Abstract elementary classes and infinitary logics

David W. Kueker

Department of Mathematics, University of Maryland, College Park, MD 20742, USA

1. Introduction

In this paper we prove various results relating abstract elementary classes to infinitary logics. In Section 2 we prove that if \((\mathcal{K}, \prec_{\mathcal{K}})\) is an a.e.c. with Löwenheim–Skolem number \(\kappa\) then \(\mathcal{K}\) is closed under \(L_{\kappa, \kappa^+}\)-elementary equivalence. If \(\kappa = \omega\) and \((\mathcal{K}, \prec_{\mathcal{K}})\) has finite character then \(\mathcal{K}\) is closed under \(L_{\infty, \omega}\)-elementary equivalence. Analogous results are established for \(\prec_{\mathcal{K}}\). Galois types, saturation, and categoricity are also studied. We prove, for example, that if \((\mathcal{K}, \prec_{\mathcal{K}})\) is finitary and \(\lambda\)-categorical for some infinite \(\lambda\) then there is some \(\sigma \in L_{\omega_1, \omega}\) such that \(\mathcal{K}\) and \(\text{Mod}(\sigma)\) contain precisely the same models of cardinality at least \(\lambda\).

© 2008 Elsevier B.V. All rights reserved.
A1 (Closure under isomorphism) If $\mathcal{M} \in \kappa$ and $\mathcal{M} \cong \mathcal{N}$ then $\mathcal{N} \in \kappa$; if $\mathcal{M}, \mathcal{N} \in \kappa$, $\mathcal{N} \subseteq \mathcal{M}$ and $(\mathcal{N}, \mathcal{M}) \cong (\mathcal{N}', \mathcal{M}')$ then $\mathcal{M}' \subseteq \mathcal{N}'$.

A2 (- is a strong substructure relation) If $\mathcal{M} \subseteq \mathcal{N}$ then $\mathcal{M} \subseteq \mathcal{N}$; if $\mathcal{M} \in \kappa$ then $\mathcal{M} \subseteq \mathcal{M}$; if $\mathcal{M}_0 \subseteq \mathcal{M}_1$ and $\mathcal{M}_1 \subseteq \mathcal{M}_2$ then $\mathcal{M}_0 \subseteq \mathcal{M}_2$.

A3 (Existence of Löwenheim–Skolem number) There is an infinite cardinal $\lambda(\kappa)$ such that whenever $\mathcal{M} \in \kappa$ and $\mathcal{N} \subseteq \mathcal{M}$ then there is some $\mathcal{M}' \subseteq \mathcal{M}$ such that $\mathcal{A} \subseteq \mathcal{M}'$ and $|\mathcal{M}'| \leq \max(|\mathcal{A}|, \lambda(\kappa))$.

A4 (Closure under unions of $\subseteq$ chains) Let $\{\mathcal{M}_i : i \in \mu\}$ be a chain of models in $\kappa$ under $\subseteq$ and $\mathcal{N} = \bigcup_{i \in \mu} \mathcal{M}_i$. Then:

(i) $\mathcal{N} \in \kappa$
(ii) $\mathcal{M}_j \subseteq \mathcal{N}$ for all $j \in \mu$
(iii) If $\mathcal{M}_j \subseteq \mathcal{M}_\alpha$ for all $j \in \mu$ then $\mathcal{N} \subseteq \mathcal{M}_\alpha$.

A5 (Coherence) Let $\mathcal{M}_0, \mathcal{M}_1, \mathcal{N} \in \kappa$ and assume that $\mathcal{M}_0 \subseteq \mathcal{M}_1$ and $\mathcal{M}_0, \mathcal{M}_1 \subseteq \mathcal{N}$. Then $\mathcal{M}_0 \subseteq \mathcal{M}_1$.

Note that $\subseteq$ is defined only on structures in $\kappa$, and therefore $\mathcal{M} \subseteq \mathcal{N}$ implies that both $\mathcal{M}$ and $\mathcal{N}$ are in $\kappa$. Thus in statements of results we do not always explicitly state that a structure is in $\kappa$ if this follows from the other assertions. Note also that $(\mathcal{N}, \mathcal{M})$ is the expansion of $\mathcal{N}$ to a structure for the vocabulary with a new unary predicate symbol which is interpreted by (the universe of) $\mathcal{M}$.

We will use the following result which states that the closure properties stated in A4 for unions of chains hold also for unions of directed families.

Lemma 2.1 ([12]). Let $(\kappa, \subseteq)$ be an a.e. and let $S$ be a set of models in $\kappa$ which is directed under $\subseteq$ — that is, for any $\mathcal{M}_0, \mathcal{M}_1 \subseteq S$ there is some $\mathcal{M}_2 \in S$ such that $\mathcal{M}_0, \mathcal{M}_1 \subseteq \mathcal{M}_2$. Let $\mathcal{N} = \bigcup S$. Then:

(a) $\mathcal{N} \in \kappa$
(b) $\mathcal{M} \subseteq \mathcal{N}$ for all $\mathcal{M} \in S$
(c) If $\mathcal{M} \subseteq \mathcal{M}'$ for all $\mathcal{M} \in S$ then $\mathcal{N} \subseteq \mathcal{M}'$.

Note that by the coherence axiom the hypothesis that $S$ is directed under $\subseteq$ implies that $\mathcal{M}_0 \subseteq \mathcal{M}_1$ whenever $\mathcal{M}_0 \subseteq \mathcal{M}_1$ and $\mathcal{M}_0, \mathcal{M}_1 \subseteq \mathcal{S}$.

We will also use some of the material from [9,11] on countable approximations and the closed unbounded filter on them. We briefly review the main definitions and results here.

