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Kaluza-Klein geometry*

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Abstract: We formulate a Kaluza-Klein theory in terms of short exact sequences of vector bundles.

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1. Introduction

Kaluza-Klein theory has been developed in a number of geometrical settings and from various points of view. A most general setting involves viewing Kaluza-Klein space (the multidimensional universe) as a fibered manifold E over a manifold M (spacetime) $\pi : E \to M$. The geometry in such a setting was worked out by O'Neill [9], modulo the obvious modifications to the semi-Riemannian case, and more recently discussed by Hogan [6]. The principal fiber bundle case $E = P$ was developed by Cho [3] and Kopczynski [7], while the generalization from P to the case where the standard fiber is a homogeneous space G/H was discussed by Coquereaux and Jadczyk [4, 5] and Percacci and Randjbar [10]. The text [1] contains some of the old and recent papers. I apologize for omissions in this brief historical overview.

In all of these cases the Kaluza-Klein metric $\bar{g}$ is a fiber metric on TE and the basic assumptions are such as to force a splitting of the short exact sequence $VE \hookrightarrow TE \to E \times TM$ of vector bundles over E. Then $\bar{g}$ splits into a gauge field potential σ (which is the splitting map) and fiber metrics $\bar{g}, g$ on $VE, E \times TM$, and g eventually gets identified with a metric on M.

In this paper we look at the above situation in the general framework of splittings of short exact sequences $A \hookrightarrow B \to C$ of vector bundles over some manifold and derive results on (1) the splitting of fiber metrics on B into their constituent parts and (2) the relation of the invariances of the parts to those of $\bar{g}$. After applying this to the settings mentioned above, we also formulate the Kaluza-Klein theory in terms of the

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short exact sequence:

$$VP/G \hookrightarrow TP/G \to TM$$

of vector bundles over M. The theory here has a number of advantages.

2. Preliminary generalities

As a preface to the discussion of Kaluza-Klein theory we present here some generalities on its underlying geometrical structure. The mechanism for splitting the extended gravity metric into a gauge field, a spacetime metric, and a fiber metric arises in the more general setting of splittings of short exact sequences of vector bundles.

2.1. Short Exact Sequences of Vector Bundles. Suppose N is a manifold and $0 \to A \hookrightarrow B \to C \to 0$ is a short exact sequence of vector bundles over N:

$$\begin{array}{ccccc} A & \xrightarrow{\ i\ } & B & \xrightarrow{\ \beta\ } & C \\ & {\scriptstyle \pi_A} \searrow & \downarrow {\scriptstyle \pi_B} & \swarrow {\scriptstyle \pi_C} & \\ & & N. & & \end{array}$$

Here π_A, π_B, π_C are the projections on the base space N, i and β are linear fibered morphisms over N with i injective, β surjective and Ker $\beta =$ Im i. In the sequel (for convenience) we will always consider A as a subbundle of $B : A \subset B$, and i as the inclusion map. In general a linear fibered morphism $\alpha : A \to B$ is fiber preserving map which is linear on the fibers. Denote the induced map on the base by $\alpha_N : N \to N$ and then $\alpha(x, a_x) = (\alpha_N(x), \alpha_x a_x)$ where $\alpha_x : A_x \to B_{\alpha_N(x)}$ is linear. When α induces the identity on the base: $\alpha_N = 1$, then α is called a linear fibered morphism over N. There is a natural category whose objects are such short exact sequences (A, B, C) and whose morphisms $\phi : (A, B, C) \to (A', B', C')$ are triples $\phi = (\phi_A, \phi_B, \phi_C)$ of linear fibered morphisms such that the following diagram commutes:

$$\begin{array}{ccccc} A & \xrightarrow{\ i\ } & B & \xrightarrow{\ \beta\ } & C \\ {\scriptstyle \phi_A} \downarrow & & {\scriptstyle \phi_B} \downarrow & & \downarrow {\scriptstyle \phi_C} \\ A' & \xrightarrow[\ i'\ ]{} & B' & \xrightarrow[\ \beta'\ ]{} & C'. \end{array}$$

In particular a morphism from (A, B, C) to itself consists of a pair of maps ϕ_B and ϕ_C which intertwine $\beta : \beta'\phi_B = \phi_C \beta$ and such that ϕ_B leaves A invariant, i.e., $\phi_B(A) \subset A$. Then $\phi = (\phi_B|_A, \phi_B, \phi_C) \equiv (\phi_B, \phi_C)$ is a morphism in the above sense.

