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Lipschitz Functions on Classical Spaces

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We show that, for every $\varepsilon > 0$ and every Lipschitz function f from the unit sphere of the Banach space c_0 to $\mathbb{R}$, there is an infinite-dimensional subspace of c_0, on the unit sphere of which f varies by at most ε. This result is closely related to a theorem of Hindman, and a well known open problem in Banach space theory.

A famous question in Banach space theory, known as the distortion problem, asks whether, given $1 < p < \infty$ and $\varepsilon > 0$, it is true that every space isomorphic to l_p has a subspace which is $(1 + \varepsilon)$-isomorphic to l_p. While a straightforward result of James [10] shows that the corresponding question for l_1 or c_0 has a positive answer, there has been very little progress on the distortion problem itself. The reason the question is easier for l_1 and c_0 is that it is possible to use the triangle inequality in a very strong way to obtain a bound in one direction for the equivalent norm. On the other hand, when $1 < p < \infty$, it is known, and not hard to show, that the problem is equivalent to the following question. Given $\varepsilon > 0$ and a real-valued Lipschitz function f on the unit sphere of l_p, does there exist an infinite-dimensional subspace of l_p on the unit sphere of which f varies by at most ε? Since arbitrary Lipschitz functions need not obey the triangle inequality, it is clear that this is a very different problem from the l_1 or c_0 case.

It also suggests very naturally the question of whether Lipschitz functions on the unit spheres of l_1 and c_0 can be restricted to unit spheres of infinite-dimensional subspaces on which they are almost constant. This question is certainly not equivalent to the result of James. Indeed, a positive answer for l_1 would imply a positive answer when $1 < p < \infty$. In this paper we shall give a positive answer for c_0. This suggests that it is not unreasonable to hope for a positive answer in general. On the other hand, there are of course special features of c_0 that make it easy to deal with. We shall discuss this at the end of the paper.

The main idea of the proof is to exploit analogies with an important result of Hindman [8]. In 1975 he proved a conjecture of Graham and Rothschild by showing that, given any finite colouring of $\mathbb{N}$, there exists an infinite sequence $n_1, n_2, \ldots$ such that, for any finite set $A \subset \mathbb{N}$, the colour of $\sum_{i \in A} n_i$ is the same. This theorem has an equivalent formulation in terms of colourings of $\mathbb{N}^{(<\omega)}$, the set of finite subsets of $\mathbb{N}$. Given $X, Y \in \mathbb{N}^{(<\omega)}$, let us write $X < Y$ if $\max X < \min Y$. Then, given any finite colouring of $\mathbb{N}^{(<\omega)}$, there exists an infinite sequence $X_1 < X_2 < \cdots$ of elements of $\mathbb{N}^{(<\omega)}$ such that, for any $A \in \mathbb{N}^{(<\omega)}$, the colour of $\bigcup_{i \in A} X_i$ is the same. Hindman's theorem in this form can be regarded as the natural discrete analogue of the distortion problem.

The 'finite unions version' of Hindman's theorem can be regarded as a theorem about finite words in an alphabet consisting of the two letters 0 and 1, and as such has been generalized by Carlson and Simpson [4], Carlson [3] and Furstenberg and Katznelson [6] to theorems concerning larger alphabets. The theorem of Carlson and Simpson is an infinite version of the Hales–Jewett theorem, while that of Furstenberg and Katznelson is a refinement of the Carlson–Simpson theorem. In both these theorems, the alphabet concerned is a finite set with no order (although for Furstenberg and Katznelson it has preferred elements). In this paper we prove another

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natural generalization of Hindman's theorem, this time using a totally ordered alphabet and proving a result which respects the order. Our proof has certain obvious similarities with the methods of [3] and [6], and generalizes Glazer's remarkable proof of Hindman's theorem (cf. [7]). In particular, the inductive step of our Lemma 3 is standard (cf. [6, Theorem 1.3]). Since our proof is quite short, we give it in full, except for Lemma 2, which is well known, and a number of elementary facts that need to be checked, and have been checked elsewhere (see, e.g., [9]).

As a consequence, we shall deduce easily that a Lipschitz function on the unit sphere of c_0 which does not depend on the signs of the co-ordinates of any vector can be restricted to the unit sphere of an infinite-dimensional subspace on which it is almost constant. The general case is a little harder, because the obvious candidate for a combinatorial result which would imply it is false. Instead, we prove an 'approximate Ramsey result' which states, roughly speaking, that if a certain discrete structure is coloured with finitely many colours, then it has an infinite substructure, all of the points of which are close to a point of one particular colour. This turns out to be sufficient for our purposes.

Before stating our first theorem, we shall introduce some notation. Let us write $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. Then, for any $k \in \mathbb{N}$, let the shift $T\colon \mathbb{N}_0^k \to \mathbb{N}_0^k$ be defined by

$$T\colon (n_1, n_2, \ldots, n_k) \mapsto (0, n_1, \ldots, n_{k-1})$$

and let $X_k = \mathbb{N}_0^k \setminus T\mathbb{N}_0^k = \{(n_1, \ldots, n_k)\colon n_1 \neq 0\}$. Given a subset $A = \{\mathbf{n}_i\colon i \in I\} \subset X_k$ indexed by a set I, we shall say that *the subspace generated by* A is the set of elements of $\mathbb{N}_0^k$ of the form

$$\sum_{j=1}^{k} \sum_{i \in B_j} T^{j-1}\mathbf{n}_i,$$

where $B_1, \ldots, B_k$ are disjoint subsets of I and B_1 is non-empty. Note that the conditions that $A \subset X_k$ and that B_1 is non-empty ensure that the subspace generated by A is in fact a subset of X_k. Later, it will become clear why we use the word 'subspace', when we use this combinatorial structure to obtain results about the unit sphere of c_0.

Given any set X, a *finite colouring* of X is a partition of X into finitely many subsets $c_1 \cup \cdots \cup c_r$. We shall refer to a finite colouring as simply a colouring. The subsets $c_1, \ldots, c_r$ are said to be *colour classes*, and a subset $Y \subset X$ is said to be *monochromatic* if Y is contained in a single colour class. We can now state our first result.

