$ invariance is required; this makes $SU_5 \times \mathbb{D}_4 \times U_1^{\perp}$ model with two quark generations accommodated into a $\mathbb{D}_4$-doublet non interesting phenomenologically. Monodromy invariant couplings require implementation of flavons $\vartheta_{p,q}$ by thinking of $5^{H_u} \sim \left(5_{-,+}\right)_0 \otimes \left(\vartheta_{-,+}\right)_0$ leading therefore to a superpotential of order 4. The same property appears with the down-Higgs couplings where $\mathbb{D}_4 \times U_1^{\perp}$ invariance of $10_3 \otimes \bar{5}_3^M \otimes \bar{5}^{H_d}$ requires: (α) a matter $\bar{5}_3^M \equiv (\bar{5}_{-,+}^M)_0$ in a $U_1^{\perp}$ chargeless $\mathbb{D}_4$-singlet with character $(-1, 1)$; and (β) a curve $\bar{5}^{H_d}$ with a $\mathbb{D}_4$-character like $(\bar{5}_{-,-})_{+t_5}$ composed with a charged flavon $(\vartheta_{++})_{-t_5}$; that is as

$$(\bar{5}_{-,-})_{+t_5} \otimes (\vartheta_{++})_{-t_5}$$

By analysing the conditions that a $\mathbb{D}_4 \times U_1^{\perp}$-spectrum has to fulfil in order to have a tri-Yukawa coupling for top-quark family $\mathbf{10}_3$, we end with the constraint that the character of 5^{H_u} up-Higgs should be equal to (1, 1) as clearly seen on $10_{+,-} \otimes 10_{+,-} \otimes 5^{H_u}$. This constraint is valid even if 10_3 was chosen like $10_{+,+}$. By inspecting the spectrum of the three studied $SU_5 \times \mathbb{D}_4 \times U_1^{\perp}$ models; it results that the spectrum of the third model given by eqs. (4.14)–(4.15) which allow tri-Yukawa coupling; for details on contents and couplings of models II and III; see appendix B.

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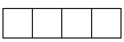
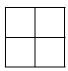
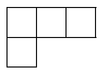
Appendix A. Characters in $\mathbb{S}_4$-models

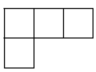
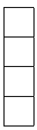
In this appendix, we give details on some useful properties of Γ-models studied in this paper, in particular on the representations of $\mathbb{S}_4$ and their characters.

A.1. Irreducible representations of $\mathbb{S}_4$

First, recall that $\mathbb{S}_4$ has five *irreducible* representations; as shown on the character formula $24 = 1^2 + 1'^2 + 2^2 + 3^2 + 3'^2$; these are the 1-dim representations including the trivial **1** and the sign $\epsilon = \mathbf{1}'$; a 2-dim representation **2**; and the 3-dim representations **3** and **3**′, obeying some "duality relation". This duality may be stated in different manners, but, in simple words, it may be put in parallel with polar and axial vectors of 3-dim Euclidean space. In the language of Young diagrams, these five irreducible representations are given by

1 :  , **2** : , **3** :  (A.1)

and

3′ : , **1**′ :  (A.2)

This diagrammatic description is very helpful in dealing with $\mathbb{S}_4$ representation theory [40–42], it teaches us a set of useful information, in particular helpful data on the three following:

i) *Expressions of* (3.5)

In the representation 3 of the permutation group $\mathbb{S}_4$, the three x_i-weights in (3.5) read in terms of the t_i's as

$$\vec{x} = \frac{1}{2}\begin{pmatrix} t_1 - t_2 - t_3 + t_4 \\ t_1 + t_2 - t_3 - t_4 \\ t_1 - t_2 + t_3 - t_4 \end{pmatrix} = \begin{pmatrix} x_4 - t_2 - t_3 \\ x_4 - t_3 - t_4 \\ x_4 - t_4 - t_2 \end{pmatrix} \tag{A.3}$$

where $x_4 = \frac{1}{2}(t_1 + t_2 + t_3 + t_4)$ is the completely symmetric term. The normalisation coefficient $\frac{1}{2}$ is fixed by requiring the transformation $x_i = U_{ij} t_j$ as follows

$$U = \frac{1}{2}\begin{pmatrix} 1 & -1 & -1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & 1 & 1 \end{pmatrix}, \qquad \det U = 1 \tag{A.4}$$

For the representation $3'$, we have

$$\vec{x}' = \frac{1}{\sqrt{8}} \begin{pmatrix} t_1 - 3t_2 + t_3 + t_4 \\ t_1 + t_2 - 3t_3 + t_4 \\ t_1 + t_2 + t_3 - 3t_4 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} x_4 - 2t_2 \\ x_4 - 2t_3 \\ x_4 - 2t_4 \end{pmatrix} \tag{A.5}$$

The entries of these triplets are cyclically rotated by the (234) permutation.

ii) $\mathbb{S}_4$*-triplets as 3-cycle* (234)
The $\{|t_i\rangle\}$ and $\{|x_i\rangle\}$ weight bases are related by the orthogonal 5×5 matrix

$$\begin{pmatrix} U & 0 \\ 0 & 1 \end{pmatrix}, \qquad |x_i\rangle = U_{ij} \left|t_j\right\rangle \tag{A.6}$$

with U as in (A.4); and then

$$\begin{aligned} t_1 &= \tfrac{1}{2}(x_4 + x_1 + x_2 + x_3) \\ t_2 &= \tfrac{1}{2}(x_4 - x_1 + x_2 - x_3) \\ t_3 &= \tfrac{1}{2}(x_4 - x_1 - x_2 + x_3) \\ t_4 &= \tfrac{1}{2}(x_4 + x_1 - x_2 - x_3) \end{aligned} \tag{A.7}$$

From these transformations, we learn $t_i = U_{ki} x_k$; and then $t_i \pm t_j = \left(U_{ki} \pm U_{kj}\right) x_k$ which can be also expressed $t_i \pm t_j = V_{ij}^{\pm kl} X_{kl}^{\pm}$. Similar relations can be written down for $\left\{\left|x_i'\right\rangle\right\}$.