2.2. Connections (Splitting maps). For any bundle B over N we let $\Gamma(B)$ denote the space of sections $s : N \to B$. Observe that any linear morphism $\alpha : A \to B$, over N can also be viewed, in the obvious way, as a section $\alpha : N \to \mathrm{Hom}(A, B)$ of the bundle of linear fibered morphisms from A to B, i.e., $\alpha \in \Gamma(\mathrm{Hom}(A, B))$.

A connection for the short exact sequence (A, B, C), is a linear fibered morphism $\sigma : C \to B$ over N such that $\beta\sigma = 1$ (the identity map on C). Each connection σ (also called a splitting map) induces a splitting (decomposition):

$$B = A \oplus \sigma(C)$$

where the splitting of an element b_x in the fiber B_x over $x \in N$ is given by

$$b_x = (b_x - \sigma_x\beta_x b_x) + \sigma_x\beta_x b_x.$$

Note that there is also a corresponding decomposition of the sections, i.e., a bijection $\Gamma(B) \to \Gamma(A) \times \Gamma(C)$. Let $\mathcal{C}(C, B) \subset \mathrm{Hom}(C, B)$ denote the subbundle of $\mathrm{Hom}(C, B)$ whose fiber over $x \in N$ consists of those linear maps $\sigma_x : C_x \to B_x$ such that $\beta_x\sigma_x = 1$. Then we can alternatively view a connection $\sigma : C \to B$ as a section $\sigma : N \to \mathcal{C}(C, B)$. Note that each isomorphism $\phi = (\phi_B, \phi_C)$ of the short exact sequence (A, B, C) induces a map $\phi_{\mathcal{C}} : \mathcal{C}(C, B) \to \mathcal{C}(C, B)$ on the bundle of connections:

$$\phi_{\mathcal{C}}(\sigma) \equiv \phi_B\sigma\phi_C^{-1}.$$

2.3. Fiber metrics. Let $S^2(A)$ be the vector bundle of symmetric forms on A, i.e., an element $g_x^A \in S^2(A)_x$ in the fiber over x is a symmetric bilinear map $g_x^A : A_x \times A_x \to \mathbf{R}$. The bundle of metrics on A is the subbundle $\mathcal{M}(A) \subset S^2(A)$ with fibers consisting of non-degenerate forms ($g_x^A(a_x, a'_x) = 0$ for every a_x implies that $a'_x = 0$). The sections $g^A : N \to \mathcal{M}(A)$ are the fiber metrics on A. For each linear fibered morphism $\alpha : A \to B$ such that the base map α_N is bijective one gets a corresponding linear fibered morphism $\alpha^* : S^2(B) \to S^2(A)$, defined by $\alpha^*(x, g_x^B) = (y, \alpha_x^* g_x^B)$ where $y = \alpha_N^{-1}(x)$ and

$$\alpha_x^*(g_x^B)(a_y, a'_y) \equiv g_x^B(\alpha_y a_y, \alpha_y a'_y).$$

This also gives a map (pullback map) on the sections $\alpha^* : \Gamma S^2(B) \to \Gamma S^2(A)$. Now S^2 is a contravariant functor which (for obvious reasons) does not preserve exactness (likewise for the functor $\mathcal{M}$), i.e.,

$$S^2(A) \xleftarrow{i^*} S^2(B) \xleftarrow{\beta^*} S^2(C)$$

is not exact. However one can achieve a splitting (decomposition) of a certain subbundle $\mathcal{M}_A(B)$ of $S^2(B)$ by the following Kaluza-Klein mechanism: For the exact sequence $A \xrightarrow{i} B \xrightarrow{\beta} C$ define a fibered morphism

$$\mathfrak{g} : \mathcal{C}(C, B) \oplus S^2(A) \oplus S^2(C) \to S^2(B)$$

by

$$\mathfrak{g}_x(\sigma_x, g_x^A, g_x^C) = (1 - \sigma_x\beta_x)^* g_x^A + \beta_x^* g_x^C.$$

Otherwise said,

$$\mathfrak{g}_x(\sigma_x, g_x^A, g_x^C)(b_x, b'_x) = g_x^A(b_x - \sigma_x\beta_x b_x, b'_x - \sigma_x\beta_x b'_x) + g_x^C(\beta_x b_x, \beta_x b'_x).$$

