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Hybrid Legendre polynomials and Block-Pulse functions approach for nonlinear Volterra–Fredholm integro-differential equations

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ABSTRACT

This paper introduces an approach for obtaining the numerical solution of the nonlinear Volterra–Fredholm integro-differential (NVFID) equations using hybrid Legendre polynomials and Block-Pulse functions. These hybrid functions and their operational matrices are used for representing matrix form of these equations. The main characteristic of this approach is that it reduces NVFID equations to a system of algebraic equations, which greatly simplifying the problem. Numerical examples illustrate the validity and applicability of the proposed method.

1. Introduction

Integral equations have been one of the principal tools in various areas of applied mathematics, physics and engineering. In this paper we are concerned with the NVFID equations [1]. These types of equations were introduced by Volterra for the first time. Volterra investigated the population growth on the topic of integro-differential equations [2]. Scientists have investigated the topic of integro-differential equations through their work in many scientific applications such as heat transfer, diffusion process in general, neutron diffusion and biological species coexisting together with increasing and decreasing rates of generating [1,2]. The NVFID equations arise in neurosciences [3].

The aim of this work is to present a numerical method for approximating the solution of NVFID equations of the form:

$$\begin{cases} u'(x) + q(x)u(x) + \lambda_1 \displaystyle\int_0^1 k_1(x,s)\psi_1(s,u(s))\mathrm{d}s + \lambda_2 \int_0^x k_2(x,s)\psi_2(s,u(s))\mathrm{d}s = f(x), \\ u(0) = u_0, \quad 0 \le s < 1, \end{cases} \tag{1}$$

where the parameters λ_1, λ_2 and functions $q(x), f(x), \psi_1(s,u(s)), \psi_2(s,u(s)), k_1(x,s)$ and $k_2(x,s)$ are known and belong to $L^2[0,1)$. $u(x)$ is the unknown function. In this work we suppose $\psi_1(s,u(s)) = u(s)^\alpha$ and $\psi_2(s,u(s)) = u(s)^\beta$ where α, β are positive integers.

In recent years, many different basic functions have been used to estimate the solution of integral equations, such as orthogonal functions and wavelets. Three families of the orthogonal functions are classified: 1-(PCOF) Piecewise Constant Orthogonal Functions (e.g., Walsh, Block-Pulse, Haar, etc.), 2-Orthogonal polynomials (e.g., Legendre, Laguerre, Chebyshev, etc.) and 3-Sine–Cosin functions in the Fourier series. For more information on these orthogonal functions, see [4–13].

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In this paper, we use the hybrid Legendre polynomials and Block-Pulse functions to solve NVFID equations of the form Eq. (1). These hybrid function methods had been used for some Volterra and Fredholm integral equations [13], control problems [14,15], linear time-varying descriptor systems [16] and simple form of integro-differential equations [10,17] beforehand. We present hybrid function's useful properties such as product matrix, integration of the cross product, operational matrix of integration and coefficient matrix to solve NVFID equations.

This paper is organized as follows. In Section 2, we introduce hybrid functions and its properties. In Section 3, we apply these sets of hybrid functions for approximating the solution of NVFID equations. Numerical results are reported in Section 4. Finally, Section 5 concludes the paper.

2. Hybrid functions and some of their properties

The orthogonal set of hybrid functions $h_{ij}(x),\ i = 1, 2, \ldots, n,\ j = 0, 1, \ldots, m-1$, where i is the order for Block-Pulse functions, j is the order for Legendre polynomials and x is the normalized time, is defined on the interval $[0, 1)$ as

$$h_{ij}(x) = \begin{cases} L_j(2nx - 2i + 1), & \dfrac{i-1}{n} \le x < \dfrac{i}{n}, \\ 0, & \text{otherwise}. \end{cases} \tag{2}$$

Here, the Legendre polynomials $L_m(x)$ defined in the interval $[-1, 1]$ are given by

$$L_0(x) = 1, \qquad L_1(x) = x,$$
$$(m+1)L_{m+1}(x) = (2m+1)xL_m(x) - mL_{m-1}(x), \quad m = 1, 2, 3, \ldots.$$

The set of $\{L_m(x) : m = 0, 1, \ldots\}$ in Hilbert space $L^2[-1, 1]$ is a complete orthogonal system.