For any set $\mathcal{C}$, $\mathcal{M}'$ is the substructure of $\mathcal{M}$ generated by $(\mathcal{M} \cap \mathcal{S})$. If $\mathcal{S}$ is countable then $\mathcal{M}'$ is called a countable approximation to $\mathcal{M}$. (Countable approximations $\varphi^\mathcal{M}$ are defined recursively [see [9,11] for details]. For any set $\mathcal{C}$ we define a filter $\mathcal{P}_{\omega}(\mathcal{C})$ on the set $\mathcal{P}_{\omega}(\mathcal{C})$ of all countable subsets of $\mathcal{C}$ which gives a notion of almost all countable $\mathcal{C} \subseteq \mathcal{C}$.

Definition 1.3. Let $X \subseteq \mathcal{P}_{\omega}(\mathcal{C})$.
(a) $X$ is closed if $X$ is closed under unions of countable chains.
(b) $X$ is unbounded if for every $\mathcal{S}_0 \in \mathcal{P}_{\omega}(\mathcal{C})$ there is some $\mathcal{S} \in X$ such that $\mathcal{S}_0 \subseteq X$.

Definition 1.4. $\mathcal{D}_{\omega}(\mathcal{C})$ is the set of all $X \subseteq \mathcal{P}_{\omega}(\mathcal{C})$ such that $X$ contains some closed unbounded subset.

Lemma 1.5 ([6,9,11]). $\mathcal{D}_{\omega}(\mathcal{C})$ is a countably complete filter.

We will need the following game characterization of these filters.

Fix $\mathcal{C}$. For any $X \subseteq \mathcal{P}_{\omega}(\mathcal{C})$ we define the infinite two person game $G(\mathcal{C})$ as follows: at each stage $n$ player I chooses some $\mathcal{A}_n \in \mathcal{C}$ and player II responds by choosing some $\mathcal{B}_n \in \mathcal{C}$. Player II wins if $\{\mathcal{A}_n : n \in \omega\} \cap X$.

Theorem 1.6 ([9,11]). Let $X \subseteq \mathcal{P}_{\omega}(\mathcal{C})$. Then $X \in \mathcal{D}_{\omega}(\mathcal{C})$ if and only if $X$ has a winning strategy in the game $G(\mathcal{C})$.

A set $\mathcal{C}$ is large enough to approximate $\mathcal{M}$ if $\mathcal{M} \subseteq \mathcal{C}$. A similar definition is given for formulas. We say that a property of countable approximations to one or more structures and/or formulas holds almost everywhere (a.e.) if it holds for all $X \subseteq \mathcal{C}$ for some $X \in \mathcal{D}_{\omega}(\mathcal{C})$ where $\mathcal{C}$ is large enough to approximate each of the structures and formulas involved. This is independent of the choice of $\mathcal{C}$.

Theorem 1.7 ([9,11]). (a) For any $\mathcal{M}$ and $\sigma \in L_{\omega,\omega}, \mathcal{M} \models \sigma$ if $\mathcal{M} \models \sigma^\mathcal{C}$ a.e.
(b) For any $\mathcal{M}$ and $\mathcal{N}, \mathcal{M} \equiv_{\omega,\omega} \mathcal{N}$ if $\mathcal{M} \models \sigma^\mathcal{N}$ a.e.

We also require a filter $\mathcal{D}_{\omega}(\mathcal{C})$ on $\mathcal{P}_{\omega}(\mathcal{C})$ which will yield a corresponding notion of $\kappa$-a.e. for uncountable $\kappa$. There are three possible candidates. Although we will choose one of them for our definition, a few proofs will also involve the other concepts. The first candidate is the obvious generalization of closed unbounded sets.

Definition 1.8. Let $X \subseteq \mathcal{P}_{\omega}(\mathcal{C}), X$ is $\kappa$-club if $X$ is closed under unions of chains of length at most $\kappa$ and for every $\mathcal{S}_0 \in \mathcal{P}_{\omega}(\mathcal{C})$ there is some $\mathcal{S} \in X$ such that $\mathcal{S}_0 \subseteq \mathcal{S}$.

Note that the case $\kappa = \omega$ of this definition is exactly closed unbounded as in Definition 1.3. Unfortunately the filter generated by the $\kappa$-club sets does not usually have a game characterization like Theorem 1.6, which some proofs require. The filter which gives our definition of $\kappa$-a.e. will be defined in terms of the appropriate game.

Fix a set $\mathcal{C}$. For any $X \subseteq \mathcal{P}_{\omega}(\mathcal{C})$ we define a two person game $G(\mathcal{C})$, with $\omega$ moves, as follows: at each stage $n$ player I chooses some $\mathcal{S}_n \in \mathcal{P}_{\omega}(\mathcal{C})$ and player II responds by choosing some $\mathcal{S}_{n+1} \in \mathcal{P}_{\omega}(\mathcal{C})$. Player II wins if $\{\mathcal{S}_n : n \in \omega\} \in X$. 


275
Definition 1.9. $D_κ^+(C)$ is the set of all $X \subseteq P_κ^+(C)$ such that player II has a winning strategy in the game $G_κ^+(X)$.

The proof of Theorem 1.6 shows that the case $κ = ω$ of this definition agrees with the earlier Definition 1.4. Note that if $X \subseteq P_κ^+(C)$ is $κ$-club then $X \in D_κ^+(C)$, but the converse does not hold. Note also that $D_κ^+(C)$ is not the filter defined in section 7 of [11], which is the third candidate. We will refer to this last filter as $D_κ^*(C)$ (although the notation used in [11] does not have the star) and use the fact that every $κ$-club set belongs also to $D_κ^*(C)$.

We use $κ$-a.e. for the notion of almost all defined using the filter $D_κ^+(C)$. The following generalization of Theorem 1.7 is proved just like that result.

Theorem 1.10. (a) For any $M$ and $σ ∈ L_{∞,ω}$, $M \models σ$ iff $M \models σ^+$ $κ$-a.e.
(b) For any $M$ and $N$, $M \equiv_{∞,ω} N$ iff $M \equiv_{∞,ω} N^+$ $κ$-a.e.