A.2. Characters

The discrete symmetry group $\mathbb{S}_4$ model has *24* elements arranged into five conjugacy classes $\mathfrak{C}_1, \ldots, \mathfrak{C}_5$ as in table (A.8); it has five irreducible representations $\boldsymbol{R}_1, \ldots, \boldsymbol{R}_5$ with dimensions given by the relation $24 = 1^2 + 1^{2\prime} + 2^2 + 3^2 + 3^{2\prime}$; their character table $\chi_{ij} = \chi_{\boldsymbol{R}_j}(\mathfrak{C}_i)$ is as given below

$\mathfrak{C}_i \backslash \mathrm{irrep}\boldsymbol{R}_j$	$\chi_{\boldsymbol{1}}$	$\chi_{\boldsymbol{3}'}$	$\chi_{\boldsymbol{2}}$	$\chi_{\boldsymbol{3}}$	χ_{ϵ}	Number
$\mathfrak{C}_1 \equiv \mathrm{e}$	1	3	2	3	1	1
$\mathfrak{C}_2 \equiv (\alpha\beta)$	1	−1	0	1	−1	6
$\mathfrak{C}_3 \equiv (\alpha\beta)(\gamma\delta)$	1	−1	2	−1	1	3
$\mathfrak{C}_4 \equiv (\alpha\beta\gamma)$	1	0	−1	0	1	8
$\mathfrak{C}_5 \equiv (\alpha\beta\gamma\delta)$	1	1	0	−1	−1	6

(A.8)

The $\mathbb{S}_4$ group has *3* non-commuting generators (a, b, c) which can be chosen as given by the 2-, 3- and 4-cycles obeying amongst others the cyclic relations $a^2 = b^3 = c^4 = I_{id}$. In our approach the character of these generators have been used in the engineering of GUT models with $\mathbb{S}_4$ monodromy, they are as follows

χ_{ij}	$\chi_{\boldsymbol{1}}$	$\chi_{\boldsymbol{3}'}$	$\chi_{\boldsymbol{2}}$	$\chi_{\boldsymbol{3}}$	χ_{ϵ}
a	1	−1	0	1	−1
b	1	0	−1	0	1
c	1	1	0	−1	−1

(A.9)

In the $SU_5 \times \mathbb{S}_4$ theory considered in paper, the various curves of the spectrum of the GUT-model belong to $\mathbb{S}_4$-multiplets which can be decomposed into irreducible representation of $\mathbb{S}_4$. In doing so, one ends with curves indexed by the characters of the generators of $\mathbb{S}_4$ as follows

$$\begin{aligned} \mathbf{4} &= \mathbf{1}_{(1,1,1)} \oplus \mathbf{3}_{(1,0,-1)} \\ \mathbf{6} &= \mathbf{3}_{(1,0,-1)} \oplus \mathbf{3}'_{(-1,0,1)} \end{aligned} \tag{A.10}$$

Appendix B. Results on $SU_5 \times \mathbb{D}_4$ models II and III

In this appendix, we collect results regarding the $SU_5 \times \mathbb{D}_4 \times U_1^{\perp}$ models II and III of subsections 4.2.2 and 4.2.3. In addition to higher order terms, we also study when couplings like

Couplings	SU_5	$\mathbb{D}_4$	$U_1^{\perp}$	Parity
$10_i \otimes 10_j \otimes 5_{H_u}$	1	$1_{+,+}$	0	$+$
$10_i \otimes \overline{5}_j \otimes \overline{5}_{H_d}$	1	$1_{+,+}$	0	$+$
$\nu_i^c \otimes \overline{5}_M \otimes 5_{H_u}$	1	$1_{+,+}$	0	$+$
$m\nu_i^c \otimes \nu_j^c$	1	$1_{+,+}$	0	$+$

(B.1)

can be generated.

B.1. $SU_5 \times \mathbb{D}_4$ model II

The spectrum of the $SU_5 \times \mathbb{D}_4 \times U_1^{\perp}$ model II under breaking $SU_5 \times \mathbb{D}_4 \times U_1^{\perp}$ to MSSM is given by:

Curve in D_4 model II	$U_1^{\perp}$	Spectrum in MSSM
$10_1 = 10_{+,-}$	0	$M_1 Q_L + u_L^c (M_1 - N - P) + e_L^c (M_1 + N + P)$
$10_2 = 10_{-,+}$	0	$M_2 Q_L + u_L^c M_2 + e_L^c M_2$
$10_3 = 10_{+,-}$	0	$M_3 Q_L + u_L^c M_3 + e_L^c M_3$
$10_4 = 10_{+,+}$	0	$M_4 Q_L + u_L^c (M_4 + P) + e_L^c (M_4 - P)$
$10_5 = 10_{+,+}$	1	$M_5 Q_L + u_L^c (M_5 + N) + e_L^c (M_5 - N)$
$5_1 = 5_{+,-}$	0	$M'_1 \overline{d}_L^c + (M'_1 + N + P)\overline{L}$
$5_2 = 5_{-,+}$	0	$M'_2 \overline{d}_L^c + M'_2 \overline{L}$
$5_3 = 5_{-,+}$	0	$M'_3 \overline{D}_d + (M'_3 - N)\overline{H}_d$
$5_4 = 5_{+,-}$	0	$M'_4 D_u + (M'_4 - P) H_u$
$5_5 = 5_{-,+}$	0	$M'_5 \overline{d}_L^c + M'_5 \overline{L}$
$5_6 = 5_{+,-}$	0	$M'_6 \overline{d}_L^c + M'_6 \overline{L}$
$5_7 = 5_{+,-}^{t_5}$	-1	$M'_7 \overline{d_L^c} + (M'_7 - P)\overline{L}$
$5_8 = 5_{-,+}^{t_5}$	-1	$M'_8 \overline{d_L^c} + M'_8 \overline{L}$
$5_9 = 5_{+,-}^{t_5}$	-1	$M'_9 \overline{d}_L^c + (M'_9 - N)\overline{L}$
$5_{10} = 5_{+,+}^{t_5}$	-1	$M'_{10} \overline{d}_L^c + (M'_{10} + N + P)\overline{L}$

(B.2)

To get 3 generations of matter curves and 2 Higgs doublets of MSSM, taking into account the constraints in subsection 5.1, we make the following choice of the flux parameters, $P = -N = 1$, and

$$\begin{aligned} &M_1 = M_2 = M_3 = M_4 = -M_5 = 1 \\ &M'_1 = M'_3 = M'_4 = M'_8 = M'_{10} = 0 \\ &M'_2 = M'_5 = M'_6 = M'_9 = -M'_7 = -1 \end{aligned} \tag{B.3}$$

Using the property $\sum_i M_5^i = -\sum_i M_{10}^i = -3$, the localisation of Higgs curves are as $5^{H_u} = 5_{-,+}$, $\overline{5}^{H_d} = 5_{+,-}$, and the third generation like $10_1 = 10^{M_3}$, and $5_2 = 5^{M_3}$. The distribution of the matter curves is collected in the following table:

Curve in D_4 model II	$U_1^{\perp}$	Spectrum in MSSM	Z_2 parity
$10_1 = 10^{M_3} = (10_{+,-})_0$	0	$Q_L + u_L^c + e_L^c$	–
$10_2 = (10_{-,+})_0$	0	$Q_L + u_L^c + e_L^c$	–
$10_3 = (10_{+,-})_0$	0	$Q_L + u_L^c + e_L^c$	–
$10_4 = (10_{+,+})_0$	0	$Q_L + 2u_L^c$	$+$
$10_5 = (10_{+,+})_{t_5}$	1	$-Q_L - 2u_L^c$	–
$5_1 = (5_{+,-})_0$	0	–	$+$
$5_2 = 5^{M_3} = (5_{-,+})_0$	0	$-\overline{d}_L^c - \overline{L}$	–
$5_3 = (5^{H_u}_{-,+})_0$	0	H_u	$+$
$5_4 = (5^{H_d}_{+,-})_0$	0	$-\overline{H}_d$	$+$
$5_5 = 5^{M_1} = (5_{-,+})_0$	0	$-\overline{d}_L^c - \overline{L}$	–
$5_6 = 5^{M_2} = (5_{+,-})_0$	0	$-\overline{d}_L^c - \overline{L}$	–
$5_7 = (5_{+,-})_{-t_5}$	-1	$\overline{d}_L^c$	$+$
$5_8 = (5_{-,+})_{-t_5}$	-1	–	$+$
$5_9 = (5_{+,-})_{-t_5}$	-1	$-\overline{d}_L^c$	$+$
$5_{10} = (5_{+,+})_{-t_5}$	-1	–	$+$

(B.4)

From this spectrum, we learn that we have three families of fermions, an extra vector like pairs, $d_L^c + \overline{d}_L^c$, $Q_L + \overline{Q}_L$; and two $2(u_L^c + \overline{u}_L^c)$ which are expected to get a large mass if some of the singlet states acquire large VEV's. In this $\mathbb{D}_4$ model; there are only singlet flavons transforming in the representations $1_{+,+}$, $1_{+,-}$, $1_{-,+}$; with and without t_5 charges, they are classified as $(\vartheta_{p,q})_{0,\pm t_5}$ with $p, q = \pm 1$; they lead to the following order 4-couplings

- *Up-type quark Yukawa couplings*

  The allowed Yukawa couplings that are invariant under $\mathbb{D}_4 \times U_1^{\perp}$ are:

$$\begin{aligned} &(10_{+,-})_0 \otimes (10_{+,-})_0 \otimes (5^{H_u}_{-,+})_0 \otimes (\vartheta_{-,+})_0 \\ &(10_{-,+})_0 \otimes (10_{-,+})_0 \otimes (5^{H_u}_{-,+})_0 \otimes (\vartheta_{-,+})_0 \\ &(10_{+,-})_0 \otimes (10_{+,-})_0 \otimes (5^{H_u}_{-,+})_0 \otimes (\vartheta_{-,+})_0 \\ &(10_{+,-})_0 \otimes (10_{-,+})_0 \otimes (5^{H_u}_{-,+})_0 \otimes (\vartheta_{+,-})_0 \\ &(10_{-,+})_0 \otimes (10_{+,-})_0 \otimes (5^{H_u}_{-,+})_0 \otimes (\vartheta_{+,-})_0 \end{aligned} \tag{B.5}$$