THEOREM 1. *Let $k, r \in \mathbb{N}$ and let $X_k = c_1 \cup \cdots \cup c_r$ be a colouring of X_k with r colours. Then there exists a monochromatic subspace of X_k generated by an infinite set.*

The case $k = 1$ of Theorem 1 is simply Hindman's theorem. In order to prove the result in general, we rely heavily on the following lemma, which was also used by Glazer.

LEMMA 2. *Let $(S, +)$ be a compact Hausdorff semigroup such that the function $y \mapsto y + x$ on S is continuous for every $x \in S$. Then there exists an idempotent; that is, an element $x \in S$ such that $x + x = x$.*

The compact semigroup used by Glazer was the set of ultrafilters on $\mathbb{N}$, with the product topology and an addition which we shall soon describe. We shall also use ultrafilters a great deal, and the following notation will be very useful for simplifying the presentation of proofs (cf. [1]). Given a set X and an ultrafilter α on X, let the symbol Λ_α be defined as follows. If $P(x)$ is any proposition involving the elements of

X, then when we write $(\Lambda_\alpha x)P(x)$ we mean $\{x \in X: P(x)\} \in \alpha$. ('$(\Lambda_\alpha x)P(x)$' can be read 'for a lot$_\alpha$ of x, $P(x)$'.) Syntactically, Λ_α behaves like a quantifier, and there seems to be no harm in calling it one. If X is a semigroup, then the set $U(X)$ of ultrafilters on X can be turned into a compact semigroup by giving it the product topology and setting

$$\alpha + \beta = \{A \subset X: (\Lambda_\alpha x)(\Lambda_\beta y)x + y \in A\}.$$

In the case $X = \mathbb{N}$, this was the addition used by Glazer. It is not hard to verify that this operation on $U(X)$ is right-continuous. By Lemma 2, it follows that $U(X)$ contains an idempotent.

Given $k \geqslant 2$, let a 'shift' operator $S: U(X_k) \to U(X_{k-1})$ be defined as follows. For any $\alpha \in U(X_k)$ we define $S(\alpha) \in U(X_{k-1})$ to be the set

$$\{A \subset X_{k-1}: (\Lambda_\alpha x)Tx \in A\}.$$

It is not hard to check that the operator S is continuous.

Given any $j < k \in \mathbb{N}$, there is an obvious identification between X_j and $T^{k-j}X_k$. We may therefore define a map $+: (X_j, X_k) \to X_k$ by

$$(m_1, \ldots, m_j) + (n_1, \ldots, n_k) = (n_1, \ldots, n_{k-j}, m_1 + n_{k-j+1}, \ldots, m_j + n_k).$$

We also define the map $+: (X_k, X_j) \to X_k$ in a similar way. We can define corresponding maps on the spaces of ultrafilters. For example, the map $+: (U(X_k), U(X_j)) \to U(X_k)$ is defined by setting

$$\alpha + \beta = \{A \subset X_k: (\Lambda_\alpha x \in X_k)(\Lambda_\beta y \in X_j)x + y \in A\}.$$

It is not hard to verify that these maps are all right-continuous. It is also not hard to verify that, if $j, k \geqslant 2$, then $S(\alpha + \beta) = S\alpha + S\beta$ for any $\alpha \in U(X_j)$, $\beta \in U(X_k)$, and, if $1 \leqslant j < k$, then $S(\alpha + \beta) = S(\beta + \alpha) = S\beta$ for any $\alpha \in U(X_j)$, $\beta \in U(X_k)$. Finally, one can check that $S^{-1}\alpha$ is non-empty for every $\alpha \in U(X_{k-1})$.

We may now state and prove the main lemma upon which Theorem 1 depends. As we mentioned in the introduction, the basic idea of the proof is now fairly standard.

LEMMA 3. *For every $k \in \mathbb{N}$ there exists an ultrafilter $\alpha \in U(X_k)$ such that $S^j\alpha + \alpha = \alpha + S^j\alpha = \alpha$ for each $0 \leqslant j \leqslant k-1$.*

PROOF. We use induction on k. The case $k = 1$ follows from Lemma 2 and the remarks just after it. Indeed, we take the semigroup to be $U(X_1) = U(\mathbb{N})$, with the addition as defined above. So now let us suppose that there exists $\beta \in U(X_{k-1})$ such that $S^j\beta + \beta = \beta + S^j\beta = \beta$ for every $0 \leqslant j \leqslant k-2$. Then, since S is continuous and $U(X_k)$ is compact, $S^{-1}\beta$ is a compact subset of $U(X_k)$. Since addition is right-continuous, the set $S^{-1}\beta + \beta$ is also compact. It is closed under addition, since, if $S\gamma_1 = S\gamma_2 = \beta$, then $S(\gamma_1 + \beta + \gamma_2) = S\gamma_1 + S\beta + S\gamma_2 = \beta + S\beta + \beta = \beta + \beta = \beta$. Therefore, by Lemma 2, there exists $\gamma \in S^{-1}\beta$ such that $(\gamma + \beta) + (\gamma + \beta) = \gamma + \beta$.

To complete the proof, set $\alpha = \beta + \gamma + \beta$. Then, if $1 \leqslant j \leqslant k-1$, we have

$$\alpha + S^j\alpha = \beta + \gamma + \beta + S^j\beta + S^{j-1}\beta + S^j\beta = \beta + \gamma + \beta = \alpha.$$

Similarly, $S^j\alpha + \alpha = \alpha$. Furthermore,

$$\alpha + \alpha = \beta + \gamma + \beta + \beta + \gamma + \beta = \beta + (\gamma + \beta) + (\gamma + \beta) = \beta + (\gamma + \beta) = \alpha.$$

Therefore, α will do.