A set of Block-Pulse functions $b_i(x),\ i = 1, 2, \ldots, n$ on the interval $[0, 1)$ is defined as follows

$$b_i(x) = \begin{cases} 1, & \dfrac{i-1}{n} \le x < \dfrac{i}{n}, \\ 0, & \text{otherwise}. \end{cases} \tag{3}$$

The Block-Pulse functions on $[0, 1)$ are disjoint, so for $i, j = 1, 2, \ldots, n$, we have $b_i(x)b_j(x) = \delta_{ij}b_i(x)$, also these functions have the property of orthogonality on $[0, 1)$.

Since $h_{ij}(x)$ is the combination of Legendre polynomials and Block-Pulse functions which are both complete and orthogonal, then the set of hybrid functions is a complete orthogonal system in $L^2[0, 1)$.

2.1. Function approximation

Any function $u(x) \in L^2[0, 1)$ can be expanded in a hybrid function

$$u(x) = \sum_{i=1}^{\infty}\sum_{j=0}^{\infty} c_{ij}h_{ij}(x), \tag{4}$$

where the hybrid coefficients are given by $c_{ij} = \frac{(u(x),h_{ij}(x))}{(h_{ij}(x),h_{ij}(x))}$ for $i = 1, 2, \ldots, \infty,\ j = 0, 1, \ldots, \infty$, such that $(\cdot, \cdot)$ denotes the inner product.

Usually, the series expansion Eq. (4) contains an infinite number of terms for a smooth $u(x)$. If $u(x)$ is piecewise constant or may be approximated as piecewise constant, then the sum in Eq. (4) may be terminated after nm terms, that is

$$u(x) \simeq \sum_{i=1}^{n}\sum_{j=0}^{m-1} c_{ij}h_{ij}(x) = C^T\mathbf{h}(x), \tag{5}$$

where

$$C = [c_{10}, \ldots, c_{1,m-1}, c_{20}, \ldots, c_{2,m-1}, \ldots, c_{n0}, \ldots, c_{n,m-1}]^T, \tag{6}$$
$$\mathbf{h}(x) = [h_{10}(x), \ldots, h_{1m-1}(x), h_{20}(x), \ldots, h_{2m-1}(x), \ldots, h_{nm-1}(x)]^T. \tag{7}$$

We can also approximate the function $k(x, s) \in L^2([0, 1) \times [0, 1))$ as follows

$$k(x, s) \simeq \mathbf{h}^T(x)K\mathbf{h}(s), \tag{8}$$

where K is an $nm \times nm$ matrix that $K_{ij} = \frac{\left(\mathbf{h}_{(i)}(x), \left(k(x,s), \mathbf{h}_{(j)}(s)\right)\right)}{\left(\mathbf{h}_{(i)}(x), \mathbf{h}_{(i)}(x)\right)\left(\mathbf{h}_{(j)}(s), \mathbf{h}_{(j)}(s)\right)}$ for $i, j = 1, 2, \ldots, nm$.

2.2. Operational matrix of integration

The integration of the vector $\mathbf{h}(x)$ defined in Eq. (7) is given by

$$\int_0^x \mathbf{h}(x')\mathrm{d}x' \simeq P\mathbf{h}(x), \tag{9}$$

where P is the $nm \times nm$ operational matrix for integration and is given in [18] as

$$P = \begin{bmatrix} E & H & H & \cdots & H \\ O & E & H & \cdots & H \\ O & O & E & \cdots & H \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ O & O & O & \cdots & E \end{bmatrix}, \tag{10}$$

that E and H are $m \times m$ matrices that have the following shapes,

$$H = \frac{1}{n}\begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}, \tag{11}$$

$$E = \frac{1}{2n}\begin{bmatrix} 1 & 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\ -\frac{1}{3} & 0 & \frac{1}{3} & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\ 0 & -\frac{1}{5} & 0 & \frac{1}{5} & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{7} & 0 & \frac{1}{7} & \cdots & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{9} & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & \frac{1}{2m-9} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & \frac{-1}{2m-7} & 0 & \frac{1}{2m-7} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & \frac{-1}{2m-5} & 0 & \frac{1}{2m-5} & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & \frac{-1}{2m-3} & 0 & \frac{1}{2m-3} \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \frac{-1}{2m-1} & 0 \end{bmatrix}. \tag{12}$$

Pattern of operational matrix P by $n = 4$ and $m = 8$ appears in Fig. 1.

