Stochastic Processes and their Applications 41 (1992) 33-44
North-Holland

A central limit theorem for functions of a Markov chain with applications to shifts

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Received 7 August 1989
Revised 17 July 1990 and 24 January 1991

A sufficient condition is developed for partial sums of a function of a stationary, ergodic Markov chain to be asymptotically normal. For Bernoulli and Lebesgue shifts, the condition may be related to the Fourier coefficients of the given function; and the latter condition is shown to be satisfied by most square integrable functions in the case of Bernoulli shifts.

AMS 1980 Subject Classification: 60G10, 60F05.

Fourier coefficients * maximal inequalities * martingale central limit theorem * potential theory

1. Introduction

Let $X_0, X_1, X_2, \ldots$ denote a strictly stationary, ergodic Markov chain with values in a Polish space $\mathscr{X}$, a transition function Q, and a (stationary) initial distribution π. Let $\mathscr{L}$ denote the collection of all $\xi \in L^2(\pi)$ for which $\int \xi \, d\pi = 0$; and, given a $\xi \in \mathscr{L}$, consider the central limit question for

$$S_n = S_n(\xi) = \xi(X_1) + \cdots + \xi(X_n), \quad n \geq 1.$$

In this context, Gordin and Lifšic (1978) have obtained the following result: if there is an $h \in \mathscr{L}$ for which $\xi = h - Qh$, then $S_n/\sqrt{n}$ is asymptotically normal with mean 0 and variance $\sigma^2 = \|h\|^2 - \|Qh\|^2$, where $\|\cdot\|$ denotes the norm in $L^2(\pi)$ and Q denotes the contraction of $L^2(\pi)$ defined by $Qg(x) = \int g(y) Q(x; dy)$ for a.e. $x(\pi)$ for all $g \in L^2(\pi)$. Observe that the condition is satisfied if

$$h_n := \sum_{k=0}^{n} Q^k \xi \to h \quad \text{in } L^2(\pi) \quad \text{as } n \to \infty. \tag{1}$$

Conversely, if the two-sided extension $X_0, X_1, \ldots$ has a trivial left tail field, then (1) is necessary for the solution to $\xi = h - Qh$ with $h \in \mathscr{L}$. See Remark 2, below.

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Research supported by the National Science Foundation under DMS-8902188.

0304-4149/92/\$05.00

Let π_1 denote the joint distribution of X_0 and X_1. In Section 2, a related theorem is established in which (1) is replaced by the condition that there is a $g \in L^2(\pi_1)$ for which

$$g_n(x, y) := \sum_{k=1}^{n} [Q^{k-1}\xi(y) - Q^k\xi(x)] \to g(x, y) \quad \text{in } L^2(\pi_1), \tag{2}$$

as $n \to \infty$. If (1) holds, then so does (2), in which case $g(x, y) = h(y) - Qh(x)$ for a.e. (x, y); but the converse is false (Remark 2 and Example 2, below). Under the condition (2), it is shown that

$$S_n^{\#} = S_n^{\#}(\xi) = \frac{S_n - E(S_n \mid X_0)}{\sqrt{n}} \tag{3}$$

is asymptotically normal, as $n \to \infty$. In fact, a limit is obtained for the conditional distributions given X_0.

In Sections 3 and 4, the theorem is applied to the Bernoulli shift

$$X_n = \sum_{k=0}^{\infty} (\tfrac{1}{2})^{k+1} \varepsilon_{n-k}, \quad n \geq 0, \tag{4}$$

where ε_k, $k \in \mathbb{Z}$, are i.i.d. random variables which take the values 0 and 1 with probability $\frac{1}{2}$ each and $\mathbb{Z}$ denotes the integers. This is perhaps the simplest process which is mixing, but not strongly mixing or irreducible. In this case $\mathscr{X} = [0, 1)$ and $\pi = \lambda$ is the restriction of Lebesgue measure to $[0, 1)$, so that $L^2(\pi)$ is a familiar space. A condition on the Fourier coefficients of ξ is shown to be sufficient for asymptotic normality of $S_n^{\#}(\xi)$, and the condition is shown to be satisfied by most ξ in the following sense: let ξ_x denote the translate, $\xi_x(y) = \xi(x+y)$ for $x, y \in [0, 1)$, where addition is mod 1; then for *every* $\xi \in L^2(\pi)$, $S_n^{\#}(\xi_x)$ is asymptotically normal for a.e. x (Lebesgue). So, asymptotic normality is the rule, not the exception.