PROOF OF THEOREM 1. Given a finite sequence $\mathbf{n}_1, \ldots, \mathbf{n}_r$ of elements of X_k, let $\langle \mathbf{n}_1, \ldots, \mathbf{n}_r \rangle$ denote the subspace generated by $\{\mathbf{n}_1, \ldots, \mathbf{n}_r\}$. Let $\alpha \in U(X_k)$ be an ultrafilter of the kind guaranteed to exist by Lemma 3, and let us write Λ for Λ_α. It follows immediately from Lemma 3 that, given any subset $A \subset X_k$, we have $(\Lambda x)x \in A$ iff $(\Lambda x)(\Lambda y)\langle x, y\rangle \subset A$. Given an r-colouring of X_k, let c_s be the unique colour for which $(\Lambda x)x \in c_s$ and set $A_1 = c_s$. By the above remarks, we have $(\Lambda x)(\Lambda y)\langle x, y\rangle \subset A_1$. Pick $x_1 \in X_k$ such that $(\Lambda y)\langle x_1, y\rangle \subset A_1$ and set $A_2 = A_1 \cap \{y: \langle x_1, y\rangle \subset A_1\}$. Then, since α is a filter, $(\Lambda x)x \in A_2$, and this implies that $(\Lambda x)(\Lambda y)\langle x, y\rangle \subset A_2$. Pick x_2 such that $(\Lambda y)\langle x_2, y\rangle \subset A_2$ and note that this implies that $(\Lambda y)\langle x_1, x_2, y\rangle \subset A_1$. Continuing this process, we produce an infinite sequence $x_1, x_2, \ldots$ such that the subspace it generates is contained in A_1; that is, is monochromatic of colour c_s. □

We come now to a 'finite unions version' of Theorem 1. Let us set Y_k to be the set of functions $f: \mathbb{N} \to \{0, 1, \ldots, k\}$ which are finitely supported and take the value k at least once. Given $f \in Y_k$, we write supp(f) for its support. Define a shift operator $T: Y_k \to Y_{k-1}$ by setting $(Tf)(n) = (f(n) - 1) \vee 0$. If $I = \mathbb{N}$ or $[n]$ for some integer n, and $A = \{f_i: i \in I\} \subset Y_k$ is a set of functions with the property that $\max \operatorname{supp}(f_i) < \min \operatorname{supp}(f_j)$ whenever $i < j$, then *the subspace generated by* A is the collection of functions of the form $\sum_{i \in I} T^{r_i} f_i$, where $r_i = k$ in all but finitely many cases and $r_i = 0$ at least once. There is an obvious isomorphism between Y_k and any infinitely generated subspace. Indeed, if the subspace is generated by $f_1, f_2, \ldots$, then the isomorphism is given by

$$(x_1, x_2, \ldots) \mapsto \sum_{i=1}^{\infty} T^{k-x_i} f_i.$$

It will be helpful to identify $\mathbb{N}^{(<\omega)}$ with Y_1 in the obvious way, i.e. by associating each set in $\mathbb{N}^{(<\omega)}$ with its characteristic function.

In order to deduce our next theorem from Theorem 1 we shall need a simple lemma.

LEMMA 4. *Let* $\mathbf{n}_1, \mathbf{n}_2, \ldots$ *be an infinite sequence of elements of* $\mathbb{N}_0^k$. *Then, for every* $M \in \mathbb{N}$, *there exists a finite subset* $A \subset \mathbb{N}$ *such that every co-ordinate of* $\sum_{i \in A} \mathbf{n}_i$ *is divisible by* M.

PROOF. By the pigeonhole principle, there exists integers $t_1, \ldots, t_k$ and an infinite subset $S \subset \mathbb{N}$ such that the jth co-ordinate of $\mathbf{n}_i$ is t_j, modulo 2^m, for every $1 \leq j \leq k$ and $i \in S$. Let A be any subset of S of cardinality 2^m. □

THEOREM 5. *Let* $k, n \in \mathbb{N}$, *and let* Y_k *be coloured with* r *colours* $c_1, \ldots, c_r$. *Then* Y_k *contains a monochromatic subspace generated by an infinite set.*

PROOF. Let $\tilde{\phi}: \mathbb{N} \to \mathbb{N}^{(<\omega)}$ be the usual binary correspondence and let $\phi: X_k \to Y_k$ be defined by

$$\phi: (n_1, \ldots, n_k) \mapsto \max\{k\tilde{\phi}(n_1), (k-1)\tilde{\phi}(n_2), \ldots, \tilde{\phi}(n_k)\}$$

where this is, of course, a pointwise maximum of functions.

Let X_k be as defined before Theorem 1, and let it be coloured by setting the colour of $\mathbf{n}$ to be the colour of $\phi(\mathbf{n})$ in Y_k. By Theorem 1, X_k contains a monochromatic subspace generated by an infinite sequence $\mathbf{n}_1, \mathbf{n}_2, \ldots$. Note that, given any sequence $A_1, A_2, \ldots$ of disjoint finite subsets of $\mathbb{N}$, if we set $\mathbf{n}_i' = \sum_{j \in A_i} \mathbf{n}_j$, then the subspace generated by $\mathbf{n}_1', \mathbf{n}_2', \ldots$ is contained in the subspace generated by $\mathbf{n}_1, \mathbf{n}_2, \ldots$ and hence is monochromatic of the same colour. We shall choose such a subspace inductively as follows. Let $A_1 = 1$, and, having chosen $A_1, \ldots, A_r$, let $\mathbf{n}_r' = \sum_{j \in A_r} \mathbf{n}_j$. Then $\phi(\mathbf{n}_r')$ is certainly supported on $\{1, 2, \ldots, m\}$ for some m. By Lemma 4, we may choose A_{r+1}

so that every co-ordinate of $\mathbf{n}'_{r+1} \equiv \sum_{j \in A_{r+1}} \mathbf{n}_j$ is divisible by 2^m. It is easy to see that $\min \operatorname{supp}(\phi(\mathbf{n}'_{r+1}))$ is therefore greater than m. Setting $f_i = \phi(\mathbf{n}'_i)$, we now have $\max \operatorname{supp}(f_i) > \min \operatorname{supp}(f_j)$ whenever $i < j$. It is also easy to check that the image of the subspace of X_k generated by $\mathbf{n}'_1, \mathbf{n}'_2, \ldots$ is the subspace of Y_k generated by $f_1, f_2, \ldots$ The result follows. □

