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Total irredundance in graphs

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Abstract

A set S of vertices in a graph G is called a total irredundant set if, for each vertex v in G, v or one of its neighbors has no neighbor in $S-\{v\}$. We investigate the minimum and maximum cardinalities of maximal total irredundant sets.
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1. Introduction

Let $G=(V,E)$ be a graph with order $|V|=n$. For any vertex $v\in V$, the *open neighborhood of* v, denoted by $N(v)$, is the set $\{u\in V|uv\in E\}$ and its *closed neighborhood* $N[v]=N(v)\cup\{v\}$. For a set $S\subseteq V$, its *open neighborhood* $N(S)=\bigcup_{v\in S}N(v)$ and its *closed neighborhood* $N[S]=N(S)\cup S$. The *private neighbor set of a vertex* v *with respect to a set* S, denoted as $PN[v,S]$, is the set $N[v]-N[S-\{v\}]$. If $PN[v,S]\neq\emptyset$ for some vertex v and some $S\subseteq V$, then every vertex of $PN[v,S]$ is called a *private neighbor of* v *with respect to* S, or just an S-pn.

A set S is a *dominating set* if $N[S]=V$, or equivalently, every vertex in $V-S$ has a neighbor in S; and S is a *total dominating set* if $N(S)=V$, or equivalently, every vertex in V has a neighbor in S. The *domination number* $\gamma(G)$ (*total domination number* $\gamma_t(G)$, respectively) is the minimum cardinality of any dominating set (total dominating

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set, respectively) of G; while the *upper (total) domination number* $\Gamma(G)$ ($\Gamma_t(G)$) is the maximum cardinality of any minimal (total) dominating set.

A set S is an *irredundant set* if for every vertex $v \in S$, $PN[v,S] \neq \emptyset$ (every vertex in S has a private neighbor with respect to S). The *irredundance number* $ir(G)$ is the minimum cardinality of any maximal irredundant set of G, while the *upper irredundance number* $IR(G)$ is the maximum cardinality of any such set.

To date, an estimated 1500 papers have been written on domination in graphs. The literature on this subject has been surveyed and detailed in the two books by Haynes et al. [11,12].

It is felt by some researchers that a deeper understanding of the concept of domination in graphs can best be obtained by studying the more general concept of irredundance in graphs. This belief stems from the observation, first made in 1978 by Cockayne et al. [4], that a set is minimal dominating if and only if it is irredundant and dominating. It was subsequently observed by Bollobás and Cockayne [2] that the class of minimal dominating sets in G is contained within the broader class of maximal irredundant sets in G. To date, an estimated 100 research papers have explored the properties of irredundant sets in graphs (e.g. [1,3,6–9,14]). In fact, a research monograph on irredundance in graphs, by Cockayne and Mynhardt, is currently in preparation [5]. This paper is also motivated by the belief that a better understanding of irredundant sets will shed light on properties of dominating sets, and in particular, that the study of total irredundance will shed light on the concept of total domination.

Hedetniemi et al. [13] defined a set S to be a *total irredundant set* if for every vertex $v \in V$, $PN[v,S] \neq \emptyset$ (every vertex in V has a private neighbor with respect to S). The *total irredundance number* $ir_t(G)$ and the *upper total irredundance number* $IR_t(G)$ are defined as expected. If a graph G has no total irredundant set, then we define $ir_t(G)$ and $IR_t(G)$ to be 0. Note that an irredundant set S is a maximal total irredundant set if and only if every vertex of $N(S) - S$ has a neighbor in $V - N[S]$. In particular, a total irredundant set is a dominating set if and only if $ir_t(G) = IR_t(G) = n$ if and only if $G = \bar{K}_n$.

Algorithmic aspects of the total irredundance numbers were studied in [13]. In particular, a linear algorithm for computing the value of $IR_t(T)$ for any tree T is presented in [13], and it is shown that UPPER TOTAL IRREDUNDANT SET is an NP-complete problem for arbitrary graphs. In this paper, we study theoretical aspects of total irredundance numbers. We first consider $ir_t(G)$ and $IR_t(G)$ for selected graphs G in Section 2. From these examples, we observe some rather unexpected results concerning total irredundance numbers. For instance, the total irredundance number of cycles is not a monotonic function of their length. Furthermore, we shall see that $ir_t(G)$ can equal zero and that $ir_t(G)$ and $\gamma_t(G)$ are incomparable. (Both these results are somewhat surprising considering the fact that $1 \leqslant ir(G) \leqslant \gamma(G)$ for all graphs G.) Regarding these properties, we characterize the graphs G for which $ir_t(G) = IR_t(G) = 0$ in Section 3, and then study regular graphs G for which $ir_t(G) \geqslant 1$ in Section 4. In Section 5, the trees T having $ir_t(T) = 1$ are characterized. Sharp upper bounds on the total irredundance numbers are given in Section 6. In particular, we show that $ir_t(G) \leqslant IR_t(G) \leqslant n - \gamma_t(G)$ for any graph G. Finally, we close with some open problems in Section 7.