2.3. The integration of the cross product

The integration of the cross product of two hybrid function vectors $\mathbf{h}(x)$ in Eq. (7) can be obtained as

$$D = \int_0^1 \mathbf{h}(x)\mathbf{h}^T(x)\mathrm{d}x = \begin{bmatrix} L & O & \cdots & O \\ O & L & \cdots & O \\ \vdots & \vdots & \ddots & \vdots \\ O & O & \cdots & L \end{bmatrix}, \tag{13}$$

where L is an $m \times m$ diagonal matrix that is given by

$$L = \frac{1}{n}\begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & \frac{1}{3} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \frac{1}{2m-1} \end{bmatrix}. \tag{14}$$

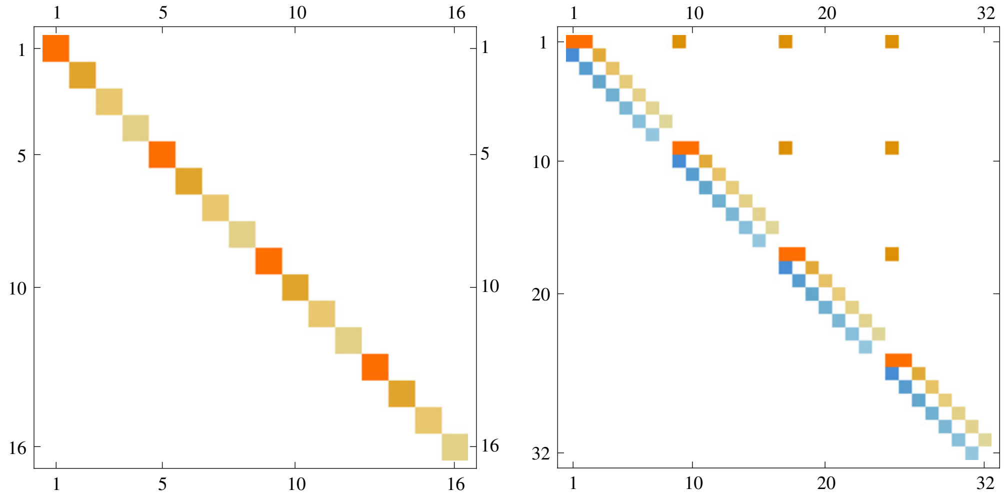

Fig. 1. Patterns of the matrices D (left) and P (right).

The efficacy of matrix D is used for converting the Fredholm part of NVFID equations to an algebraic equation. Because of its diagonal shape it can increase the calculating speed. Fig. 1 shows the pattern of matrix D when $n = 4$ and $m = 4$.

2.4. Product operational matrix

It is always necessary to evaluate the product of $\mathbf{h}(x)$ and $\mathbf{h}^T(x)$, that is called the product matrix of hybrid functions. Let

$$\mathbf{H}(x) = \mathbf{h}(x)\mathbf{h}^T(x), \tag{15}$$

where $\mathbf{H}(x)$ is $nm \times nm$ matrix. Multiplying the matrix $\mathbf{H}(x)$ by vector C that defined in Eq. (6) we obtain

$$\mathbf{H}(x)C = \widetilde{C}\mathbf{h}(x), \tag{16}$$

where $\widetilde{C}$ is $nm \times nm$ matrix and called the coefficient matrix. To illustrate the calculation procedure in Eq. (16), we consider that $n = 2$, $m = 8$ [13] we have

$$\widetilde{C} = \begin{bmatrix} \widetilde{C}_1 & O \\ O & \widetilde{C}_2 \end{bmatrix},$$