We are only a short step away from the first of our results about Lipschitz functions on c_0. Let us recall and introduce some definitions concerning normed spaces and bases. If X is a normed space, let $S(X)$ denote its *unit sphere*; that is, the set of vectors in X of norm 1. If $x_1, x_2, \ldots$ is a given basis for X, and $\mathbf{a} \in X$, then $\mathbf{a} = \sum_{i=1}^{\infty} a_i x_i$ for some $a_1, a_2, \ldots$: the support of $\mathbf{a}$, written $\operatorname{supp}(\mathbf{a})$, is defined to be the set $\{i \in \mathbb{N}: a_i \neq 0\}$. A vector $\mathbf{a}$ is said to be *finitely supported* if its support is finite. A *block basis* of $x_1, x_2, \ldots$ is a sequence $y_1, y_2, \ldots$ of finitely supported vectors with the property that $\max \operatorname{supp}(y_i) < \min \operatorname{supp}(y_j)$ whenever $1 \leqslant i < j < \infty$. In particular, the supports of two distinct vectors in a block basis are disjoint. A basis $x_1, x_2, \ldots$ is said to be *normalized* if all the vectors in it are of norm 1. If $\mathbf{a} = \sum_{i=1}^{\infty} a_i x_i \in X$, we write $|\mathbf{a}|$ for $\sum_{i=1}^{\infty} |a_i|\, x_i \in X$. Then $x_1, x_2, \ldots$ is said to be *unconditional* if $|\mathbf{a}| \in X$ whenever $\mathbf{a} \in X$, and *1-unconditional* if $\|\mathbf{a}\| = \||\mathbf{a}|\|$ for every $\mathbf{a} \in X$. Note that the obvious bases of c_0 and l_p $(1 \leqslant p < \infty)$ are normalized and 1-unconditional. If X is a normed space with a given 1-unconditional basis $x_1, x_2, \ldots$ and $F: S(X) \to \mathbb{R}$ is a real-valued function, we shall say that it is *unconditional* if $F(\mathbf{a}) = F(|\mathbf{a}|)$ for every $\mathbf{a} \in S(X)$, and we shall call the set of vectors in $S(X)$ with non-negative co-ordinates the *positive part* of $S(X)$, denoted $PS(X)$. If $\mathbf{a} \in X$, we shall say that it is *positive* if all its co-ordinates (with respect to $x_1, x_2, \ldots$) are non-negative; and if $y_1, y_2, \ldots$ is a block basis of $x_1, x_2, \ldots$ we shall say that it is *positive* if all the vectors in it are positive. Finally, a subspace of X generated by a (positive) block basis will be called a (*positive*) *block subspace.*

Our immediate aim is to show that, given any unconditional Lipschitz function on $S(c_0)$ and $\varepsilon > 0$, there is an infinite-dimensional block subspace on the unit sphere of which it varies by at most ε. Now the condition that the function is unconditional allows us to restrict our attention to $PS(c_0)$. The importance of the collection of functions Y_k that we have been discussing is that there is a natural bijection between Y_k and a δ-net of $PS(c_0)$ (where, of course, δ depends on k) with the property that the subspaces of Y_k, as defined earlier, correspond to δ-nets of the positive parts of positive block subspaces of c_0. This explains our use of the word 'subspace'. To deduce the next theorem from Theorem 5, we have to do little more than exhibit the bijection.

THEOREM 6. *Let F be an unconditional Lipschitz function on $S(c_0)$. Then, for any $\varepsilon > 0$, there exists an infinite-dimensional positive block subspace X of c_0 such that F varies by at most ε on $S(X)$.*

PROOF. Without loss of generality, F has Lipschitz constant 1. Let $\delta = \varepsilon/2$. There is a natural δ-net of $PS(c_0)$; namely, the collection of functions $f: \mathbb{N} \to \{1, (1+\delta)^{-1}, \ldots, (1+\delta)^{-(k-1)}\}$ which are finitely supported and take the value 1 at least once, where k is chosen so that $(1+\delta)^{-(k-1)} < \delta$. Let us write Δ for this collection of functions.

Since F is Lipschitz, there exists an interval $[a, b) \subset \mathbb{R}$ such that $F(S(c_0)) \subset [a, b)$. Let r be such that $a + r\delta \geqslant b$ and let the intervals $I_1, \ldots, I_r$ be defined by $I_j = [a + (j-1)\delta, a + j\delta)$ for each $1 \leqslant j \leqslant r$. Given $f \in \Delta$, let us colour f according to the interval I_j in which $F(f)$ falls.

Now there is an obvious bijection between Δ and Y_k. Indeed, let us define a map $\psi: \Delta \to Y_k$ by

$$(\psi f)(n) = \begin{cases} k + \log_{1+\delta} f(n) & f(n) \neq 0 \\ 0 & f(n) = 0 \end{cases}$$

The colouring on Δ induces a colouring on Y_k. By Theorem 5, Y_k contains a monochromatic subspace in this colouring, generated by an infinite set. This set corresponds to a block basis of c_0, and it is not hard to see that the subspace generated by the set corresponds to a δ-net of the positive part of the unit sphere of the subspace generated by the block basis in c_0. Therefore, since F is unconditional and varies by at most δ on this set, it can vary by at most $2\delta = \varepsilon$ on the whole of the unit sphere of the subspace. This completes the proof of Theorem 6. □

We shall now extend Theorem 6 to arbitrary Lipschitz functions. The proof becomes harder, because, as we commented earlier, the most obvious combinatorial approach does not work, and must be replaced with an 'approximate Ramsey result', rather than an exact one.

To begin with, let us define a new discrete structure, which will be in a natural 1–1 correspondence with a δ-net of the whole of the unit sphere of c_0, rather than just the positive part. Given $k \in \mathbb{N}$, let Z_k be the set of functions $f: \mathbb{N} \to \{-k, -(k-1), \ldots, k\}$ which take the value 0 all but finitely many times and one of the values $\pm k$ at least once. Let the shift T be defined by

$$(Tf)(n) = \operatorname{sign}(f(n))((|f(n)| - 1) \vee 0).$$

If the functions $\{f_i: i \in I\}$ are disjointly supported, then let *the subspace generated by* $\{f_i: i \in I\}$ be defined to be the set of functions of the form

$$\sum_{j=1}^{k} \sum_{i \in A_j} T^{k-j} f_i - \sum_{j=1}^{k} \sum_{i \in B_j} T^{k-j} f_i,$$

where $A_1, \ldots, A_k, B_1, \ldots, B_k$ are all finite and disjoint, and at least one of A_k and B_k is non-empty. Note that the subspace generated by $\{f_i: i \in I\}$ is, as we would wish, a subset of Z_k.