We assume familiarity with infinitary logics — for more detail, especially on the connection to partial isomorphisms and the back-and-forth method, see [10].

Recall that formulas of $L_{∞,ω}$ are restricted to have just finitely many free variables. This restriction leads to the following definition of $L_{∞,ω}$-equivalence of infinite sequences.

Definition 1.11. Given $M, N$ let $\vec{a}, \vec{b}$ be sequences of the same length from $M, N$ respectively. Then $(M, \vec{a}) \equiv_{∞,ω} (N, \vec{b})$ iff for every $n ∈ ω$, every $ϕ(x_0, \ldots, x_{n-1}) ∈ L_{∞,ω}$ and every $i_0 < \cdots < i_{n-1} < lh(\vec{a}), M \models ϕ(a_{i_0}, \ldots, a_{i_{n-1}})$ iff $N \models ϕ(b_{i_0}, \ldots, b_{i_{n-1}})$.

In particular, $(M, \vec{a}) \equiv_{∞,ω} (N, \vec{b})$ holds iff $(M, \vec{a} \upharpoonright i) \equiv_{∞,ω} (N, \vec{b} \upharpoonright i)$ holds for all finite $i \subseteq lh(\vec{a})$. Further, the extension property normally fails for infinite sequences — that is, if $(M, \vec{a}) \equiv_{∞,ω} (N, \vec{b})$ and $a' ∈ M$ there need not be $b' ∈ N$ such that $(M, \vec{a}a') \equiv_{∞,ω} (N, \vec{b}b')$.

For some purposes we will want to extend $L_{∞,ω}$ to allow formulas with countably many free variables. We use $L^*_{∞,ω}$ for the logic defined exactly like $L_{∞,ω}$, but allowing formulas to have countably many free variables. Note that $L^*_{∞,ω}$ adds no new sentences to $L_{∞,ω}$, $(M, \vec{a}) \equiv_{∞,ω} (N, \vec{b})$ is defined as above except using formulas of $L^*_{∞,ω}$, which may have countably many free variables. Thus, unlike the first notion, this is not determined by restriction to finite subsequences. When $\vec{a}, \vec{b}$ are countable sequences the extension property holds — that is, if $(M, \vec{a}) \equiv^{*}_{∞,ω} (N, \vec{b})$ and $a' ∈ M$ then there is a $b' ∈ N$ such that $(M, \vec{a}a') \equiv^{*}_{∞,ω} (N, \vec{b}b')$. Note also that if $(M, \vec{a}) \equiv^{*}_{∞,ω} (N, \vec{b})$ then $(M, \vec{a}) \equiv_{∞,ω} (N, \vec{b})$.

We also consider extensions of $L_{∞,ω}$ and $L^*_{∞,ω}$ allowing game quantifiers. We refer to [8] for a detailed account of such quantifiers.

A game quantifier, of length $ω$, is an infinite string $Q_0x_0Q_1x_1 \cdots Q_nx_n \cdots$ where, for each $n ∈ ω$, $Q_n$ is either $∀$ or $∃$. We may assume the quantifiers alternate.

Given a formula $ϕ(\vec{x}, \vec{y})$ where $\vec{x}$ is an $ω$-sequence and $\vec{y}$ is either finite or an $ω$-sequence, we consider the game formula $θ(\vec{y})$ defined by

∀x_0∃x_1 \cdots ∀x_N∃x_{N+1} \cdots ϕ(\vec{x}, \vec{y}).

Given $M$ and $\vec{b}$ from $M$ with the same length as $\vec{y}$, we define $M \models θ(\vec{b})$ by a game: players I and II alternately choose elements of $M$, generating an $ω$-sequence $\vec{a}$; player II wins iff $M \models ϕ(\vec{a}, \vec{b})$; and $M \models θ(\vec{b})$ iff player II has a winning strategy in this game.

Due to Theorem 1.6, formulas with game quantifiers can be used to say that almost all countable submodels of a model have a certain property — see [11] for more detail.

The logic $L(ω)$, extending $L_{∞,ω}$, was defined by Keisler in [8].

Definition 1.12 ([8]). The set of formulas of $L(ω)$ is defined as follows:

(i) Every atomic formula of $L$ belongs to $L(ω)$.
(ii) If $ϕ(ω)$ then $¬ϕ$ is in $L(ω)$.
(iii) If $ϕ(ω)$ is a set then $ϕ \upharpoonright ω \subseteq L(ω)$.
(iv) If $ϕ(ω)$ then $∀xϕ, ∃xϕ, ϕ^x$ is in $L(ω)$ provided they have just finitely many free variables.
(v) If $ϕ(ω)$ then $Q_nϕ(x_0)\cdots ϕ(x_n)$ in $L(ω)$ provided it has just finitely many free variables.

Note that $L_{∞,ω} ⊆ L(ω)$, and formulas of $L(ω)$ may have infinitely many free variables but they are all boolean combinations of formulas with just finitely many free variables. By convention we will restrict the formulas of $L(ω)$ to have just countably many free variables. Keisler proved that formulas of $L(ω)$ are preserved by $L_{∞,ω}$-elementary equivalence.

Theorem 1.13 ([8]). (a) Given $M, N$ let $\vec{a}, \vec{b}$ be $ω$-sequences from $M, N$ respectively. Assume that $(M, \vec{a}) \equiv_{∞,ω} (N, \vec{b})$. Then for every $ϕ(\vec{x}) ∈ L(ω)$, where $\vec{x}$ is an $ω$-sequence, we have $M \models ϕ(\vec{a})$ iff $N \models ϕ(\vec{b})$.

(b) If $M \equiv_{∞,ω} N$ then $M \equiv_{L(ω)} N$.

