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On the Cohen–Macaulayness of the conormal module of an ideal

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ABSTRACT

In the present paper we investigate a question stemming from a long-standing conjecture of Vasconcelos: given a generically complete intersection perfect ideal I in a regular local ring R, when is it true that the Cohen–Macaulayness of I/I^2 (or R/I^2) implies that R/I is Gorenstein? This property is known to hold for licci ideals and, essentially, squarefree monomial ideals. We show that a positive answer actually holds for every monomial ideal. We then give a positive answer for several special classes of ideals and provide application to algebroid curves with low multiplicity. We also exhibit prime ideals in regular local rings and homogeneous level ideals in polynomial rings for which the answer is negative and use them to show the sharpness of our main result, as they lie in the first class of ideals not covered by it. The homogeneous examples have been found thanks to the help of J.C. Migliore. As a by-product, we exhibit several classes of Cohen–Macaulay ideals whose square is not Cohen–Macaulay. Our methods work both in the homogeneous and in the local settings.

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1. Introduction

For an ideal I in a Noetherian local ring R, the first conormal module I/I^2 plays an important role in the process of understanding the structure of I. One (and the first) main result in this sense is a celebrated theorem of Ferrand and Vasconcelos stating that, if the ideal I has finite projective dimension, the freeness of I/I^2 (as R/I-module) is equivalent to I being a complete intersection [9,22]. Later, Vasconcelos [23] proposed that one could relax the assumptions in the above statement,

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namely he conjectured that if the projective dimension of I/I^2 over R/I is finite, i.e., $\mathrm{pd}_{R/I}(I/I^2) < \infty$, then I must be a complete intersection. This conjecture has been proved for some classes of ideals in a Noetherian local ring by Vasconcelos (see for instance [23] and [24]) and, in the characteristic zero case, for any homogeneous ideal by Avramov and Herzog [3].

In the present paper, we investigate a question in this spirit, stemming out from two old-standing conjectures of Vasconcelos, see [25, Conjecture B] and [20, Conjecture 3.12]. Historically, these conjectures are the reason why this question (for short, we will call it 'the question') was first considered, in [15]. The question is the following: *Let R be a regular local ring and I a perfect R-ideal that is generically a complete intersection, for which ideals I does the Cohen–Macaulayness of I/I^2 imply that R/I is Gorenstein?*. Equivalently, one could ask for which perfect, generically complete intersection ideals I in a regular local ring R does the Cohen–Macaulayness of R/I^2 imply that R/I is Gorenstein. Ideals enjoying this property, indeed, highlight an intriguing connection between the Cohen–Macaulayness of the square of an ideal I and the Gorenstein property of I.

Earlier work by several authors showed that the following ideals have the property conjectured in the question:

- perfect *prime* ideals of height 2 (see work of Herzog [11]);
- licci ideals (proved by Huneke and Ulrich in [15]);
- squarefree monomial ideals in a polynomial ring whose square is Cohen–Macaulay over any field (proved recently by Rinaldo, Terai and Yoshida [18]).

Also, work of Minh–Trung and Trung–Tuan gives a complete description of all 2- or 3-dimensional Stanley–Reisner ideals whose second power is Cohen–Macaulay whereas the third power is not Cohen–Macaulay and it is shown that in all these cases, the ideal is Gorenstein (independent of the base field) [17,21]. However, in general, the question is wide open.

In the present paper we prove that the question has a positive answer if I is *any* monomial ideal (extending a result of Rinaldo–Terai–Yoshida) and for ideals defining stretched algebras, short algebras or algebras with low multiplicity (precise definitions will be given later in the paper). We use several different techniques and *ad hoc* methods, including careful estimates (both in the local and homogeneous settings) of the Hilbert function of the square of the ideal I by means of the Hilbert function of I.

We also provide examples for which the question has a negative answer (to our best knowledge these are the first negative examples in the literature) and employ them to prove the sharpness of our main result (see below).

Combining together the main results of the first five sections, we prove the following:

Theorem 1.1. *Let R be a regular local ring containing a field k (resp. a polynomial ring over a field k) of characteristic $\neq 2$ and I a Cohen–Macaulay (resp. homogeneous Cohen–Macaulay) ideal that is generically a complete intersection. Assume that one of the following conditions holds*:

(a) *I is a monomial ideal (Theorem 4.2)*;
(b) *R/I is a stretched algebra (Theorem 3.5)*;
(c) *R/I is a short algebra with socle degree at least 3 (Theorem 5.3)*;
(d) *R/I is a short algebra with socle degree 2 and its multiplicity satisfies some numerical conditions (Proposition 5.5)*;
(e) *$e(R/I) < \mathrm{ecodim}(R/I) + 5$, where $\mathrm{ecodim}(R/I)$ is the embedding codimension of R/I (Theorems 6.2 and 6.4).*

If I/I^2 (equivalently, R/I^2) is Cohen–Macaulay then R/I is Gorenstein.

Parts (d) and (e) are sharp. In fact, with the help of J.C. Migliore, we produced – using CoCoA [5] – a computer-generated homogeneous level ideal defined by a set of 10 (general) points in $\mathbb{P}^5$ that is a counterexample to the question. Remarkably, this ideal I belongs to the first class of ideals not covered by Theorem 1.1. Indeed, it defines a short algebra with socle degree 2 (showing the sharpness of the numerical conditions of Theorem 1.1(d)) and $e(R/I) = c + 5$ (which shows the sharpness of Theorem 1.1(e))

We then apply the theory of universal linkage to modify the above example and produce a prime ideal $\mathfrak{p}$ in a regular local ring S with $e(S/\mathfrak{p}) = \operatorname{ht}\mathfrak{p} + 5$ such that $S/\mathfrak{p}$ and $\mathfrak{p}/\mathfrak{p}^2$ are Cohen–Macaulay but $S/\mathfrak{p}$ is not Gorenstein. It shows that the question of Vasconcelos has a negative answer even for *prime* ideals. Furthermore, this method shows that the whole question can be reduced to the case of prime ideals (at least in the local setting) – see Proposition 2.8 and the discussion after it.

Finally, notice that in the cases where R/I is not Gorenstein (e.g. most short algebras), the question reduces to proving that R/I^2 cannot be Cohen–Macaulay.

The structure of the paper is the following. In Section 1, we prove basic facts that will be used throughout the paper, and we reduce the question to the case of non-degenerate prime ideals. At the end of the section we explain the structure of the proof of our main results. In Section 2, we prove the case of stretched ideals. In Section 3 we generalize the result of Rinaldo–Terai–Yoshida to every monomial ideal. In Section 4 we prove that almost every ideal I defining a short algebra has the property that I^2 is not Cohen–Macaulay. Somewhat surprisingly, the few missing cases provide counterexamples to the question. In Section 5, we employ results from Sections 2 and 4 to deal with the case of ideals defining algebras whose multiplicity is at most the embedding codimension plus 4. We then provide an application to algebroid curves (e.g. monomial curves). In Section 6, we provide both local and homogeneous examples showing that the general answer to Vasconcelos questions is negative, even for prime ideals. They rely on a computer-generated example found with the help of J.C. Migliore. These examples also yield the sharpness of Theorem 1.1. We also make the following conjecture: For *any* embedding codimension $c \geqslant 5$, the homogeneous ideal I defined by a set of $c + 1 + \lceil \frac{c(c-1)}{6} \rceil$ general points in $\mathbb{P}^c$ has the property that I/I^2 is Cohen–Macaulay, but R/I is not Gorenstein. This would provide an entire class of counterexamples. We checked this conjecture with the help of J.C. Migliore and CoCoA [5] for small values of c, namely, for $5 \leqslant c \leqslant 10$.

2. Preliminaries and reduction to the non-degenerate case

In this section we fix some notation and prove basic facts that will be used throughout the paper. In particular, we show that in Question 2.1 one can always assume the ideal I to be 'non-degenerate', i.e., $I \subseteq \mathfrak{m}^2$ (see Proposition 2.6), and prime (see Proposition 2.8).

Recall that in a Noetherian ring R, an ideal I is *generically a complete intersection* if $I_\mathfrak{p}$ is a complete intersection $R_\mathfrak{p}$-ideal for every $\mathfrak{p} \in \operatorname{Ass}_R(R/I)$, the set of associated prime ideals of R/I. We are now able to state Vasconcelos's question in the form that appears in [15].

Question 2.1. *(See [15].) Let R be a regular local ring (resp. a polynomial ring over a field) and I a Cohen–Macaulay (resp. homogeneous Cohen–Macaulay) generically complete intersection ideal. When is it true that the Cohen–Macaulayness of I/I^2 implies that R/I is Gorenstein?*

Notice that from the Cohen–Macaulayness of I and the short exact sequence

$$0 \longrightarrow I/I^2 \longrightarrow R/I^2 \longrightarrow R/I \longrightarrow 0$$

it follows that I/I^2 is Cohen–Macaulay if and only if R/I^2 is. Hence one can phrase Question 2.1 in the following slightly different way, involving only I and I^2.

Question 2.2. *Let R be a regular local ring (resp. a polynomial ring over a field) and I a generically complete intersection, Cohen–Macaulay (resp. homogeneous Cohen–Macaulay) ideal. For which I does the Cohen–Macaulayness of R/I^2 imply that R/I is Gorenstein?*

The classes of ideals for which Question 2.2 has a positive answer, show an interesting connection between the Cohen–Macaulayness of I^2 and the Gorenstein property of I. As it was mentioned in the introduction, the only known cases were licci ideals [15, 2.8] and, essentially, squarefree monomial ideals [18]. Furthermore, in the case of 2- and 3-dimensional squarefree monomial ideals, one can actually give a description of the ideals having the desired property [17,21].

Since Questions 2.1 and 2.2 are equivalent, from now on we will only refer to Question 2.1. To study this question, we first reduce it to the 'non-degenerate' case, i.e., $I \subseteq \mathfrak{m}^2$, where $\mathfrak{m}$ is the maximal ideal of R. This is accomplished in Proposition 2.6. In order to prove it, we first need to collect a couple of lemmas. The first one gives a condition for the conormal module I/I^2 to contain a free summand as R/I-module.

Lemma 2.3. *Let R be a Noetherian ring and $I = (\underline{x}) + J$, where J is an R-ideal and $\underline{x} = x_1, \ldots, x_t$ forms a regular sequence on R/J. Then I/I^2 has a free summand (as R/I-module) of rank t. In fact, $(\underline{x} + I^2)/I^2 \simeq (R/I)^t$ and $I/I^2 = (\underline{x} + I^2)/I^2 \oplus K$ for some submodule K of I/I^2.*

Proof. If $I = R$ then we are done. So we may assume that I is a proper R-ideal. Set $\overline{R} = R/J$ and denote by $^-$ the images in $\overline{R}$, then $\overline{I} = (\overline{x_1}, \ldots, \overline{x_t})$ is a complete intersection $\overline{R}$-ideal. Hence $\overline{I}/\overline{I}^2$ is a free $\overline{R}/\overline{I}$-module of rank t. Now the first part of the lemma follows by the exact sequence $I/I^2 \to I/(J + I^2) \to 0$ and the fact that $I/(J + I^2) \simeq (I/J)/(I/J)^2 \simeq \overline{I}/\overline{I}^2 \simeq (\overline{R}/\overline{I})^t \simeq (R/I)^t$. Finally since $(\underline{x} + I^2)/I^2 \simeq I/(J + I^2) \simeq (R/I)^t$, the second part of the lemma follows from the split exact sequence $0 \to K \to I/I^2 \to (\underline{x} + I^2)/I^2 \to 0$. □

Recall that a Noetherian local ring R is said to be *equicharacteristic* if it contains a field. R has *mixed characteristic* if it does not contain a field. We are now able to prove the following result.