In Section 5, the theorem is applied to the Lebesgue shift process

$$X_n = (\ldots, U_{n-2}, U_{n-1}, U_n), \quad n \geq 0, \tag{5}$$

where U_k, $k \in \mathbb{Z}$, are i.i.d. uniformly distributed random variables on $[0, 1)$. Then $\mathscr{X} = [0, 1)^{\mathbb{M}}$, where $\mathbb{M}$ denotes the nonpositive integers; and $\pi = \lambda^{\mathbb{M}}$. Since $\mathscr{X}$ is the countable product of the circle group, any $\xi \in L^2(\pi)$ has a Fourier expansion; and it is possible to develop conditions for normality in terms of the Fourier coefficients.

Bhattacharrya and Lee (1988) have also developed conditions for normality, which are applicable to many non-irreducible models, including (4). They too make use of the Gordon Lifšic Theorem; but otherwise their methods are quite different. When specialized to processes of the form (4), their conditions require more regularity of ξ, but get stronger conclusions. Two other recent contributions are those of Guivarc'h and Hardy (1988) and Touati (1990).

If X_n, $n \in \mathbb{Z}$, is as in (5), then any process of the form $\xi(X_n)$, $n \in \mathbb{Z}$, is a stationary sequence with a trivial left tail field; and there is a partial converse, due to Hanson (1964). For recent surveys of central limit theory for stationary sequences under strong mixing conditions, see Bradley (1986) and Peligrad (1986).

2. A central limit theorem

In this section X_k, $k=0, 1, 2, \ldots$, denotes a strictly stationary, ergodic Markov chain, as in the Introduction. The probability space on which X_k, $k=0, 1, 2, \ldots$, are defined is denoted by $(\Omega, \mathscr{A}, P)$; $\mathscr{A}_j = \sigma\{X_0, \ldots, X_j\}$ for $j=0, 1, 2, \ldots$; it is assumed that $\mathscr{A} = \bigvee_{j=0}^{\infty} \mathscr{A}_j$; and P_x denotes the regular conditional probability given $X_0 = x$, obtained from the Markov Property.

For $x \in \mathscr{X}$, $z \in \mathbb{R}$, and $n=1, 2, \ldots$, let $F_n^*(x; z) = P_x\{S_n^* \leq z\}$, where S_n^* is defined by (3); let Δ denote the Levy metric (that is, $\Delta(F, G) = \inf\{\varepsilon > 0\colon F(z-\varepsilon) - \varepsilon \leq G(z) \leq F(z+\varepsilon) + \varepsilon$, for all $z \in \mathbb{R}\}$ for distribution functions F and G); and let Φ_σ denote the normal distribution with mean 0 and variance σ^2.

Theorem 1. *If $\xi \in \mathscr{L}$ and relation* (2) *holds, then*

$$\lim_{n\to\infty} \int \Delta[\Phi_\sigma, F_n^*(x; \cdot)] \pi(\mathrm{d}x) = 0 \tag{6}$$

where

$$\sigma^2 = \int g^2 \, \mathrm{d}\pi_1.$$

Proof. Writing $S_n - E(S_n \mid \mathscr{A}_0) = \sum_{j=1}^n [E(S_n \mid \mathscr{A}_j) - E(S_n \mid \mathscr{A}_{j-1})]$, one finds

$$S_n^* = \frac{1}{\sqrt{n}} \sum_{j=1}^n g_{n-j+1}(X_{j-1}, X_j) = \frac{1}{\sqrt{n}} (S_n' + S_n'')$$

where

$$S_n' = \sum_{j=1}^n g(X_{j-1}, X_j) \quad \text{and} \quad S_n'' = \sum_{j=1}^n [g_{n-j+1}(X_{j-1}, X_j) - g(X_{j-1}, X_j)]$$

for all $n = 1, 2, \ldots$. Let $F_n'(x; \cdot)$ denote the conditional distribution of $S_n'/\sqrt{n}$, given $X_0 = x$ for $x \in \mathscr{X}$ and $n \geq 1$. Then

$$\Delta[\Phi_\sigma, F_n^*(x; \cdot)] \leq \Delta[\Phi_\sigma, F_n'(x; \cdot)] + \sqrt{E_x |S_n''/\sqrt{n}|} \tag{7}$$

for all such n and x, by an easy application of Markov's Inequality.

First it is shown that $\Delta[\Phi_\sigma, F_n'(x; \cdot)] \to 0$ as $n \to \infty$ for a.e. x (π). To see this first observe that $g(X_{j-1}, X_j)$, $j = 1, 2, \ldots$, are martingale differences w.r.t. $\mathscr{A}_j = \sigma\{X_0, \ldots, X_j\}$, $j = 1, 2, \ldots$, on the probability space $(\Omega, \mathscr{A}, P_x)$ for a.e. x (π), by the Markov Property. Next, let

$$\sigma_c^2(x) = \int_{\{y : |g(x,y)| > c\}} g(x, y)^2 Q(x; \mathrm{d}y)$$

for $x \in \mathscr{X}$ and $0 < c < \infty$, so that $\sigma_0^2(X_{j-1}) = E_x\{g(X_{j-1}, X_j)^2 | \mathscr{A}_{j-1}\}$ w.p.1 (P_x) for all $j = 1, 2, \ldots$ and a.e. x (π). Then