We define $L^*(ω)$ extending $L^*_{∞,ω}$ just like $L(ω)$ except that in clauses (iv) and (v) the resulting formula is allowed to have countably many free variables and in (v) $ϕ$ is restricted to be a boolean combination of formulas with just finitely many free variables. By convention we restrict the formulas of $L^*(ω)$ to have just countably many free variables. Note that every sentence of $L^*(ω)$ belongs to $L(ω)$.

The following result may be proved like Theorem 1.13(a).

Theorem 1.14. Given $M, N$ let $\vec{a}, \vec{b}$ be $ω$-sequences from $M, N$ respectively. Assume that $(M, \vec{a}) \equiv^*_{∞,ω} (N, \vec{b})$. Then for every $ϕ(\vec{x}) ∈ L^*(ω)$, where $\vec{x}$ is an $ω$-sequence, we have $M \models ϕ(\vec{a})$ iff $N \models ϕ(\vec{b})$.
2. Löwenheim-Skolem Number $\omega$

Until further notice we assume the following:

\[(\mathbb{K}, <_{\mathbb{K}})\] is an a.e.c. in a countable vocabulary and $LS(\mathbb{K}) = \omega$.

The main result of this section, Theorem 2.5, states that $\mathbb{K}$ is closed under $L_{\omega_1,1}$-elementary equivalence. We also obtain (Theorem 2.11) a sufficient condition for $\mathbb{K}$ to be $L_{\omega_1,1}$-axiomatizable. A simple example (Example 2.10) shows that $\mathbb{K}$ need not be closed under $L_{\omega_1,1}$-elementary equivalence. We prove Theorem 2.5 by showing that if $M \in \mathbb{K}$ and $M \equiv_{\omega_1,1} N$ then $N$ is the union of a family of countable structures in $\mathbb{K}$ which is directed under $<_{\mathbb{K}}$, and so $N \subseteq \mathbb{K}$ by Lemma 1.2(a).

In fact, $\mathbb{K}$ will be definable by a sentence using game quantification (Theorem 2.9), which has Theorem 2.5 as a consequence. The sentence axiomatizing $\mathbb{K}$ will also imply that its models are unions of (definable) families of countable models in $\mathbb{K}$ which are directed under $<_{\mathbb{K}}$.

We will make heavy use of countable approximations in the proof of Theorem 2.5 and of most other results in this paper. We first prove the following easy consequence of the definitions. Note that $N^2 <_{\mathbb{K}} N$ implies that $N^2 \subseteq \mathbb{K}$ by the remark following Definition 1.1.

**Lemma 2.1.** If $N \in \mathbb{K}$ then $N^2 <_{\mathbb{K}} N$ a.e.

**Proof.** Assume $N \in \mathbb{K}$ and let $X = \{ s \in P_{\omega_1}(N) : N^2 s <_{\mathbb{K}} N \}$. Then $X$ is unbounded since $LS(\mathbb{K}) = \omega$. $X$ is closed by the chains and coherence axioms. Thus $X \subseteq D_{\omega_1}(N)$ and so $N^2 s <_{\mathbb{K}} N$ a.e. since $C = N$ is large enough to approximate $N$. \(\square\)

More generally, if $N \in \mathbb{K}$ and $|N| > \lambda$ then $X = \{ s \in P_{\lambda^+}(N) : N^2 s <_{\mathbb{K}} N \}$ is a.e.-club, hence belongs both to $D_{\lambda^+}(N)$ and to $D'_{\lambda^+}(N)$.

We can now characterize $M <_{\mathbb{K}} N$, where $M$ is countable, using countable approximations to $N$.

**Lemma 2.2.** Assume $N \in \mathbb{K}$ and let $M \subseteq N$ be countable. Then $M <_{\mathbb{K}} N$ if and only if $M <_{\mathbb{K}} N^2$ a.e.

**Proof.** First assume $M <_{\mathbb{K}} N$. By Lemma 2.1, $N^2 <_{\mathbb{K}} N$ a.e. But clearly $M \subseteq N^2$ a.e. and so $M <_{\mathbb{K}} N^2$ a.e. by coherence. Conversely, if $M <_{\mathbb{K}} N^2$ a.e. then $M <_{\mathbb{K}} N$ since, by Lemma 2.1 again, $N^2 <_{\mathbb{K}} N$ a.e. \(\square\)

In the rest of this section $\bar{x}, \bar{a},$ etc. are used exclusively for $\omega$-sequences, and $\text{ran}(\bar{a}) = \{ a_n : n \in \omega \}$.

The following lemma is the key step in the proof of Theorem 2.5. Note that we do not assume that $\mathbb{K} \subseteq \mathbb{K}$.

**Lemma 2.3.** Let $N \in \mathbb{K}$, $M_0 <_{\mathbb{K}} M$ be countable, and assume $\bar{a}$ is an $\omega$-sequence with $\text{ran}(\bar{a}) = M_0$. Let $N$ be arbitrary, let $\bar{b}$ be an $\omega$-sequence from $N$, and assume that $(M, \bar{a}) \equiv_{\omega, \omega} (N, \bar{b})$. Then $\text{ran}(\bar{b}) = N_0$ where $N_0 <_{\mathbb{K}} N^2$ a.e.

**Proof.** First note that $\text{ran}(\bar{b}) = N_0$ where $N_0 \subseteq N$ and $f : M_0 \equiv N_0$ where $f(a_i) = b_i$ for all $i \in \omega$. Without loss of generality we may assume $a_i = b_i$ for all $i \in \omega$, so $M_0 \equiv N_0$. Thus our hypothesis is that $N_0 <_{\mathbb{K}} M$ and $(M, \bar{a}) \equiv_{\omega, \omega} (N, \bar{b})$.