Proposition 2.4. *Let $(R, \mathfrak{m})$ be an equicharacteristic regular local and I an R-ideal containing an element $x \in \mathfrak{m} \backslash \mathfrak{m}^2$. Then I/I^2 contains a free summand (as R/I-module) of rank 1.*

Proof. The statement is trivial if $I = R$, hence we may assume that I is a proper R-ideal. We first prove that one can always reduce to the case where R is complete. Set $S = R/I$, $M = I/I^2$, let $N = (xR + I^2)/I^2 = \bar{x}S \subseteq M$ be the cyclic S-submodule of I/I^2 generated by the image of x and assume that $\widehat{N}$ is a free direct summand of $\widehat{M}$ of rank 1. We need to show that N is a free direct summand of M of rank 1. Let K be the kernel of the presentation map $\phi : S \to N \to 0$ defined by $\phi(1_S) = \bar{x}$. Since the induced map $\hat{\phi} : \widehat{S} \to \widehat{N} \to 0$ is an isomorphism of $\widehat{S}$-modules, it follows that $K = 0$, proving the freeness of N. Next, notice that, by assumption, we have the epimorphism of $\widehat{S}$-modules $\hat{j} : \widehat{M} \to \widehat{N}$ with $\hat{j}(\bar{x}) = \bar{x}$ and $\hat{j}(\tilde{f}_i) = 0$, $1 \leqslant i \leqslant r$, where $\bar{x}, \tilde{f}_1, \ldots, \tilde{f}_r$ form a set of minimal generators of $\widehat{M}$. Since $\mathrm{Hom}_{\widehat{S}}(\widehat{M}, \widehat{N}) \simeq \widehat{\mathrm{Hom}_S(M, N)}$, there exists a homomorphism of S-modules $\pi : M \to N$ such that $\hat{j} - \hat{\pi} \in \mathfrak{m}\widehat{\mathrm{Hom}_S(M, N)} \simeq \mathfrak{m}\mathrm{Hom}_{\widehat{S}}(\widehat{M}, \widehat{N})$, where $\hat{\pi} : \widehat{M} \to \widehat{N}$ denotes the map induced by π. Hence $(\hat{j} - \hat{\pi})(\widehat{M}) \subseteq \mathfrak{m}\widehat{N}$ and thus $\hat{\pi}(\widehat{M}) + \mathfrak{m}\widehat{N} = \hat{j}(\widehat{M}) = \widehat{N}$. By Nakayama's lemma, $\hat{\pi}(\widehat{M}) = \widehat{N}$, i.e., $\hat{\pi} : \widehat{M} \to \widehat{N}$ is surjective. Since $\widehat{S}$ is faithfully flat over S, the map $\pi : M \to N$ is also surjective. Therefore $\pi(\bar{x}) = \mu\bar{x}$ for some unit μ in S. If one denotes by ι the injective map $N \hookrightarrow M$ defined by $\iota(\bar{x}) = \bar{x}/\mu$, then we have $\pi \circ \iota \equiv \mathrm{id}_N$, and hence N is a free direct summand of M.

We may then assume that R is a complete regular local ring containing a field k. Write $R \simeq k[[x, x_1, \ldots, x_s]]$ for some s and $I = (x, J)$ for some R-ideal $J = (f_1, \ldots, f_r)$. Now use x to write $I = (x, J')$ with $J' = (f'_1, \ldots, f'_r)$ and each f'_i is in $k[[x_1, \ldots, x_s]]$. Then x is clearly regular on R/J' and by Lemma 2.3, I/I^2 contains a free direct summand of rank 1, finishing the proof. □

Notice that the assumption of R containing a field cannot be removed. In [12], Hochster constructed an example to provide an obstruction to Grothendieck's Lifting Problem. We employ it here to show that the assumption of R being equicharacteristic is needed: if we assume R to have mixed characteristic, Proposition 2.4 fails to be true even in the unramified case.

Example 2.5. (See [12, Example 1].) There exists an unramified complete regular local ring $(A, \mathfrak{n})$ and an A-ideal I with $x \in I$ and $x \notin \mathfrak{n}^2$ such that the A/I-submodule N of I/I^2 generated by the image of x has non-zero annihilator. In particular, N is not even a free A/I-submodule of I/I^2.

Proof. Let V be a complete discrete valuation ring with maximal ideal generated by 2, and let $A = V[[X_1, X_2, Y_1, Y_2, Z_1, Z_2]]$ be a power series ring in 6 analytic variables over V. Set $q = X_1X_2 + Y_1Y_2 + Z_1Z_2$, $x = 2$ and $I = (2, q, X_1^2, X_2^2, Y_1^2, Y_2^2, Z_1^2, Z_2^2)A$. We show that $I^2 : 2A \nsubseteq I$. In fact, set $D = X_1X_2Y_1Y_2 + Y_1Y_2Z_1Z_2 + X_1X_2Z_1Z_2$, then $2D = q^2 - X_1^2X_2^2 - Y_1^2Y_2^2 - Z_1^2Z_2^2 \in I^2$ (since all the summands are in I^2). To show that $D \notin I$, set $K = V/2V$, $H = (2, X_1^2, X_2^2, Y_1^2, Y_2^2, Z_1^2, Z_2^2)A$ and denote by $'$ residues modulo H. Then I' is a principal ideal generated by q' and any degree 4 element in I' is in the K-span of $q'f'$ where $f \in V$ runs over all squarefree monomials of degree 4. It is not hard to see that D' is not in this span, proving that $D \notin I$. $\square$

Similar examples can be produced in any positive characteristic, showing that Proposition 2.4 does not hold true in the mixed characteristic case.

As a result of Proposition 2.4 and Lemma 2.3, we can now reduce Vasconcelos's question to the non-degenerate case.

Proposition 2.6 *(Reduction to the non-degenerate case). Let $(R, \mathfrak{m})$ be an equicharacteristic regular local ring and I an R-ideal containing an element $x \in \mathfrak{m}\backslash\mathfrak{m}^2$. Set $\overline{R} = R/(x)$ and $\overline{I} = I\overline{R}$. Then Question 2.1 holds true for I if and only if it holds true for $\overline{I}$.*

Proof. Since $R/I \simeq \overline{R}/\overline{I}$, R/I is Cohen–Macaulay or Gorenstein if and only if $\overline{R}/\overline{I}$ is. It is also easy to see that I is generically a complete intersection if and only if $\overline{I}$ is. Also, by Proposition 2.4 and Lemma 2.3, $(x + I^2)/I^2 \simeq R/I$ and $I/I^2 = (x + I^2)/I^2 \oplus K$ for some submodule K of I/I^2. Therefore $\overline{I}/\overline{I}^2 \simeq K$ and it follows that I/I^2 is Cohen–Macaulay if and only if $\overline{I}/\overline{I}^2$ is. $\square$

Recall that for an ideal I in a regular local ring R, the embedding codimension of $\overline{R} = R/I$ is defined to be $\operatorname{ecodim}(\overline{R}) = \mu(\overline{\mathfrak{m}}) - \dim \overline{R}$, where $\mu(\overline{\mathfrak{m}})$ denotes the minimal number of generators of the maximal ideal $\overline{\mathfrak{m}}$ of $\overline{R}$. Notice that in general the height of I is greater than or equal to the embedding codimension, and if I is non-degenerate, i.e., $I \subseteq \mathfrak{m}^2$, then $\operatorname{ht} I = \operatorname{ecodim}(R/I)$.

The following lemma, consequence of work of Huneke and Ulrich, gives a positive answer to Question 2.1 for all ideals with $\operatorname{ecodim}(R/I) \leqslant 2$. Therefore, throughout the paper we will always deal with the case where the embedding codimension is at least 3.

Lemma 2.7. *(See [15, 2.8].) Question 2.1 has a positive answer if $\operatorname{ecodim}(R/I) \leqslant 2$ and either R is equicharacteristic or $I \subseteq \mathfrak{m}^2$.*

Proof. Let $\mathfrak{m}$ be the maximal ideal or the homogeneous maximal ideal of R. By Proposition 2.6, one can assume $I \subseteq \mathfrak{m}^2$. Then $\operatorname{ht} I = \operatorname{ecodim}(R/I) \leqslant 2$. If $\operatorname{ht} I = 1$, it is easy to see that I is a licci ideal. If $\operatorname{ht} I = 2$, I is licci by a classical result of Apèry–Gaèta (see [2,10]). In both cases, [15, 2.8] implies that Question 2.1 is true for I. $\square$

Next, we want to show that Question 2.1 can be reduced to the case of prime ideals. To do it, we first recall some definitions from linkage theory. For more details we recommend [14].

Let (S, J) and (R, I) be pairs, where $(S, \mathfrak{n})$ and $(R, \mathfrak{m})$ are Noetherian local rings and $J \subseteq S$ and $I \subseteq R$ are ideals. One says that (S, J) and (R, I) are *isomorphic*, written $(S, J) \simeq (R, I)$, if there is an isomorphism of rings $\phi : S \to R$ such that $\phi(J) = I$. (S, J) is called a *deformation* of (R, I) if there is a sequence $\underline{x} = x_1, \dots, x_t$ in S regular both on S and on S/J such that $(S/(\underline{x}), J + (\underline{x})/(\underline{x})) \simeq (R, I)$. Let $\underline{Z} = Z_1, \dots, Z_t$ be variables over R, $\underline{a} = a_1, \dots, a_t$ elements in R and $\underline{Z} - \underline{a} = Z_1 - a_1, \dots, Z_t - a_t$. Let $J \subseteq R[\underline{Z}]_{(\mathfrak{m}, \underline{Z}-\underline{a})}$ be an ideal. One says that $(R[\underline{Z}]_{(\mathfrak{m}, \underline{Z}-\underline{a})}, J)$ is a *specialization* of (R, I) if $(R[\underline{Z}]_{(\mathfrak{m}, \underline{Z}-\underline{a})}, J)$ is a deformation of (R, I) via the regular sequence $\underline{Z} - \underline{a}$.

Let R be a Gorenstein ring and $I = (f_1, \dots, f_n)$ an unmixed R-ideal of grade $g > 0$. Let X be a generic $g \times n$ matrix of variables over R and $\underline{\alpha} = \alpha_1, \dots \alpha_g$ the regular sequence in $R[X]$ defined by $(\underline{\alpha})^t = X(\underline{f})^t$, where $\underline{f} = f_1, \dots, f_n$. Then $L_1(\underline{f}) = (\underline{\alpha})R[X] :_{R[X]} IR[X]$ is the *first generic link* of I. Since $L_1(\underline{f})$ is independent of the generating set $\underline{f} = f_1, \dots, f_n$, up to isomorphism of pairs (after adjoining variables) [14, 2.11], we then write $L_1(I) = L_1(\underline{f})$ and by iteration of this process one defines the *i-th generic link* $L_i(I)$ as $L_i(I) = L_1(L_{i-1}(I))$ for any $i \geqslant 1$.

Using tools from linkage, we now prove that, when dealing with Question 2.1, one can always work with *prime* ideals instead of generically complete intersection ideals. More precisely:

Proposition 2.8 *(Reduction to the case of prime ideals). Any generically complete intersection ideal that gives a negative answer to Question 2.1 admits a deformation that is a prime ideal of the same height as the original ideal and gives a negative answer to Question 2.1 too.*

Proof. Let I be generically a complete intersection with R/I and I/I^2 Cohen–Macaulay, and assume R/I is not Gorenstein. Let $L_2(I) \subseteq R[\underline{Z}]$ be the second generic link of I in a polynomial extension of R. By [14, Theorem 2.17(a)], there exists a sequence $\underline{a}$ of R such that if one sets $S = R[\underline{Z}]_{(\mathfrak{m}, \underline{Z}-\underline{a})}$ and $\mathfrak{p} = L_2(I)S$, then $\mathfrak{p}$ is a prime ideal of the same height as I and $(S, \mathfrak{p})$ is a deformation of (R, I). By [13, 2.1], one has that $\mathfrak{p}/\mathfrak{p}^2$ is also Cohen–Macaulay. Since $(S, \mathfrak{p})$ is a deformation of (R, I), it is easy to see that $S/\mathfrak{p}$ is Cohen–Macaulay but not Gorenstein. □

As a consequence, if one could prove that Question 2.1 has a positive answer for any prime ideal then, by Proposition 2.8, the question would automatically be settled in full generality. At the same time, if there exists a counterexample to the question, by 2.8 there exists also a prime ideal that is a counterexample. Therefore, one can always assume that the ideal I in Question 2.1 is *prime*.