This induces a corresponding map on the sections

$$\mathfrak{g} : \Gamma\mathcal{C}(B, C) \times \Gamma S^2(A) \times \Gamma S^2(C) \to \Gamma S^2(B)$$

given by

$$\mathfrak{g}(\sigma, g^A, g^C) = (1 - \sigma\beta)^* g^A + \beta^* g^C.$$

Thus the mapping $\mathfrak{g}$ generates symmetric forms on B from symmetric forms on A, C and a connection (splitting) of

$$A \xrightarrow{i} B \underset{\sigma}{\overset{\beta}{\rightleftarrows}} C.$$

Let $\mathcal{M}_A(B)$ be the subbundle of $\mathcal{M}(A)$ defined by $\mathcal{M}_A(B) = (i^*)^{-1}\mathcal{M}(A)$. This is the bundle of metrics on B whose restrictions to A are nondegenerate. For $h \in \Gamma\mathcal{M}_A(B)$ let $A^\perp(h)$ be the subbundle of B which is orthogonal to A. Then $B = A \oplus A^\perp(h)$. Define a map $s : \Gamma\mathcal{M}_A(B) \to \Gamma\mathcal{C}(C, B)$ as follows:

$$s(h)_x(c_x) = P_x b_x,$$

where $b_x \in \beta_x^{-1}\{c_x\}$ and $P_x : B_x \to A^\perp(h)_x$ is the orthogonal projection.

Theorem 1. *The mapping*

$$\mathfrak{g} : \Gamma\mathcal{C}(B, C) \times \Gamma S^2(A) \times \Gamma S^2(C) \to \Gamma S^2(B)$$

has the following properties:

(1) $i^*\mathfrak{g}(\sigma, g^A, g^C) = g^A$.

(2) $\sigma^*\mathfrak{g}(\sigma, g^A, g^C) = g^C$.

(3) $\sigma(C) \subseteq A^\perp(\mathfrak{g}(\sigma, g^A, g^C))$ *and equality holds if and only if* g^A *is nondegenerate.*

(4) $\mathfrak{g}(\sigma, g^A, g^C)$ *is nondegenerate if and only if both* g^A *and* g^C *are nondegenerate. In this case* $s(\mathfrak{g}(\sigma, g^A, g^C)) = \sigma$.

(5) $\mathfrak{g} : \Gamma\mathcal{C}(B, C) \times \Gamma\mathcal{M}(A) \times \Gamma\mathcal{M}(C) \to \Gamma\mathcal{M}_A(B)$ *is a bijection.* (Indeed (1), (2) and (4) show that $\mathfrak{g}$ is injective. On the other hand if g^B is a metric on B which is nondegenerate on A then it's easy to check that

$$\mathfrak{g}(s(g^B), i^*g^B, s(g^B)^*g^B) = g^B$$

and thus $\mathfrak{g}$ is surjective.) *The connection* $\sigma \equiv s(g^B)$ *and metrics* $g^A \equiv i^*g^B$, $g^C \equiv s(g^B)^*g^B$ *are the Kaluza-Klein components of the extended gravity metric* g^B.

(6) *Suppose $\phi = (\phi_A, \phi_B, \phi_C) : (A, B, C) \to (A, B, C)$ is an isomorphism of the exact sequence, then*

$$\phi_B^* \mathfrak{g}(\sigma, g^A, g^C) = \mathfrak{g}(\phi_C^{-1*}\sigma, \phi_A^* g^A, \phi_C^* g^C)$$

where $\phi_C^{-1*}\sigma = \phi_B^{-1} \circ \sigma \circ \phi_C$ (and of course $\phi_A = \phi_B|_A$, the restriction of ϕ_B to A). Thus in particular if g^A and g^C are metrics then ϕ_B is an isometry of the Kaluza-Klein metric $g^B \equiv \mathfrak{g}(\sigma, g^A, g^C)$ if and only if $\phi_A = \phi_B|_A$ and ϕ_C are isometries of g^A and g^C, and ϕ_C^{-1} leaves σ invariant:

$$\phi_C^{-1*}\sigma = \sigma.$$

3. Kaluza-Klein spaces

In a most general setting, a Kaluza-Klein space (multidimensional universe) is a fibered manifold E over a base manifold M (spacetime) with $\pi : E \to M$ denoting the projection. For our purposes it suffices to assume that E is a fiber bundle. The metric structure arises from the foregoing generalities by specializing the exact sequence $A \to B \to C$ of vector bundles over N to the exact sequence

$$\begin{array}{ccccc} VE & \longrightarrow & TE & \longrightarrow & E \times TM \\ & \searrow & \downarrow & \swarrow & \\ & & E & & \end{array}$$

of vector bundles over E. Here VE is the usual vertical subbundle of TE ($Z_e \in V_eE$ iff $d\pi\,|_e\, Z_e = 0$), and $E \times TM = \{(e, X_x) \mid e \in E, X_x \in T_xM \text{ and } \pi(e) = x\}$ is the fibered product of E and TM as bundles over M. The map $\beta : TE \to E \times TM$ is given by $\beta(e, Z_e) = (e, d\pi\,|_e\, Z_e)$.