If every colouring of Z_k yielded an infinite monochromatic subspace, then we would be done, by imitating the deduction of Theorem 6 from Theorem 5. However, it does not take long to see that this is not the case. For example, if we colour each function f by the sign of its first non-zero co-ordinate, then f and $-f$ are always coloured differently. Alternatively, if we color f RED if the first and last non-zero co-ordinates have the same sign and BLUE otherwise, then, given any two disjointly supported functions $f, g \in Z_k$, the colours of $f+g$ and $f-g$ are different. This second colouring shows that it is no use dealing with the first one by restricting our attention to colourings for which the colour of f is always the same as the colour of $-f$.

One might think that a small adaptation of the above examples would be enough to produce a Lipschitz function on $S(c_0)$ that varied by at least ε on the unit sphere of any infinite-dimensional subspace, for some $\varepsilon > 0$. However, a little experiment should convince the reader that this is not so. The next result is the 'approximate Ramsey result' which will enable us to show that there is no such function. The proof will take up most of the rest of the paper, and will need several lemmas. We shall need a small piece of notation for the statement. Given a set $A \subset Z_k$, we define $\bar{A}$ to be

$$\{f \in Z_k: (\exists g \in A)\, \|f - g\|_\infty \leq 1\}.$$

THEOREM 7. *Let Z_k be finitely coloured. Then there exists a colour class $A \subset Z_k$ and an infinite subspace $W \subset Z_k$ such that $W \subset \bar{A}$.*

Thus, loosely speaking, the theorem claims that, given any finite colouring of Z_k, there exists a colour A and an infinite subspace of Z_k every point of which is close to a point in A. In order to prove this approximation of a Ramsey result, we shall construct a filter that approximates an ultrafilter, in a sense that will become clear.

The next lemma is in a sense the finite-dimensional version of what we want. It is perhaps surprising that the finite-dimensional version should be useful for providing the infinite-dimensional version, but this seems to be the case. The meanings of the terms *block basis* and *block subspace* in a finite-dimensional context are the obvious ones. Also, if A is a subset of a metric space Y, then A_ε stands for the set $\{y \in Y: d(y, A) \leqslant \varepsilon\}$.

LEMMA 8. *Let $k, n \in \mathbb{N}$ and $\varepsilon > 0$. If $N = N(n, \varepsilon)$ is sufficiently large, then, given any decomposition $S(l_\infty^N) = A_1 \cup \cdots \cup A_k$, there exists a block subspace $X \subset l_\infty^N$ and $1 \leqslant i \leqslant k$ such that* $\dim(X) = n$ *and $X \subset (A_i)_\varepsilon$.*

PROOF. We begin by proving a similar result for a decomposition of $S(c_0)$ in the case $k = 2$. Let $S(c_0) = A \cup B$ and let a Lipschitz function F be defined on $S(c_0)$ by $F(x) = d(x, A)$. By an obvious adaptation from norms to Lipschitz functions of the methods of [2] or [5], one can find, for any $M \in \mathbb{N}$, a block subspace $X = \{x_1, \ldots, x_M\rangle$ of c_0 such that, whenever $x = \sum_{i=1}^M a_i x_i$, $y = \sum_{i=1}^M b_i x_i$ are in $S(X)$ and $|a_i| = |b_i|$ for each i, we have $|F(x) - F(y)| \leqslant \varepsilon/2$. Let us pick such a subspace X and let G be the unconditional Lipschitz function defined on $S(l_\infty^M)$ by

$$G(\mathbf{a}) = \max\left\{F\left(\sum_{i=1}^M a_i' x_i\right): |a_i'| = |a_i| \text{ for each } i\right\}.$$

By the finite version of Theorem 6, which follows easily from the compactness of the unit sphere of l_∞^n for every n (cf., e.g., [7]), we have that if M is large enough then l_∞^M has an n-dimensional block subspace, on the unit sphere of which G varies by at most $\varepsilon/2$. It follows easily that X has a block subspace on which F varies by at most ε. Hence, we must either have $Y \subset A_\varepsilon$ or $Y \subset A^c \subset B$.

This establishes the result for $S(c_0)$ when $k = 2$. By compactness once again, we deduce the lemma as stated for $k = 2$. The general case follows easily. $\square$

Let us say that a subset $A \subset S(c_0)$ is *n-large* if, for every n-dimensional block subspace X of c_0, $A \cap X \neq \emptyset$. We shall call a subset that is n-large for some n *finitely large*. Given a set S of elements of Y_k, Z_k, let $\langle S\rangle$ denote the subspace generated by S, when this is defined, and if $S \subset X$ for a normed space X, let $\langle S\rangle$ denote the unit sphere of the subspace generated by S. We shall suppress set brackets when they appear; so, for example, if $\{x_1, \ldots, x_n\}$ is a subset of c_0, then $\langle x_1, \ldots, x_n\rangle$ denotes the unit sphere of the subspace generated by $\{x_1, \ldots, x_n\}$. Also, if the space X under discussion is c_0 or l_p for $1 \leqslant p < \infty$., let $\mathbf{e}_1, \mathbf{e}_2, \ldots$ be the standard basis of X. Let us write X_n for the set $\langle \mathbf{e}_n, \mathbf{e}_{n+1}, \ldots\rangle$. Given a filter α on $S(c_0)$ or $S(l_p)$, we shall say that it is *cofinite* if, for every $n \in \mathbb{N}$, $X_n \in \alpha$. Note that the sets X_n are all finitely large.