where C_i, $i = 1, 2$ are 8×8 matrices given by

$$\widetilde{C}_i = \begin{bmatrix}
c_{i0} & c_{i1} & c_{i2} & c_{i3} & \cdots & c_{i7} \\
1/3c_{i1} & \begin{matrix} c_{i0} \\ +2/5c_{i2} \end{matrix} & \begin{matrix} 2/3c_{i1} \\ +3/7c_{i3} \end{matrix} & \begin{matrix} 3/5c_{i2} \\ +4/9c_{i4} \end{matrix} & \cdots & 7/13c_{i6} \\
1/5c_{i2} & \begin{matrix} 2/5c_{i1} \\ +9/35c_{i3} \end{matrix} & \begin{matrix} c_{i0} \\ +2/7c_{i2} \\ +2/7c_{i4} \end{matrix} & \begin{matrix} 3/5c_{i1} \\ +4/15c_{i3} \\ +10/33c_{i5} \end{matrix} & \cdots & \begin{matrix} 63/143c_{i5} \\ +56/221c_{i7} \end{matrix} \\
1/7c_{i3} & \begin{matrix} 9/35c_{i2} \\ +4/21c_{i4} \end{matrix} & \begin{matrix} 3/7c_{i1} \\ +4/21c_{i3} \\ +50/231c_{i5} \end{matrix} & \begin{matrix} c_{i0} \\ +4/15c_{i2} \\ +2/11c_{i4} \\ +100/429c_{i6} \end{matrix} & \cdots & \begin{matrix} 175/429c_{i4} \\ +504/2431c_{i6} \end{matrix} \\
\vdots & \vdots & \vdots & \vdots & \cdots & \vdots \\
1/15c_{i7} & 7/65c_{i6} & \begin{matrix} 21/143c_{i5} \\ +56/663c_{i7} \end{matrix} & \begin{matrix} 245/1287c_{i4} \\ +1176/12\,155c_{i6} \end{matrix} & \cdots & \begin{matrix} c_{i0} \\ +56/221c_{i2} \\ +6804/46\,189c_{i4} \\ +5000/46\,189c_{i6} \end{matrix}
\end{bmatrix}. \tag{17}$$

With the powerful properties of Eq. (16) we can convert the Volterra part of NVFID equations to an algebraic equation.

3. Numerical solution of NVFID equations using hybrid functions

Consider the NVFID equation (1). The unknown function $u(x)$ can be expanded as

$$u(x) \simeq U^T\mathbf{h}(x), \tag{18}$$

where U is the unknown nm-vector and $\mathbf{h}(x)$ is given by Eq. (7). Likewise, $k_1(x,s)$, $k_2(x,s)$, $q(x)$ and $f(x)$ are also expanded into the hybrid functions

$$k_1(x,s) \simeq \mathbf{h}^T(x)K_1\mathbf{h}(s), \qquad k_2(x,s) \simeq \mathbf{h}^T(x)K_2\mathbf{h}(s), \tag{19}$$

$$f(x) \simeq F^T\mathbf{h}(x), \qquad q(x) \simeq Q^T\mathbf{h}(x), \tag{20}$$

where K_1, K_2 are $nm \times nm$-matrices and F is an nm-vector. We approximate $u'(x)$ as follows

$$u'(x) \simeq U'^T\mathbf{h}(x), \tag{21}$$

which U' will be evaluated in terms of U.

$$u(x) - u(0) = \int_0^x u'(\eta)\mathrm{d}\eta \simeq \int_0^x U'^T\mathbf{h}(\eta)\mathrm{d}\eta \simeq U'^T P\mathbf{h}(x). \tag{22}$$

If we expand $u(0)$ with hybrid basis i.e. $u(0) = U_0\mathbf{h}(x)$, then U_0 is obtained as follows:

$$U_0 = [\underbrace{\overbrace{u(0),0,\ldots,0}^{m}, \overbrace{u(0),0,\ldots,0}^{m}, \ldots, \overbrace{u(0),0,\ldots,0}^{m}}_{nm}]^T,$$

and we have

$$u(x) \simeq U'^T P\mathbf{h}(x) + U_0^T\mathbf{h}(x). \tag{23}$$

Therefore,

$$U \simeq P^T U' + U_0. \tag{24}$$

After substituting the approximate equations (18)–(20) into (1) we get

$$\begin{aligned} &U'^T\mathbf{h}(x) + Q^T\mathbf{h}(x)\mathbf{h}^T(x)U + \lambda_1\mathbf{h}^T(x)K_1\int_0^1 \mathbf{h}(s)\psi_1(s, U^T\mathbf{h}(s))\mathrm{d}s \\ &\quad + \lambda_2\mathbf{h}^T(x)K_2\int_0^x \mathbf{h}(s)\psi_2(s, U^T\mathbf{h}(s))\mathrm{d}s \simeq F^T\mathbf{h}(x). \end{aligned} \tag{25}$$