3.1. Connections on E. In keeping with the foregoing a connection on E is linear fibered morphism $\sigma : E \times TM \to TE$ over E such that $\beta\sigma = 1$, i.e., with the notation $\sigma(e, X_e) = (e, \sigma_e X_x)$, $\sigma_e : T_xM \to T_eE$ is a linear map such that $d\pi|_e \sigma_e X_x = X_x$. Thus σ gives rise to a horizontal lifting map $\sigma : \Gamma TM \to \Gamma TE$ defined by $\sigma(X)(e) = (e, \sigma_e X_{\pi(e)})$. The previous notation for the bundle $\mathcal{C}(E \times TM, TE)$ of connections associated with the short exact sequence will be abbreviated to $\mathcal{C}(TE) \equiv \mathcal{C}(E \times TM, TE)$, and is referred to as the bundle of connections on E. $\mathcal{C}(TE)$ can be thought of as a subbundle of the bundle $A^1(M, TE) = T^*M \otimes TE$ and a connection σ on E is also viewed as a section $\sigma : E \to \mathcal{C}(TE) \subseteq (A^1M, TE)$, i.e., a horizontal, tangent-valued form which projects to the identity.

3.2. Kaluza-Klein metrics. In keeping with the customery terminology, a metric g^B on E is just a fiber metric on TE, i.e., $g^B \in \Gamma\mathcal{M}(TE)$. A Kaluza-Klein metric on E is a metric g^B on E which is nondegenerate on VE, i.e., $g^B \in \Gamma\mathcal{M}_{VE}(TE)$. The general theory gives us that g^B is represented by:

$$g^B = \mathfrak{g}(\sigma, g^A, g^C)$$

where σ is a connection on E (viewed as a gauge field potential), g^A is a fiber metric on $A = VE$ and g^C is a fiber metric on $C = E \times TM$. Without further assumptions g^C is not a fiber metric on TM, i.e., a metric on M. Thus g^C is viewed as a generalized spacetime metric, since in realistic physical theories the standard fiber F of Kaluza-Klein space is small and thus locally and approximately $E \times TM \cong (M \times F) \times TM \cong M \times TM \cong TM$. Alternately and more precisely we will see below that the requirement that the Kaluza-Klein metric g^B possesses certain types of isometries forces g^C to arise from a metric on M. We note here however that the map $\pi' : E \times TM \to TM$ given by $\pi'(e, X_x) = (\pi(e), X_x)$ induces a pullback $\pi'^* : \Gamma\mathcal{M}(TM) \to \Gamma\mathcal{M}(E \times TM)$ which is injective and defining $\mathfrak{g}_\# = \mathfrak{g} \circ (1 \times 1 \times \pi'^*)$ gives an injection $\mathfrak{g}_\# : \Gamma\mathcal{C}(E) \times \Gamma\mathcal{M}(VE) \times \Gamma\mathcal{M}(TM) \to \Gamma\mathcal{M}_{VE}(TE)$. One could thus restrict attention to those Kaluza-Klein metrics which are in the image of $\mathfrak{g}_\#$.

3.3. Isomorphisms of $(VE, TE, E \times TM)$. There is a natural group of isomorphisms for the short exact sequence $(VE, TE, E \times TM)$. Namely consider those fiber morphisms $f : E \to E$ which are diffeomorphisms and let $f_M\ M \to M$ denote the induced diffeomorphism on the base space. Then $(Tf|_{VE}, Tf, f \times Tf_M)$ is easily seen to be an isomorphism of $(VE, TE, E \times TM)$. For the corresponding pullback maps $(Tf|_{VE}{}^*, Tf^*, (f \times Tf_M)^*)$ acting on the fiber metrics, it is customary to write Tf^* as f^*, and so likewise we will abbreviate the notation for the other pullbacks induced by $f : Tf|_{VE}{}^* \equiv f^*$, $(f \times Tf_M)^* \equiv f^*$. Also note that for a fiber metric g^C on $E \times TM$ the pullback induced by f works out to be:

$$(f^* g^C)_e(X_x, Y_x) = g^C_{f(e)}(df_M|_x\, X_x, df_M|_x\, Y_x).$$

The pullback action induced by f on connections $\sigma : E \times TM \to TE$ ($\sigma(e, X_x) \equiv (e, \sigma_e X_e)$, and $\sigma_e : T_x M \to T_e E$) has the following formula. First by definition $f^*(\sigma) \equiv Tf \circ \sigma \circ (f \times Tf_M)^{-1}$ and so (using f^{-1} instead of f for notational convenience):