2. Special families

In this section, we study the total irredundance numbers of selected graphs. We have four immediate aims: first, to show that the total irredundance number of a graph can be zero; second, to show that the total irredundance number can differ substantially from the irredundance number and the total domination number; third, to show that the total irredundance number of a cycle is not monotonic; and fourth, to compute the total irredundance numbers of paths and cycles. These examples also show that certain bounds obtained in subsequent sections are sharp.

Proposition 1. *For* $n \geqslant 2$, $ir_t(K_n) = IR_t(K_n) = 0$.

This example serves to illustrate that, although $ir(G) \geqslant 1$ for any graph G, $ir_t(G)$ and $IR_t(G)$ can equal zero. We show next that the total irredundance numbers can be arbitrarily large. Since the sets consisting of all except one of the leaves are the only maximal total irredundant sets in a star of order at least 3, we have the following result.

Proposition 2. *For* $k \geqslant 2$, $ir_t(K_{1,k}) = IR_t(K_{1,k}) = k - 1$.

The subdivided star $K^*_{1,k}$ is obtained from the star $K_{1,k}$ by subdividing every edge exactly once.

Proposition 3. *For* $k \geqslant 1$, $ir_t(K^*_{1,k}) = 1$.

It is well known that $ir(G) \leqslant \gamma(G)$, so one might expect that $ir_t(G) \leqslant \gamma_t(G)$. However, from Proposition 2, we see that $\gamma_t(K_{1,k}) = \Gamma_t(K_{1,k}) = 2$ and the difference $ir_t(G) - \gamma_t(G)$ can be arbitrarily large. On the other hand, since $\gamma_t(K^*_{1,k}) = k + 1$, Proposition 3 shows that the difference $\gamma_t(G) - ir_t(G)$ can also be arbitrarily large. Hence, there is no general relation between $ir_t(G)$ and $\gamma_t(G)$. Since $ir(K_{1,k}) = 1$ and $ir(K^*_{1,k}) = k$, Propositions 2 and 3 also show that the differences $ir(G) - ir_t(G)$ and $ir_t(G) - ir(G)$ can be arbitrarily large.

Let O_{2k} denote the graph obtained from K_{2k} by deleting a perfect matching (sometimes known as a generalized octahedron). Then every vertex of O_{2k} forms a total irredundant set of O_{2k}, and so $ir_t(O_{2k}) \geqslant 1$. However, any two vertices form a dominating set of O_{2k} and therefore cannot be a total irredundant set. Consequently, we have the following result.

Proposition 4. *For* $k \geqslant 2$, $ir_t(O_{2k}) = 1$.

Next, we determine the total irredundance numbers of paths and cycles.

Proposition 5. *For* $n \geqslant 4$,

$$ir_t(C_n) = \begin{cases} \lceil \frac{n}{4} \rceil + 1 & \textit{if } n \equiv 2 \textit{ or } 3 \pmod 8, \\ \lceil \frac{n}{4} \rceil & \textit{otherwise.} \end{cases}$$

Proof. For $n=4$, the sets consisting of a single vertex are the only maximal total irredundant sets of C_n, and so $ir_t(C_n)=1=\lceil n/4 \rceil$. For $5 \leqslant n \leqslant 7$, the sets consisting of two adjacent vertices are the only maximal total irredundant sets in C_n, and so $ir_t(C_n)=2=\lceil n/4 \rceil$.

Let C be the n-cycle $v_1, v_2, \ldots, v_n, v_1$ for $n \geqslant 8$. (We note that there is an implied orientation to C, so that we may refer to the next vertices on C.) Let S be a maximal total irredundant set of C. Then each component in $\langle S \rangle$ is either K_1 or K_2. Let S_1 denote the set of isolated vertices in $\langle S \rangle$, and let S_2 denote the set of all pairs of adjacent vertices in $\langle S \rangle$. Letting $|S|=s$ and $|S_i|=s_i$, we have $s=s_1+2s_2$. Let v_i and v_j be two non-adjacent vertices of S such that v_j is the vertex of S next on C from v_i (in our implied orientation). Then the maximality of S implies that v_i and v_j are separated by exactly three vertices on C if one of v_i and v_j belongs to S_1 and by three to six vertices if both belong to S_2. Thus, $n \leqslant 4s_1 + 8s_2 = 4s$, and so $s \geqslant n/4$. Hence, $ir_t(C_n) \geqslant \lceil n/4 \rceil$.

We show that if $n \equiv 2$ or $3 \pmod 8$, then $ir_t(C_n) \geqslant \lceil n/4 \rceil + 1$. If $s_2=0$, then $n=4s_1$ and $n \equiv 0 \pmod 4$, while if $s_2=1$, then $n=4s_1+5$ and $n \equiv 1 \pmod 4$. Hence, $s_2 \geqslant 2$. If $s_1 \geqslant 1$, then at least one pair in S_2 is separated from another vertex in S by exactly three vertices on C (that are not in S), whence $n \leqslant 4s_1 + 8(s_2-1)+5 = 4s-3$ and $s \geqslant (n+3)/4$. On the other hand, suppose $s_1=0$. If $n=8s_2$, then $n \equiv 0 \pmod 4$. If $n=8s_2-1$, then $n \equiv 7 \pmod 8$. If $n=8s_2-2$, then $n \equiv 6 \pmod 8$. Hence we must have $n \leqslant 8s_2-3=4s-3$, and so once again $s \geqslant (n+3)/4$.