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} \sigma_c^2(X_{j-1}) = \sigma_c^2 := \int \sigma_c^2(x)\, \mathrm{d}\pi(x) \tag{8}$$

w.p.1 (P), by the Ergodic Theorem. So, (8) holds w.p.1 (P_x) for a.e. x (π) for all $0 \leqslant c < \infty$, since $P(A) = \int P_x(A)\pi(\mathrm{d}x)$ for all $A \in \mathscr{A}$. Letting $c = 0$ in (8) shows the relative stability of the conditional variances; and letting $c \to \infty$ in (8) verifies the Lindeberg–Feller Condition. The desired conclusion now follows from the martingale central limit theorem. See Hall and Heyde (1980, pp. 58–59).

For the second term on the right side of (7), it is easily seen that $g_{n-j+1}(X_{j-1}, X_j)$, $j = 1, \ldots, n$, are martingale differences for each n. So,

$$\int \sqrt{E_x \left| \frac{S_n''}{\sqrt{n}} \right|}\, \pi(\mathrm{d}x) \leqslant E\left(\frac{S_n''^2}{n}\right)^{1/4} = \left\{\frac{1}{n}\sum_{j=1}^{n} \int |g_j - g|^2\, \mathrm{d}\pi_1\right\}^{1/4} \to 0$$

as $n \to \infty$ to complete the proof. $\square$

When the (1) holds, a stronger conclusion is possible. For $0 \leqslant t \leqslant 1$ and $n = 1, 2, \ldots$, let $\mathbb{B}_n(t) = S_{[nt]}/\sigma\sqrt{n}$, where $[x]$ denotes the greatest integer which is less than or equal to x.

Theorem 2. *If* (1) *holds, then the conditional distribution of* $\mathbb{B}_n$ *given* $X_0 = x$ *converges weakly to the distribution of Brownian motion as* $n \to \infty$ *in* $D[0, 1]$ *for a.e.* x (π).

Proof. Letting $B_n'(t) = S_{[nt]}'/\sigma\sqrt{n}$, $0 \leqslant t \leqslant 1$, $n = 1, 2, \ldots$, where S_n', $n = 1, 2, \ldots$, are as in the proof of Theorem 1 (with $g(x, y) = h(y) - Qh(x)$), it is easily seen that

$$\mathbb{B}_n(t) = \mathbb{B}_n'(t) + \frac{1}{\sqrt{n}}[Qh(X_0) - Qh(X_{[nt]})]$$

for all $0 \leqslant t \leqslant 1$ and $n = 1, 2, \ldots$. The proof of Theorem 1 and Theorem 2.5 of Durrett and Resnick (1978), show that the conditional distribution of $\mathbb{B}_n'$ given $X_0 = x$ converges to the distribution of standard Brownian motion for a.e. x (π); and $\max_{k \leqslant n} |Qh(X_0) - Qh(X_k)|/\sqrt{n} \to 0$ as $n \to \infty$ w.p.1 (P) and, therefore w.p.1 (P_x) for a.e. x (π). $\square$

Remark 1. Theorem 1 is related to Theorem 5.3 of Hall and Heyde (1980). A continuous time version of Theorem 2 appears in Bhattacharya (1982).

Remark 2. The stationary sequence $X_0, X_1, \ldots$ has a two sided extension X_k', $k \in \mathbb{Z}$, (Breiman, 1968, p. 105). If this process has a trivial left tail field, then (1) is necessary for the representation $\xi = h - Qh$ with $h \in \mathscr{L}$. For then $h = h_n + Q^{n+1}\xi$ for all $n = 1, 2, \ldots$, and $Q^n\xi \to 0$ in $L^2(\pi)$ as $n \to \infty$, since $Q^n\xi(X_{-n}') = E\{\xi(X_0') | X_k'\ \forall k \leqslant -n\}$ w.p.1 for all $n = 1, 2, \ldots$.

3. Bernoulli shifts

In this section X_k, $k=1, 2, \ldots$, denotes the Bernoulli shift (4). Then, letting

$$W_k = \sum_{j=1}^{k} 2^{-j}\varepsilon_j, \qquad Q^k(x; B) = P\left\{\frac{x}{2^k} + W_k \in B\right\}$$

for $0 \leqslant x < 1$, Borel sets $B \subseteq [0, 1)$, and $k = 1, 2, \ldots$.