$$(f^{-1*}\sigma)_e X_x = df^{-1}|_{f(e)}\, \sigma_{f(e)} df_M|_x\, X_x.$$

Now suppose that K is a group of fibered bundle isomorphisms $f E \to E$. Let $\Gamma_K\mathcal{M}TE$ denote the set of metrics g^B on E which are invariant under K (K-invariant metrics): $f^* g^B = g^B$ for every $f \in K$ (i.e. each f is an isometry of g^B). Similarly $\mathcal{C}_K(TE)$ denotes the set of K-invariant connections on E, etc. Then by Theorem 1, part (5), the restriction $\mathfrak{g}_K$ of $\mathfrak{g}$ gives us a bijection

$$\mathfrak{g}_K : \Gamma_K\mathcal{C}(TE) \times \Gamma_K\mathcal{M}(VE) \times \Gamma_K\mathcal{M}(E \times TM) \to \Gamma_K\mathcal{M}(TE).$$

Hence in particular we obtain the following special case of this.

Proposition 1. *Suppose K is a group of vertical, fibered isomorphisms $f : E \to E$ which is transitive, i.e., each f induces the identity on the base $f_M = 1$ and the restriction of K to each fiber E_x is transitive. Then each K-invariant, Kaluza-Klein metric g^B on E is uniquely represented by*

$$g^B = \mathfrak{g}_K(\sigma, g^A, g)$$

where

(1) *σ is a K-invariant connection on E, i.e. $\sigma_{f(e)} = df\,|_e\,\sigma_e$ for every $f \in K$;*
(2) *g^A is a K-invariant fiber metric on VE;*
(3) *g is a metric on M.*

3.4. Examples (1) *Principal fiber bundles over M.* Let $K = \{R_a \mid a \in G\}$ where $R_a : P \to P$ is right multiplication by a: $R_a u = ua$. The K-invariant metrics on P are, in this case called equivariant metrics, or G-invariant metrics:

$$g^B_{ua}(dR_a|_u\, Z_u, dR_a|_u\, Z'_u) = g_u(Z_u, Z'_u).$$

The G-invariant connections on P are precisely the principal connections on P. The content of Proposition 1 can also be improved in this case since VP has a canonical vertical splitting $\alpha : P \times \mathcal{G} \to VP$ given by $\alpha(u,\xi) \equiv (u, d\lambda\,|_e\,\xi_e)$ where $\lambda_u : G \to P$ is defined by $\lambda_u(a) = ua$. Here $\mathcal{G}$ is the Lie algebra of G (the set of left invariant vector fields on G, $\mathcal{G} \cong T_eG$, where e is the identity element of G). Note that G has a right action on $P\times\mathcal{G} : (u,\xi)g \equiv (ug, \mathrm{Ad}_{g^{-1}}\xi)$ and relative to this action $\alpha : P\times\mathcal{G} \to VP$ is an equivariant map (and also a linear fibered isomorphism over P). Thus $\alpha^* : \Gamma\mathcal{M}(VP) \to \Gamma\mathcal{M}(P \times \mathcal{G})$ is a bijection establishing a one to one correspondence between the fiber metrics on VP and $P \times \mathcal{G}$. Additionally g^A is a G-invariant (equivariant) fiber metric on VP if and only if $\alpha^* g^A$ is an equivariant fiber metric on $P \times \mathcal{G}$ (sometimes thought of as invariance with respect to the adjoint action $\mathrm{Ad}_a : \mathcal{G} \to \mathcal{G}$). This is so since if g^A is equivariant then

$$\begin{aligned}
(\alpha^* g^A)_{ua}(\mathrm{Ad}_{a^{-1}}\xi, \mathrm{Ad}_{a^{-1}}\xi') &= g^A_{ua}(d\lambda_{ua}|_e\,(\mathrm{Ad}_a\xi)_e, d\lambda_{ua}|_e\,(\mathrm{Ad}_a\xi')_e) \\
&= g^A_{ua}(dR_u|_a\, d\lambda_u|_e\,\xi_e, dR_u|_a\, d\lambda_u|_e\,\xi'_e) \\
&= g^A_u(d\lambda_u|_e\,\xi_e, d\lambda_u|_e\,\xi'_e) \\
&= (\alpha^* g^A)_u(\xi, \xi').
\end{aligned}$$

And similarly, equivariance of $\alpha^* g^A$ implies equivariance of g^A.

(2) *Riemannian submersions.* The paper [6] of Hogan, based on the work of O'Neill [9] provides a general setting for the Kaluza-Klein geometry. They consider a fibered manifold $\pi : E \to M$ over M and assume that E and M are Riemannian manifolds with Riemannian metrics $\bar{g}$ and g, respectively, and that $\pi^* g = \bar{g}$ on $(VE)^{\perp}$. In this case then $\bar{g} = g_{\#}(s(\bar{g}), i^*\bar{g}, g)$. They give (among other things) a calculation of the Ricci scalars of g and $i^* g$, and the curvature of the gauge field potential $s(\bar{g})$. (All of this relies upon the fibers $E_x = \pi^{-1}\{x\}$ being totally geodesic submanifolds).