We now show the reverse inequalities by exhibiting sets S that achieve the bounds. Let $k=\lfloor n/4 \rfloor$ and $l=\lfloor n/8 \rfloor$. Let

$$\begin{aligned} &S=\{v_1, v_5, \ldots, v_{4k-3}\} && \text{if } n \equiv 0 \pmod 4, \\ &S=\{v_1, v_5, \ldots, v_{4k-3}, v_{4k-2}\} && \text{if } n \equiv 1 \pmod 4, \\ &S=\{v_1, v_9, \ldots, v_{8l+1}, v_2, v_{10}, \ldots, v_{8l+2}\} && \text{if } n \equiv 6 \text{ or } 7 \pmod 8, \\ &S=\{v_1, v_5, \ldots, v_{4k-3}, v_{4k-2}, v_n\} && \text{if } n \equiv 2 \text{ or } 3 \pmod 8. \end{aligned}$$

This suffices to complete the proof. □

Proposition 6. *For* $n \geqslant 4$,

$$IR_t(C_n) = \begin{cases} \lfloor \frac{2n}{5} \rfloor - 1 & \textit{if } n \equiv 3 \pmod 5, \\ \lfloor \frac{2n}{5} \rfloor & \textit{otherwise.} \end{cases}$$

Proof. Using the notation employed in the proof of Proposition 5, we have $n \geqslant 4s_1 + 5s_2$, and so $2n/5 \geqslant s + 3s_1/5 \geqslant s$. Hence, $IR_t(C_n) \leqslant \lfloor 2n/5 \rfloor$. Suppose $n \equiv 3 \pmod 5$. If $s_1 \geqslant 2$, then $2n/5 \geqslant s + 3s_1/5 \geqslant s + 6/5$, and so $s \leqslant (2n-6)/5 = \lfloor 2n/5 \rfloor - 1$. On the other hand, if $s_1 \leqslant 1$, then $n \geqslant 4s_1 + 5s_2 + 1$, whence $s \leqslant (2n-2)/5 = \lfloor 2n/5 \rfloor - 1$.

We again show the reverse inequalities by exhibiting sets S that achieve the bounds. Let $k = \lfloor n/5 \rfloor - 1$ and let $S = \{v_1\}$ if $n = 4$. For $n \geqslant 5$, let

$$S = \{v_1, v_2, v_6, v_7, \ldots, v_{5k+1}, v_{5k+2}, v_{n-3}\} \quad \text{if } n \equiv 4 \pmod 5,$$

and

$$S = \{v_1, v_2, v_6, v_7, \ldots, v_{5k+1}, v_{5k+2}\} \quad \text{otherwise.}$$

This suffices to complete the proof. □

Our next result gives the total irredundance numbers of a path. The proof of this result is very similar to those of Propositions 5 and 6 and is therefore omitted.

Proposition 7. *For* $n \geqslant 3$,

$$ir_t(P_n) = \left\lceil \frac{n-1}{4} \right\rceil,$$

and

$$IR_t(P_n) = \left\lfloor \frac{2n}{5} \right\rfloor.$$

3. Graphs G with $ir_t(G) = IR_t(G) = 0$

In this section, our aim is to study graphs whose total irredundance numbers are zero. For this purpose, we first make some more definitions.

For vertices u and v, if $N[u] = N[v]$, we say that u and v are *clones* (also called *twins* in the literature). If $N[u] \subset N[v]$, that is, the neighborhood of v properly contains the neighborhood of u, then we say that v is *superior* to u and u is *inferior* to v. An inferior vertex that is not superior or clone to any vertex is called a *subvertex*. If a vertex v is not inferior, superior, or clone to any other vertex, then we say that v is *normal*. If v is superior (inferior) to every vertex in a set S, then we say that v is superior (inferior) to set S.

Notice that the relation 'is a clone' partitions the vertex set V into sets of clones and singletons (vertices with no clones). We call the sets of the partition *clone-sets* (note that each singleton, although not a clone, is referred to as a clone-set). Observe also that between any pair of clone-sets, either no edge is present or every possible edge is present. In the latter case, we say that the clone-sets are *clone-adjacent*.

Observation 8. *Every clone-set induces a complete subgraph.*

Observation 9. *If a graph G is connected and not complete, then the subgraph induced by a clone-set of G is not a maximal complete subgraph, that is, is not a clique of G.*

Proof. Let C be a clone-set of G. Then the induced subgraph $\langle C \rangle$ is complete. Since G is connected and not complete, some vertex in C must have a neighbor x in $V - C$, and, since C is a clone-set, it follows that $\langle C \cup \{x\} \rangle$ is complete. □

If a vertex u is superior or clone to another vertex v, then v has no private neighbor with respect to any set containing u. Thus, we have the following result.