Functions $\psi_1(s, U^T\mathbf{h}(s)) = (U^T\mathbf{h}(s))^\alpha$ and $\psi_2(s, U^T\mathbf{h}(s)) = (U^T\mathbf{h}(s))^\beta$ are known and can be expanded into the hybrid functions as

$$\begin{aligned} (u(s))^\alpha &\simeq U_\alpha^T\mathbf{h}(s), \\ (u(s))^\beta &\simeq U_\beta^T\mathbf{h}(s). \end{aligned} \tag{26}$$

In the next subsection, we consider computing U_α and U_β in terms of U, where U_α, U_β are mn-vectors whose elements are nonlinear combination of the elements of the vector U. Substituting Eq. (26) in Eq. (25) produces

$$U'^T\mathbf{h}(x) + Q^T\mathbf{h}(x)\mathbf{h}^T(x)U + \lambda_1\mathbf{h}^T(x)K_1\int_0^1 \mathbf{h}(s)\mathbf{h}^T(s)U_\alpha\mathrm{d}s + \lambda_2\mathbf{h}^T(x)K_2\int_0^x \mathbf{h}(s)\mathbf{h}^T(s)U_\beta\mathrm{d}s \simeq F^T\mathbf{h}(x), \tag{27}$$

where $\int_0^x \mathbf{h}(s)\mathbf{h}^T(s)U_\beta\mathrm{d}s = \int_0^x \widetilde{U_\beta}\mathbf{h}(s)\mathrm{d}s = \widetilde{U_\beta}P\mathbf{h}(x)$, making use of Eqs. (16) and (13) and operational matrix P, we get

$$U'^T\mathbf{h}(x) + Q^T\widetilde{U}\mathbf{h}(x) + \lambda_1\mathbf{h}^T(x)(K_1DU_\alpha) + \lambda_2\mathbf{h}^T(x)K_2\widetilde{U_\beta}P\mathbf{h}(x) \simeq F^T\mathbf{h}(x). \tag{28}$$

If we approximate the fourth term of Eq. (28) with hybrid basis we achieve

$$\mathbf{h}^T(x)(K_2\widetilde{U_\beta}P)\mathbf{h}(x) \simeq \widehat{U_\beta}\mathbf{h}(x). \tag{29}$$

We can achieve $\widehat{U_\beta}$ by a way like $\widetilde{C}$ and we see that each element of $\widehat{U_\beta}$ is obtained by the sum of column elements of $(K_2\widetilde{U_\beta}P)$ with respect to coefficient $\widetilde{C}$ in Eq. (16) at each column. By using this property and omitting hybrid vector functions in Eq. (28), we will have

$$U'^T + Q^T\widetilde{U} + \lambda_1(K_1DU_\alpha)^T + \lambda_2\widehat{U_\beta} \simeq F^T. \tag{30}$$

Another equivalent form is

$$U' + \widetilde{U}^T Q + \lambda_1(K_1DU_\alpha) + \lambda_2\widehat{U_\beta}^T \simeq F, \tag{31}$$

multiplying matrix P^T on both sides of Eq. (31) and applying Eq. (24) in Eq. (31) we get

$$U - U_0 + P^T\widetilde{U}^T Q + \lambda_1 P^T(K_1DU_\alpha) + \lambda_2 P^T\widehat{U_\beta}^T \simeq P^T F. \tag{32}$$

After replacing $\simeq$ with $=$, we have a nonlinear system that can be solved with Newton's method for the unknown vector U, then by the use of $u(x) \simeq U^T\mathbf{h}(x)$ the approximated solution is given.