4. Kaluza-Klein theory on the adjoint bundle

In this section we advocate the use of the adjoint bundle sequence as a convenient framework for the Kaluza-Klein theory. The main advantage in this is that all the differential forms of interest, like the gauge fields F^σ and their potentials σ, as well as the Riemann curvature tensor $\Omega^{\sigma,g}$ are actual differential forms on M with values in a vector bundle over M. This is not the case if one uses the sequence $VE \to TE \to E \times TM$ to formulate the theory, since one has to deal with the forms on E with values in bundles over E. In this case one can use the calculus of tangent valued forms and the Frölicher-Nijenhuis bracket advocated by Mangiarotti and Modugno [8] to develope the Kaluza-Klein theory. This will be presented in a forthcoming paper.

The adjoint bundle approach to Kaluza-Klein theory arises from taking a principal bundle sequence: $VP \to TP \to P \times TM$ and taking quotients by G to obtain the adjoint bundle sequence:

$$\begin{array}{ccccc} VP/G & \longrightarrow & TP/G & \longrightarrow & E \times TM \\ & \searrow & \downarrow & \swarrow & \\ & & M. & & \end{array}$$

As described in a previous paper [2] there is a basic functor from the category of equivariant bundles over P into the category of bundles over M which takes various equivariant geometric structures and forms on P over into their counterparts on M. In particular TP is an equivariant bundle over P with right action by $a \in G$ given by $(u, Z_u)a \equiv (ua, dR_a|_u Z_u)$. Then TP/G is just the bundle of equivalence classes: $\langle(u, Z_u)\rangle \in TP/G$ determined by this equivalence relation on TP. As such there is a one-to-one correspondence between sections $\tau : M \to TP/G$ of TP/G and equivariant vector fields $Z : P \to TP$ on P. Thus each τ is represented by $\tau(x) = \langle(u, Z_u)\rangle$, $u \in \pi^{-1}\{x\}, Z$ equivariant. Note that for $f \in C^\infty(M)$ the section $f\tau$ corresponds to $(f \circ \pi)Z$, i.e. $f(x)\tau(x) = \langle(u, f \circ \pi(u)Z_u)\rangle$. The sections of TP/G form a Lie algebra with Lie bracket defined by $[\tau, \tau'](x) \equiv \langle(u, [Z, Z']_u)\rangle$. The map $\beta : TP/G \to TM$ is defined by $\beta\langle(u, Z_u)\rangle = (\pi(u), d\pi|_u Z_u)$ and gives rise to a Lie algebra epimorphism $\# : \Gamma(TP/G) \to \Gamma(TM)$ defined by $\tau \to \tau^\# = \beta \circ \tau$. Note that for $f \in C^\infty(M)$, $(f\tau)^\# = f(\tau^\#)$ and as a vector field on M acting on C^∞-functions: $\tau^\#(f)(x) = Z_u(f \circ \pi)$ where $u \in \pi^{-1}\{x\}$ and Z is the equivariant vector field on P representing τ. Hence the sections of VP/G form the kernel of $\# : \Gamma(VP/G) = \text{Ker}(\#)$, and as such $\Gamma(VP/G)$ is a Lie subalgebra of $\Gamma(TP/G)$. Thus $\nu \in \Gamma(VP/G)$ if and only if $\nu^\#(f) = 0$ for every f. $\text{Ad}P \equiv VP/G \cong (P \times \mathcal{G})/G$ is known as the adjoint bundle of P. Finally it is important to note that the Lie bracket on the sections of TP/G has the property

$$[\tau, f\theta] = \tau^\#(f)\theta + f[\tau, \theta].$$

4.1. Connections on $VP/G \to TP/G \to TM$. Here, as before, a connection σ is a linear fibered morphism $\sigma : TM \to TP/G$ over M such that $\beta\sigma = 1$. Thus for each vector field X on M, $\sigma(X)$ on M, $\sigma(X)$ is the section of TP/G defined by $\sigma(X) = \sigma \circ X$. Otherwise said σ is a differential 1-form on M with values in the vector bundle TP/G and such that $\sigma(X)^{\#} = X$ for every X. The notation for the connection bundle in this case is abbreviated to $\mathcal{C}(M) \equiv \mathcal{C}(TP/G, TM)$. Thus σ is a section of the subbundle $\mathcal{C}(M) \subseteq A^1(M, TP/G)$.