Let $\hat{\xi}(k)$, $k \in \mathbb{Z}$, denote the (complex) Fourier coefficients of $\xi \in \mathscr{L}$, so that $\hat{\xi}(r) = \int_0^1 \mathrm{e}^{-2\pi i r x}\xi(x)\,\mathrm{d}x\ \forall 0 \neq r \in \mathbb{Z}$,

$$\|\xi\|^2 = \sum_{r\neq 0}^{\infty} |\hat{\xi}(r)|^2 < \infty \quad \text{and} \quad \xi(x) = \sum_{r\neq 0} \hat{\xi}(r) \times \mathrm{e}^{2\pi i r x}$$

for a.e. x (λ), using Carleson's (1966) Theorem.

Let $\mathbb{E}$ and $\mathbb{O}$ denote the even and odd integers; let $\mathbb{O}^+$ denote the positive odd integers; and observe that any integer $r \neq 0$ may be written uniquely in the form $r = s2^k$, where $s \in \mathbb{O}$ and $k \geqslant 0$.

Lemma 1.

$$Q^k\xi(X_0) = \sum_{r\neq 0} \hat{\xi}(r2^k)\,\mathrm{e}^{2\pi i r X_0}$$

and

$$Q^k\xi(X_1) - Q^{k+1}\xi(X_0) = \sum_{r\in\mathbb{O}} \hat{\xi}(r2^k)\,\mathrm{e}^{2\pi i r X_1}$$

*w.p.*1 (P) *for all* $k = 1, 2, \ldots$ *and all* $\xi \in \mathscr{L}$.

Proof. Let $e(x) = \mathrm{e}^{2\pi i x}$ for $x \in \mathbb{R}$. Then, for a.e. x,

$$\xi\left(\frac{x}{2^k} + W_k\right) = \sum_{r\neq 0} \hat{\xi}(r) e\left(\frac{rx}{2^k} + rW_k\right) = \sum_{r\neq 0} \hat{\xi}(r) e\left(\frac{rx}{2^k}\right) \prod_{j=1}^{k} e\left(\frac{r\varepsilon_j}{2^j}\right),$$

so that

$$Q^k\xi(x) = \sum_{r\neq 0} \hat{\xi}(r) b_{k,r} e\left(\frac{rx}{2^k}\right),$$

where

$$b_{k,r} = 2^{-k} \prod_{j=1}^{k} \left[1 + e\left(\frac{r}{2^j}\right)\right]$$

for $r = 1, 2, \ldots$ and $k = 1, 2, \ldots$. Now, $b_{k,r} = 0$ unless r is a integral multiple of 2^k, in which case $b_{k,r} = 1$. To see this write $r = s2^l$, where l is a nonnegative integer and s is an odd integer. If $l < k$, then $e(r/2^{l+1}) = e(s/2) = -1$, so that $b_{k,r} = 0$; and if $l \geqslant k$, then $e(r/2^j) = 1$ for all $j = 1, \ldots, k$. This establishes the first assertion of the lemma.

For the second, observe that $X_1 = \frac{1}{2}(X_0 + \varepsilon_1)$. So, $e(2rX_1) = e(rX_0)e(r\varepsilon_1) = e(rX_0)$ for all $r \neq 0$. It follows that

$$\sum_{0 \neq r \in \mathbb{E}} \hat{\xi}(r2^k)e(rX_1) = \sum_{r \neq 0} \hat{\xi}(r2^{k+1})e(2rX_1) = \sum_{r \neq 0} \hat{\xi}(r2^{k+1})e(rX_0) = Q^{k+1}\xi(X_0),$$

and therefore

$$Q^k\xi(X_1) - Q^{k+1}\xi(X_0) = \sum_{r \in \mathbb{O}} \hat{\xi}(r2^k)e(rX_1)$$

w.p.1 for all $k = 1, 2, \ldots$. □

Theorem 3. *If* $\xi \in \mathscr{L}$, *then* (2), *respectively*, (1), *holds iff*

$$\lim_{m,n\to\infty} \sum_{r \in \mathbb{O}^+} \left| \sum_{k=m}^{n} \hat{\xi}(r2^k) \right|^2 = 0, \tag{9}$$

resp.,

$$\lim_{m,n\to\infty} \sum_{r=1}^{\infty} \left| \sum_{k=m}^{n} \hat{\xi}(r2^k) \right|^2 = 0. \tag{10}$$

Proof. By Lemma 1, for all $1 \leq m < n < \infty$,

$$\begin{aligned} g_n(X_0, X_1) - g_m(X_0, X_1) &= \sum_{k=m}^{n-1} [Q^k\xi(X_1) - Q^{k+1}\xi(X_0)] \\ &= \sum_{r \in \mathbb{O}} \left[\sum_{k=m}^{n-1} \hat{\xi}(r2^k)] \right] e(rX_1) \quad \text{w.p.1.} \end{aligned}$$