COROLLARY 9. *Let $\beta = \{A_\varepsilon \cap X_n: \varepsilon > 0, A \subset S(c_0)$ is finitely large, $n \in \mathbb{N}\}$. Then β is a filter-base.*

Proof. Since we can restrict our attention to an appropriate X_n, it is clearly enough to show that if A and B are finitely large and $\varepsilon > 0$, then there exist $C \subset S(c_0)$ and $\delta > 0$ such that C is finitely large and $C_\delta \subset A_\varepsilon \cap B_\varepsilon$. Pick n such that A and B are both n-large. Let $N = N(n, \varepsilon/4)$ be as given by Lemma 8 and let X be any n-dimensional block subspace of c_0. Then X is isometric to l_∞^N and $X \subset A_{\varepsilon/2} \cup A^c_{\varepsilon/2}$. By Lemma 8, X has an n-dimensional block subspace Y which is either contained in $A_{3\varepsilon/4}$ or $A^c_{\varepsilon/4}$. Since A is n-large, the former must be the case. But then, since B is also n-large, $B \cap Y \neq \varnothing$. However, X was arbitrary, so $A_{3\varepsilon/4} \cap B$ is N-large. But$(A_{3\varepsilon/4} \cap B)_{\varepsilon/4} \subset A_\varepsilon \cap B_\varepsilon$, so we may set $C = A_{3\varepsilon/4} \cap B$ and $\delta = \varepsilon/4$. □

Corollary 10. *There exists a cofinite filter α on $S(c_0)$ with the following two properties. First, whenever $\bigcup_{i=1}^n A_i = S(c_0)$ and $\varepsilon > 0$, there exists $1 \leqslant i \leqslant n$ such that $(A_i)_\varepsilon$ is in α; and, second, whenever $A \in \alpha$, the set $-A$ is also in α.*

Proof. Let α be a maximal filter on $S(c_0)$ with the following two properties. First, α extends the filter generated by β (and is therefore cofinite); and, second, $-A_\varepsilon \in \alpha$ whenever $A \in \alpha$ and $\varepsilon > 0$. That such a maximal filter exists follows easily from Zorn's lemma. We shall show that α has the first of the two properties in the statement of the theorem in the case $n = 2$. Indeed, suppose that $A \cup B = S(c_0)$ and let $X \subset S(c_0)$ be any 2-dimensional block subspace. We claim that, for every $\delta > 0$, there exists $x \in X$ such that $x \in (A_\delta \cap -A_\delta) \cup (B_\delta \cap -B_\delta)$. This is easy to see. If $X \subset A$ or $X \subset B$ then we are done. Otherwise, since X is connected, $X \cap \bar{A} \cap \bar{B} \neq \varnothing$, and therefore there exists $x \in X \cap A \cap B_\delta$. We are then obviously done, whether $-x$ is in A or B. Thus, the set $E(\delta) = (A_\delta \cap -A_\delta) \cup (B_\delta \cap -B_\delta)$ is 2-large, for every $\varepsilon > 0$. It follows that $E(\delta)_\eta \in \beta$ for every δ, $\eta > 0$ and hence that $E(\delta)_\eta \in \alpha$.

Now suppose that neither A_ε nor B_ε is in α for some $\varepsilon > 0$. Since α is maximal, there must be some $C \in \alpha$ such that

$$(C \cap A_\varepsilon) \cap -(C \cap A_\varepsilon) = \varnothing$$

and some $D \in \alpha$ such that

$$(D \cap B_\varepsilon) \cap -(D \cap B_\varepsilon) = \varnothing.$$

Without loss of generality, $C = D$ since we can replace C and D by their intersection.

It follows that

$$(C \cap -C) \cap ((A_\varepsilon \cap -A_\varepsilon) \cup (B_\varepsilon \cap -B_\varepsilon)) = \varnothing.$$

It is not hard to check that $(A_\varepsilon \cap -A_\varepsilon) \supset (A_{\varepsilon/2} \cap -A_{\varepsilon/2})_{\varepsilon/2}$ and $(B_\varepsilon \cap -B_\varepsilon) \supset (B_{\varepsilon/2} \cap -B_{\varepsilon/2})_{\varepsilon/2}$. It follows that

$$(A_\varepsilon \cap -A_\varepsilon) \cup (B_\varepsilon \cap -B_\varepsilon) \supset ((A_{\varepsilon/2} \cap -A_{\varepsilon/2}) \cup (B_{\varepsilon/2} \cap -B_{\varepsilon/2}))_{\varepsilon/2}.$$

By our earlier remarks, this shows that $(A_\varepsilon \cap -A_\varepsilon) \cup (B_\varepsilon \cap -B_\varepsilon) \in \alpha$. But $C \cap -C \in \alpha$ by hypothesis, so we have contradicted the fact that α was a filter. The result for general n follows easily. □

Given a filter on Z_k, there is an obvious notion of cofiniteness corresponding to the case of filters on $S(c_0)$. We shall say that a filter α is *cofinite* if, for every $n \in \mathbb{N}$, the subspace $\langle k\mathbf{e}_n, k\mathbf{e}_{n+1}, \ldots \rangle$ is in α.

Corollary 11. *For every $k \in \mathbb{N}$ there exists a cofinite filter $\tilde{\alpha}$ on Z_k such that, whenever $\bigcup_{i=1}^n A_i = Z_k$, there exists $1 \leqslant i \leqslant n$ such that $\bar{A}_i \in \tilde{\alpha}$ and whenever $A \in \tilde{\alpha}$, $-A \in \tilde{\alpha}$.*

PROOF. Let $\phi\colon S(c_0)\to Z_k$ *be defined by*

$$(\phi f)(n)=\operatorname{sign}(f(n))(\max\{j\in\mathbb{Z}\colon (1+\delta)^{-(k-j)}\leqslant |f(n)|\}\vee 0).$$

In other words, if one takes the obvious analogue of Δ in the whole of the unit sphere of c_0, the map ϕ 'rounds down' to the nearest point in the net Δ and takes the corresponding point of Z_k.

The filter $\tilde{\alpha}$ is defined by taking $A\subset Z_k$ to be in $\tilde{\alpha}$ iff $\phi^{-1}(A)\in\alpha$, where α is the filter constructed in Corollary 9. Now, if $A\subset Z_k$ and ε is sufficiently small, then $\phi((\phi^{-1}(A))_\varepsilon)=\bar{A}$. Also, if $\bigcup_{i=1}^n A_i=Z_k$, then $\bigcup_{i=1}^n\phi^{-1}(A_i)=S(c_0)$: so, for every $\varepsilon>0$, there exists $1\leqslant i\leqslant n$ such that $(\phi^{-1}(A_i))_\varepsilon$ is in α. It follows that, for some $1\leqslant i\leqslant n$, $\bar{A}_i$ is in $\tilde{\alpha}$. Finally, if $A\subset Z_k$, then $\phi(-\phi^{-1}(A))=-A$, so clearly $-A\in\tilde{\alpha}$ whenever $A\in\tilde{\alpha}$. □