Lemma 10. *A vertex that is superior or clone to another vertex cannot be in any total irredundant set.*

As a special case of Lemma 10, a vertex adjacent to all others cannot belong to any total irredundant set. Furthermore, no vertex adjacent to an endvertex is in any total irredundant set.

Lemma 11. *If a graph G has either a subvertex or a normal vertex, then $ir_t(G) \geqslant 1$.*

Proof. Let u be a subvertex or a normal vertex and let v be any other vertex. Clearly, u is its own private neighbor with respect to $S = \{u\}$. Also, $N[v] - N[u] \neq \emptyset$ since u is neither superior nor clone to v. Hence, $PN[v, S] \neq \emptyset$. Thus, S is a total irredundant set (although certainly not maximal in general). Thus, $ir_t(G) \geqslant 1$. □

A vertex that is not normal or subvertex must be a superior vertex or clone to some other vertex. Hence an immediate consequence of Lemmas 10 and 11 now follows.

Theorem 12. *A graph G has $ir_t(G) = 0$ if and only if every vertex is either superior or clone to another vertex.*

Since no vertex of minimum degree is superior, Theorem 12 implies the following properties of a graph with total irredundance number zero.

Corollary 13. *If G is a graph with $ir_t(G) = 0$, then every vertex of minimum degree must be clone to another vertex.*

Corollary 14. *The only connected graph G which has $\delta(G) = 1$ and $ir_t(G) = 0$ is K_2.*

Corollary 15. *If G is a regular graph with $ir_t(G) = 0$, then it has no singleton clone-sets.*

We close this section by defining a unary operation on a graph to construct a family of graphs H satisfying $ir_t(H) = 0$. A graph H is a *clone-inflation* of a graph G if

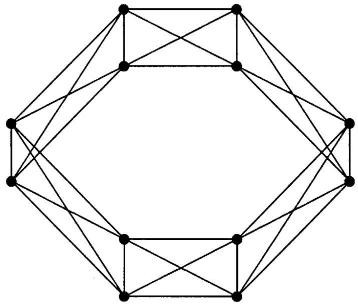


Fig. 1. The clone-inflation of C_6 with clone-sets of cardinality 2.

each vertex of G is replaced with a non-trivial complete subgraph and vertices in different subgraphs (clone-sets) are adjacent if and only if the corresponding vertices are adjacent in G. (Note that if the clone-sets have the same cardinality p, then the clone-inflation of G is the composition graph of G with K_p [10].) For example, the graph in Fig. 1 is a clone-inflation of the cycle C_6 where each clone-set has cardinality 2, that is, G is the composition graph of C_6 with K_2.

Observe that the graph G in Fig. 1 is a regular graph with $ir_t(G)=0$. In general, not every regular graph G with $ir_t(G)=0$ has clone-sets of the same cardinality. For example, consider the clone-inflation of C_6 where for $p \neq t$ the clone-sets have cardinality t, p, t, t, p, t, respectively. This clone-inflation graph has total irredundance number 0 and is regular of degree $2t+p-1$. Since every vertex in a clone-inflation graph is clone to another vertex, we have the following consequence of Theorem 12.

Corollary 16. *If G is the clone-inflation of some graph, then $ir_t(G)=0$.*

4. Regular graphs G with $ir_t(G) \geqslant 1$

Since a regular graph G contains no superior vertices, every vertex x of G which is not clone to another vertex forms a total irredundant set (since each neighbor of x has a neighbor in $V-N[x]$). Thus, we have the following result.

Observation 17. *If G is a regular graph with a normal vertex, then $ir_t(G) \geqslant 1$.*

Fig. 2 shows two 4-regular graphs G_1 and G_2; in each, x is a superior vertex and $\{x\}$ is a maximal total irredundant set.

The family O_{2k} for $k \geqslant 2$ is an example of $(2k-2)$-regular graphs with $ir_t(O_{2k})=1$. The next result shows that there exist r-regular graphs with $ir_t(G)=1$ for all $r \geqslant 4$.

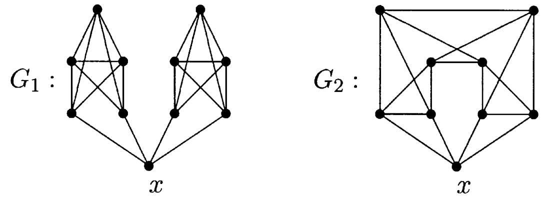


Fig. 2. Examples of 4-regular graphs $G \in \{G_1, G_2\}$ with $ir_t(G) = 1$.