However, in the following we will not use the extra assumption that I is a prime ideal. In fact, in this paper we prove results regarding the conormal module (or the square) of some classes of ideals and it seems natural to present them in the most general context, that is for generically complete intersection ideals. Proposition 2.8 will be employed in the last section to produce a prime ideal providing a negative answer to the general version of Question 2.1, see Example 7.1(b).

The last result of this section is a useful formula for computing the multiplicity of the square of any generically complete intersection ideal in terms of the multiplicity of the ideal itself.

Proposition 2.9. *Let R be an equidimensional catenary Noetherian local ring and I an ideal that is generically a complete intersection of height c. Then $e(R/I^2) = (c+1)e(R/I)$.*

Proof. By the associativity formula, one has

$$e\big(R/I^2\big) = \sum_P \lambda\big(R_P/I_P^2\big)e(R/P),$$

where P ranges over all minimal primes of I of maximal dimension. Since R is equidimensional and catenary, I_P is a complete intersection of height c and hence I_P/I_P^2 is a free R_P/I_P-module of rank c. Now consider the short exact sequence

$$0 \longrightarrow I_P/I_P^2 \longrightarrow R_P/I_P^2 \longrightarrow R_p/I_P \longrightarrow 0,$$

by additivity of lengths, we obtain that $\lambda(R_P/I_P^2) = (c+1)\lambda(R_P/I_P)$. Therefore

$$\begin{aligned} e\big(R/I^2\big) &= \sum_P \lambda\big(R_P/I_P^2\big)e(R/P) \\ &= \sum_P (c+1)\lambda(R_P/I_P)e(R/P) \end{aligned}$$

$$= (c+1)\sum_{P} \lambda(R_P/I_P)e(R/P)$$
$$= (c+1)e(R/I)$$

where the last equality follows again by the associativity formula. □

Proposition 2.9 will be used frequently to prove the main results of the next several sections. Also, since some of the proofs of these results may look technical, we want to sketch here the underlying ideas. The main steps are:

I. using the fact that I is generically a complete intersection and R/I and R/I^2 are Cohen–Macaulay, reduce Question 2.1 to proving that $e(R/I^2) = (c+1)e(R/I)$, where $c = \operatorname{ecodim}(R/I)$;
II. after going modulo a (special) minimal reduction of the maximal ideal of R/I, reduce to the case where I is $\mathfrak{m}$-primary (I may no longer be generically a complete intersection, but this is not needed in the rest of the proof);
III. use the Hilbert function of R/I to estimate the Hilbert function of R/I^2 – this step is the longest and hardest, especially in the local setting. It requires the assumptions on the structure of R/I (e.g. stretched, short, etc.) and involves several *ad hoc* methods and strategies;
IV. use step III to estimate the multiplicity of R/I^2 in terms of the multiplicity of R/I and show that it leads to a contradiction with the equality found in step I.

We would also like to remark that in this paper we mainly focus on the local case, since it is the most problematic one. In fact, all our proofs work also in the homogeneous settings, and for some results (e.g. stretched algebra case or the reduction to the non-degenerate case) the homogeneous setting actually allows one to give much simpler proofs.

3. Stretched algebras

In this section, we study the conormal module of ideals defining stretched algebras and show that it is almost never Cohen–Macaulay. We start by setting up some notation.

Let $(A,\mathfrak{n})$ be an Artinian local ring. We recall that the *socle degree* of A, $\operatorname{socdeg}(A)$, is the unique positive integer s with $\mathfrak{n}^{s+1} = 0$ and $\mathfrak{n}^s \neq 0$. Notice that with this definition, $\operatorname{socdeg}(A)$ is exactly one unit smaller than the Loewy length of A. The *socle* of A, $\operatorname{soc}(A) = 0 :_A \mathfrak{n}$, is a subset of A consisting of the elements of A annihilated by the maximal ideal $\mathfrak{n}$. $\operatorname{Soc}(A)$ is a non-trivial $A/\mathfrak{n}$-vector space whose dimension, $\tau(A) = \dim_{A/\mathfrak{n}} \operatorname{soc}(A)$, is called the (*Cohen–Macaulay*) *type* of A. In general if $(R,\mathfrak{m})$ is a Cohen–Macaulay local ring with infinite residue field, the *socle degree*, $\operatorname{socdeg}(R)$, the *socle*, $\operatorname{soc}(R)$, and the *type*, $\tau(R)$, are defined respectively to be $\operatorname{socdeg}(R/J)$, $\operatorname{soc}(R/J)$ and $\tau(R/J)$, where J is any minimal reduction of $\mathfrak{m}$. The ring R is *Gorenstein* if in addition $\tau(R) = 1$. Finally for any Noetherian local ring R, we will use $e = e(R)$ to denote the Hilbert–Samuel multiplicity of R with respect to its maximal ideal $\mathfrak{m}$.

We now recall the definition of stretched algebras which is one of the main classes of ideals that we will study in this paper.

Definition 3.1. An Artinian local ring $(A,\mathfrak{n})$ is said to be *stretched* if $\mathfrak{n}^2$ is a principal ideal. In general, for a Cohen–Macaulay local ring $(R,\mathfrak{m})$ we say that R is *stretched* if R/J is stretched for some minimal reduction J of $\mathfrak{m}$ generated by a system of parameters for R. A Cohen–Macaulay ideal I in a Noetherian local ring R is *stretched* if R/I is.

Notice that if $(R,\mathfrak{m})$ is a Cohen–Macaulay local ring with infinite residue field, then every minimal reduction J of $\mathfrak{m}$ is generated by a system of parameters.

Recall that the *Hilbert function* of an Artinian local ring $(A,\mathfrak{n})$ is defined to be the function $HF_A : \mathbb{N}_0 \to \mathbb{N}_0$ where $HF_A(t) = \dim_{A/\mathfrak{n}} \mathfrak{n}^t/\mathfrak{n}^{t+1}$ for every $t \in mathbbN_0$. An immediate consequence of Definition 3.1 is that one can easily describe the Hilbert function of a stretched Artinian local ring $(A, \mathfrak{n})$:

$$HF_A: \quad 1 \quad c \quad 1 \quad 1 \quad \ldots \quad 1 \quad 0 \quad \ldots \quad 0 \quad \ldots.$$

In fact, if A has the above Hilbert function then $\mu(\mathfrak{n}^2) = 1$, which implies that A is stretched. On the other hand, if $\mu(\mathfrak{n}^2) = 1$, then $HF_A(2) = HF_{\mathrm{gr}_{\mathfrak{n}}(A)}(2) = 1$. Now Macaulay's bound (see for instance [4, Theorem 4.2.10]) applied to $\mathrm{gr}_{\mathfrak{n}}(A)$ shows that $HF_A(i) = HF_{\mathrm{gr}_{\mathfrak{n}}(A)}(i)$ is at most 1 for any $i \geqslant 2$, proving that A has the above Hilbert function.

Another easy consequence is that in a stretched Artinian algebra $(A, \mathfrak{n})$ if the initial degree of a socle element f does not equal $\mathrm{socdeg}(A)$, then f must be part of a minimal generating set of $\mathfrak{n}$. This is proved in the following lemma.

Lemma 3.2. *(See [8, 3.2].) Let $(A, \mathfrak{n})$ be a stretched Artinian local ring with socle degree s and let $f \in \mathfrak{n}$ be a socle element. If $f \notin \mathfrak{n}^s$ then $f \notin \mathfrak{n}^2$.*

Proof. If $s \leqslant 2$ we are done. So we may assume that $s \geqslant 3$. By assumption of stretchedness, one has the following Hilbert function for A:

$$HF_A: \quad 1 \quad c \quad 1 \quad 1 \quad \ldots \quad 1 \quad 0_{\longrightarrow}.$$

Assume $f \in \mathfrak{n}^2$ and let i be the smallest integer with $f \notin \mathfrak{n}^{i+1}$. Since $f \notin \mathfrak{n}^s$ we have $2 \leqslant i \leqslant s-1$. Then $HF_A(i) = 1$, which implies $\mathfrak{n}^i = (f)$. Since $f \cdot \mathfrak{n} = (0)$, then $\mathfrak{n}^{i+1} = \mathfrak{n}^i \cdot \mathfrak{n} = f \cdot \mathfrak{n} = (0)$. Hence $i+1 \geqslant s+1$, contradicting that $i < s$. Therefore $f \notin \mathfrak{n}^2$. □

A classical result of Sally [19] gives the structure of stretched Gorenstein Artinian local algebras. Recently, Elias and Valla [8] generalized it to *any* stretched Artinian local algebras as follows:

Theorem 3.3. *(See [19, 1.2], [8, 3.1].) Let $(R, \mathfrak{m})$ be a c-dimensional regular local ring such that the characteristic of the residue field is 0. Let $I \subseteq \mathfrak{m}^2$ be an $\mathfrak{m}$-primary ideal with R/I stretched of socle degree s. Write $\tau(R/I) = r+1$ for some $0 \leqslant r \leqslant c-1$. Then there exist minimal generators $x_1, \ldots, x_c$ for the maximal ideal $\mathfrak{m}$ and elements $u_{r+1}, \ldots, u_{c-1} \notin \mathfrak{m}$ with*

$$I = (x_1\mathfrak{m}, \ldots, x_r\mathfrak{m}) + J$$

where

$$J = \begin{cases} (x_{r+i}x_{r+j} \mid 1 \leqslant i < j \leqslant c-r) + (x_c^s - u_{r+i}x_{r+i}^2 \mid 1 \leqslant i \leqslant c-r-1), & \text{if } r < c-1; \\ (x_c^{s+1}), & \text{if } r = c-1. \end{cases}$$

In a forthcoming paper [16] we are actually able to slightly improve 3.3, showing that the structure theorem holds for any regular local ring whose residue field has characteristic $\neq 2$. Hence, this will be our only requirement on the residue field in the following.

In a Noetherian local ring $(R, \mathfrak{m})$, let $x_1, \ldots, x_r$ be a regular sequence in $\mathfrak{m}$; we recall that a *monomial* in $x_1, \ldots, x_r$ is an element in R of the shape $x_1^{a_1}x_2^{a_2}\cdots x_r^{a_r}$. An R-ideal I is dubbed *monomial* (in $x_1, \ldots, x_r$) if it can be generated by monomials in $x_1, \ldots, x_r$.

As an application of Theorem 3.3 we prove that the square of a stretched $\mathfrak{m}$-primary ideal I can be very tightly estimated by using a monomial ideal. This fact will be crucial as it will allow us to compute explicitly, or estimate tightly, $e(R/I^2)$ (see the comment after 3.5, or the proof of 6.4).

Proposition 3.4. *Let $(R, \mathfrak{m})$ be a c-dimensional regular local ring with* $\operatorname{char} R/\mathfrak{m} \neq 2$. *Let $I \subseteq \mathfrak{m}^2$ be an $\mathfrak{m}$-primary ideal with R/I stretched of socle degree s. Write $\tau(R/I) = r + 1$ for some $0 \leqslant r \leqslant c - 1$. Then there exist minimal generators $x_1, \ldots, x_c$ for the maximal ideal $\mathfrak{m}$ such that*

$$I^2 \subseteq L = (x_1, \ldots, x_{c-1})H + (x_1, \ldots, x_{c-1})x_c^{s+1} + \left(x_c^{2s}\right),$$

where H is the monomial ideal generated by all monomials of degree 3 in $x_1, \ldots, x_c$ except for x_c^3. The above inclusion is strict if and only if $r \geqslant c - 2$, and in this case $\lambda(R/I^2) \geqslant \lambda(R/L) + 2$.

Proof. By Theorem 3.3, there exist minimal generators $x_1, \ldots, x_c$ for $\mathfrak{m}$ and units $u_{r+1}, \ldots, u_{c-1}$ with $I = (x_1\mathfrak{m}, \ldots, x_r\mathfrak{m}) + J$, where J is as in Theorem 3.3. It is now easy to verify that every element in I^2 is a linear combination of the elements of L, proving the inclusion $I^2 \subseteq L$.