So,

$$\begin{aligned} \int |g_n - g_m|^2 \, d\pi_1 &= \int \left| \sum_{r \in \mathbb{O}} \left[\sum_{k=m}^{n-1} \hat{\xi}(r2^k) \right] e(rx) \right|^2 dx \\ &= \sum_{r \in \mathbb{O}} \left| \sum_{k=m}^{n-1} \hat{\xi}(r2^k) \right|^2 = 2 \sum_{r \in \mathbb{O}^+} \left| \sum_{k=m}^{n-1} \hat{\xi}(r2^k) \right|^2 \end{aligned}$$

for all such m and n, by the orthogonality of the complex exponentials. The first assertion follows immediately; and the second may be proved similarly. □

Combining Propositions 1 and 2 with Theorems 1 and 2 yields:

Corollary. *If* (9) *holds, then so does* (6); *and if* (10) *holds, then the conditional distribution of* $\mathbb{B}_n$ *given* $X_0 = x$ *converges to that of Brownian motion for a.e. x.* □

Remark 3. The condition imposed on the Fourier coefficients is not very restrictive. In fact, using Schwarz' Inequality, it is easily seen that

$$\begin{aligned} \sum_{r \in \mathbb{O}^+} \left| \sum_{k=m}^{n} \hat{\xi}(r2^k) \right|^2 &= \sum\sum_{j,k=m}^{n} \sum_{r \in \mathbb{O}^+} \hat{\xi}(r2^j)\hat{\xi}(r2^k) \\ &\leq \left\{ \sum_{k=m}^{n} \sqrt{\sum_{r \in \mathbb{O}^+} |\hat{\xi}(r2^k)|^2} \right\}^2. \end{aligned}$$

So (6) and (9) hold if the last sum is finite when m and n are replaced by 1 and ∞. In particular, (6) and (9) hold if $\sum_{n=m}^{\infty}|\hat{\xi}(n)|^2 = \mathrm{O}[(\log m)^{-\alpha}]$ as $m \to \infty$ for some $\alpha > 2$. Similar sufficient conditions may be given for (10).

Remark 4. According to Zygmund (1968, p. 45), $|\hat{\xi}(n)| \leq \omega(1/n)$ for all $n = 1, 2, \ldots$, where ω denotes the modulus of continuity of ξ. So, (6) and (9) hold if ξ satisfies a Lipschitz condition of order $\alpha > \frac{1}{2}$. By way of contrast, the conditions of Bhattacharrya and Lee (1988) are satisfied if ξ is Lipschitz condition of order $\alpha = 1$. Relation (9) does not even require continuity, however.

Example 1. If

$$\xi(x) = \frac{1}{|x-\frac{1}{2}|^{\alpha}} \sin\left(\frac{1}{2\pi|x-\frac{1}{2}|}\right) \quad \forall x \neq \tfrac{1}{2},$$

for some $0 \leq \alpha < \frac{1}{2}$, then it may be shown that $|\hat{\xi}(n)| = \mathrm{O}(n^{-\beta})$, where $\beta = \frac{1}{4}(3 - 2\alpha) > \frac{1}{2}$. So, (6) and (9) are satisfied.

Example 2. If a_r, $r \in \mathbb{O}^+$, and b_k, $k = 0, 1, 2, \ldots$, are two real, square summable sequences, then there is a $\xi \in \mathscr{L}$ for which

$$\hat{\xi}(r2^k) = \tfrac{1}{2} a_{|r|} b_k$$

for all $r \in \mathbb{O}$ and $k = 0, 1, 2, \ldots$. In this case, it is easily seen that (9) holds iff $b_0 + b_1 + b_2 + \cdots$ converges and that (10) is satisfied only if $b_0 + b_1 + b_2 + \cdots$ converges and $\sum_{k=1}^{\infty}|\sum_{j=k}^{\infty} b_j|^2 < \infty$. Since the extended Bernoulli Shift has a trivial left tail field, by the zero-one law, this example shows that (2) need not imply (1).

Example 3. If ξ is as in Example 2, then it is easily seen that

$$g_n(X_0, X_1) = (b_0 + \cdots + b_{n-1})Y, \quad n \geq 1,$$

where

$$Y = \sum_{r \in \mathbb{O}^+} a_r \cos(2\pi r X_1).$$

Let

$$\sigma_n^2 = \sum_{k=1}^{n} (b_0 + \cdots + b_{k-1})^2, \quad n \geq 1,$$

and suppose that $\sigma_n^2 > 0$ for all sufficiently large n. If $|a_1| + |a_2| + \cdots < \infty$, then (using the arguments of Section 2), it is easily seen that

$$S_n^{**} = \frac{1}{\sigma_n}\{S_n - E(S_n \mid \mathscr{A}_0)\} \Rightarrow \mathrm{N}(0, 1), \quad \text{as } n \to \infty,$$

if

$$\lim_{n\to\infty} \max_{j\leq n} \frac{|b_0+\cdots+b_{j-1}|}{\sigma_n} = 0,$$

where $\Rightarrow$ denotes convergence in distribution. In particular, if $|b_0+\cdots+b_k|$, $k=0,1,2,\ldots$, are bounded and $\sigma_n^2 \sim n$, as $n\to\infty$, then $S_n^{\#} \Rightarrow \mathrm{N}(0,1)$, so that (9) is not a necessary condition for asymptotic normality of $S_n^{\#}$.