Proposition 18. *For every $r \geqslant 4$, there exists an r-regular graph G with $ir_t(G) = 1$.*

Proof. For $r \geqslant 4$, let H be a graph of order r and size $r(r-3)/2$ with $\Delta(H) = r-2$, and let $A = K_r$. Let G be the graph obtained by adding two new (non-adjacent) vertices x and y and all edges between $\{x, y\}$ and H, and then adding r edges between A and H in such a way as to construct an r-regular graph. Then $\{x\}$ is a total irredundant set in G. If $z \in N(x)$, then y has no $\{x, z\}$-pn. If z is a vertex of $N(x)$ of degree $r-2$ in H, then z has no $\{x, y\}$-pn. If $a \in A$, then no other vertex of A has a $\{a, x\}$-pn. Hence, $\{x\}$ is in fact a maximal total irredundant set in G. Thus, $ir_t(G) = 1$. □

Theorem 19. *If G is a triangle-free r-regular graph with $r \geqslant 3$, then $ir_t(G) \geqslant 2$.*

Proof. Let x be a vertex of G and let $N(x) = \{x_1, x_2, \ldots, x_r\}$. Since G is triangle-free, $N(x)$ is independent. We show that x belongs to a maximal total irredundant set of cardinality at least 2. If no two neighbors of x have a common neighbor different from x, then $\{x, x_1\}$ is a total irredundant set for any $x_1 \in N(x)$, a contradiction. Hence we may assume there exists a vertex y, different from x, that is a common neighbor of x_1 and x_2. We claim that $\{x, y\}$ is a total irredundant set. Since G is triangle-free, each of x_1 and x_2 has an $\{x, y\}$-pn. For each x_i, $3 \leqslant i \leqslant r$, y has at most $r-2$ neighbors in $N(x_i)$ and thus, x_i has an $\{x, y\}$-pn. Each neighbor z of y different from x_1 and x_2 is adjacent to no other neighbor of y and to at most the $r-2$ neighbors $x_3, \ldots, x_r$ of x. Hence, z has an $\{x, y\}$-pn. Therefore, $\{x, y\}$ is a total irredundant set. Thus, every vertex of G belongs to a maximal total irredundant set of cardinality at least 2. □

We show next that every cubic graph, different from K_4, has total irredundance number of at least 2.

Theorem 20. *For every cubic graph $G \neq K_4$, $ir_t(G) \geqslant 2$.*

Proof. It is straightforward to see that if G is cubic and $ir_t(G) = 0$, then $G = K_4$. Suppose $ir_t(G) = 1$ and let $\{x\}$ be a maximal total irredundant set and let $N(x) = \{x_1, x_2, x_3\}$. Since x is not clone to another vertex, $\langle N(x) \rangle$ contains at most one edge. If any $x_i \in N(x)$ has two external $N(x)$-pns (that is, private neighbors with respect to $N(x)$), then $\{x,x_i\}$ is a total irredundant set containing x, contradicting the fact that $\{x\}$ is a maximal total irredundant set. Hence each $x_i, 1 \leqslant i \leqslant 3$, has at most one external $N(x)$-pn.

If each $x_i \in N(x)$ has exactly one external $N(x)$-pn, then $N(x)$ is independent and there must exist a vertex y such that $y \in N(x_1) \cap N(x_2) \cap N(x_3) - \{x\}$. Then $\{x, y\}$ is a total irredundant set, a contradiction. Thus at least one x_i, say x_1, has no external $N(x)$-pn.

We show next that $N(x)$ is an independent set. If $x_1x_2 \in E(G)$, let $\{y\} = N(x_1) - \{x,x_2\}$. If $N(y) = N(x)$, let $z = N(x_3) - \{x, y\}$. Then $\{x,z\}$ is a total irredundant set, again contradicting the maximality of $\{x\}$. If $N(y) \cap N(x) = \{x_1,x_3\}$, then $\{x,x_2\}$ is a total irredundant set, a contradiction. If $N(y) \cap N(x) = \{x_1,x_2\}$, then $\{x,x_3\}$ is a total irredundant set, a contradiction. Hence, $x_1x_2 \notin E(G)$. Similarly, $x_1x_3 \notin E(G)$. If $x_2x_3 \in E(G)$, then the two neighbors of x_1 different from x are adjacent to x_2 and x_3, respectively. But then $\{x,x_2\}$ is a total irredundant set, a contradiction. Hence, $N(x)$ is an independent set.