To prove the second part of the statement, observe that if $r \leqslant c - 3$ then both $x_c^s - u_{c-2}x_{c-2}^2$ and $x_c^s - u_{c-1}x_{c-1}^2$ are in I, hence their product is in I^2. Since $x_{c-2}^2 x_{c-1}^2 \in I^2$ and $x_{c-i}^2 x_c^s = (x_{c-i}x_c)^2 x_c^{s-2} \in I^2$ for any $i = 1, 2$, we have $x_c^{2s} \in I^2$ as well. Therefore

$$(x_1\mathfrak{m}, \ldots, x_r\mathfrak{m}, x_{r+i}x_{r+j} \mid 1 \leqslant i < j \leqslant c - r)^2 + \left(x_c^{2s}\right) \subseteq I^2.$$

Again, for any $1 \leqslant i \leqslant c - r - 1$, the fact that $(x_c^s - u_{r+i}x_{r+i}^2)^2 \in I^2$ implies that $x_{r+i}^4 \in I^2$, since both x_c^{2s} and $x_{r+i}^2 x_c^s$ are in I^2. Now let $1 \leqslant i \leqslant c - 1$, there exists $1 \leqslant j \leqslant c - r - 1$ such that $i \neq r + j$. Since both $x_i x_c(x_c^s - u_{r+j}x_{r+j}^2) \in I^2$ and $x_i x_c x_{r+j}^2 = (x_i x_{r+j})(x_{r+j}x_c) \in I^2$, we must have that $x_i x_c^{s+1} \in I^2$. Finally, for any $1 \leqslant i \leqslant c - r - 1$ and $j \neq r + i$, since $x_{r+i}x_j(x_c^s - u_{r+i}x_{r+i}^2) \in I^2$, we have $x_{r+i}^3 x_j \in I^2$. These arguments show that all the generators of L are in I^2 thus proving the equality in the case $r \leqslant c - 3$.

Now assume $r = c - 2$. Then $L = (M, x_{c-1}^4, x_{c-1}^3 x_c, x_{c-1}x_c^{s+1}, x_c^{2s})$ and $I^2 = (M, x_{c-1}x_c^{s+1} - ux_{c-1}^3 x_c, x_c^{2s} + u^2 x_{c-1}^4)$, where u is a unit and $M = (x_1, \ldots, x_{c-2})H + (x_1, \ldots, x_{c-2})x_c^{s+1} + (x_{c-1}^2 x_c^2)$ is a monomial ideal. Observe that $L/I^2 = (\overline{x_{c-1}^4}, \overline{x_{c-1}^3 x_c})$ and none of the two generators of L/I^2 is redundant. This shows that $I^2 \neq L$ and $\lambda(R/I^2) \geqslant \lambda(R/L) + 2$.

Finally if $r = c - 1$ then the claim is easily proved, since $I = (x_1, \ldots, x_{c-1})\mathfrak{m} + (x_c^{s+1})$ and one can easily compare the two monomial ideals I^2 and L. $\square$

The next theorem is the main result of this section. It shows that unless the embedding codimension is 3, the first conormal module (hence square) of any generically complete intersection stretched Cohen–Macaulay ideal is not Cohen–Macaulay.

Theorem 3.5. *Let $(R, \mathfrak{m})$ be a regular local ring with* $\operatorname{char} R/\mathfrak{m} \neq 2$ *and I a stretched Cohen–Macaulay ideal that is generically a complete intersection. Assume that either R is equicharacteristic or $I \subseteq \mathfrak{m}^2$, and write* $c = \operatorname{ecodim}(R/I)$.

(a) *If $c = 3$ and I/I^2 (or R/I^2) is Cohen–Macaulay then R/I is Gorenstein.*
(b) *If $c \geqslant 4$ then I/I^2 and R/I^2 are not Cohen–Macaulay.*

Proof. If R contains a field, by Proposition 2.6 we can reduce at once to the case where $I \subseteq \mathfrak{m}^2$. Hence, in either case we may assume that $I \subseteq \mathfrak{m}^2$ and then $c = \operatorname{ht} I$. Let $s = \operatorname{socdeg}(R/I)$. Since R/I is stretched, we have that $e(R/I) = c + s$. Hence, by Proposition 2.9, $e(R/I^2) = (c + 1)(c + s)$. We will show by contradiction that I/I^2 is not Cohen–Macaulay if $c \geqslant 4$ or if $c = 3$ and R/I is not Gorenstein. So assume I/I^2 is Cohen–Macaulay. Since R/I is also Cohen–Macaulay, the short exact sequence

$$0 \longrightarrow I/I^2 \longrightarrow R/I^2 \longrightarrow R/I \longrightarrow 0$$

shows that R/I^2 is Cohen–Macaulay as well. Hence, after possibly extending the residue field of R, there exists an R-ideal J whose images in R/I and R/I^2 are minimal reductions of $\mathfrak{m}_{R/I}$ and $\mathfrak{m}_{R/I^2}$, respectively, generated by systems of parameters. Let $\overline{R}$ be the regular local ring $\overline{R} = R/J$. Then $\lambda(\overline{R}/\overline{I}^2) = \lambda(R/I^2 + J) = e(R/I^2) = (c+1)(c+s)$. However, by assumption $\overline{R}/\overline{I}$ is a stretched Artinian algebra of socle degree s, hence by Proposition 3.4 there exist minimal generators $x_1, \ldots, x_c$ of $\overline{\mathfrak{m}}$ such that $\overline{I}^2 \subseteq L$, where L is the $\overline{R}$-ideal defined as $L = (x_1, \ldots, x_{c-1})H + (x_1, \ldots, x_{c-1})x_c^{s+1} + (x_c^{2s})$, and H is the monomial ideal in $\overline{R}$ generated by all the monomials of degree 3 in $x_1, \ldots, x_c$ except for x_c^3. Now, it is easy to see that the monomials $x_i x_c^t$ and x_c^j are not in L for any $1 \leqslant i \leqslant c-1$, $3 \leqslant t \leqslant s$ and $4 \leqslant j \leqslant 2s-1$. Since L is a monomial ideal in $x_1, \ldots, x_c$, the initial forms of the monomials $x_i x_c^t$ and x_c^j, where $1 \leqslant i \leqslant c-1$, $3 \leqslant t \leqslant s$ and $4 \leqslant j \leqslant 2s-1$, are actually $R/\mathfrak{m}$-linearly independent of $\mathrm{gr}_{\overline{\mathfrak{m}}}(\overline{R}/L)$ (see also the comment at the end of the proof). Therefore, if one writes

$$1 \quad c \quad h_2 \quad h_3 \quad \ldots$$

for the Hilbert function of $\overline{R}/L$, the above statement shows that $h_{j+1} \geqslant c$ for every $3 \leqslant j \leqslant s$ and $h_j \geqslant 1$ for every $s+2 \leqslant j \leqslant 2s-1$. Then $\lambda(\overline{R}/L) \geqslant 1 + c + \binom{c+1}{2} + \binom{c+2}{3} + c(s-2) + s - 2$. An easy computation shows that, if $c \geqslant 4$ then $1 + c + \binom{c+1}{2} + \binom{c+2}{3} + c(s-2) + s - 2 > (c+1)(c+s)$. Hence we obtain $\lambda(\overline{R}/\overline{I}^2) \geqslant \lambda(\overline{R}/L) > (c+1)(c+s)$ which is a contradiction. Assume then $c = 3$. Since R/I is not Gorenstein, then $\overline{R}/\overline{I}$ is not Gorenstein as well and we have $c - 2 = 1 \leqslant r = \tau(\overline{R}/\overline{I}) - 1$. Now, by Proposition 3.4, we have that $\lambda(\overline{R}/\overline{I}^2) > \lambda(\overline{R}/L) \geqslant (c+1)(c+s)$, which gives the desired contradiction. □

Remark 3.6. In the proof, the key point is to estimate the Hilbert function of $\overline{R}/\overline{I}^2$ using the existence of some $R/\mathfrak{m}$-linearly independent elements in $\overline{R}/L$. We would like to explain this step better.

In the local case it is usually hard to estimate the Hilbert function at a given degree using linearly independent elements. In fact, even if there are $A/\mathfrak{n}$-linearly independent elements $a_1, \ldots, a_n$ of $\mathfrak{n}^i \backslash \mathfrak{n}^{i+1}$ in an Artinian local ring $(A, \mathfrak{n})$, their initial forms in the associated graded ring $A^* = \mathrm{gr}_{\mathfrak{n}}(A)$ need not be $A/\mathfrak{n}$-linearly independent. For instance, the elements $x + y^2$, $x + xy$, $x + x^2$ are clearly k-linearly independent in $A = k[x, y]_{(x,y)}/(x, y)^3$, but their initial forms in A^* are all equal! However, let $(R, \mathfrak{m})$ be a regular local ring and fix a regular system of parameters $\underline{x} = x_1, \ldots, x_c$, so that every 'monomial' item in the following is referred to the regular sequence $\underline{x}$. If L is an $\mathfrak{m}$-primary monomial ideal and there are distinct monomial elements $a_1, \ldots, a_n$ in R with $a_j \notin L$ for every j and all a_j's have the same initial degree i, then their initial forms are $R/\mathfrak{m}$-linearly independent, proving that $HF_{R/L}(i) \geqslant n$. In the proof of Theorem 3.5, we use that $\overline{I}^2$ is contained in the monomial ideal L to translate the problem into estimating the Hilbert function of L, which is a much easier task by the above. This approach is successful because L estimates very tightly $\overline{I}^2$, as explained in Proposition 3.4.

As a consequence of Theorem 3.5 and Lemma 2.7, we can now give a positive answer to Question 2.1 for stretched algebras.

Corollary 3.7. *Assume* $\mathrm{char} R/\mathfrak{m} \neq 2$. *Then Question 2.1 is true for I defining a stretched algebra.*

4. Monomial ideals

In the present section we prove that Question 2.1 has a positive answer for monomial ideals.

Recall that in a regular local ring $(R, \mathfrak{m})$, fixed a regular system of parameters $\underline{x} = x_1, \ldots, x_c$, an element $a \in R$ is said to be a *monomial* in $\underline{x}$, if there exist non-negative integers $a_1, \ldots, a_c$ with $a = x_1^{a_1} x_2^{a_2} \cdots x_c^{a_c}$. An R-ideal I is said to be a *monomial ideal* in $\underline{x}$, if it can be generated by monomial elements in $\underline{x}$.

Rinaldo, Terai and Yoshida recently proved the following result.

Theorem 4.1. *(See [18].) Let Δ be a simplicial complex on $V = [n]$, and $I_\Delta \subseteq S = K[X_1, \dots, X_n]$ be the Stanley–Reisner ideal of Δ. If S/I_Δ^2 is Cohen–Macaulay for every field K, then I_Δ is Gorenstein.*

Equivalently, Question 2.1 holds true for any squarefree monomial ideal J with the property that S/J^2 is Cohen–Macaulay for any field K.

We can now prove that Question 2.1 holds true for *any* monomial ideal.

Theorem 4.2. *In the setting of Question 2.1, further assume that R is equicharacteristic and I is a monomial ideal in a regular system of parameters $\underline{x}$ whose corresponding monomial ideal $\tilde{I}$ in $\tilde{R} = K[[\underline{x}]]$ is Cohen–Macaulay for every field K. If either $\tilde{I}/\tilde{I}^2$ or $\tilde{R}/\tilde{I}^2$ is Cohen–Macaulay for any field K then R/I is Gorenstein.*

Proof. Without loss of generality we may assume that R is complete, hence we may assume that $R = k[[\underline{x}]]$ is a power series over a field k. Let $J \subseteq S = k[[\underline{x}, \underline{y}]]$ be a squarefree monomial ideal obtained by polarization from I. Then, (S, J) is a deformation of (R, I). Fix a field K. Set $\tilde{S} = K[[\underline{x}, \underline{y}]]$ and let $\tilde{I}$ and $\tilde{J}$ be the corresponding monomial ideals in $\tilde{R}$ and $\tilde{S}$, respectively. Notice that $\tilde{R}/\tilde{I}$ and $\tilde{R}/\tilde{I}^2$ are Cohen–Macaulay, by our assumption on I and I^2 (if $\tilde{I}/\tilde{I}^2$ is Cohen–Macaulay, then by the short exact sequence displayed after the statement of Question 2.1 we also have that $\tilde{R}/\tilde{I}^2$ is Cohen–Macaulay).