Let $Y = \{ s \in P_{\omega_1}(N) : N_0 <_{\mathbb{K}} N^2 s \equiv s \}$. We will show that $\text{I}_Y$, player II in the game $G(Y)$, has a winning strategy. This will imply $Y \subseteq D_{\omega_1}(N)$ by Theorem 1.6 and hence that $N_0 <_{\mathbb{K}} N^2$ a.e., since $C = N$ is large enough to approximate $N$.

Similarly, defining $X = \{ s \in P_{\omega_1}(M) : N_0 <_{\mathbb{K}} M^2 s \equiv s \}$, we know that $\text{I}_X$, player II in the game $G(X)$, has a winning strategy, since $N_0 <_{\mathbb{K}} M^2$ a.e. by Lemma 2.2. \(\square\)

We use the winning strategy of $\text{I}_X$ and the back-and-forth properties of $L_{\omega_1,1}$-elementary equivalence to define a winning strategy for $\text{I}_Y$.

In the first round of $G(Y)$ suppose that $\text{I}_Y$ chooses $d_0 \in N$. We first pick $c_0 \in M$ such that $(M, \bar{c}_0) \equiv_{\omega, \omega} (N, \bar{b}d_0)$. We next have $\text{I}_X$ choose $c_0$ in the game $G(X)$; using his winning strategy, $\text{I}_X$ chooses $c_1$. We now pick $d_1 \in N$ such that $(M, \bar{b}c_0c_1) \equiv_{\omega} (N, \bar{b}d_0d_1)$. Finally, the strategy of $\text{I}_Y$ is to choose $d_i$ as the response to $\text{I}_X$’s choice of $d_0$.

Continuing in this way we obtain $\omega$-sequences $\bar{c}$ from $M$ and $\bar{d}$ from $N$ satisfying the following:

1. $(\bar{M}, \bar{b}c) \equiv_{\omega, \omega} (N, \bar{b}d)$
2. $\text{ran}(\bar{c}) = s_0 \in X$, hence $N_0 <_{\mathbb{K}} M^{s_0}$ and $M^{s_0} = s_0$, in particular $\text{ran}(\bar{b}) = N_0 \subseteq s_0 = \text{ran}(\bar{c})$.

Define $s_1 = \text{ran}(\bar{d})$, we claim that $s_1 \in Y$, which will complete the proof.

First, define $g : M^{s_1} \to N$ by $g(c_i) = d_i$ for all $i \in \omega$. By (1) and (2) this is an isomorphism of $M^{s_0}$ onto $N^{s_1} \subseteq N$ where $N^{s_1} = s_1$. Similarly, $g(b_i) = b_i$ for all $i \in \omega$, so $g : (M^{s_0}, N_0) \cong (N^{s_1}, N_0)$. Since $<_{\mathbb{K}}$ is preserved by isomorphism we conclude that $N_0 <_{\mathbb{K}} N^{s_1}$ and thus $s_1 \in Y$. \(\square\)

As a consequence note that if $N \in \mathbb{K}$ then we may conclude $N_0 <_{\mathbb{K}} N$ by Lemma 2.2. Using Lemma 2.3 we can establish the following.

**Lemma 2.4.** Let $M \in \mathbb{K}$ and assume that $M \equiv_{\omega, \omega} N$. Let $B_0 \subseteq N$ be countable. Then there is some countable $N_0 \subseteq N$ such that $B_0 \subseteq N_0$ and $N_0 <_{\mathbb{K}} N^2$ a.e.

**Proof.** Let $B_0 = \{ b_{2n} : n \in \omega \}$. Since $M \equiv_{\omega, \omega} N$ we can find $a_{2n} \in M$ for all $n \in \omega$ such that $(M, (a_{2n}))_{n \in \omega} \equiv_{\omega, \omega} (N, (b_{2n}))_{n \in \omega}$. Since $LS(\mathbb{K}) = \omega$ we can pick a countable $M_0$ such that $(a_{2n} : n \in \omega) \subseteq M_0$ and $M_0 <_{\mathbb{K}} M$. Choose $a_{2n+1}$ for $n \in \omega$ such that $M_{0} = \{ a_k : k \in \omega \}$ and find $b_{2n+1} \in N$ for all $n \in \omega$ so that (re-arranging the $b_k$’s and $a_k$’s into $\omega$-sequences) we have $(\bar{M}, \bar{a}) \equiv_{\omega, \omega} (N, \bar{b})$. In particular, $(\bar{M}, \bar{a}) \equiv_{\omega} (N, \bar{b})$, hence $\text{ran}(\bar{b}) = N_0$ is as desired, by Lemma 2.3. \(\square\)


277
We now easily obtain the desired theorem.

**Theorem 2.5.** Let $\mathcal{M} \in \mathbb{K}$ and assume that $\mathcal{M} \equiv_{\infty \omega} \mathcal{N}$. Then $\mathcal{N} \in \mathbb{K}$.

**Proof.** Define $S$ to be $\{ N_0 \subseteq \mathcal{N} : N_0$ is countable and $N_0 \not\prec \mathcal{N}^3 \text{ a.e.}\}$. We first note that if $N_0, N_1 \subseteq S$ and $N_0 \subseteq N_1$ then $N_0 \not\prec \mathcal{N}, N_1$ by coherence, since there will be some $N^* \subseteq N$ such that both $N_0 \not\prec \mathcal{N}^* , N_1 \not\prec \mathcal{N}^*$. Secondly, by Lemma 2.4, $S$ is non-empty, $\mathcal{N} = \bigcup S$, and $S$ is directed under $\subseteq$. But, by our first remark, $S$ will then be directed under $\not\prec$ and so $\mathcal{N} \in \mathbb{K}$ by Lemma 1.2(a).

We also obtain the following result asserting that $\not\prec$ is preserved under $L^*_{\infty \omega}$-elementary equivalence generalizing the remark following Lemma 2.3.