Note some important facts about the filter constructed in Corollary 11. First, the set of filters on Z_k which satisfy the conditions in the corollary is easily seen to be a closed subset of $2^{2^{Z_k}}$, so it is compact. Let us denote it by $V(Z_k)$. Given $j,k\in\mathbb{N}$, $\alpha\in V(Z_j)$ and $\beta\in V(Z_k)$, let us set

$$\alpha+\beta=\{A\subset Z_{j\vee k}\colon (\Lambda_\alpha x)(\Lambda_\beta y)\operatorname{supp}(x)\cap\operatorname{supp}(y)=\varnothing,\ x+y\in A\}.$$

Note that, since α and β are cofinite, $(\Lambda_\alpha x)(\Lambda_\beta y)\operatorname{supp}(x)\cap\operatorname{supp}(y)=\varnothing$. In order to avoid having to write $\operatorname{supp}(x)\cap\operatorname{supp}(y)=\varnothing$, throughout, we shall now adopt the convention that the operation $+$ is only defined on elements of Z_j and Z_k when they are disjointly supported. We shall show that $\alpha+\beta\in V(Z_{j\vee k})$.

To show this, let us assume that $j\leqslant k$. (The case $j\geqslant k$ is similar.) We must show that $\bigcup_{i=1}^n A_i=Z_k$ implies that $\bar{A}_i\in\alpha+\beta$ for some $1\leqslant i\leqslant n$, and that $A\in\alpha+\beta$ implies that $-A\in\alpha+\beta$. It will be convenient to write $N(x,y)$ for the statement $\|y-x\|_\infty\leqslant 1$. (This can be loosely read 'x is near to y'.)

Now let $\bigcup_{i=1}^n A_i=Z_k$ and, for every $x\in Z_j$, $1\leqslant i\leqslant n$, let $A_i^x\subset Z_k$ be $\{y\in Z_k\colon x+y\in A_i\}$. Then, for every $x\in Z_j$, we have $\bigcup_{i=1}^n A_i^x=Z_k$, so at least one $\bar{A}_i^x$ is in β. We can rewrite this statement as

$$(\forall x\in Z_j)(\exists i\in[n])(\Lambda_\beta y)(\exists y')N(y,y'),\qquad x+y'\in A_i.$$

Since $\alpha\in V(Z_j)$, it follows that, for some $1\leqslant i\leqslant n$,

$$(\Lambda_\alpha x)(\exists x')N(x,x')(\Lambda_\beta y)(\exists y')N(y,y'),\qquad x'+y'\in A_i.$$

But $x'+y'\in A_i$ implies that $x+y\in\bar{A}_i$, so for some i we have $(\Lambda_\alpha x)(\Lambda_\beta y)x+y\in\bar{A}_i$.

It is easy to see that if

$$(\Lambda_\alpha x)(\Lambda_\beta y)x+y\in A$$

then

$$(\Lambda_\alpha x)(\Lambda_\beta y)-x-y\in A,$$

which implies that $(\Lambda_\alpha x)(\Lambda_\beta y)x+y\in -A$ whenever $A\in\alpha+\beta$. This completes the verification that $\alpha+\beta\in V(Z_k)$. It is not hard to check also that the map from $V(Z_j)$ to $V(Z_{j\vee k})$ defined by $\alpha\mapsto\alpha+\beta$ is continuous for every $\beta\in Z_k$.

We define a shift operator $T\colon V(Z_k)\to V(Z_{k-1})$ just as before; that is, by setting

$$T(\alpha)=\{A\in Z_{k-1}\colon (\Lambda_\alpha x)Tx\in A\}.$$

It is easy to check that $T(V(Z_k))=V(Z_{k-1})$ and that T is continuous.

We state the next lemma without proof, since the simple facts we have just checked are all that one needs to prove it exactly as we proved Lemma 3.

LEMMA 12. *For every $k \in \mathbb{N}$ there exists a filter $\alpha \in V(Z_k)$ such that $T^j\alpha + \alpha = \alpha + T^j\alpha = \alpha$ for each $0 \leq j \leq k-1$.*

We are now ready to prove Theorem 7. The proof is very similar to that of Theorem 1, but we shall nevertheless give it in some detail.

PROOF OF THEOREM 7. The filter α given by Lemma 12 has the property that, for every $A \in \alpha$ and every $0 \leq j \leq k$,

$$(\Lambda_\alpha x)(\Lambda_\alpha y)x + T^j y \in A \qquad \text{and} \qquad T^j x + y \in A.$$

Since $\alpha \in V(Z_k)$, this also implies that

$$(\Lambda_\alpha x)(\Lambda_\alpha y) \pm x \pm T^j y \in A \qquad \text{and} \qquad \pm T^j(x) \pm y \in A.$$

In other words, $A \in \alpha$ implies that $(\Lambda_\alpha x)(\Lambda_\alpha y)\langle x, y\rangle \subset A$.

Given a finite colouring of Z_k, let $A \subset Z_k$ be the colour such that $\bar{A} \in \alpha$ and let $A_1 = \bar{A}$. Then we have $(\Lambda_\alpha x)(\Lambda_\alpha y)\langle x, y\rangle \subset A_1$. Let x_1 be such that $(\Lambda_\alpha y)\langle x_1, y\rangle \subset A_1$, and let A_2 be the set of $y \in A_1$ for which $\langle x_1, y\rangle \subset A_1$. In general, having picked sets $A_1 \supset \cdots \supset A_n$ in α and vectors $x_1, \ldots, x_n$, we have $(\Lambda_\alpha x)(\Lambda_\alpha y)\langle x, y\rangle \subset A_n$, so pick x_n such that $(\Lambda_\alpha y)\langle x_n, y\rangle \subset A_n$, and let A_{n+1} be the set of y for which $\langle x_n, y\rangle \subset A_n$. Then, certainly, $A_{n+1} \subset A_n$ and $A_{n+1} \in \alpha$. We claim that $\langle x_1, \ldots, x_n\rangle \subset A_1$ for every $n \in \mathbb{N}$. This implies that $\langle x_1, \ldots, x_n\rangle \subset A_1$ for every n, which implies that $\langle x_1, x_2, \ldots\rangle \subset A_1 = \bar{A}$ as desired.