Since $\tilde{I}$ is generically a complete intersection, a proof similar to the one of [13, Theorem 2.1] shows that $\tilde{S}/\tilde{J}$ and $\tilde{S}/\tilde{J}^2$ are Cohen–Macaulay for any field K, because $\tilde{R}/\tilde{I}$ and $\tilde{R}/\tilde{I}^2$ are. We have then just proved that J is a squarefree monomial ideal with $\tilde{S}/\tilde{J}^2$ Cohen–Macaulay for any field K. Then, by Theorem 4.1 (applied to $R = k[[\underline{x}]]$), S/J is Gorenstein. Finally, since (S, J) is a deformation of (R, I) and S/J is Gorenstein then R/I is Gorenstein, finishing the proof. □

Corollary 4.3. *With notation as in Theorem 4.2, Question 2.1 holds true for all monomial ideals I with the property that $\tilde{R}/\tilde{I}$ and $\tilde{I}/\tilde{I}^2$ are Cohen–Macaulay for every field K.*

5. Short algebras

In this section we will study the conormal module of ideals defining short algebras. We will prove that for these ideals the conormal module (equivalently, the square) is almost never Cohen–Macaulay. Again, the information on the Hilbert function of such algebras will play a crucial role.

Recall that in a Noetherian local ring $(R, \mathfrak{m})$, the *associated graded ring* of R with respect to $\mathfrak{m}$ is defined as the graded algebra $R^* = \mathrm{gr}_{\mathfrak{m}}(R) = \bigoplus_{i \geqslant 0} \mathfrak{m}^i/\mathfrak{m}^{i+1}$. For an element $0 \neq a \in R$, there is a unique integer i with $a \in \mathfrak{m}^i \backslash \mathfrak{m}^{i+1}$. This integer is called the *initial degree* of a, denoted $\mathrm{indeg}(a)$. The *initial form* of a is $a^* = (a + \mathfrak{m}^{i+1})/\mathfrak{m}^{i+1} \in R^*$. Similarly, the *ideal of initial forms* of an R-ideal J is defined as $J^* = (a^* \mid a \in J)R^*$. It is well known that $\mathrm{gr}_{\mathfrak{m}}(R/J) \simeq R^*/J^*$. The definition of short algebra can be found, for instance, in [7].

Definition 5.1. An Artinian local algebra $(A, \mathfrak{n})$ is short if there exist integers c and s with $HF_A(j) = HF_{A/\mathfrak{n}[X_1,\dots,X_c]}(j)$ for every $j < s$, and $HF_A(s+1) = 0$ (i.e., if A^* is a short graded algebra). A Cohen–Macaulay local ring $(R, \mathfrak{m})$ is short if R/J is short for some minimal reduction J of $\mathfrak{m}$ generated by a system of parameters of R.

Notice that if $(R, \mathfrak{m})$ is a Cohen–Macaulay local ring with infinite residue field, then every minimal reduction J of $\mathfrak{m}$ is generated by a system of parameters.

The second part of the next remark will be used to deal with Artinian short algebras. It shows that one can reduce any computation on the length of R/I or R/I^2 to the homogeneous case.

Remark 5.2. Let $(R, \mathfrak{m}, k)$ be a d-dimensional regular local ring containing a field and I an $\mathfrak{m}$-primary ideal. Then $R/I \simeq \tilde{R}/\tilde{I}$, where $\tilde{R} = k[x_1, \dots, x_d]$ and $\tilde{I}$ is generated by polynomials of degree at most $s+1$, where $s = \mathrm{socdeg}(R/I)$. Furthermore, if I defines a short algebra, then $\tilde{I}$ can be taken to be generated by homogeneous polynomials. In particular, $(I^*)^n = (I^n)^*$ for any $n \geqslant 1$ and $\lambda(R/I^n) = \lambda(R^*/(I^*)^n)$.

Proof. For the first part of the statement, since R/I is Artinian, then $R/I \simeq \widehat{R/I} \simeq \widehat{R}/\hat{I}$. Since R contains a field, one has $\widehat{R} \simeq k[[x_1, \dots, x_d]]$, and since I is $\mathfrak{m}$-primary with $\mathfrak{m}^{s+1} \subseteq I$, one may assume that $\hat{I} = (f_1, \dots, f_r, \mathfrak{m}^{s+1})$, where $f_1, \dots, f_r$ are of degrees at most s. However, since $k[x_1, \dots, x_d]/(f_1, \dots, f_r, \mathfrak{m}^{s+1})$ is Artinian, one has that $k[x_1, \dots, x_d]/(f_1, \dots, f_r, \mathfrak{m}^{s+1}) \simeq k[[x_1, \dots, x_d]]/(f_1, \dots, f_r, \mathfrak{m}^{s+1})$, finishing the proof.

The second part follows from the first one and the definition of short algebra. □

For instance, let $(R, \mathfrak{m})$ be a regular local ring of dimension 4 with $\mathfrak{m} = (x_1, \dots, x_4)$ and coefficient field k. Assume $I = (x_1x_2^3, x_2x_3^2x_4, x_1^3x_2, x_4^4, \mathfrak{m}^5)$. Then R/I is a short algebra, with

$$k[X_1, \dots, X_4]/\big(X_1X_2^3, X_2X_3^2X_4, X_1^3X_2, X_4^4, M^5\big) \simeq R/I,$$

where $M = (X_1, \dots, X_4)$.

The following theorem is the main result of this section. It shows that the conormal module (equivalently, the square) of a short algebra is not Cohen–Macaulay if the socle degree is at least 3. Somewhat surprisingly, in 7.1 we exhibit examples showing that, if the socle degree is 2, the conormal module (equivalently, the square) can actually be Cohen–Macaulay. These examples give a negative answer to Question 2.1.

Theorem 5.3. *Let R be an equicharacteristic regular local ring and I a Cohen–Macaulay ideal that is generically a complete intersection. Let $c = \operatorname{ecodim}(R/I) \geqslant 2$ and $s = \operatorname{socdeg}(R/I) \geqslant 3$. If R/I is short then I/I^2 (equivalently, R/I^2) is not Cohen–Macaulay.*

Proof. As in Theorem 3.5, one can always assume that $I \subseteq \mathfrak{m}^2$. Let J be an R-ideal, whose image in R/I is a minimal reduction of $\mathfrak{m}_{R/I}$, generated by a system of parameters (after possibly extending the residue field of R, the existence of J is guaranteed). Now, write $\mathfrak{m} = (x_1, \dots, x_c) + J$ for the maximal ideal of R, where $x_1, \dots, x_c$ is part of a regular system of parameters of R. Let n_i be the number of monomials of degree i in a polynomial ring in c variables and N_i the number of monomials of degree i in $c + 1$ variables. An elementary combinatorial argument shows that

$$\sum_{i=0}^{s-1} n_i = N_{s-1} \quad \text{for any } s \geqslant 1.$$

Hence $e(R/I) = \sum_{i=0}^{s-1} n_i + \binom{c+s-1}{s} - q = \sum_{i=0}^{s} n_i - q = N_s - q$, where $0 \leqslant q < \binom{c+s-1}{s}$ is the number of minimal generators of I lying in $\mathfrak{m}^s \backslash \mathfrak{m}^{s+1}$. To show that I/I^2 is not Cohen–Macaulay, it is enough to show that

$$e\big(R/I^2\big) = \lambda\big(R/I^2 + J\big) > (c+1)(N_s - q).$$

Write $\mathfrak{n} = (x_1, \dots, x_c)$. Since $R/I + J$ is short, then $I + J = (f_1, \dots, f_q, \mathfrak{n}^{s+1}) + J$ where the f_i's are all homogeneous of degree s in the variables $x_1, \dots, x_c$ in the sense of Remark 5.2. Therefore $I^2 + J \subseteq \mathfrak{n}^{2s} + J$ and

$$\lambda\big(R/I^2 + J\big) \geqslant \lambda\big(R/\mathfrak{n}^{2s} + J\big) = \sum_{i=0}^{2s-1} n_i = N_{2s-1}.$$

Hence, to finish the proof, it is enough to show that $N_{2s-1} > (c+1)(N_s - q)$ for all $c \geqslant 2$ and $s \geqslant 3$. To do this, we write $\frac{N_{2s-1}}{c+1} = \frac{\binom{c+2s-1}{2s-1}}{c+1} = \frac{\prod_{j=0}^{c-1}(\frac{2s+j}{j+1})}{c+1} = \prod_{j=0}^{c-1}(\frac{2s+j}{j+2})$ and similarly, $N_s = \prod_{j=0}^{c-1}(\frac{s+j+1}{j+1})$.

Then we are down to show that $\prod_{j=0}^{c-1}(\frac{2s+j}{j+2}) - \prod_{j=0}^{c-1}(\frac{s+j+1}{j+1}) + q > 0$. Set $Q(c,s) = \prod_{j=0}^{c-1}(\frac{2s+j}{j+2}) - \prod_{j=0}^{c-1}(\frac{s+j+1}{j+1})$, we will show that $Q(c,s) + q > 0$.

Claim. *If $Q(\bar{c}, \bar{s}) > 0$ for some positive integers $\bar{c}$ and $\bar{s}$, then $Q(c,s) > 0$ for every $c \geqslant \bar{c}$ and $s \geqslant \bar{s}$.*

Assume the claim is proved. Since $Q(5,4) = 6 > 0$, $Q(4,5) = 17 > 0$, $Q(3,6) = 7 > 0$ and $Q(2,8) = \frac{1}{3} > 0$, it follows from the claim that $Q(c,s) > 0$ either if $c \geqslant 5$, $s \geqslant 4$, or if $c \geqslant 4$, $s \geqslant 5$, or if $c \geqslant 3$, $s \geqslant 6$ or if $c \geqslant 2$, $s \geqslant 8$. We are left to check the following several cases.

If $c = 3$, $s = 5$ then $Q(3,5) = -1 < 0$. However, if $q \geqslant 2$ then $\lambda(R/I^2 + J) - (3+1)(N_5 - q) \geqslant (3+1)(Q(3,5) + q) > 0$ and we are done. Assume now $q \leqslant 1$. One has

$$\lambda\big(R/I^2 + J\big) \geqslant \lambda\big(R/\mathfrak{n}^{10} + J\big) + \left[\binom{c-1+10}{10} - q(q+1)/2\right] \geqslant \lambda\big(R/\mathfrak{n}^{10} + J\big) + 65.$$

Thus $\lambda(R/I^2 + J) - (3+1)(N_5 - q) \geqslant (3+1)(Q(3,5) + q) + 65 \geqslant -4 + 65 = 61 > 0$.

If $c = 3$, $s = 4$ then $Q(3,4) = -5$. In this case, if $q \geqslant 6$ then $\lambda(R/I^2 + J) - (3+1)(N_4 - q) \geqslant (3+1)(Q(3,4) + q) > 0$ and we are done. So we may assume $q \leqslant 5$. Then, $\lambda(R/I^2 + J) - (3+1)(N_4 - q) \geqslant (3+1)(Q(3,4) + q) + (45 - q(q+1)/2) = -1/2(q^2 - 7q - 50) > 0$ whenever $0 \leqslant q \leqslant 5$.

The cases $c = 3$, $s = 3$, or $c = 4$, $s = 3, 4$, or $c = 5$, $s = 3$, or $c = 2$ and $3 \leqslant s \leqslant 7$ can be proved similarly. Thus, our result is proved modulo the proof of the above claim. This is accomplished in the next lemma. □

Lemma 5.4. *For any $c \in \mathbb{Z}_+$ and $s \in \mathbb{Z}_+$, define $Q(c,s) = \prod_{j=0}^{c-1}(\frac{2s+j}{j+2}) - \prod_{j=0}^{c-1}(\frac{s+j+1}{j+1})$. Then*

(a) $Q(c, s+1) \geqslant Q(c,s)$;
(b) *if $c \geqslant 2$ and $s \geqslant 3$ then $Q(c+1, s) \geqslant Q(c,s)$.*

In particular, if $Q(\bar{c}, \bar{s}) > 0$ for some fixed positive integers $\bar{c} \geqslant 2$ and $\bar{s} \geqslant 3$ then $Q(c,s) > 0$ for all $c \geqslant \bar{c}$ and $s \geqslant \bar{s}$.