**Theorem 2.6.** Let $\mathcal{M}_0, \mathcal{M}, \mathcal{N} \in \mathbb{K}$ and assume that $\mathcal{M}_0 \not\prec \mathcal{M}$. Assume also that $h : \mathcal{M}_0 \rightarrow \mathcal{N}$ is $L^*_{\infty \omega}$-elementary as a map of $\mathcal{M}$ (not $\mathcal{M}_0$) to $\mathcal{N}$. Then for every $\mathcal{N}$-sequence $\bar{a}$ from $\mathcal{M}_0$, $(\mathcal{M}, \bar{a}) \equiv_{\infty \omega} (\mathcal{N}, h(\bar{a}))$. Then $h$ is a $\mathbb{K}$-embedding — that is $h : \mathcal{M}_0 \equiv \mathcal{N}_0$, where $\mathcal{N}_0 \not\prec \mathcal{N}$. Proof. If $\mathcal{M}_0$ is countable this is immediate from Lemmas 2.2 and 2.3. If $\mathcal{M}_0$ is not countable, it is the union of a family of countable $\mathbb{K}$-substructures directed under $\not\prec$. By the first case, the images under $h$ of the models in this family are all $\mathbb{K}$-structures of $\mathcal{N}$, are therefore directed under $\not\prec$ (by coherence), and their union is $\mathcal{N}_0 = h(\mathcal{M}_0)$. Therefore $\mathcal{N}_0 \not\prec \mathcal{N}$ by Lemma 1.2(c).

**Corollary 2.7.** Assume that $\mathcal{M}, \mathcal{N} \in \mathbb{K}$ and $\not\prec_{L^*_{\infty \omega}} \mathcal{N}$. Then $\mathcal{M} \not\prec \mathcal{N}.

**Proof.** Let $\mathcal{M}_0 = \mathcal{M}$ and $h = id_{\mathcal{M}}$ in the preceding theorem.

We briefly describe how to axiomatize $\mathbb{K}$. Most of the work comes in showing the following Lemma.

**Lemma 2.8.** There is a formula $\varphi(x) \in L^*(\omega)$ such that for every $\mathcal{N}$ and every $\omega$-sequence $\bar{a}$ from $\mathcal{N}$, $\mathcal{N} \models \varphi(\bar{a})$ iff $\text{ran}(\bar{a}) = \mathcal{M}_0$ for some $\mathcal{M}_0$ such that $\mathcal{M}_0 \not\prec \mathcal{N}$. Proof (outline). We first claim that there is a quantifier-free formula $\delta(x, y) \in L^*(\omega)$ such that for every $\mathcal{N}$ and $\omega$-sequences $\bar{a}, \bar{b}$ from $\mathcal{N}$, $\mathcal{N} \models \delta(\bar{a}, \bar{b})$ iff $\text{ran}(\bar{a}) = \mathcal{M}_0$ and $\text{ran}(\bar{b}) = \mathcal{N}_0$ where $\mathcal{M}_0 \not\prec \mathcal{N}_0$.

Let $\{(N_i, \mathcal{M}_i) : i \in I\}$ list, up to isomorphism, all pairs of countable models in $\mathbb{K}$ such that $\mathcal{M}_i \not\prec \mathcal{N}_i$. For each $i \in I$, list $\{(\bar{a}_i, j) : j \in J\}$ and $\{(\bar{b}_i, k) : k \in K\}$ list all $\omega$-sequences whose ranges are $\mathcal{M}_i, \mathcal{N}_i$ respectively. For each $i, j \in I$, and $k \in K$, let $\delta_{i,j,k}(\bar{x}, \bar{y})$ be the conjunction of all atomic and negated atomic formulas satisfied in $\mathcal{N}$ by $\bar{a}_i, \bar{b}_j$. Then $\delta(\bar{x}, \bar{y}) = \bigvee_{i, j, k} \delta_{i,j,k}$. Finally, we define $\varphi(x)$ to be $\bigwedge_{i \in I} \exists y_0 \exists y_1 \ldots \exists y_{2^n} \exists x_{2^n+1} \ldots \delta(\bar{x}, \bar{y})$, which is as desired by Theorem 2.3.

Let $\theta$ be $\forall x_0 \forall x_2 \ldots \exists x_1 \exists x_3 \ldots \varphi(x)$. 

**Theorem 2.9.** $\mathbb{K} = \text{Mod}(\theta)$.

**Proof (Outline).** If $\mathcal{N} \in \mathbb{K}$ then $\mathcal{N} \models \theta$ by Lemmas 2.2 and 2.8 since $L\mathbb{S}(\kappa) = \omega$. Conversely, if $\mathcal{N} \not\models \theta$ then $\mathcal{N}$ is the union of a family $S$ as in the proof of Theorem 2.5, hence $\mathcal{N} \in \mathbb{K}$. 

Using Theorem 1.14 it is easy to see that $\theta$ is preserved under $L^*_{\infty \omega}$-elementary equivalence, and so Theorem 2.5 is a consequence.

Two obvious questions arise:

(1) Is $\mathbb{K}$ closed under $L^*_{\infty \omega}$-elementary equivalence?

(2) Is $\mathbb{K}$ axiomatizable by a sentence of $L^*_{\infty \omega}$?

A simple example shows the first question has a negative answer (but see the next section).

**Example 2.10.** There is a totally categorical $(\mathbb{K}, \not\prec)$ satisfying amalgamation and joint embedding such that $\mathbb{K}$ is not closed under $L^*_{\infty \omega}$-elementary equivalence.

The vocabulary of $\mathbb{K}$ consists of just a unary predicate symbol $P$. $\mathbb{K}$ is the class of all structures $\mathcal{M}$ such that $|P^\mathcal{M}| = \omega$ and $|\mathcal{M} \setminus P^\mathcal{M}| ≥ \omega$. For $\mathcal{M}, \mathcal{N} \in \mathbb{K}$ we define $\mathcal{M} \not\prec \mathcal{N}$ iff $\mathcal{M} \subseteq \mathcal{N}$ and $P^\mathcal{N} = P^\mathcal{M}$. $(\mathbb{K}, \not\prec)$ clearly has the required properties.