4. Translates

The results of this section depend on the following lemma. In its statement $f\in\mathscr{L}$ has complex Fourier coefficients $c_j=\hat{f}(j)$, $j\in\mathbb{Z}$,

$$S_n(f;x)=\sum_{j=0}^{n} c_j\,\mathrm{e}^{2\pi \mathrm{i} jx} \quad\text{and}\quad \Sigma_n(f;x)=\frac{1}{n}\sum_{k=0}^{n-1} S_k(f;x)$$

for all $0\leq x<1$ and $n=1,2,\ldots$.

Lemma 2. *There is an absolute constant C for which*

$$\int_0^1 \sup_{n\geq 1} |\Sigma_n(f;x)|^2\,\mathrm{d}x \leq C\int_0^1 f(x)^2\,\mathrm{d}x$$

for all $f\in\mathscr{L}$.

Proof. The lemma follows easily from Zygmund (1968, IV, 7.8 and VII, 7.32); but it requires some notation to explain why. Let $\alpha_k=2\int_0^1\cos(2\pi kx)f(x)\,\mathrm{d}x$ and $\beta_k=2\int_0^1\sin(2\pi kx)f(x)\,\mathrm{d}x$, $k=0,1,2,\ldots$, denote the (real) Fourier coefficients of f, so that $c_k=\frac{1}{2}(\alpha_k-\mathrm{i}\beta_k)$ for all $k=0,1,2,\ldots$; let

$$s_n(f;x)=\tfrac{1}{2}\alpha_0+\sum_{k=1}^{n}\tfrac{1}{2}[\alpha_k\cos(2\pi kx)+\beta_k\sin(2\pi kx)]$$

and

$$\tilde{s}_n(f;x)=\sum_{k=0}^{n}\tfrac{1}{2}[\alpha_k\sin(2\pi kx)-\beta_k\cos(2\pi kx)]$$

for $0\leq x\leq 1$ and $n=1,2,\ldots$; and let σ_n and $\tilde{\sigma}_n$, $n=1,2,\ldots$, denote the Cesaro averages of s_k and $\tilde{s}_k$, $k=1,2,\ldots$. Then $2\Sigma_n(f;x)=\sigma_n(f;x)+\mathrm{i}\tilde{\sigma}_n(f;x)$ for all x and n, so that

$$\int_0^1 \sup_{n\geq 1}|\Sigma_n(f;x)|^2\,\mathrm{d}x \leq \tfrac{1}{4}\int_0^1\left[\sup_{n\geq 1}|\sigma_n(f;x)|^2+\sup_{n\geq 1}|\tilde{\sigma}_n(f;x)|^2\right]\mathrm{d}x;$$

and if follows from Zygmund (op. cit.) that the right side of (10) is bounded by $C\int_0^1 f(x)^2\,\mathrm{d}x$ for some absolute constant C. $\square$

If $\xi \in \mathscr{L}$ and $0 \leqslant x < 1$, then the translate of ξ is defined by $\xi_x(y) = \xi(x+y)$, $0 \leqslant y < 1$, where addition is understood modulo one. Clearly $\xi_x \in \mathscr{L}$ and $\hat{\xi}_x(r) = \hat{\xi}(r)\exp(2\pi i r x)$ for $r \in \mathbb{Z}$ and $x \in [0, 1)$.

Theorem 4.

$$\lim_{m,n\to\infty} \sum_{r\in\mathbb{O}^+} \left| \sum_{k=m}^{n} \hat{\xi}_x(r2^k) \right|^2 = 0$$

for a.e. x for every $\xi \in \mathscr{L}$.

Proof. For $r \in \mathbb{O}^+$ and $m \geqslant 1$, the (real) lacunary series

$$f_{r,m}(x) = \sum_{k=m}^{\infty} [\hat{\xi}(r2^k)e(r2^kx) + \hat{\xi}(-r2^k)e(-r2^kx)]$$

converges for a.e. x (λ). See Zygmund (1968, p. 203). For $n \geqslant m$,

$$\sup_{n_1,n_2\geqslant m} \sum_{r\in\mathbb{O}^+} \left| \sum_{k=n_1}^{n_2} \hat{\xi}_x(r2^k) \right|^2 \leqslant 2 \sum_{r\in\mathbb{O}^+} \sup_{n\geqslant m} \left| \sum_{k=m}^{n} \hat{\xi}_x(r2^k) \right|^2.$$