• *Down-type quark Yukawa couplings*
The Yukawa couplings down-type are:

$$\begin{array}{l}
(10_{+,-})_0 \otimes (\overline{5}_{-,+})_0 \otimes (5^{H_d}_{+,-})_0 \otimes (\vartheta_{-,+})_0 \\
(10_{+,-})_0 \otimes (\overline{5}_{-,+})_0 \otimes (5^{H_d}_{+,-})_0 \otimes (\vartheta_{-,+})_0 \\
(10_{+,-})_0 \otimes (\overline{5}_{+,-})_0 \otimes (5^{H_d}_{+,-})_0 \otimes (\vartheta_{+,-})_0 \\
(10_{-,+})_0 \otimes (\overline{5}_{-,+})_0 \otimes (5^{H_d}_{+,-})_0 \otimes (\vartheta_{+,-})_0 \\
(10_{-,+})_0 \otimes (\overline{5}_{-,+})_0 \otimes (5^{H_d}_{+,-})_0 \otimes (\vartheta_{+,-})_0 \\
(10_{-,+})_0 \otimes (\overline{5}_{+,-})_0 \otimes (5^{H_d}_{+,-})_0 \otimes (\vartheta_{-,+})_0 \\
(10_{+,-})_0 \otimes (\overline{5}_{-,+})_0 \otimes (5^{H_d}_{+,-})_0 \otimes (\vartheta_{-,+})_0 \\
(10_{+,-})_0 \otimes (\overline{5}_{-,+})_0 \otimes (5^{H_d}_{+,-})_0 \otimes (\vartheta_{-,+})_0 \\
(10_{+,-})_0 \otimes (\overline{5}_{+,-})_0 \otimes (5^{H_d}_{+,-})_0 \otimes (\vartheta_{+,-})_0
\end{array} \tag{B.6}$$

B.2. $SU_5 \times \mathbb{D}_4$ model III

The spectrum of the model $SU_5 \times \mathbb{D}_4 \times U_1^{\perp}$ Model III is as follows

(B.7)

Curves in D_4 model III	$U_1^{\perp}$	Spectrum in MSSM
$10_1 = 10_{+,+}$	0	$M_1 Q_L + u^c_L(M_1 - N - P) + e^c_L(M_1 + N + P)$
$10_2 = 10_{-,-}$	0	$M_2 Q_L + u^c_L M_2 + e^c_L M_2$
$10_3 = 10_{+,-}$	0	$M_3 Q_L + u^c_L M_3 + e^c_L M_3$
$10_4 = 10_{+,+}$	0	$M_4 Q_L + u^c_L(M_4 + P) + e^c_L(M_4 - P)$
$10_5 = 10_{+,+}$	1	$M_5 Q_L + u^c_L(M_5 + N) + e^c_L(M_5 - N)$
$5_1 = 5_{+,+}$	0	$M'_1 \overline{d}^c_L + (M'_1 + N + P)\overline{L}$
$5_2 = 5_{-,-}$	0	$M'_2 \overline{d}^c_L + (M'_2 - \kappa_1 P)\overline{L}$
$5_3 = 5_{-,+}$	0	$M'_3 D_u + (M'_3 - N)\overline{H}_d$
$5_4 = 5_{+,+}$	0	$M'_4 \overline{D}_d + (M'_4 - \kappa_2 P) H_u$
$5_5 = 5_{-,-}$	0	$M'_5 \overline{d}^c_L + M'_5 \overline{L}$
$5_6 = 5_{+,-}$	0	$M'_6 \overline{d}^c_L + M'_6 \overline{L}$
$5_7 = 5^{t_5}_{+,+}$	-1	$M'_7 \overline{d}^c_L + (M'_7 - \kappa_1 P)\overline{L}$
$5_8 = 5^{t_5}_{-,-}$	-1	$M'_8 \overline{d}^c_L + {}^{(}M'_8 - \kappa_2 P)\overline{L}$
$5_9 = 5^{t_5}_{+,-}$	-1	$M'_9 \overline{d}^c_L + (M'_9 - N)\overline{L}$
$5_{10} = 5^{t_5}_{+,+}$	-1	$M'_{10} \overline{d}^c_L + (M'_{10} + N + P)\overline{L}$

The 3 generations of fermions and the 2 Higgs H_u, H_d are obtained by taking the fluxes like $N = -P = -1$ with $\kappa_1 = 0$, $\kappa_2 = 1$, and

$$M_1 = M_2 = M_3 = M_4 = -M_5 = 1$$

$$M'_1 = M'_3 = M'_4 = M'_7 = M'_{10} = 0$$
$$M'_2 = M'_5 = M'_6 = M'_8 = -M'_9 = -1 \tag{B.8}$$

We choose the Higgs curves as $5^{H_u} = (5^{H_u}_{+,+})_0$, $5^{H_d} = (5^{H_d}_{-,+})_0$ and the third 10^{M_3}, $\overline{5}^{M_3}$ generation as follow