To prove this claim, observe that x_n was chosen so that $(\Lambda_\alpha y)\langle x_n, y\rangle \subset A_n$, so certainly $x_n \in A_n$. But A_n is the set of y such that $\langle x_{n-1}, y\rangle \subset A_{n-1}$, so $\langle x_{n-1}, x_n\rangle \subset A_{n-1}$. Now A_{n-1} is the set of y such that $\langle x_{n-2}, y\rangle \subset A_{n-2}$, so, for every $x \in \langle x_{n-1}, x_n\rangle$, we have $\langle x_{n-2}, x\rangle \subset A_{n-2}$. However,

$$\bigcup_{x \in \langle x_{n-1}, x_n\rangle} \langle x_{n-2}, x\rangle = \langle x_{n-2}, x_{n-1}, x_n\rangle,$$

so we have shown that $\langle x_{n-2}, x_{n-1}, x_n\rangle \subset A_{n-2}$. Continuing in this way, we obtain $\langle x_1, \ldots, x_n\rangle \subset A_1$, which proves the theorem. □

We have essentially proved our main theorem.

THEOREM 13. *Let $\varepsilon > 0$ and let F be any real-valued Lipschitz function on the unit sphere of c_0. Then there is an infinite-dimensional subspace $X \subset c_0$ on the unit sphere of which F varies by at most ε.*

PROOF. Let $\delta = \varepsilon/4$ and let Δ_1 be the set of functions $f: \mathbb{N} \to \{\pm 1, \pm(1+\delta)^{-1}, \ldots, \pm(1+\delta)^{-(k-1)}\}$ which are finitely supported and take one of the values ± 1 at least once, where k is chosen such that $(1+\delta)^{-(k-1)} \leq \delta$. This is a 2δ-net of $S(c_0)$. Let $S(c_0)$ be coloured as in the proof of Theorem 6, and let ψ be the map defined there. Let ψ_1 be the natural bijection between Δ_1 and Z_k, defined by

$$(\psi_1 f)(n) = \operatorname{sign}(f(n))(\psi\,|f|)(n).$$

The colouring on Δ_1 induces a colouring on Z_k. It is now easy to see that Theorem 13 follows directly from Theorem 7. □

We shall now discuss very briefly the distortion problem itself. It is easy to show that a positive answer to it would follow from the existence of certain ultrafilters. For example, suppose that, for every $\varepsilon > 0$, there is a cofinite filter α on the unit sphere of l_p with the following two properties. First, whenever $\bigcup_{i=1}^n A_i = S(l_p)$, at least one of the $(A_i)_\varepsilon$ is in α; and, second, $A \in \alpha$ implies that $(\Lambda_\alpha x)(\Lambda_\alpha y)\langle x, y\rangle \subset A$. Then one can prove that whenever $\bigcup_{i=1}^{n} A_i = S(l_p)$ at least one of the $(A_i)_\varepsilon$ contains an infinite-dimensional block subspace. The proof is almost identical to that of Theorem 7.

It is not hard to see that if there is an ultrafilter α on $S(l_p)$ with the property that $A \in \alpha$ implies, for every $\varepsilon > 0$, that $(\Lambda_\alpha x)(\Lambda_\alpha y)\langle x, y\rangle \subset A_\varepsilon$, then it also gives a positive answer to the distortion problem.

It is worth noting that, given any cofinite ultrafilter α on $S(l_p)$ and any $\varepsilon > 0$, there exist N and $\lambda_1, \ldots, \lambda_N$ such that the ultrafilter $\beta = \lambda_1\alpha + \cdots + \lambda_n\alpha$ satisfies

$$(\Lambda_\beta x)(\Lambda_\beta y)\langle x, y\rangle \subset B_\varepsilon$$

whenever $B \in \beta$. This is essentially a special case of a well known theorem of Krivine [11] (both the statement and the proof).

The difficulty with trying to extend the methods of this paper to the distortion problem proper is that the unit sphere of l_p does not admit a useful semigroup structure in the way that the unit sphere of c_0 does. (To be a little more accurate: the cofinite ultrafilters on the unit sphere of c_0 can be made into a semigroup in a natural way which reflects the addition of disjointly supported vectors). This appears to be a serious difficulty, at least in the absence of more flexible methods of producing 'idempotent' ultrafilters than are known at present. However, the cofinite ultrafilters on $S(l_p)$ do have a structure which reflects many of the properties of l_p, and there is a good chance that an ultrafilter with the weaker property above could be shown to exist, giving a positive answer to the distortion problem.

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References

1. P. Aczel, Quantifiers, games and inductive definitions, in: *Proceedings of the Third Scandinavian Logic Symposium*, S. Kanger (ed.), North-Holland, Amsterdam, pp. 1–14.
2. D. Amir and V. D. Milman, Unconditional and symmetric sets in n-dimensional normed spaces, *Israel J. Math.*, **37** (1980), 3–20.
3. T. J. Carlson, Some unifying principles in Ramsey theory, *Discr. Math.*, **68** (1988), 117–169.
4. T. J. Carlson and S. G. Simpson, A dual form of Ramsey's theorem, *Adv. Math.*, **53** (1984), 265–290.
5. J. Elton, On the existence of finite-dimensional subspaces with good unconditional basis, in: *Texas Functional Analysis Seminar*, 1982–3, Longhorn Notes, pp. 139–142.
6. H. Furstenberg and Y. Katznelson, Idempotents in compact semigroups and Ramsey theory, *Israel J. Math.*, **68** (1989), 257–270.
7. R. L. Graham, B. L. Rothschild and J. H. Spencer, *Ramsey Theory*, John Wiley, New York, 1980.
8. N. Hindman, Finite sums from sequences within cells of a partition of $\mathbb{N}$, *J. Combin. Theory, Ser. A*, **17** (1974), 1–11.
9. N. Hindman, Ultrafilters and combinatorial number theory, in: *Number Theory Carbondale* 1979, M. Nathanson (ed.), LNM 751, Springer-Verlag, Berlin, 1979, pp. 119–184.
10. R. C. James, Uniformly non-square Banach spaces, *Ann. Math.*, **80** (1964), 542–550.
11. J. L. Krivine, Sous-espaces de dimension finie des espaces de Banach réticulés, *Ann. Math.*, **104** (1976), 1–29.

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