Without loss of generality, we may assume that one of the following three cases occurs, where all the vertices y_i are distinct: (1) $N(x_1) - \{x\} = \{y_1, y_2\}$, $N(x_2) - \{x\} = \{y_1, y_3\}$, and $N(x_3) - \{x\} = \{y_2, y_4\}$; (2) $N(x_1) - \{x\} = \{y_1, y_2\}$, $N(x_2) - \{x\} = \{y_1, y_3\}$, and $N(x_3) - \{x\} = \{y_2, y_3\}$; (3) $N(x_1) - \{x\} = N(x_2) - \{x\} = \{y_1, y_2\}$ and $N(x_3) - \{x\} = \{y_1, y_2\}$ or $\{y_1, y_3\}$. If Case (1) holds, then either $y_1y_2 \in E(G)$, in which case $\{x,x_2\}$ is a total irredundant set, or $y_1y_2 \notin E(G)$, in which case $\{x,x_1\}$ is a total irredundant set. If Case (2) holds, then the subgraph $\langle\{y_1, y_2, y_3\}\rangle$ either has no edges or contains a unique edge, say y_1y_2. In any event, $\{x,x_3\}$ is a total irredundant set. If Case (3) holds, then $\{x, y_2\}$ is a total irredundant set. All the three cases contradict the fact that $\{x\}$ is a maximal total irredundant set. Hence, $ir_t(G) \geqslant 2$. $\square$

We close this section with lower bounds on the total irredundance numbers of an r-regular graph, $r \geqslant 4$, with girth at least 5.

Theorem 21. *If G is an r-regular graph of degree $r \geqslant 4$ and girth $g \geqslant 5$, then $IR_t(G) \geqslant r$ and $ir_t(G) \geqslant 3$.*

Proof. Let x be any vertex of G. Since G is triangle-free, x is not a clone and $\{x\}$ is a total irredundant set (not maximal by Theorem 19). Let $N(x) = \{x_1,x_2,\dots,x_r\}$, and for each $1 \leqslant i \leqslant r$, let y_i^j, $1 \leqslant j \leqslant r-1$, be the neighbors of x_i different from x. Since $g \geqslant 5$, the $r(r-1)$ vertices y_i^j are all distinct.

First, consider the set $A = \{x,x_1,x_2,\dots,x_{r-1}\}$. The vertex x_r is an A-pn of x, y_i^1 is an A-pn of x_i for $1 \leqslant i \leqslant r-1$, and y_r^1 is an A-pn of x_r. Each neighbor y_i^j of a vertex x_i, $1 \leqslant i \leqslant r-1$, has no neighbor in $N(x_i)$, since $g > 3$; and at most one neighbor in each $N(x_k)$, $k \neq i$, since $g > 4$. Hence, y_i^j has at most $r-1$ neighbors in $N[A]$ and thus has an A-pn. Therefore, A is a total irredundant set and thus, $IR_t(G) \geqslant r$.

We choose now x belonging to an ir_t-set I. By Theorem 19, we have that $|I| \geqslant 2$. If I contains some x_i, then, since $\{x, x_i\}$ is not maximal (as the above argument shows), $|I| \geqslant 3$. Hence, we may assume that I contains a vertex $t \notin N[x]$.

Let y be the predecessor of t on a shortest path P_{xt} from x to t (y is possibly an x_i). Consider the set $B = \{x, y, t\}$. By the choice of P_{xt} and since $g \geqslant 5$, the three vertices of B have pairwise no common neighbor not in P_{xt}. Since $r \geqslant 4$, B is an irredundant set. From the definition of x, y, t and since $g \geqslant 5$, any neighbor z of y (respectively, t) which is not in B is adjacent to at most one vertex of $N[x] \cup N[t]$ (respectively, $N[x] \cup N[y]$). Hence, since $r \geqslant 4$, z has a B-pn. Any neighbor x_i of x, with $x_i \neq y$ in the case of $d(x,t) = 2$, is adjacent to at most two vertices of $N[y] \cup N[t]$. Therefore, x_i has a B-pn. Hence, B is a total irredundant set implying that $\{x, t\}$ is not maximal and thus, $IR_t(G) \geqslant ir_t(G) > 2$. □

5. Trees T with $ir_t(T) = 1$

Since every tree T with $n \geqslant 3$ vertices has at least one inferior vertex, it follows that $ir_t(T) \geqslant 1$. By Proposition 3, we know that the subdivided star has total irredundance number 1. Next we characterize those trees T having $ir_t(T) = 1$.

Theorem 22. *A nontrivial tree T satisfies $ir_t(T) = 1$ if and only if it can be obtained from a star $K_{1,m}$ by*

(a) *subdividing one edge for $m \geqslant 1$,*
(b) *subdividing $m - 1$ edges for $m \geqslant 3$, or*
(c) *subdividing m edges for $m \geqslant 2$.*

Proof. Let T be obtained from a star $K_{1,m}$ by one the operations of the theorem. If T is obtained by (a), let x be the endvertex of the subdivided edge. If T is obtained by (b), let x be the endvertex adjacent to the center. Finally, if T is obtained by (c), let x be the center. For each tree T, $\{x\}$ is a maximal total irredundant set and $ir_t(T) = 1$.

Conversely, let T be a non-trivial tree such that $ir_t(T) = 1$, and let $\{x\}$ be a maximal irredundant set. Note first that $T \neq K_2$ for otherwise $ir_t(T) = 0$. Also note that x is not adjacent to an endvertex. Moreover, if T contains a vertex at distance of at least 4 from x, then it contains a leaf y at distance of at least 4 from x and $\{x, y\}$ is a total irredundant set, a contradiction. Hence, $ecc(x) \leqslant 3$ and every vertex at distance 3 from x is a leaf.