Curves in D_4 model III	$U_1^{\perp}$	Spectrum in MSSM	Z_2 parity
$10_1 = 10^{M_3} = (10_{+,+})_0$	0	$Q_L + u^c_L + e^c_L$	$-$
$10_2 = (10_{-,-})_0$	0	$Q_L + u^c_L + e^c_L$	$-$
$10_3 = (10_{+,-})_0$	0	$Q_L + u^c_L + e^c_L$	$-$
$10_4 = (10_{+,+})_0$	0	$Q_L + 2e^c_L$	$+$
$10_5 = (10_{+,+})_{t_5}$	1	$-Q_L - 2e^c_L$	$-$
$5_1 = (5_{+,+})_0$	0	$-$	$+$
$5_2 = 5^{M_3} = (5_{-,-})_0$	0	$-\overline{d}^c_L - \overline{L}$	$-$
$5_3 = (5^{H_d}_{-,+})_0$	0	$-\overline{H}_d$	$+$
$5_4 = (5^{H_u}_{+,+})_0$	0	H_u	$+$
$5_5 = 5^{M_1} = (5_{-,-})_0$	0	$-\overline{d}^c_L - \overline{L}$	$-$
$5_6 = 5^{M_2} = (5_{+,-})_0$	0	$-\overline{d}^c_L - \overline{L}$	$-$
$5_7 = (5_{+,+})_{-t_5}$	-1	$-$	$+$
$5_8 = (5_{-,-})_{-t_5}$	-1	$-\overline{d}^c_L$	$+$
$5_9 = (5_{+,-})_{-t_5}$	-1	$\overline{d}^c_L$	$+$
$5_{10} = (5_{+,+})_{-t_5}$	-1	$-$	$+$

(B.9)

• *Up-type quark Yukawa couplings*

The allowed Yukawa couplings that are invariant under $\mathbb{D}_4 \times U_1^{\perp}$ and preserving parity symmetry are:

$$(10_{+,-})_0 \otimes (10_{+,-})_0 \otimes (5^{H_u}_{+,+})_0 \tag{B.10}$$

for third generation; and

$$(10_{-,-})_0 \otimes (10_{-,-})_0 \otimes (5^{H_u}_{+,+})_0$$
$$(10_{+,+})_0 \otimes (10_{+,+})_0 \otimes (5^{H_u}_{+,+})_0$$
$$(10_{+,+})_0 \otimes (10_{-,-})_0 \otimes (5^{H_u}_{+,+})_0 \otimes (\vartheta_{-,-})_0$$
$$(10_{+,+})_0 \otimes (10_{+,-})_0 \otimes (5^{H_u}_{+,+})_0 \otimes (\vartheta_{+,-})_0$$
$$(10_{-,-})_0 \otimes (10_{+,-})_0 \otimes (5^{H_u}_{+,+})_0 \otimes (\vartheta_{-,+})_0 \tag{B.11}$$

• *Down-type quark Yukawa couplings*

The Yukawa couplings down-type are:

$$(10_{+,+})_0 \otimes (\overline{5}_{-,-})_0 \otimes (5^{H_d}_{-,+})_0 \otimes (\vartheta_{+,-})_0 \tag{B.12}$$

for third generation, and

$$
\begin{aligned}
&(10_{+,+})_0 \otimes (\overline{5}_{-,-})_0 \otimes (5^{H_d}_{-,+})_0 \otimes (\vartheta_{+,-})_0\\
&(10_{+,+})_0 \otimes (\overline{5}_{+,-})_0 \otimes (5^{H_d}_{-,+})_0 \otimes (\vartheta_{-,-})_0\\
&(10_{-,-})_0 \otimes (\overline{5}_{-,-})_0 \otimes (5^{H_d}_{-,+})_0 \otimes (\vartheta_{-,+})_0\\
&(10_{-,-})_0 \otimes (\overline{5}_{-,-})_0 \otimes (5^{H_d}_{-,+})_0 \otimes (\vartheta_{-,+})_0\\
&(10_{-,-})_0 \otimes (\overline{5}_{+,-})_0 \otimes (5^{H_d}_{-,+})_0\\
&(10_{+,-})_0 \otimes (\overline{5}_{-,-})_0 \otimes (5^{H_d}_{-,+})_0\\
&(10_{+,-})_0 \otimes (\overline{5}_{-,-})_0 \otimes (5^{H_d}_{-,+})_0\\
&(10_{+,-})_0 \otimes (\overline{5}_{+,-})_0 \otimes (5^{H_d}_{-,+})_0 \otimes (\vartheta_{-,+})_0
\end{aligned}
\tag{B.13}
$$

For the neutrino sectors in both models II and III, the couplings are embedded in the Dirac and Majorana operators as for model I; their mass matrix depend on the choice of the localisation of right neutrino in the singlet curves $\vartheta_{\pm,\pm}$.

Appendix C. Monodromy and flavor symmetry

We begin by recalling that in F-theory GUTs, quantum numbers of particle fields and their gauge invariant interactions descend from an affine E_8 singularity in the internal Calabi–Yau Geometry: $CY4 \sim \mathcal{E} \to \mathcal{B}_3$. The observed gauge bosons, the 4D matter generations and the Yukawa couplings of standard model arise from symmetry breaking of the underlying E_8 gauge symmetry of compactification of F-theory to 4D space time.

In this appendix, we use known results on F-theory GUTs to exhibit the link between non-abelian monodromy and flavor symmetry which relates the three flavor generations of SM. First, we briefly describe how abelian monodromy like $\mathbb{Z}_p$ appear in F-GUT models; then we study the extension to non-abelian discrete symmetries such the dihedral $\mathbb{D}_4$ we have considered in present study.