4.2. Kaluza-Klein calculus on TP/G. Specializing Theorem 1 to the present case one sees that each fiber metric $\bar{g}$ on TP/G which is nondegenerate on VP/G (a Kaluza-Klein fiber metric on TP/G) is represented by

$$\bar{g} = \mathfrak{g}(\sigma, \tilde{g}, g)$$

where σ is a connection on M, $\tilde{g}$ is a fiber metric on VP/G and g is a metric on M. In this setting the Kaluza-Klein space (multidimensional universe) is the vector bundle TP/G together with the fiber metric $\bar{g}$. The Einstein-Yang-Mills equations for g and σ can be formulated in this setting as follows.

We use the standard calculus associated with a vector bundle E over M and the bundles $A^p(ME) = \Lambda^p T^*M \otimes E$ of differential p-forms on M with values in E (cf., for example, Tóth's book [11]; the calculus developed in [8] could also be used but is different from our approach). Thus a covariant derivative ∇ on E (as a morphism $\nabla : \Gamma E \to \Gamma \mathrm{Hom}(TM, E)$) gives for each vector field X on M a differential operator $\nabla_X : \Gamma E \to \Gamma E$, $\nabla_X \tau \in \Gamma E$ which is $C^\infty(M)$-linear in X, $\mathbf{R}$-linear in τ and $\nabla_X f\tau = X(f)\tau + f\nabla_X \tau$. Also ∇ gives rise to an exterior derivative d on differential forms with values in E. In addition if a covariant derivative ∇' on TM is given then ∇ extends to a covariant derivative on each $A^p(M, E)$ and furthermore gives rise to an exterior co-derivative operator $\partial : \Gamma A^p(M, E) \to \Gamma A^{p-1}(M, E)$ (simply defined without use of the Hodge star operator, cf. [11]).

The Kaluza-Klein fiber metric $\bar{g}$ on TP/G with connection component σ gives rise to covariant derivatives $\nabla^{\bar{g}}$ and ∇^{σ} on TP/G and VP/G, respectively. The covariant derivative $\nabla^{\bar{g}}$ arises from a differential operator $\bar{\nabla}_\gamma$ which is not a covariant derivative but which is the analog of the Levi-Civita covariant derivative for $\bar{g}$. This is the content of the following theorem.

Definition. For notational simplicity we use a dot for the inner product, i.e., if $\tau, \theta \in \Gamma(TP/G)$ then $\tau \cdot \theta = \bar{g}(\tau, \theta)$ is the C^∞-function on M defined by $(\tau \cdot \theta)(x) = \bar{g}_x(\tau_x, \theta_x)$. Likewise $X \cdot Y = g(X, Y)$ for vector fields on M. Note in particular that $\sigma(X) \cdot \sigma(Y) = X \cdot Y$ and $\sigma(X) \cdot \tau = 0$ for every $\tau \in \Gamma(VP/G)$.

Theorem 2. *For each $\gamma, \tau \in \Gamma(TP/G)$ let $\bar{\nabla}_\gamma \tau$ be given by the Koszul formula, i.e., $\bar{\nabla}_\gamma \tau$ is the unique element of $\Gamma(TP/G)$ which satisfies*

$$\begin{aligned} 2(\bar{\nabla}_\gamma \tau) \cdot \theta &= \gamma^{\#}(\tau \cdot \theta) + \tau^{\#}(\gamma \cdot \theta) - \theta^{\#}(\tau \cdot \gamma) \\ &\quad + [\theta, \tau] \cdot \gamma + [\theta, \gamma] \cdot \tau + [\gamma, \tau] \cdot \theta \end{aligned}$$

for every θ. Then $\bar{\nabla}$ has the following properties:

(1) $\bar{\nabla}_\gamma \tau$ *is* $C^\infty(M)$*-linear in* γ *and* $\mathbf{R}$*-linear in* τ;

(2) $\bar{\nabla}_\gamma(f\tau) = \gamma^\#(f)\tau + f\bar{\nabla}_\gamma \tau$;

(3) $\gamma^\#(\tau \cdot \theta) = (\bar{\nabla}_\gamma) \cdot \theta + \tau \cdot (\bar{\nabla}_\gamma \theta)$.