The first term is nonincreasing in $m = 1, 2, \ldots$ and, therefore, has a limit as $m \to \infty$ for all $x \in [0, 1]$. So, it suffices to show that the last term approaches zero in the mean as $m \to \infty$. For fixed $r \in \mathbb{O}^+$ and $m \geqslant 1$,

$$\sup_{n\geqslant m} \left| \sum_{k=m}^{n} \hat{\xi}_x(r2^k) \right| \leqslant \sup_{n\geqslant 1} |S_n(f_{r,m}; x)| \leqslant 3 \sup_{n\geqslant 1} |\Sigma_n(f_{r,m}; x)|$$

for all $x \in [0, 1]$, where S_n and Σ_n are as in Lemma 2, since $f_{r,m}$ is a lacunary series. See Zygmund (1968, p. 79). It follows that

$$\begin{aligned}
\int_0^1 \left\{ \sum_{r\in\mathbb{O}^+} \sup_{n\geqslant m} \left| \sum_{k=m}^{n} \hat{\xi}_x(r2^k) \right|^2 \right\} \mathrm{d}x &\leqslant 9 \sum_{r\in\mathbb{O}^+} \int_0^1 \sup_{n\geqslant m} |\Sigma_n(f_{r,m}; x)|^2 \,\mathrm{d}x \\
&\leqslant 9C \sum_{r\in\mathbb{O}^+} \int_0^1 f_{r,m}(x)^2 \,\mathrm{d}x \\
&\leqslant 18C \sum_{r\in\mathbb{O}^+} \left\{ \sum_{k=m}^{\infty} |\hat{\xi}(r2^k)|^2 \right\},
\end{aligned}$$

which approaches zero as $m \to \infty$, since $|\hat{\xi}(k)|^2$, $k \in \mathbb{Z}$, are summable. The theorem follows. $\square$

5. Lebesgue shifts

In this section, $X_n = (\ldots, U_{n-2}, U_{n-1}, U_n)$, $n \in \mathbb{Z}$, denotes the Lebesgue shift process (5). Let $\mathbb{M}$ and $\mathbb{N}$ denote the nonpositive and nonnegative integers. Then $\mathscr{X} = [0, 1)^{\mathbb{M}}$,

$\pi = \lambda^{\mathbb{M}}$, and

$$Q^k g(x) = \int_{[0,1)^k} g(x, u)\lambda^k(\mathrm{d}u), \quad x \in \mathscr{X}, \tag{11}$$

for bounded measurable functions $g: \mathscr{X} \to \mathbb{R}$ and $k = 1, 2, \ldots$.

If $\mathscr{X}$ is viewed as the direct product of circles, then $\mathscr{X}$ is a compact commutative group; and the character group Γ consists of all $\gamma = (\gamma_0, \gamma_1, \gamma_2, \ldots) \in \mathbb{Z}^{\mathbb{N}}$ for which $\gamma_k = 0$ for all but a finite number of k with the convention that

$$\gamma(x) = \mathrm{e}^{2\pi \mathrm{i} \gamma \cdot x}$$

where $\gamma \cdot x = \gamma_0 u_0 + \gamma_1 u_{-1} + \cdots$ for all $x = (\ldots, u_{-2}, u_{-1}, u_0) \in \mathscr{X}$ and all γ. Any $\xi \in L^2(\pi)$ has the Fourier expansion

$$\xi(x) = \sum_{\gamma \in \Gamma} \hat{\xi}(\gamma) \times \gamma(x),$$

where $\hat{\xi}(\gamma) = \int \bar{\gamma}\xi \, \mathrm{d}\pi$, for all $\gamma \in \Gamma$, the sum converges in $L^2(\pi)$, and the bar denotes complex conjugate. See Hewitt and Ross (1979, pp. 364 and 382).

For $k = 1, 2, \ldots$, let $\Gamma_k = \{\gamma \in \Gamma\colon \gamma_j = 0 \text{ for } j = 0, \ldots, k-1\}$ and $\Delta_k = \Gamma_k - \Gamma_{k+1}$; and let $\Delta = \{\gamma \in \Gamma\colon \gamma_0 \neq 0\}$. Further, let $\gamma^k = (\gamma_k, \gamma_{k+1}, \ldots)$ for $\gamma \in \Gamma$ and $k \geq 1$; and let T denote the shift operator $T(\ldots, u_{-1}, u_0) = (\ldots, u_{-2}, u_{-1})$ for $x = (\ldots, u_{-1}, u_0) \in \mathscr{X}$.

Lemma 3. *If $\xi \in L^2(\pi)$, then*

$$Q^k \xi = \sum_{\gamma \in \Gamma} \hat{\xi}(\theta_k, \gamma) \times \gamma$$

and

$$Q^k \xi - Q^{k+1}\xi \circ T = \sum_{\gamma \in \Delta} \hat{\xi}(\theta_k, \gamma) \times \gamma,$$

where $\theta_k = (0, \ldots, 0) \in \mathbb{Z}^k$ for all $k = 1, 2, \ldots$.