C.1. Abelian monodromy

One of the interesting field realisations of the F-theory approach to GUT is given by the remarkable $SU_5 \times SU_5^{\perp}$ model with basic features encoded in the internal geometry; in particular the two following useful ones: (i) the $SU_5 \times SU_5^{\perp}$ invariance follows from a particular breaking way of E_8; and (ii) the full spectrum of the field representations of the model is as in eq. (2.1). From the internal CY4 geometry view, SU_5 and $SU_5^{\perp}$ have interpretation in terms of singularities; the SU_5 lives on the so called GUT surface $\mathcal{S}_{GUT}$; it appears in terms of the singular locus of the following Tate form of the elliptic fibration $y^2 = x^3 + b_5xy + b_4x^2z + b_3yz^2 + b_2xz^3 + b_0z^5$; it is the gauge symmetry visible in 4D space time of the GUT model. Quite similarly, the $SU_5^{\perp}$ may be also imagined to have an analogous geometric representation in the internal geometry, but with different physical interpretation it lives as well on a complex surface $\mathcal{S}'$, another divisor of the base $\mathcal{B}_3$ of the complex four dimensional elliptic CY4 fibration. Obviously these two divisors are different, but intersect. Here, we want to focus on aspects of the representations of $SU_5^{\perp}$ appearing in eq. (2.1) and too particulary on the associated matter curves Σ_{t_i}, $\Sigma_{t_i+t_j}$, $\Sigma_{t_i-t_j}$, which are nicely described in the spectral cover method using an extra spectral parameter s. If thinking of the hidden $SU_5^{\perp}$ in terms of a broken symmetry by an abelian flux or Higgsing down to its Cartan subgroup, the resulting symmetry of the GUT model becomes $U(1)^4 \times SU_5$ with[4]

$$U(1)^4 = U(1)_1 \times U(1)_2 \times U(1)_3 \times U(1)_4 \equiv \prod_{i=1}^{4} U(1)_i \tag{C.1}$$

The extra $U(1)$'s in the breaking $U(1)^4 \times SU_5$ put constraints on the superpotential couplings of the effective low energy model; the simultaneous existence of $U(1)^4$ is phenomenologically undesirable since it does not allow a tree-level Yukawa coupling for the top quark. This ambiguity is overcome by imposing abelian monodromies among the $U(1)$'s allowing the emergence of a rank one fermion mass matrix structure, see eqs. (C.4)–(C.5) given below.

Following the presentation of section 2 of this paper, the spectral covers describing the above invariance are given by polynomials with an affine variable s as in eq. (2.8), see also (2.9), (2.12), (2.13). To fix the ideas, we consider monodromy properties of 10-plets Σ_{t_i} encoded in the spectral cover equation

$$\mathcal{C}_5 : b_5 + b_3 s^2 + b_2 s^3 + b_4 s^4 + b_0 s^5 = 0 \tag{C.2}$$

The location of the seven branes on GUT surface associated to this SU_5 representation is given by $b_5 = 0$. Using the method of [18,27,30,31], the possible abelian monodromies are $\mathbb{Z}_2$, $\mathbb{Z}_3$, $\mathbb{Z}_4$, $\mathbb{Z}_2 \times \mathbb{Z}_3$ and $\mathbb{Z}_2 \times \mathbb{Z}_2$; they lead to factorisations of the $\mathcal{C}_5$ spectral cover as

$$\mathcal{C}_2 \times (\mathcal{C}_1)^3 \quad , \quad \mathcal{C}_3 \times (\mathcal{C}_1)^2 \quad , \quad \mathcal{C}_4 \times \mathcal{C}_1 \quad , \quad \mathcal{C}_3 \times \mathcal{C}_2 \quad , \quad (\mathcal{C}_2)^2 \times \mathcal{C}_1 \tag{C.3}$$

and to the respective identification of the weights $\{t_1, t_2\}$, $\{t_1, t_2, t_3\}$, $\{t_1, t_2, t_3, t_4\}$, $\{t_1, t_2\} \cup \{t_3, t_4, t_5\}$ and $\{t_1, t_2\} \cup \{t_3, t_4\}$.

The algebraic equations for the matter curves Σ_{t_i}, $\Sigma_{t_i+t_j}$, $\Sigma_{t_i-t_j}$ in terms of the t_i weights associated with the $SU_5^{\perp}$ fundamental representation are respectively given by $t_i = 0$, $\left(t_i + t_j\right)_{i<j} = 0$ and $\pm\left(t_i - t_j\right)_{i<j} = 0$, they are denoted like 10_{t_i}, $\bar{5}_{t_i+t_j}$ and $1_{\pm(t_i-t_j)}$, see eq. (2.2).

As a first step to approach non-abelian monodromies we are interested in here, it is helpful to notice the two useful following things: (a) the homology 2-cycles in the CY4 underlying $SU_5 \times U(1)^4$ invariance has monodromies captured by a finite discrete group that can be used as a constraint in the modelling. (b) From the view of phenomenology, these monodromies must be at least $\mathbb{Z}_2$ in order to have top-quark Yukawa coupling at tree level as noticed before. Notice moreover that under this $\mathbb{Z}_2$, matter multiplets of the SU_5 model split into two $\mathbb{Z}_2$ sectors[5]: even and odd; for example the two tenplets $\left\{10_{t_1}, 10_{t_2}\right\}$ are interchanged under $t_1 \leftrightarrow t_2$; the corresponding eigenstates are given by $10_{t_\pm}$ with eigenvalues ± 1. By requiring the identification

---

[4] Recall the three useful relations: (a) Let $\vec{H} = (H_1, \ldots, H_4)$ the generators of the $U(1)_i$ charge factors and $E_{\pm\alpha_i}$ the step operators associated with the simple roots $\vec{\alpha}_i$, then we have $\left[E_{+\alpha_i}, E_{-\alpha_i}\right] = \vec{\alpha}_i.\vec{H}$. ($b$) If denoting by $|\vec{\mu}\rangle$ a weight vector of the fundamental representation of $SU_5^{\perp}$, then we have $\vec{\alpha}_i.\vec{H}\,|\vec{\mu}\rangle = \lambda_i\,|\vec{\mu}\rangle$ with $\lambda_i = \vec{\alpha}_i.\vec{\mu}$. ($c$) using the 4 usual fundamental weight vectors $\vec{\omega}_i$ dual to the 4 simple roots, the 5 weight vectors $\{\vec{\mu}_k\}$ of the representation are: $\vec{\mu}_1 = \vec{\omega}_1$, $\vec{\mu}_2 = \vec{\omega}_2 - \vec{\omega}_1$, $\vec{\mu}_3 = \vec{\omega}_3 - \vec{\omega}_2$, $\vec{\mu}_4 = \vec{\omega}_4 - \vec{\omega}_3$, $\vec{\mu}_5 = -\vec{\omega}_4$.