Proof. For any two positive integers c and s,

$$\begin{aligned}
Q(c, s+1) &= \prod_{j=0}^{c-1}\left(\frac{2s+2+j}{j+2}\right) - \prod_{j=0}^{c-1}\left(\frac{s+1+j+1}{j+1}\right) \\
&= \left[\prod_{j=0}^{c-1}\left(\frac{2s+2+j}{2s+j}\right)\cdot\left(\frac{2s+j}{j+2}\right)\right] - \left[\prod_{j=0}^{c-1}\left(\frac{s+1+j+1}{s+j+1}\right)\cdot\left(\frac{s+j+1}{j+1}\right)\right] \\
&\geqslant \left[\prod_{j=0}^{c-1}\left(\frac{s+1+j+1}{s+j+1}\right)\cdot\left(\frac{2s+j}{j+2}\right)\right] - \left[\prod_{j=0}^{c-1}\left(\frac{s+1+j+1}{s+j+1}\right)\cdot\left(\frac{s+j+1}{j+1}\right)\right] \\
&= \left[\prod_{j=0}^{c-1}\frac{s+1+j+1}{s+j+1}\right]\cdot\left[\left(\prod_{j=0}^{c-1}\frac{2s+j}{j+2}\right) - \left(\prod_{j=0}^{c-1}\frac{s+j+1}{j+1}\right)\right] \\
&\geqslant Q(c,s)
\end{aligned}$$

where the first inequality holds since $\frac{2s+2+j}{2s+j} \geqslant \frac{s+1+j+1}{s+j+1}$, for all $j \geqslant 0$.

Similarly to show part (b), write

$$Q(c,s) = \overbrace{\prod_{j=0}^{c-1}\left(\frac{2s+j}{j+2}\right)}^{Q_1(c,s)} - \overbrace{\prod_{j=0}^{c-1}\left(\frac{s+j+1}{j+1}\right)}^{Q_2(c,s)}.$$

Then

$$\begin{aligned} Q(c+1,s) &= Q_1(c+1,s) - Q_2(c+1,s) \\ &= Q_1(c,s)\frac{c+2s}{c+2} - Q_2(c,s)\frac{c+1+s}{c+1} \\ &\geqslant Q_1(c,s) - Q_2(c,s) \\ &= Q(c,s) \end{aligned}$$

where the inequality holds since $\frac{c+2s}{c+2} \geqslant \frac{c+s+1}{c+1} \geqslant 1$, for all $s \geqslant 3$ and $c \geqslant 2$. □

The following proposition deals with the case of socle degree 2. Sharp numerical conditions on the multiplicity are provided to ensure that the conormal module is not Cohen–Macaulay.

The idea of the proof is to assume that I/I^2 is Cohen–Macaulay, reduce to the case where I is $\mathfrak{m}$-primary and produce a contradiction showing that

$$e(R/I^2) > (c+1)e(R/I). \tag{1}$$

If the number q of quadric minimal generators of I is 'large' (part (a)), then the multiplicity of R/I is 'small', hence the right-hand side in (1) is 'small' and we can prove the above strict inequality. On the other hand, if the number q of quadric minimal generators of I is 'small' (parts (b)–(c)), then I^2 has a fairly small amount of minimal generators in degree 4, hence the multiplicity of R/I^2 becomes 'large'. In this case, then, the left-hand side of (1) is 'large' and we are again able to prove the above strict inequality. Finally, notice that if q is 'intermediate' there may be no contradiction, as I/I^2 may actually be Cohen–Macaulay – see Example 7.1.

We are grateful to A. Conca for bringing our attention to [6, Theorem 2.4], that finishes the proof of 5.5.

Proposition 5.5. *Let $(R,\mathfrak{m})$ be an equicharacteristic regular local ring with $|R/\mathfrak{m}| = \infty$ and I a Cohen–Macaulay ideal that is generically a complete intersection with $s = \operatorname{socdeg}(R/I) = 2$ and $c = \operatorname{ecodim}(R/I) \geqslant 3$. Write $e(R/I) = 1 + c + \binom{c+1}{2} - q$, for some non-negative integer $q < \binom{c+1}{2}$. Then I/I^2 is not Cohen–Macaulay if q satisfies one of the following numerical conditions*:

(a) $q > \frac{c^2+2c}{3}$,
(b) $\frac{2c+1-\sqrt{K}}{2} < q < \frac{2c+1+\sqrt{K}}{2}$, *where $K = (2c+1)^2 + 8(N_4 - (c+1)N_2)$ and N_2, N_4 are defined as in the proof of Theorem 5.3*,
(c) $\frac{2c+1-2n_3-\sqrt{K}}{2} < q < \frac{2c+1-2n_3+\sqrt{K}}{2}$, *where $K = (2n_3 - 2c - 1)^2 + 8(N_5 - (c+1)N_2)$, where N_2, N_5, n_3 are defined as in the proof of Theorem 5.3.*

Furthermore, I/I^2 and R/I^2 are not Cohen–Macaulay if $c = 3$, R/I is not Gorenstein and $\operatorname{char} R/\mathfrak{m} \neq 2$, or if $c = 4$.

Proof. By Proposition 2.6, we may assume that $I \subseteq \mathfrak{m}^2$ and then $c = \operatorname{ht} I$. By Proposition 2.9, $e(R/I^2) = (c+1)e(R/I) = (c+1)(N_2 - q)$. As before, let J be an ideal whose image in R/I gives a minimal reduction of $\mathfrak{m}_{R/I}$, and notice that, since $|R/\mathfrak{m}| = \infty$, J is generated by a system of parameters. Then $\lambda(R/I^2 + J) \geqslant (c+1)(N_2 - q)$, where equality holds if and only if I/I^2 is Cohen–Macaulay. A computation shows that

$$\lambda\big(R/I^2 + J\big) - (c+1)(N_2 - q) \geqslant N_3 - (c+1)N_2 + (c+1)q + \max\big\{0, n_4 - q(q+1)/2\big\} + \max\{0, n_5 - qn_3\}.$$

Now, (a) follows from the fact that

$$\lambda\big(R/I^2 + J\big) - (c+1)(N_2 - q) \geqslant N_3 - (c+1)N_2 + (c+1)q = \frac{1}{6}(c+1)\big(-2c^2 - 4c + 6q\big)$$

which is strictly greater than zero if $q > \frac{c^2+2c}{3}$.

Since

$$\begin{aligned}\lambda\big(R/I^2 + J\big) - (c+1)(N_2 - q) &\geqslant N_3 - (c+1)N_2 + (c+1)q + n_4 - q(q+1)/2\\ &= -\frac{1}{2}\big[q^2 - (2c+1)q - 2\big(N_4 - (c+1)N_2\big)\big],\end{aligned}$$

it is easy to see that $\lambda(R/I^2 + J) - (c+1)(N_2 - q) > 0$ if q satisfies the numerical condition given in (b). Part (c) follows from the inequality

$$\begin{aligned}\lambda\big(R/I^2 + J\big) - (c+1)(N_2 - q) &\geqslant -\frac{1}{2}\big[q^2 - (2c+1)q - 2\big(N_4 - (c+1)N_2\big)\big] + n_5 - qn_3\\ &= -\frac{1}{2}\big[q^2 + (2n_3 - 2c - 1)q - 2\big(N_5 - (c+1)N_2\big)\big].\end{aligned}$$

Finally, it follows from parts (a), (b) and (c) that $\lambda(R/I^2 + J) - (c+1)(N_2 - q) > 0$ if $c = 3$, $q \neq 5$ or if $c = 4$, $q \neq 8$. If $c = 3$, $q = 5$, we are done by Theorem 3.5(a). Assume $c = 4$, $q = 8$. Set $\overline{R} = R/J$. By Remark 5.2, we may assume that $\overline{R} \simeq k[\![X_1, \dots, X_4]\!]$ and $\overline{I}$ is a homogeneous ideal generated by 8 quadric polynomials. Since $e(R/I^2) = \lambda(\overline{R}/\overline{I}^2) \geqslant \lambda(\overline{R}/\overline{\mathfrak{m}}^4) = 35 = (c+1)\lambda(\overline{R}/\overline{I}) = (c+1)e(R/I)$, if we show that $\overline{I}^2 \neq \overline{\mathfrak{m}}^4$, then $e(R/I^2) > (c+1)e(R/I)$, that gives a contradiction. However, since $\overline{I}$ is generated by polynomials, it is enough to show that the ideal L in $k[X_1, \dots, X_4]$ generated by the same 8 quadrics generating $\overline{I}$ has the property that $L^2 \neq \mathfrak{m}^4$, where $\mathfrak{m} = (X_1, \dots, X_4)$. This is achieved in [6, Theorem 2.4], where it was shown that any homogeneous ideal L in $k[X_1, \dots, X_4]$ generated by 8 quadrics has the property that $L^2 \neq \mathfrak{m}^4$, if $\operatorname{char} k = 0$. Similar methods can be actually employed to show that the same result holds if $\operatorname{char} k \neq 2$; this finishes the proof. □

Example 7.1 shows that the numerical conditions given in Proposition 5.5 are actually very sharp. When $c = 5$, indeed, I/I^2 cannot be Cohen–Macaulay if either $q > \frac{c^2+2c}{3} = 11\frac{2}{3}$ (by Proposition 5.5(a)) or $q < \frac{2c+1+\sqrt{K}}{2} = 11$ (by Proposition 5.5(b)). Hence, the only case left out by our numerical criteria is $q = 11$. However, Example 7.1 has exactly $c = 5$ and $q = 11$ and gives a negative answer to Question 2.1!

A combination of Theorem 5.3, Proposition 5.5 and Lemma 2.7 shows that the question has a positive answer for most short algebras.

Corollary 5.6. *Question 2.1 is true for any ideal defining an equicharacteristic short algebra with socle degree at least 3, or socle degree 2 and multiplicity satisfying one of the numerical conditions in Proposition 5.5.*

6. Algebras having low multiplicity

For a Cohen–Macaulay local ring $(R, \mathfrak{m})$, it is well known (by Abhyankar's inequality) that the Hilbert–Samuel multiplicity of R with respect to $\mathfrak{m}$ is at least $c+1$, where c denotes as usual the embedding codimension of R. In this section we show that Question 2.1 holds true for any ideal I with $e(R/I) \leqslant c+4$. In Section 5, we then show that this result is sharp by exhibiting an ideal I with $e(R/I) = c+5$ that is a counterexample to Question 2.1. We will call ideals I with $e(R/I) \leqslant c+4$ 'ideals with low multiplicity'.

We begin this section by proving a general statement that will be used in the next results. After fixing the multiplicity of a Cohen–Macaulay algebra R/I, we provide a lower bound on the embedding codimension of R/I that ensures that I/I^2 is *not* Cohen–Macaulay.

Proposition 6.1. *Let R be a regular local ring and I a Cohen–Macaulay R-ideal that is generically a complete intersection. Assume that either R is equicharacteristic or $I \subseteq \mathfrak{m}^2$. Write $e(R/I) = c+t$ for some positive integer t, where $c = \operatorname{ecodim}(R/I)$. If $c > \frac{1+\sqrt{1+24(t-1)}}{2}$ then I/I^2 (equivalently, R/I^2) is not Cohen–Macaulay.*

Proof. Let $\mathfrak{m}$ be the maximal ideal of R. After possibly invoking Proposition 2.6 (if R contains a field), we may assume $I \subseteq \mathfrak{m}^2$ and then $c = \operatorname{ht} I$. Assume I/I^2 is Cohen–Macaulay, in which case R/I^2 is Cohen–Macaulay. Let J be an R-ideal whose image in R/I is generated by a system of parameters of $\mathfrak{m}_{R/I}$. By Proposition 2.9,

$$\lambda\big(R/I^2+J\big) = e\big(R/I^2\big) = (c+1)e(R/I) = (c+1)(c+t) = c^2+(t+1)c+t.$$

Notice that $I^2 \subseteq \mathfrak{m}^4$, so $\lambda(R/I^2+J) \geqslant \lambda(R/\mathfrak{m}^4+J) = 1+c+\binom{c+1}{2}+\binom{c+2}{3} = \frac{1}{6}(c^3+11c)+c^2+1$. However, $\frac{1}{6}(c^3+11c)+c^2+1 > c^2+(t+1)c+t$ if $\frac{1}{6}(c^3-c(6t-5)-6(t-1)) > 0$, or equivalently, if $(c+1)(c^2-c-6(t-1)) > 0$. Since c is positive, it is enough to have $c^2-c-6(t-1) > 0$, which holds true if $c > \frac{1+\sqrt{1+24(t-1)}}{2}$. Hence if $c > \frac{1+\sqrt{1+24(t-1)}}{2}$, then $e(R/I^2) = \lambda(R/I^2+J) \geqslant \lambda(R/\mathfrak{m}^4+J) > (c+1)(c+t)$ which gives a contradiction. Therefore I/I^2 cannot be Cohen–Macaulay. $\square$

We now employ results from Sections 2 and 3 together with Proposition 6.1 to show that Question 2.1 has a positive answer for ideals with multiplicity at most $c+4$. We first deal with ideals of multiplicity at most $c+3$.