While we do not know the answer to the second question we do have a sufficient condition for such axiomatizability.

**Theorem 2.11.** Assume that $\mathbb{K}$ contains at most $\lambda$ models of cardinality $\lambda$ for some $\lambda$ such that $\lambda^\omega = \lambda$. Then $\mathbb{K} = \text{Mod}(\theta)$ for some $\theta \in L^*_{\infty \omega}$. If $\mathbb{K}$ also contains at most $\lambda$ models of cardinality $< \lambda$ then we can find $\theta \in L^*_{\lambda^+ \omega}$.

**Proof.** Let $\{\mathcal{M}_i : i \in I\}$ list all models in $\mathbb{K}$ of cardinality $\lambda$, up to isomorphism. For each $i \in I$ there is some $\sigma_i \in L^*_{\lambda^+ \omega}$ determining $\mathcal{M}_i$ up to $L^*_{\infty \omega}$-elementary equivalence, since $\lambda^\omega = \lambda$. Let $\theta_1 = \bigvee_{i \in I} \sigma_i$. Then $\theta_1 \in L^*_{\lambda^+ \omega}$ since $|I| \leq \lambda$.

Let $\mathcal{N} \in \mathbb{K}$ have cardinality $> \lambda$. We claim that $\mathcal{N} \models \theta_1$. If not, then by Corollary 7.4 from [11] and the remark following Lemma 2.1, $\{ s \in P = \mathcal{N} : \mathcal{N} \models \sigma_i \}$ and $\{ s \in P : \mathcal{N} \models \sigma_i \}$ both belong to $D^+_{\lambda^+ \omega}(\mathcal{N})$ and hence there is some $N_0 \not\prec_{\mathbb{K}} N$ such that $|N_0| = \lambda$ and $N_0 \models \neg \sigma_i$, contradicting the definition of $\sigma_i$.

Similarly every model in $\mathbb{K}$ of cardinality $< \lambda$ is also determined up to $L^*_{\infty \omega}$-elementary equivalence by a sentence of $L^*_{\lambda^+ \omega}$. Let $\theta_0$ be the disjunction of all these sentences. Finally let $\theta = (\theta_0 \lor \theta_1)$. Then every model in $\mathbb{K}$ satisfies $\theta$, by construction, and every model of $\theta$ belongs to $\mathbb{K}$ since $\mathbb{K}$ is closed under $L^*_{\infty \omega}$-elementary equivalence by Theorem 2.5.
Note that the hypothesis of this theorem could be weakened to say that $K$ contains at most $\lambda$ pairwise $L_{\infty,\omega}$-elementarily inequivalent structures of cardinality $\lambda$.

Although Example 2.10 shows that $K$ need not be closed under $L_{\infty,\omega}$-elementary equivalence the last result in this section, Theorem 2.13, gives a relative closure property with respect to $L_{\infty,\omega}$-elementary equivalence. We first need the following refinement of Lemma 2.2. Note that we do not assume that $M \in K$.

Lemma 2.12. Let $N \in K$, $M \subseteq N$. Then $M \prec_{K} N$ iff $M^3 \prec_{K} N^3$ a.e.

Proof. Assume $M \prec_{K} N$. Then $M^3 \prec_{K} N$ a.e. by Lemma 2.1 and $N^3 \prec_{K} N$ a.e. by the same Lemma, so $M^3 \prec_{K} N^3$ a.e. by coherence.

For the converse, assume that $M^3 \prec_{K} N^3$ a.e. Let $S = \{\text{countable } M_0 \subseteq M : M_0 \prec_{K} N\}$. By assumption $M^3 \in S$ a.e., so in particular $S$ is directed under $\subseteq$ and its union is $M$. But by coherence if $M_0, M_1 \in S$ and $M_0 \subseteq M_1$ then $M_0 \prec_{K} M_1$, so $S$ is directed under $\prec_{K}$. Thus $M \in K$ and $M \prec_{K} N$ by Lemma 1.2(a) and (c).

Theorem 2.13. Let $M_0, M, N \in K$, let $N_0 \subseteq N$ and assume that $M_0 \prec_{K} M$ and $(M, M_0) \equiv_{\infty,\omega} (N, N_0)$. Then $N_0 \in K$ and $N_0 \prec_{K} N$.

Proof. By hypothesis and Theorem 1.7(b), $(M^3, M_0^3) \equiv (N^3, N_0^3)$ a.e. By Lemma 2.12, $M_0^3 \prec_{K} M^3$ a.e., hence $N_0^3 \prec_{K} N^3$ a.e. since $K$-substructure is preserved by isomorphism. Therefore $N_0 \in K$ and $N_0 \prec_{K} N$ by Lemma 2.12 again.

We illustrate the possible uses of this theorem by the following example.

We consider structures for the vocabulary $L$ which contains just a unary predicate symbol $P$. Let $K^*$ be the class of all $L$-structures $M$ such that both $P^M$ and $\neg(P)^M$ are infinite. Let $(K, \prec_{K})$ be an a.e.c. with $LS(K) = \omega$. We know (see Example 2.10) that $K \neq K^*$ can happen if either $|P^M| = \omega$ for all $M \in K$ or $|\neg(P)^M| = \omega$ for all $\in K$. But these are the only ways in which this can happen.

Claim 2.14. Assume that there are $M, N \in K$ such that $|P^M| > \omega$ and $|\neg(P)^N| > \omega$. Then $(K, \prec_{K}) = (K^*, \subseteq)$.

Proof. For $M', N' \in K^*$ we define $M' \subseteq N'$ to hold iff $M' \subseteq N'$ and both $(P^{M'} \setminus P^M)$ and $(\neg(P)^{N'} \setminus (\neg(P)^M))$ are infinite.