Case 1: If x is a leaf with support vertex y, let $N(y) - \{x\} = \{y_1, y_2, \ldots, y_k\}$ and $N(y_i) - \{y\} = \{y_i^1, y_i^2, \ldots, y_i^{s_i}\}$. Since $ecc(x) \leqslant 3$, every y_i^j is a leaf. If $k = 1$, then since $ecc(x) \leqslant 3$ and $T \neq K_2$, T can be obtained from a star $K_{1,m}$ using operation (a). If $k \geqslant 2$ and $d(y_i) = 2$ for all $1 \leqslant i \leqslant k$, then, by $ecc(x) \leqslant 3$, T can be obtained from $K_{1,m}$ by operation (b). If $k \geqslant 2$ and, say, $d(y_1) = 1$, then $\{x, y_1\}$ is a total irredundant set, a contradiction. If $k \geqslant 2$ and, say $d(y_1) \geqslant 3$, then $\{x, y_1^1\}$ is a total irredundant set, a contradiction.

Case 2: If x is not a leaf, let $N(x) = \{x_1, x_2, \ldots, x_k\}$, $k \geqslant 2$, and $N(x_i) - \{x\} = \{x_i^1, x_i^2, \ldots, x_i^{s_i}\}$ with $s_i \neq 0$ since x is not adjacent to a leaf. Suppose x has a neighbor, say x_1, of degree more than 2. If x_1 is adjacent to an endvertex, say x_1^1 is a leaf, then $\{x, x_1^1\}$ is a total irredundant set, a contradiction. Hence, $d(x_1^j) \geqslant 2, 1 \leqslant j \leqslant s_i$, and $\{x, x_i\}$ is a total irredundant set, a contradiction. Thus, $d(x_i) = 2$ for all $1 \leqslant i \leqslant k$. If some x_i^1, say x_1^1, is not a leaf, then $\{x, x_1\}$ is a total irredundant set, a contradiction. Therefore, T can be obtained from $K_{1,m}$ by operation (c). □

6. Upper bounds on total irredundance numbers

We now turn our attention to upper bounds on the total irredundance numbers. As we mentioned in the introduction, $ir_t(\bar{K}_n) = IR_t(\bar{K}_n) = n$. In fact, it follows from the definition of total irredundance that $IR_t(G) = n$ (or, $ir_t(G) = n$) if and only if $G = \bar{K}_n$. We can improve the upper bound slightly for graphs with no isolated vertices. First we prove the following Gallai-type result.

Theorem 23. *If G has no isolated vertices, then*

$$IR_t(G) + \gamma_t(G) \leqslant n.$$

Proof. Let G be a graph with $\delta(G) \geqslant 1$ and let S be an IR_t-set of G. Then any isolated vertex in $\langle S \rangle$ must have a neighbor in $V - S$. Furthermore, any vertex in S that is not an isolated vertex in $\langle S \rangle$ has a private neighbor with respect to S in $V - S$. Hence, $V - S$ is a dominating set. To see that $V - S$ is in fact a total dominating set, partition the vertices of $V - S$ into two sets A and B where $A = N(S) \cap (V - S)$ and $B = V - N[S]$. Since S is a total irredundant set, each vertex in A must have a neighbor in B. And since G has no isolated vertices, each vertex in B has a neighbor in $V - S$. Hence, $V - S$ is a total dominating set, which implies that $IR_t(G) + \gamma_t(G) \leqslant |S| + |V - S| = n$. □

Corollary 24. *If G has no isolated vertices, then*

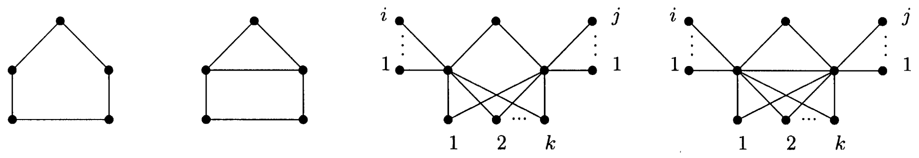
$$ir_t(G) + \gamma_t(G) \leqslant n.$$

Since $\gamma_t(G) \geqslant 2$, we now have a slightly improved bound on $IR_t(G)$.

Corollary 25. *If G has no isolated vertices, then*

$$ir_t(G) \leqslant IR_t(G) \leqslant n - 2$$

and these bounds are sharp.



(a) (b) (c) (d)

Fig. 3. The graphs G with $IR_t(G)=n-3$ where $i+j+k>0$.

These bounds are sharp as we saw with stars $K_{1,k}$ where $ir_t(K_{1,k})=IR_t(K_{1,k})=k-1$. In fact, stars are the only graphs attaining the upper bound.

Theorem 26. *For a graph G with no isolated vertices*, $ir_t(G)=IR_t(G)=n-2$ *if and only if* $G=K_{1,n-1}$.