Theorem 6.2. *Let $(R, \mathfrak{m})$ be a regular local ring containing a field of characteristic $\neq 2$ and I a Cohen–Macaulay R-ideal that is generically a complete intersection. Assume that $e(R/I) \leqslant c+3$, where $c = \operatorname{ecodim}(R/I)$. If I/I^2 (or R/I^2) is Cohen–Macaulay then R/I is Gorenstein.*

Proof. Thanks to Lemma 2.7 we can assume that $c \geqslant 3$ and, thanks to Proposition 2.6, we can always assume $I \subseteq \mathfrak{m}^2$ so that $c = \operatorname{ht} I$. By Proposition 2.9, $e(R/I^2) = (c+1)e(R/I)$. Now, let J be an ideal whose image in R/I is a minimal reduction of $\mathfrak{m}_{R/I}$ generated by a system of parameters (after extending the residue field such a J always exists). After going modulo J, we may further assume that I is $\mathfrak{m}$-primary and $\lambda(R/I^2) = e(R/I^2) = (c+1)e(R/I) = (c+1)\lambda(R/I)$. Notice that I may no longer be generically a complete intersection. We already used this assumption to establish that $e(R/I^2) = (c+1)e(R/I)$ and we will not need it for the rest of the proof.

We now show that if R/I is not Gorenstein then $\lambda(R/I^2) > (c+1)\lambda(R/I)$, giving a contradiction. The first case, $e = c+1$, simply follows from Proposition 6.1 with $t = 1$, because $c \geqslant 3$. If $e = c+2$, then R/I is a stretched algebra of socle degree 2 and the statement follows from Corollary 3.7. Assume $e = c+3$. If R/I is a stretched algebra of socle degree 3 then we are done again by Corollary 3.7. We may assume that R/I is not stretched. Then, its Hilbert function is

$$HF_{R/I}: \quad 1 \quad c \quad 2 \quad 0 \quad \ldots \quad 0 \quad \ldots$$

and Proposition 6.1 concludes the proof for all value of c, except $c = 3$ or $c = 4$. However, both these cases follow from Proposition 5.5. □

Next, we deal with the case $e(R/I) = c + 4$. We will need the following lemma, which shows how the Hilbert function and the associated graded ring are affected by factoring out a socle element.

Lemma 6.3. *Let $(R, \mathfrak{m})$ be a Noetherian local ring and $0 \neq f \in \operatorname{soc}(R)$. Then f^* is a non-zero socle element of $R^* = \operatorname{gr}_{\mathfrak{m}}(R)$ and $R^*/f^*R^* \simeq \operatorname{gr}_{\mathfrak{m}}(R/fR)$. Furthermore, if $f \in \mathfrak{m}^j \backslash \mathfrak{m}^{j+1}$, then $HF_{R/fR}(i) = HF_R(i)$ for every $i \neq j$, and $HF_{R/fR}(j) = HF_R(j) - 1$.*

Proof. Since $f^*\mathfrak{m}^* \subseteq (f\mathfrak{m})^* = (0)$, we deduce that $0 \neq f^*$ is a socle element of $\operatorname{gr}_{\mathfrak{m}}(R)$. To prove that $R^*/f^*R^* \simeq \operatorname{gr}_{\mathfrak{m}}(R/fR)$, it is enough to show the equality $(fR)^* = f^*R^*$. Clearly $f^* \in (fR)^*$ showing one inclusion. For the other inclusion, take $b \in R$, we show that $(bf)^* \in f^*R^*$. Indeed, if $b \notin \mathfrak{m}$ the statement is trivial, so we may assume that $b \in \mathfrak{m}$. But then $bf = 0$ concluding the proof. Finally, notice that for every i, $HF_{R/fR}(i) = HF_{R^*/(fR)^*}(i) = HF_{R^*/f^*R^*}(i)$ and $HF_R(i) = HF_{R^*}(i)$. Therefore, it is enough to prove the statement in the homogeneous settings, which is a well-known result. □

We now show that Question 2.1 holds true for ideals defining algebras with multiplicity $c + 4$.

Theorem 6.4. *Let $(R, \mathfrak{m})$ be a regular local ring containing a field of characteristic $\neq 2$ and I a Cohen–Macaulay R-ideal that is generically a complete intersection. Assume $e(R/I) = c + 4$, where $c = \operatorname{ecodim}(R/I)$.*

(a) *If $c \geqslant 5$, then I/I^2 (equivalently, R/I^2) is not Cohen–Macaulay.*
(b) *If $c \leqslant 4$ and I/I^2 (or R/I^2) is Cohen–Macaulay, then R/I is Gorenstein.*

Proof. First of all, we employ Lemma 2.7 and Proposition 2.6 to assume that $c \geqslant 3$, $I \subseteq \mathfrak{m}^2$ and $\operatorname{ht} I = c$. Then, by Proposition 2.9, we have that $e(R/I^2) = (c+1)e(R/I)$. As before, after factoring out a minimal reduction of the maximal ideal of R/I generated by a system of parameters, we may assume that I is an $\mathfrak{m}$-primary ideal (not necessarily generically a complete intersection) with multiplicity (or, equivalently, length) $c + 4$. Now, (a) simply follows from Proposition 6.1. For (b), since $c = 3$ or 4, the only possible Hilbert functions of R/I are:

(I)	1	c	3	0	...	...	...	0	...
(II)	1	c	2	1	0	...	...	0	...
(III)	1	c	1	1	1	0	...	0	...

Since in case (I) the algebra is short, Proposition 5.5 completes the proof. In case (III), the algebra is stretched, hence the conclusion follows by Theorem 3.5. We are then left with case (II). Assume by contradiction that R/I is not Gorenstein. We show that this forces $\lambda(R/I^2) > (c+1)(c+4)$, contradicting the previously proved equality $e(R/I^2) = (c+1)e(R/I)$.

Since R/I is not Gorenstein, R/I has a socle element of initial degree 1 or 2. We first deal with the case where there exists a socle element of initial degree 1. Since it is part of a minimal generating set of the maximal ideal, we may write $\mathfrak{m} = (x_1, \ldots, x_c)$, where x_1 is a socle element of R/I. Notice that, after passing to the completion, we may assume that R is a power series ring over k in the variables $x_1, \ldots, x_c$.

Let $c = 4$. Then $I = (x_1\mathfrak{m}, q_1, \ldots, q_4, g, \mathfrak{m}^4)$, where g is a combination (with unit coefficients) of monomials of degree 3 in x_2, x_3, x_4 and all the q_i's have initial degree 2 and involve only x_2, x_3, x_4. Since x_1 is a socle element, by Lemma 6.3 the Hilbert function of $R/(I, x_1)$ is

$$1 \quad 3 \quad 2 \quad 1 \quad 0_{\longrightarrow}.$$

Hence $(x_1\mathfrak{m}^2, (q_1, \ldots, q_4)\mathfrak{m}) \neq \mathfrak{m}^3$ and $(x_1^2\mathfrak{m}^2, x_1(q_1, \ldots, q_4)\mathfrak{m})$ is strictly contained in $x_1\mathfrak{m}^3$. Now $I^2 \subseteq (x_1^2\mathfrak{m}^2, x_1(q_1, \ldots, q_4)\mathfrak{m}, q_iq_j, 1 \leqslant i \leqslant j \leqslant 4, \mathfrak{m}^5) \subsetneq (x_1\mathfrak{m}^3, q_iq_j, 1 \leqslant i \leqslant j \leqslant 4, \mathfrak{m}^5)$,

$$\lambda\big(R/I^2\big) > \lambda\big(R/\big(x_1\mathfrak{m}^3, q_iq_j, 1 \leqslant i \leqslant j \leqslant 4, \mathfrak{m}^5\big)\big)$$
$$\geqslant 1 + 4 + 10 + 20 + 5 = 40 = (c+1)e(R/I),$$

and we are done.

Now, assume $c = 3$. Since $(c+1)e(R/I) = 28$, our goal is to show that $\lambda(R/I^2) \geqslant 29$. Similarly to the above argument, $I = (x_1\mathfrak{m}, q, g, \mathfrak{m}^4)$, where q and g involve only x_2, x_3, q has initial degree 2 and g is a combination (with unit coefficients) of monomials of degree 3 in x_2, x_3. Then $I^2 \subseteq (x_1^2\mathfrak{m}^2, x_1q\mathfrak{m}, q^2, x_1g\mathfrak{m}, qg, \mathfrak{m}^6)$ and similar to the above one gets that

$$\lambda\big(R/I^2\big) \geqslant \lambda\big(R/\big(x_1^2\mathfrak{m}^2, x_1q\mathfrak{m}, q^2, x_1g\mathfrak{m}, qg, \mathfrak{m}^6\big)\big) \geqslant 1 + 3 + 6 + 10 + 6 + 3 = 29.$$

Hence, from now on we may assume that R/I has no socle elements in $\mathfrak{m}\backslash\mathfrak{m}^2$.

First, consider $c = 4$. Then, there exists an element $q_9 \in \mathfrak{m}^2\backslash\mathfrak{m}^3$ with $q_9 \in \operatorname{soc}(R/I)$ and $q_9^* \notin I^*$. By Lemma 6.3, $K := (I, q_9)$ defines a stretched Artinian algebra of socle degree 3 and there exist $x_1, \dots, x_4$ in $\mathfrak{m}\backslash\mathfrak{m}^2$ so that a generating set of K can be described as in Theorem 3.3. Observe that $I^2 \subseteq L = (x_1, x_2, x_3)H + (x_1, x_2, x_3)x_4^4 + (x_4^6)$, where H is as in Proposition 3.4. Since

$$e\big(R/I^2\big) = \lambda\big(R/I^2\big) \geqslant \lambda(R/L) \geqslant 1 + 4 + \binom{4+1}{2} + \binom{4+2}{3} + 4 + 1 = 40 = (c+1)e(R/I),$$

to finish the proof, it is enough to prove that $I^2 \subsetneq L$. If R/K has type 3 or 4, we are done by Proposition 3.4. So assume R/K has type 1 or 2, then K has nine minimal generators, say, $q_1, \dots, q_9$. The same idea of Remark 5.2 shows that we can further assume that R is $k[x_1, \dots, x_4]$ and I is generated by polynomials whose homogeneous components have degrees 2 and 3 (this follows from $\mathfrak{m}^4 \subseteq I \subseteq \mathfrak{m}^2$ and the fact that $k[x_1, \dots, x_4]/I \simeq k[\![x_1, \dots, x_4]\!]/I$).

Notice that $I = (q_1, \dots, q_8, q_9\mathfrak{m})$. Also, the description of K yields that for every i one has $q_i = Q_i + a_ix_4^3$ for some Q_i homogeneous of degree 2 and a_i (either zero or unit) in R. Since $I^2 = (q_iq_j, q_iq_9\mathfrak{m}, 1 \leqslant i, j \leqslant 8, q_9^2\mathfrak{m}^2) \subseteq K^2 = (q_iq_j, 1 \leqslant i, j \leqslant 9) \subseteq L$, if $I^2 = K^2 = L$, an application of Nakayama's lemma – after going modulo $(q_iq_j, 1 \leqslant i, j \leqslant 8)$ – shows that $L = (q_iq_j, 1 \leqslant i, j \leqslant 8) = (q_1, \dots, q_8)^2$.