[5] In general we have two $\mathbb{Z}_2$ eigenstates: $t_\pm = \frac{1}{2}(t_1 \pm t_2)$ with eigenvalues ± 1. While any function of t_+ is $\mathbb{Z}_2$ invariant, only those functions depending on $(t_-)^2$ which are symmetric with respect to $\mathbb{Z}_2$.

$t_1 \leftrightarrow t_2$, naively realised by setting $t_1 = t_2 = t$, matter couplings in the model get restricted; therefore the off diagonal tree level Yukawa coupling

$$10_{t_1}.10_{t_2}.5_{-t_1-t_2} \tag{C.4}$$

which is invariant under $SU_5 \times U(1)^4$, becomes after $t_1 \leftrightarrow t_2$ identification a diagonal top-quark interaction invariant under $\mathbb{Z}_2$ monodromy. The resulting Yukawa coupling reads as follows [27, 30,31]

$$10_t.10_t.5_{-2t} \tag{C.5}$$

the other diagonal coupling $10_0.10_0.5_{-2t}$ is forbidden by the $U(1)$ symmetry; *see footnote 5*. Notice that for bottom-quark the typical Yukawa coupling $10_t.\bar{5}_{t_i+t_j}.\bar{5}_{t_k+t_l}$ is allowed by $\mathbb{Z}_2$ while $10_0.\bar{5}_{t_i+t_j}.\bar{5}_{t_k+t_l}$ is forbidden.

In this monodromy invariant theory, the symmetry of the model is given by $SU_5 \times U(1)^3 \times \mathbb{Z}_2$; it may be interpreted as the invariance that remains after taking the coset with respect to $\mathbb{Z}_2$; that is by a factorisation of type $G = H \times \mathbb{Z}_2$ with $H = G/\mathbb{Z}_2$. Indeed, starting from $SU_5 \times U(1)^4$ and performing the two following operations: (i) use the traceless property of the *fundamental* representation of $SU_5^\perp$ to think of (C.1) like

$$U(1)^4 = \left(\prod_{i=1}^{5} U(1)_{t_i}\right)/\mathcal{J} \tag{C.6}$$

with $\mathcal{J} = \{t_i \mid t_1 + t_2 + t_3 + t_4 + t_5 = 0\} \simeq U(1)_{diag}$. This property is a rephrasing of the usual $U(5)$ factorisation, i.e. $SU(5) = \frac{U(5)}{U(1)}$. (ii) Substitute the product $U(1)_{t_1} \times U(1)_{t_2}$ by the reduced abelian group $U(1)_t \times \mathbb{Z}_2$ where monodromy group has been explicitly exhibited. In this way of doing, one disposes of a discrete group that may be promoted to a symmetry of the fields spectrum. To that purpose, we need two more steps: first explore all allowed discrete monodromy groups; and second study how to link these groups to flavor symmetry. For the extension of above $\mathbb{Z}_2$, a similar method can be used to build other prototypes; in particular models with abelian discrete symmetries like $SU_5 \times U(1)^{5-k} \times \mathbb{Z}_k$ with $k = 3, 4, 5$; or more generally as

$$SU_5 \times U(1)^{5-p-q} \times \mathbb{Z}_p \times \mathbb{Z}_q \tag{C.7}$$

where $1 < p + q \leq 5$ and $\mathbb{Z}_1 \equiv I_{id}$, $\mathbb{Z}_0 \equiv I_{id}$. Notice that the discrete groups in eq. (C.7) are natural extensions of those of the theories with $SU_5 \times U(1)^{5-k} \times \mathbb{Z}_k$ symmetry; and that the condition $p + q \leq 5$ on allowed abelian monodromies is intimately related with the Weyl symmetry $\mathcal{W}_{SU_5^\perp}$ of $SU_5^\perp$. Therefore, we end with the conclusion that the $\mathbb{Z}_p \times \mathbb{Z}_q$ abelian discrete groups in above relation are in fact particular subgroups of the non-abelian symmetric group $\mathcal{W}_{SU_5^\perp} \simeq \mathbb{S}_5$.

C.2. Non-abelian monodromy and flavor symmetry

To begin notice that the appearance of abelian discrete symmetry in the SU_5 based GUT models with invariance (C.7) is remarkable and suggestive. It is remarkable because these finite discrete symmetries have a geometric interpretation in the internal CY4, and constitutes then a prediction of F-theory GUT. It is suggestive since such kind of discrete groups, especially their non-abelian generalisation, are highly desirable in phenomenology, particularly in playing the role of a flavor symmetry. In this regards, it is interesting to recall that it is quite well established that neutrino flavors are mixed; and this property requires non-abelian discrete group symmetries like the alternating $\mathbb{A}_4$ group which has been subject to intensive research during last decade [32–34,52,53].