Proof. By Proposition 2, $ir_t(K_{1,n-1})=IR_t(K_{1,n-1})=n-2$. Conversely, assume that $IR_t(G)=n-2$ and let S be an IR_t-set of G. Now $|V-S|=2$, and as shown in the proof of Theorem 23, $V-S$ is a total dominating set. Let $V-S=\{u,v\}$. Then $uv\in E(G)$ and every vertex in S has a neighbor in $V-S$. But since S is a total irredundant set, at least one vertex in $V-S$, say v, is not dominated by S. Hence every vertex in S is adjacent to u (and no vertex in S is adjacent to v). Now suppose S is not independent, that is, there is an edge, say xy, in $\langle S\rangle$. But then neither x nor y has a private neighbor with respect to S, contradicting the fact that S is a total irredundant set. Hence, G is a star. □

We make another slight improvement.

Theorem 27. *If G is not a star and has no isolated vertices, then*

$$IR_t(G)\leqslant n-3,$$

and this bound is attained if and only if G is one of the graphs shown in Fig. 3.

Proof. The bound follows directly from Theorem 26. Let $IR_t(G)=n-3$ and let S be an IR_t-set of G. Then $V-S$ is a total dominating set, so either $\langle V-S\rangle=P_3$ or $\langle V-S\rangle=K_3$ and every vertex in S has a neighbor in $V-S$. Furthermore, at least one vertex in $V-S$ is not dominated by S. If there is an edge xy in $\langle S\rangle$, then both x and y must have a S-pn in $V-S$, and each vertex in $V-S$ must have a S-pn in $V-S$. Thus, G is the cycle C_5 (Fig. 3(a)) or the house graph (Fig. 3(b)). Hence assume that S is independent. Assume that $V-S$ induces a $P_3=u,v,w$. If the center vertex v has a neighbor in S, then neither u nor w can have a neighbor in S (since if one of them has a neighbor in S, then it would have no S-pn). Then every vertex of S is adjacent to v and $S\cup\{w\}$ is a total irredundant set, contradicting the maximality of S. Hence vertex v has no neighbors in S. Thus, $S\subset N(u)\cup N(w)$ and G is a graph in the family of graphs shown Fig. 3(c) where at least one of i,j, and k is greater than zero. If $\langle V - S \rangle = K_3$, then G is in the family of graphs shown in Fig. 3(d) where at least one of i, j, and k is greater than zero. □

7. Open questions

We close with some questions that we have yet to settle.

(1) Which graphs have independent maximal total irredundant sets?
(2) Characterize the graphs G satisfying $ir(G) = ir_t(G) = 1$.
(3) Find bounds on $ir_t(G)$ with minimum degree conditions.
(4) Characterize the graphs G that achieve $IR_t(G) + \gamma_t(G) = n$.
(5) Characterize the regular graphs G having $ir_t(G) = 0$.
(6) Characterize the graphs G with $ir_t(G) = IR_t(G)$.

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References

[1] R.B. Allan, R.C. Laskar, On domination and independent domination numbers of a graph, Discrete Math. 23 (1978) 73–76.
[2] B. Bollobás, E.J. Cockayne, Graph theoretic parameters concerning domination, independence and irredundance, J. Graph Theory 3 (1979) 241–250.
[3] B. Bollobás, E.J. Cockayne, On the irredundance number and maximum degree of a graph, Discrete Math. 49 (1984) 197–199.
[4] E.J. Cockayne, S.T. Hedetniemi, D.J. Miller, Properties of hereditary hypergraphs and middle graphs, Canad. Math. Bull. 21 (1978) 461–468.
[5] E.J. Cockayne, C.M. Mynhardt, Irredundance in graphs, Personal Communication 2000.
[6] E.J. Cockayne, C.M. Mynhardt, Irredundance and maximum degree in graphs, Combin. Prob. Comput. 6 (1997) 153–157.
[7] O. Favaron, A note on the irredundance number after vertex-deletion, Discrete Math. 121 (1993) 51–54.
[8] O. Favaron, Two relations between the parameters of independence and irredundance, Discrete Math. 70 (1988) 17–20.
[9] M.R. Fellows, G.H. Fricke, S.T. Hedetniemi, D. Jacobs, The private neighbor cube, SIAM J. Discrete Math. 7 (1994) 41–47.
[10] F. Harary, Graph Theory, Addison-Wesley, Reading, MA, 1972.
[11] T.W. Haynes, S.T. Hedetniemi, P.J. Slater, Fundamentals of Domination in Graphs, Marcel Dekker, New York, 1998.
[12] T.W. Haynes, S.T. Hedetniemi, P.J. Slater (Eds.), Domination in Graphs, Advanced Topics, Marcel Dekker, New York, 1998.
[13] S.M. Hedetniemi, S.T. Hedetniemi, D.P. Jacobs, Total irredundance in graphs: theory and algorithms, Ars Combin. 35 (1993) 271–284.
[14] S.T. Hedetniemi, R.C. Laskar, J. Pfaff, Irredundance in graphs: a survey, Congr. Numer. 48 (1985) 183–193.