Set $\tilde{R} = R[t]$, where t is a variable over R. For $1 \leqslant i \leqslant 9$, let $\tilde{q}_i$ be obtained from q_i by replacing x_4^3 by x_4^2t, and set $\tilde{K} = (\tilde{q}_1, \dots, \tilde{q}_9)\tilde{R}$ and $\tilde{J} = (\tilde{q}_1, \dots, \tilde{q}_8)\tilde{R}$. It is easy to see that $\tilde{K}^2 = \tilde{L} = (x_1, x_2, x_3)H + (x_1, x_2, x_3)x_4^3t + (x_4^4t^2)$. Now we have $\tilde{J}^2 \subseteq \tilde{L}$.

Let $\pi(\cdot)$ denote the images modulo $t - x_4$. We have that $\pi(\tilde{J}^2) = \pi(\tilde{L}) = L$. Since $\tilde{L}$ is a monomial ideal, one can check that $t - x_4$ is regular on $\tilde{R}/\tilde{L}$. Hence $\tilde{L} = \tilde{J}^2 + (t - x_4)\tilde{R} \cap \tilde{L} = \tilde{J}^2 + (t - x_4)\tilde{L}$. By Nakayama's lemma, $\tilde{L} = \tilde{J}^2$. Now set $t = 1$. Then, $\tilde{q}_1, \dots, \tilde{q}_8$ become homogeneous of degree two, say $q_1', \dots q_8'$ and we have that $(q_1', \dots q_8')^2 = \mathfrak{m}^2$. This fact yields a contradiction to [6, Theorem 2.4] (see also the proof of Proposition 5.5).

Assume, finally, that $c = 3$. Observe that $I = (g_1, g_2, g_3, g_4, g_5\mathfrak{m})$, $(I, g_5) = (g_1, g_2, g_3, g_4, g_5)$, where all the $g_i's$ have initial degree 2 and $g_5 \in \operatorname{Soc}(R/I)$, $g_5 \notin I$. There are two cases. First case, (I, g_5) is a stretched Gorenstein ideal. Then, $I^2 = (g_ig_j, 1 \leqslant i, j \leqslant 4, g_ig_5\mathfrak{m}, 1 \leqslant i \leqslant 4, g_5^2\mathfrak{m}^2) \subseteq (I, g_5)^2 = (g_ig_j, 1 \leqslant i, j \leqslant 5) = (x_1, x_2)H + (x_1, x_2)x_3^4 + (x_3^6)$. Since the number of minimal generators of $(I, g_5)^2$ is 15, it is generated minimally by $g_ig_j, 1 \leqslant i, j \leqslant 5$. The ideal I^2 contains only 10 minimal generators of them, hence $\lambda((I, g_5)^2/I^2) \geqslant 5$. Therefore $\lambda(R/I^2) = \lambda(R/(I, g_5)^2) + \lambda((I, g_5)^2/I^2) \geqslant (1 + 3 + \binom{3+1}{2} + \binom{3+2}{3} + 3 + 1) + 5 = 29 > 28 = (c+1)e(R/I)$.

Second case, $R/(I, g_5)$ is not Gorenstein. If $\tau(R/(I, g_5)) = 2$, then $(I, g_5)^2 = (x_1, x_2)H + (x_2x_3^4 - ux_2^3x_3, x_3^6 + u^2x_2^4)$ which has 13 minimal generators. I^2 contains at most 10 of them. Hence,

$$\lambda\big(R/I^2\big) = \lambda(R/L) + \lambda\big(L/(I, g_5)^2\big) + \lambda\big((I, g_5)^2/I^2\big) \geqslant 24 + 2 + 3 = 29,$$

and the proof is finished. Finally, if $\tau(R/(I,g_5)) = 3$, then $(I,g_5) = (x_1,x_2)\mathfrak{m} + (x_3^4)$. Hence $\lambda(R/(I,g_5)^2) = 1+3+6+10+3+3+1+1 = 28$. Since (I,g_5) has 12 minimal generators, then $\lambda(R/I^2) \geqslant 28+2 = 30$, providing the desired contradiction. □

As a consequence of Theorems 6.2 and 6.4, we give a positive answer to Question 2.1 for ideals defining algebras with low multiplicity.

Corollary 6.5. *Question 2.1 holds true for I with $e(R/I) \leqslant c+4$, where $c = \operatorname{ecodim}(R/I)$.*

In particular, Question 2.1 holds true for some irreducible algebroid curves.

Corollary 6.6. *Let $S = k[[f_1(t),\dots,f_n(t)]]$ for some power series $f_1,\dots,f_n$ in $k[[t]]$ and assume none of the f_i's is redundant. Let $R = k[[X_1,\dots,X_n]]$ and $\mathfrak{p}$ the prime R-ideal defining S. If $a_i := \operatorname{in}\deg(f_i) \leqslant n+3$ for some i, then Question 2.1 holds true for $\mathfrak{p}$.*

In particular, if $S = k[[t^{a_1},\dots,t^{a_n}]] \simeq R/\mathfrak{p}$, where none of the t^{a_i} is redundant, and $a_i \leqslant n+3$ for some i, then Question 2.1 holds true for $\mathfrak{p}$.

Proof. In this setting, it is well known that $n = c+1$ and $e(R/\mathfrak{p}) \leqslant a_j$ for every j. Hence $a_i \leqslant n+3$ for some i implies that $e(R/\mathfrak{p}) \leqslant c+4$. Now apply Corollary 6.5. □

7. Counterexamples and sharpness of the main result

In this section we provide examples of ideals for which Question 2.1 has a negative answer. To our best knowledge, they are the first negative examples for the question. We employ these examples to show the sharpness of Theorem 1.1(d)–(e).

We begin by exhibiting an example found with the help of J.C. Migliore and CoCoA [5]. Then, we deform it using the theory of universal linkage (see Proposition 2.8) to obtain also a prime ideal (with the same multiplicity) that does not enjoy the property stated in Question 2.1. Finally, we conjecture that in $\mathbb{P}^c$ a certain number of points (function of c) gives a counterexample to Question 2.1 in *any* embedding codimension $c \geqslant 5$.

Example 7.1. (a) Let $S = k[a,b,c,d,e,f]$ be a polynomial ring over k, where k is either $\mathbb{Q}$ or $\mathbb{Z}/31991\mathbb{Z}$ and let $M = (a, dots, f)$. Then a homogeneous level ideal I in S defining 10 general points gives a negative answer to the homogeneous version of Question 2.1, i.e., I/I^2 and S/I^2 are Cohen–Macaulay but S/I is not Gorenstein. Also, S/I is a short algebra with $\operatorname{socdeg}(S/I) = 2$, $\tau(S/I) = 4$, $c = 5$ and $e(S/I) = c+5$.

(b) There exist a regular local ring $(R,\mathfrak{m})$ and a prime ideal $\mathfrak{p}$ such that $\mathfrak{p}/\mathfrak{p}^2$ and $R/\mathfrak{p}^2$ are Cohen–Macaulay but $R/\mathfrak{p}$ is not Gorenstein. $R/\mathfrak{p}$ has multiplicity $c+5$, where $c = \operatorname{ecodim}(R/\mathfrak{p})$.

Proof. (a) In $S = k[a,b,c,d,e,f]$, using the package 'IdealOfProjectivePoints' (see [1]), one obtains the following homogeneous ideal which defines a set of 10 general points in $\mathbb{P}^5$:

$$\begin{aligned}
I = \big(&e^2f + 2963bf^2 + 4964cf^2 + 5333df^2 - 13261ef^2,\\
&af - 13894bf + 12842cf + 4036df - 2985ef, de + 3056bf + 12160cf + 971df + 15803ef,\\
&ce - 2357bf - 14460cf + 3040df + 13776ef, be - 8504bf + 1159cf - 1581df + 8925ef,\\
&ae - 9147bf + 1379cf + 4167df + 3600ef, cd + 7380bf + 5885cf + 6255df + 12470ef,\\
&bd - 11676bf - 2833cf - 13277df - 4206ef, ad + 5555bf + 2017cf + 2100df - 9673ef,\\
&bc + 5653bf - 6596cf - 8208df + 9150ef, ac - 2335bf - 10387cf + 514df + 12207ef,
\end{aligned}$$

$$ab - 8324bf - 7688cf - 4252df - 11728ef,$$
$$b^2f + 15536bf^2 + 1265cf^2 + 9888df^2 + 5301ef^2,$$
$$d^2f + 10625bf^2 - 11725cf^2 + 9514df^2 - 8415ef^2,$$
$$c^2f + 11390bf^2 + 7112cf^2 - 10319df^2 - 8184ef^2\big).$$

Using CoCoA for instance, one can see that the H-vector of S/I is

$$1 \quad 5 \quad 4$$

proving that $c = 5$, $e(S/I) = c + 5$, and S/I is a short algebra with $\operatorname{socdeg}(S/I) = 2$. By the Betti diagram of S/I, one has that S/I is level which implies that $\tau(S/I) = 4$. Finally the Betti diagram of S/I^2 shows that S/I^2 is Cohen–Macaulay, yielding that I/I^2 is Cohen–Macaulay too.

(b) The first statement follows from part (a) and Proposition 2.8. For the second part of the statement, since $(R, \mathfrak{p})$ is a deformation of a (S_M, J_M) one gets automatically

$$e(R/\mathfrak{p}) \leqslant e(S_M/J_M) = c + 5.$$

In general, this inequality may be strict, however in this case it must be an equality. Indeed, assume by contradiction that $e(R/\mathfrak{p}) < c + 5$. Then, $\mathfrak{p}$ is an ideal satisfying the assumptions of Question 2.1 and for which the question has a negative answer. However, since the multiplicity of $R/\mathfrak{p}$ is at most $c + 4$, $\mathfrak{p}$ would contradict Theorems 1.1(d) and Theorem 6.5. Therefore, one must have

$$e(R/\mathfrak{p}) = c + 5. \qquad \square$$

Since both these examples have multiplicity $c + 5$, they show that Corollary 6.5 is sharp. Furthermore, Example 7.1 shows that the numerical estimates presented in Proposition 5.5 are very sharp too. Indeed, for the case $c = 5$, Proposition 5.5 implies that I/I^2 is not Cohen–Macaulay for any $q \neq 11$ (using the notation of Proposition 5.5). While for $q = 11$, we have the above counterexample, showing that I/I^2 is Cohen–Macaulay.

Similarly, when $c = 6$, Proposition 5.5(a) and (b) show that I/I^2 is not Cohen–Macaulay for any $q \neq 16$. However, for $q = 16$, J.C. Migliore found with CoCoA [5] a set of 12 general points in $\mathbb{P}^6$ with I/I^2 Cohen–Macaulay but R/I not Gorenstein.

It would be natural to conjecture that a set consisting of $2c$ general points in $\mathbb{P}^c$ gives a negative answer to Question 2.1 for any $c \geqslant 5$. However, this is false. Indeed, a set of 14 general points in $\mathbb{P}^7$ does *not* give a counterexample to the question.

Instead, we conjecture that the estimate of Proposition 5.5 is sharp and if we go past this number we get counterexamples to Question 2.1 in any codimension at least 5. We now state this more precisely. Let $\lceil x \rceil$ denote the smallest integer bigger than or equal to a given real number x. Then,

Conjecture 7.2. *For any $c \geqslant 5$, the homogeneous ideal I defined by a set of $1 + c + \lceil \frac{c(c-1)}{6} \rceil$ general points in $\mathbb{P}^c$ has the property that I/I^2 is Cohen–Macaulay, but R/I is not Gorenstein.*

Our conjecture is supported by data obtained by J.C. Migliore. In fact, using CoCoA, he checked that, for any $c \leqslant 10$, Conjecture 7.2 holds true and indeed provides the smallest set of (general) points in $\mathbb{P}^c$ giving a counterexample to Question 2.1.

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