Table 1
Approximate and exact solutions for Example 4.1.

x	Solution with $n=2, m=8$	Solution with $n=4, m=8$	Solution with $n=8, m=8$	Method in [19] with $m=16$	Exact
0.0	0.000000	0.000000	0.000000	0.000000	0
0.1	0.010917	0.010256	0.010031	0.010978	0.01
0.2	0.041703	0.040487	0.040075	0.040702	0.04
0.3	0.092364	0.090698	0.090171	0.090736	0.09
0.4	0.162911	0.160866	0.160094	0.161077	0.16
0.5	0.253371	0.250997	0.250228	0.250164	0.25
0.6	0.364244	0.361061	0.360502	0.361120	0.36
0.7	0.493830	0.490969	0.490583	0.490819	0.49
0.8	0.642375	0.640830	0.640374	0.640819	0.64
0.9	0.810337	0.810183	0.810047	0.811118	0.81
1.0	0.998506	0.999660	0.999986	1.000149	1

3.1. Evaluating U_α and U_β

For numerical implementation of the method explained in the previous section, we need to evaluate U_α and U_β. The elements of each one are nonlinear combination of the elements of the vector U. From (16) and (18). We have

$$\begin{aligned}(u(x))^2 &\simeq (U^T\mathbf{h}(x))(U^T\mathbf{h}(x)) = U^T\mathbf{h}(x)\mathbf{h}^T(x)U\\ &= U^T\widetilde{U}\mathbf{h}(x) = U_2\mathbf{h}(x),\end{aligned} \tag{33}$$

where the vector $U_2 = U^T\widetilde{U}$ is an mn-row vector, then for $(u(s))^3$ we get

$$\begin{aligned}(u(x))^3 &\simeq (U^T\mathbf{h}(x))(U_2\mathbf{h}(x)) = U^T\mathbf{h}(x)\mathbf{h}^T(x)U_2^T\\ &= U^T\widetilde{U_2^T}\mathbf{h}(x) = U_3\mathbf{h}(x).\end{aligned} \tag{34}$$

Therefore, with this method we can approximate $(u(s))^\alpha$ and $(u(s))^\beta$ for arbitrary α and β. Suppose that this method holds for $\alpha-1$ where $(u(x))^{\alpha-1} = U_{\alpha-1}\mathbf{h}(x)$, we obtain it for α as follows

$$\begin{aligned}(u(x))^\alpha &= u(x)u(x)^{\alpha-1} \simeq (U^T\mathbf{h}(x))(U_{\alpha-1}\mathbf{h}(x))\\ &= U^T\mathbf{h}(x)\mathbf{h}^T(x)U_{\alpha-1}^T\\ &= U^T\widetilde{U_{\alpha-1}^T}\mathbf{h}(x) = U_\alpha\mathbf{h}(x),\end{aligned} \tag{35}$$

we have similar relation for β. So, the components of U_α and U_β can be computed in terms of components of unknown vector U.

4. Numerical examples

In this section we implemented our method on four different examples. Our results achieved by a proper value for m (this feather is experimental) and different values for n. The results are tabulated in four tables, in these tables the exact solutions are compared with hybrid function solutions and also in the first example we compared hybrid functions results by triangular functions results [19] for NVFID equations. It is noticed that our method has quite acceptable results but it is clear for lower values of n we have less accuracy in some end points of the interval that by increasing n, the results become better.

We consider the following examples.

Example 4.1. Consider the NVFID equation, as follows:

$$u'(x) + u(x) + \frac{1}{2}\int_0^x xu^2(s)\mathrm{d}s - \frac{1}{4}\int_0^1 su^3(s)\mathrm{d}s = f(x), \tag{36}$$

where $f(x) = 2x + x^2 + \frac{1}{10}x^6 - \frac{1}{32}$, with the initial condition $u(0) = 0$, and the exact solution $u(x) = x^2$ [19]. The comparison among the hybrid solution with $n = 2, m = 8, n = 4, m = 8$ and $n = 8, m = 8$ besides the solutions of triangular functions [19] and exact solutions are shown in Table 1.

Example 4.2. Consider the following nonlinear Volterra integro-differential equation,

$$u'(x) - \int_0^x \cos(x-s)u^2(s)\mathrm{d}s = -2\sin x - \frac{1}{3}\cos x - \frac{2}{3}\cos(2x), \tag{37}$$

with the initial condition $u(0) = 1$, and the exact solution $u(x) = \cos x - \sin x$ [8]. The comparison among the hybrid solution with $n = 2, m = 8, n = 4, m = 8$ and $n = 8, m = 8$ besides the exact solutions are shown in Table 2.