Following the conjecture of [15,16], non-abelian discrete symmetries may be reached in F-theory GUT by assuming the existence of a non-abelian flux breaking the $SU_5^{\perp}$ down to a non-abelian group $\Gamma \subset \mathcal{W}_{SU_5^{\perp}}$. In this view, one may roughly think about the $\mathbb{Z}_p \times \mathbb{Z}_q$ group of (C.7) as special symmetries of a family of SU_5 based GUT models with invariance given by

$$SU_5 \times U(1)^{5-k} \times \Gamma_k \tag{C.8}$$

where now Γ_k is a subgroup of $\mathbb{S}_5$ that can be a non-abelian discrete group. In this way of doing, one then distinguishes several SU_5 GUT models with non-abelian discrete symmetries classified by the number of surviving $U(1)$'s. In presence of no $U(1)$ symmetry, we have prototypes like $SU_5 \times \mathbb{S}_5$ and $SU_5 \times \mathbb{A}_5$; while for a theory with one $U(1)$, we have symmetries as follows

$$\begin{aligned} &SU_5 \times U(1) \times \mathbb{S}_4 \\ &SU_5 \times U(1) \times \mathbb{A}_4 \\ &SU_5 \times U(1) \times \mathbb{D}_4 \end{aligned} \tag{C.9}$$

where the alternating $\mathbb{A}_4$ and dihedral $\mathbb{D}_4$ are the usual subgroups of $\mathbb{S}_4$ itself contained in $\mathbb{S}_5$. In the case with two $U(1)$'s, monodromy gets reduced like $SU_5 \times U(1)^2 \times \mathbb{S}_3$.

Moreover, by using non-abelian discrete monodromy groups Γ_k, one ends with an important feature; these discrete groups have, in addition to trivial representations, higher dimensional representations that are candidates to host more than one matter generation. Under transformations of Γ_k, the generations get in general mixed. Therefore the non-abelian Γ_k's in particular those having 3- and/or 2-dimensional irreducible representations may be naturally interpreted in terms of flavor symmetry.

In the end of this section, we would like to add a comment on the splitting spectral cover construction regarding non-abelian discrete monodromy groups like $\mathbb{A}_4$ and $\mathbb{D}_4$. In the models (C.9), the spectral cover for the fundamental $\mathcal{C}_5$ is factorised like $\mathcal{C}_5 = \mathcal{C}_4 \times \mathcal{C}_1$ and similarly for $\mathcal{C}_{10}$ and $\mathcal{C}_{20}$ respectively associated with the antisymmetric and the adjoint of $SU_5^{\perp}$. In the $\mathcal{C}_4 \times \mathcal{C}_1$ splitting, we have

$$\begin{aligned} &\mathcal{C}_4 = a_5 s^4 + a_4 s^3 + a_3 s^2 + a_2 s + a_1 \\ &\mathcal{C}_1 = a_7 s + a_6 \end{aligned} \tag{C.10}$$

where the a_is are complex holomorphic sections. For the generic case where the coefficients a_i are free, the splitted spectral cover $\mathcal{C}_4 \times \mathcal{C}_1$ has an $\mathbb{S}_4$ monodromy. To have splitted spectral covers with monodromies given by the subgroups $\mathbb{A}_4$ and $\mathbb{D}_4$, one needs to put constraints on the a_i's; these conditions have been studied in [14,16]; they are non linear relations given by Galois theory. Indeed, starting from $SU_5 \times SU_5^{\perp}$ model and borrowing tools from [16], the breaking of $SU_5 \times SU_5^{\perp}$ down to $SU_5 \times \mathbb{D}_4 \times U(1)$ model considered in this paper may be imagined in steps as follows: first breaking $SU_5^{\perp}$ to subgroup $SU_4^{\perp} \times U(1)$ by an abelian flux; then breaking the $SU_4^{\perp}$ part to the discrete group $\mathbb{S}_4$ by a non-abelian flux as conjectured in [15,16]; deformations of this flux lead to subgroups of $\mathbb{S}_4$. To obtain the constraints describing the $\mathbb{D}_4$ splitted spectral cover descending from $\mathcal{C}_4 \times \mathcal{C}_1$, we use Galois theory; they are given by a set of two constraints on the holomorphic sections of $\mathcal{C}_4 \times \mathcal{C}_1$; and are obtained as follows:

(i) The first constraint comes from the discriminant $\Delta_{\mathcal{C}_4}$ of the spectral cover $\mathcal{C}_4$ which should not be a perfect square; that is $\Delta_{\mathcal{C}_4} \neq \delta^2$. The explicit expression of the discriminant of $\mathcal{C}_4$ has been computed in literature; so we have

$$108a_0(\lambda a_6^2 + 4a_1a_7)(\kappa^2 a_7^2 + a_0(\lambda a_6^2 + 4a_1a_7))^2 \neq \delta^2 \tag{C.11}$$

where dependence into a_6 and a_7 is due to solving the traceless condition $b_1 = 0$ in $\mathcal{C}_5 = \mathcal{C}_4 \times \mathcal{C}_1$. ($ii$) The second constraint is given by a condition on the cubic resolvent which should be like $R_{\mathcal{C}_4}(s)\big|_{s=0} = 0$. The expression of $R_{\mathcal{C}_4}(s)$ is known; it leads to

$$a_2^2 a_7 = a_1\left(a_0a_6^2 + 4a_3a_7\right) \tag{C.12}$$

where a_0 is a parameter introduced by the solving the traceless condition $b_1 = 0$; for explicit details see [16].

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