Definition. Let $\nabla^{\bar{g}}$ be the covariant derivative on TP/G defined by

$$\nabla^{\bar{g}}_X \tau = \bar{\nabla}_{\sigma(X)} \tau.$$

Also let ∇^σ be the covariant derivative defined on VP/G by

$$\nabla^\sigma_X \tau = [\sigma(X), \tau].$$

Let ∇ be the Levi-Civita covariant derivative on TM determined by the metric g on M and let $\nabla^{\bar{g}}, \nabla^\sigma$ also denote the covariant derivatives on $A^p(M, TP/G)$, $A^p(M, VP/G)$ obtained by extending $\nabla^{\bar{g}}, \nabla^\sigma$ as usual relative to the choice of ∇ on TM. The respective exterior derivatives and exterior co-derivatives are denoted by $d^{\bar{g}}, d^\sigma$ and $\partial^{\bar{g}}, \partial^\sigma$. The curvature (gauge field) of the connection σ is the VP/G-valued, 2-form defined by

$$F^\sigma(X, Y) = [\sigma(X), \sigma(Y)] - \sigma[X, Y].$$

Proposition 2. *The Bianchi identity*

$$d^\sigma F^\sigma = 0 \qquad \text{(Bianchi)}$$

follows easily from the definition (and the Jacobi identity for the Lie bracket). The equation

$$\partial^\sigma F^\sigma = 0 \qquad \text{(YM)}$$

is identical to the Yang-Mills equation. In addition one has

$$F^\sigma = d^{\bar{g}} \sigma.$$

Consequently,

$$d^\sigma d^{\bar{g}} \sigma = 0, \qquad \text{(Bianchi)}$$

$$\partial^\sigma d^{\bar{g}} \sigma = 0. \qquad \text{(YM)}$$

Definition. Let

$$\Omega(X, X') = \nabla_X \nabla_{X'} - \nabla_{X'} \nabla_X - \nabla_{[X, X']},$$

$$\bar{\Omega}(\tau, \tau') = \bar{\nabla}_\tau \bar{\nabla}_{\tau'} - \bar{\nabla}_{\tau'} \bar{\nabla}_\tau - \bar{\nabla}_{[\tau, \tau']}.$$

Theorem 3. *It holds*

$$\bar{\nabla}_{\sigma(X)}\sigma(Y) = \sigma(\nabla_X Y) + \frac{1}{2}F^{\sigma}(X,Y).$$

Consequently,

$$\bar{\Omega}(\sigma(X),\sigma(X'))\sigma(Y) = \sigma(\Omega(X,X')Y) + \frac{1}{2}(d^{\bar{g}}F^{\sigma})(X,X',Y) - \frac{1}{2}(\nabla^{\bar{g}}_Y F^{\sigma})(X,X') + \nabla^{\sigma}_Y(F^{\sigma}(X,X')).$$

Definition. Define a map $\phi_{X,Y} : \Gamma TM \to \Gamma(TP/G)$ by

$$\phi_{X,Y}(X') \equiv \bar{\Omega}(\sigma(X),\sigma(X'))\sigma(Y)$$

and let P be the orthogonal projection operator $P_x : (TP/G)_x \to VP_x^{\perp} = \sigma_x(T_xM)$. Then the Kaluza-Klein (KK)-Ricci tensor is defined to be the $\mathbf{R}$-valued, 2-form given by

$$\overline{\mathrm{Ric}}(X,Y) \equiv \mathrm{tr}(P \circ \phi_{X,Y}) = g^{ij}\bar{\Omega}\Big(\sigma(X),\sigma\Big(\frac{\partial}{\partial x_i}\Big)\Big)\sigma(Y)\cdot\sigma\Big(\frac{\partial}{\partial x_j}\Big).$$

The KK-Ricci scalar and KK-Einstein tensor are defined as

$$\bar{R} \equiv C_1^1(\overline{\mathrm{Ric}}),$$

$$\bar{G} \equiv \overline{\mathrm{Ric}} - \frac{1}{2}\bar{R}g$$

(Note g is the metric on M).

The metric (dot product) on TP/G and the metric on M yield, in the usual fashion, a dot product on TP/G-valued forms on M. Thus in particular $F^{\sigma}\cdot F^{\sigma} = F^{\sigma}_{ij}\cdot F^{\sigma ij}$. Also let i_X denote the usual contraction operator (derivation) on forms. Then the KK-energy momentum tensor is defined by

$$T(X,Y) = \frac{1}{2}(F^{\sigma}\cdot F^{\sigma})g(X,Y) - i_X F^{\sigma}\cdot i_Y F^{\sigma}.$$

Theorem 4. *It holds*

$$\overline{\mathrm{Ric}} = \mathrm{Ric} + \frac{3}{8}(F^{\sigma}\cdot F^{\sigma})g - \frac{3}{4}T,$$

$$\bar{R} = R + \frac{3}{4}(F^{\sigma}\cdot F^{\sigma}),$$

$$\bar{G} = G - \frac{3}{4}T.$$

Thus the vacuum KK-Einstein equation $\bar{G} = 0$ yields the Einstein equation $G = \frac{3}{4}T$.

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