Table 2
Approximate and exact solutions for Example 4.2.

x	Solution with $n=2, m=8$	Solution with $n=4, m=8$	Solution with $n=8, m=8$	Exact
0.0	0.999987	0.999995	0.999999	1
0.1	0.894924	0.894912	0.895186	0.895170
0.2	0.779971	0.780797	0.781653	0.781397
0.3	0.657525	0.659114	0.659732	0.659816
0.4	0.529719	0.530699	0.530699	0.531642
0.5	0.398671	0.397870	0.398169	0.398157
0.6	0.260321	0.259787	0.260969	0.260693
0.7	0.121015	0.120360	0.120671	0.120624
0.8	−0.017300	−0.020600	−0.020638	−0.020649
0.9	−0.152906	−0.161466	−0.161638	−0.161716
1.0	−0.284295	−0.298740	−0.301983	−0.301168

Table 3
Approximate and exact solutions for Example 4.3.

x	Solution with $n=2, m=8$	Solution with $n=4, m=8$	Solution with $n=8, m=8$	Exact
0.0	0.999999	0.999999	0.999999	1
0.1	1.091923	1.098183	1.100625	1.1
0.2	1.189700	1.197715	1.200373	1.2
0.3	1.291151	1.298043	1.300626	1.3
0.4	1.393565	1.399590	1.400681	1.4
0.5	1.493661	1.498178	1.500599	1.5
0.6	1.603138	1.605163	1.601830	1.6
0.7	1.716121	1.706799	1.702132	1.7
0.8	1.827504	1.812136	1.806721	1.8
0.9	1.931639	1.920991	1.913578	1.9
1.0	2.022688	2.015233	2.009838	2

Table 4
Approximate and exact solutions for Example 4.4.

x	Solution with $n=2, m=8$	Solution with $n=4, m=8$	Solution with $n=8, m=8$	Exact
0.0	−0.000000	0.000077	0.000032	0
0.1	0.099435	0.099801	0.099825	0.099833
0.2	0.198304	0.198740	0.198678	0.198669
0.3	0.295493	0.295664	0.295603	0.295520
0.4	0.389688	0.390016	0.389605	0.389418
0.5	0.479311	0.480537	0.479398	0.479425
0.6	0.562965	0.566730	0.563598	0.564642
0.7	0.640005	0.647439	0.642606	0.644217
0.8	0.708103	0.721968	0.715049	0.717356
0.9	0.764843	0.790216	0.779882	0.783326
1.0	0.807845	0.849043	0.837683	0.841470

Example 4.3. Consider the NVFID equation, as follows:

$$u'(x) + x^2u(x) - \int_0^x (x-s)u^2(s)\mathrm{d}s + \int_0^1 \mathrm{e}^s u(s)\mathrm{d}s = f(x), \tag{38}$$

where $f(x) = 1+e+\frac{x^2}{2}+\frac{2x^3}{3}-\frac{x^4}{12}$, with the initial condition $u(0) = 1$, and the exact solution $u(x) = x+1$. The comparison among the hybrid solution with $n = 2, m = 8, n = 4, m = 8$ and $n = 8, m = 8$ besides the exact solutions are shown in Table 3.

Example 4.4. Consider the following nonlinear Volterra integro-differential equation,

$$u'(x) + u(x) - 2\int_0^x \sin(x)u^2(s)\mathrm{d}s = \cos x + (1-x)\sin x + \cos x \sin^2 x, \tag{39}$$

with the initial condition $u(0) = 0$, and the exact solution $u(x) = \sin x$. The comparison among the hybrid solution with $n = 2, m = 8, n = 4, m = 8$ and $n = 8, m = 8$ besides the exact solutions are shown in Table 4.

5. Conclusion

The hybrid Legendre polynomials and Block-Pulse functions operational matrices of integration D, operational matrix P, product matrix H and coefficient matrix $\widetilde{C}$ which are sparse matrices, are used to converting an NVFID equation to a nonlinear system of equations that can be solved by known iterative methods. By making use of these operational matrices, the problem has been reduced to solve a set of algebraic equations that can simply appeared in matrix form. The solution obtained using the suggested method shows that this approach can solve NVFID equations effectively. Although we do not claim this method shows superiority over other methods from the viewpoint of accuracy, it seems that this method is more practical, quite good accurate and has lower calculation.

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