Proof. If $k \geq 1$, $\gamma \in \Gamma$, and $x = (\ldots, x_{-1}, x_0) \in [0, 1)^{\mathbb{M}}$, then by (11),

$$\begin{aligned} Q^k \gamma(x) &= \exp\left\{\sum_{j=k}^{\infty} 2\pi \mathrm{i} \gamma_j x_{k-j}\right\} \int_{[0,1)^k} \exp\left[\sum_{j=0}^{k-1} 2\pi \mathrm{i} \gamma_j u_j\right] \lambda^k(\mathrm{d}u) \\ &= \gamma^k(x) \prod_{j=0}^{k-1} \left\{\int_0^1 \exp[2\pi \mathrm{i} \gamma_j u] \lambda(\mathrm{d}u)\right\}; \end{aligned}$$

and the last product is zero or one for $\gamma \notin \Gamma_k$ and $\gamma \in \Gamma_k$. So,

$$Q^k \xi = \sum_{\gamma \in \Gamma_k} \hat{\xi}(\gamma) \times \gamma^k$$

and

$$Q^k \xi - Q^{k+1}\xi \circ T = \sum_{\gamma \in \Gamma_k} \hat{\xi}(\gamma) \times \gamma^k - \sum_{\gamma \in \Gamma_{k+1}} \hat{\xi}(\gamma) \times \gamma^{k+1} \circ T.$$

There is some cancellation. For if $\gamma\in\Gamma_{k+1}$, then $\gamma^k(x)=\gamma^{k+1}\circ T(x)$ for all $x\in\mathscr{X}$, so that

$$\sum_{\gamma\in\Gamma_{k+1}}\hat{\xi}(\gamma)\times\gamma^k=\sum_{\gamma\in\Gamma_{k+1}}\hat{\xi}(\gamma)\times\gamma^{k+1}\circ T$$

and

$$Q^k\xi-Q^{k+1}\xi\circ T=\sum_{\gamma\in\Delta_k}\hat{\xi}(\gamma)\times\gamma^k.$$

Finally, every $\gamma\in\Gamma_k$ (respectively, Δ_k) may be written uniquely as $\gamma=(\theta_k,\beta)$, where $\beta=\gamma^k\in\Gamma$ (respectively, Δ). So,

$$Q^k\xi=\sum_{\gamma\in\Gamma_k}\hat{\xi}(\gamma)\times\gamma^k=\sum_{\gamma\in\Gamma}\hat{\xi}(\theta_k,\gamma)\times\gamma$$

and

$$Q^k\xi-Q^{k+1}\xi\circ T=\sum_{\gamma\in\Delta_k}\hat{\xi}(\gamma)\times\gamma^k=\sum_{\gamma\in\Delta}\hat{\xi}(\theta_k,\gamma)\times\gamma. \qquad\square$$

Theorem 5. *If $\xi\in\mathscr{L}$, then* (2), *respectively,* (1), *holds iff*

$$\lim_{m,n\to\infty}\sum_{\gamma\in\Delta}\left|\sum_{j=m}^{n}\hat{\xi}(\theta_j,\gamma)\right|^2=0, \tag{12}$$

resp.,

$$\lim_{m,n\to\infty}\sum_{\gamma\in\Gamma}\left|\sum_{j=m}^{n}\hat{\xi}(\theta_j,\gamma)\right|^2=0,$$

in which case the conclusion of Theorem 1, *resp., Theorem* 2, *holds.*

Proof. By Lemma 3 and the orthogonality of characters,

$$\begin{aligned}\int|g_n-g_m|^2\,\mathrm{d}\pi_1&=\int\left|\sum_{k=m}^{n-1}(Q^k\xi-Q^{k+1}\xi\circ T)\right|^2\mathrm{d}\pi\\&=\int\left|\sum_{\gamma\in\Delta}\left\{\sum_{k=m}^{n-1}\hat{\xi}(\theta_k,\gamma)\right\}\times\gamma\right|^2\mathrm{d}\pi=\sum_{\gamma\in\Delta}\left|\sum_{k=m}^{n-1}\hat{\xi}(\theta_k,\gamma)\right|^2\end{aligned}$$

for all $1\leqslant m<n<\infty$. The first half of the theorem follows easily and the respective half follows similarly. $\square$

Remark 5. As in Remark 3, it is not difficult to see that

$$\sum_{\gamma\in\Delta}\left|\sum_{j=m}^{n}\hat{\xi}(\theta_j,\gamma)\right|^2\leqslant\left\{\sum_{j=m}^{n}\sqrt{\sum_{\gamma\in\Delta}|\hat{\xi}(\theta_j,\gamma)|^2}\right\}^2, \tag{13}$$

so that (12) holds if the right side converges when $m=1$ and $n=\infty$.

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