Let $M, N \in K$ be as in the statement of the claim. We first prove that there are $M_0, N_1 \in K$ such that $M_0 \prec_{K} N_1$ and $M_0 \subseteq M_0 \subseteq N_1$. By the Löwenheim–Skolem property we can find countable $M_0, M_1, N_0$ and $N_1$ such that $M_0 \prec_{K} M_1 \prec_{K} M$. $N_0 \prec_{K} N_1 \prec_{K} N$ and both $(P^{M_1} \setminus P^{M_0})$ and $(\neg(P)^{N_1} \setminus (\neg(P)^N))$ are infinite. Since $M_1 \equiv N_0$ we may assume $M_1 = N_0$. Hence $M_0$ and $N_1$ are as claimed.

By constructing $K$-chains we can first conclude that for every $\kappa$ there is some $M' \in K^*$ such that both $P^M$ and $(\neg(P)^M)$ have cardinality less than $\kappa$.

Secondly, if $N' \in K$ and $M' \subseteq N'$ then $M' \in K$ and $M' \prec_{K} N'$ by Theorem 2.13, since $(N', M') \equiv_{\infty,\omega} (N_1, M_0)$. It follows that $K = K^*$.

Finally, if $M', N' \in K$ and $M' \subseteq N'$ then, by the above, there is some $N^* \in K$ such that $N' \subseteq N^*$. But then $M' \prec_{K} N^*$ and $N' \prec_{K} N^*$ by the preceding paragraph and hence $M' \prec_{K} N^*$ by coherence.

3. Finite character

We continue to assume that $(K, \prec_{K})$ is an a.e.c. in a countable vocabulary with $LS(K) = \omega$. When will $K$ be closed under $L_{\infty,\omega}$-elementary equivalence? We show this will happen when $(K, \prec_{K})$ has finite character.

Finite character was introduced by Hyttinen and Kesäla [4,5] to guarantee that $\prec_{K}$ is a local property. Our definition emphasizes that aspect — it differs from their definition, which implies ours, but the two definitions are equivalent assuming amalgamation (which Hyttinen and Kesäla do).

Definition 3.1. $(K, \prec_{K})$ has finite character iff for $M, N \in K$ we have $M \prec_{K} N$ whenever $M \subseteq N$ and for every finite tuple $\bar{a}$ from $M$ there is some $K$-embedding of $M$ into $N$ fixing $\bar{a}$.

We remark in passing that $(K, \prec_{K})$ has finite character provided the property in the definition holds for all countable $M, N \in K$.

This section parallels Section 2. The main result (Theorem 3.4) asserts that finite character implies that $K$ is closed under $L_{\infty,\omega}$-elementary equivalence. We obtain (Theorem 3.10) a sufficient condition for $K$ to be $L_{\infty,\omega}$ axiomatizable. In fact $K$ will be axiomatizable by a sentence of $L(\omega)$ (Theorem 3.7), which has Theorem 3.4 as a consequence.

In this section we use $\bar{x}, \bar{a},$ etc exclusively for $\omega$-sequences.

The following lemma improves Lemma 2.3, replacing $L^*_{\infty,\omega}$-elementary equivalence by $L_{\infty,\omega}$-elementary equivalence, assuming finite character.

Lemma 3.2. Assume that $(K, \prec_{K})$ has finite character. Let $M \in K$, $M_0 \prec_{K} M$ be countable, and assume $\bar{a}$ is an $\omega$-sequence with $\text{ran}(\bar{a}) = M_0$. Let $N$ be arbitrary and let $\bar{b}$ be an $\omega$-sequence from $N$, and assume $(M, \bar{a}) \equiv_{\infty,\omega} (N, \bar{b})$. Then $\text{ran}(\bar{b}) = N_0$ where $N_0 \prec_{K} N^2$ a.e.
**Proof.** Just as in the proof of Lemma 2.3 we may assume that $a_i = b_i$ for all $i \in \omega$. Thus our hypothesis is that $\mathcal{N}_0 <_K \mathcal{M}$ and $(\mathcal{M}, b_0, \ldots, b_{n-1}) \equiv_{\omega, \omega} (\mathcal{N}, b_0, \ldots, b_{n-1})$ for all $n \in \omega$.

We first show the following for every $n \in \omega$:

$(\ast_n)$ (there is a $K$-embedding of $\mathcal{N}_0$ into $\mathcal{N}^3$ fixing $b_0, \ldots, b_{n-1}$) a.e.

Fix $n \in \omega$ and define $Y_n$ as

$$\{s \in \mathcal{P}_\omega(\mathcal{N}) : \text{there is a } K \text{-embedding of } \mathcal{N}_0 \text{ into } \mathcal{N}^3 \text{ fixing } b_0, \ldots, b_{n-1}\}.$$ 

We want to show player II has a winning strategy in the game $G(Y_n)$. Defining $X = \{s \in \mathcal{P}_\omega(\mathcal{M}) : \mathcal{N}_0 <_K \mathcal{M}_0 \text{ and } \mathcal{M}_0 \equiv_{\omega, \omega} (\mathcal{N}, b_0)\}$ we know that player II has a winning strategy in the game $G(X)$ by Lemma 2.2.

We use the winning strategy of II in $G(X)$ and the back-and-forth properties of $L_{\omega, \omega}$-elementary equivalence to define a winning strategy for II in the game $G(Y_n)$. This is done exactly as in the proof of Lemma 2.3. $L_{\omega, \omega}$-elementary equivalence suffices since at every stage in the game we have just a finite sequence from each model. This establishes $(\ast_n)$.

Now let $Y = \bigcap_{n \in \omega} Y_n$. Then $Y \in D_\omega(\mathcal{N})$, since the filter is countably complete by Lemma 1.5. But if $s \in Y$ then $(\ast_n)$ holds for every $n \in \omega$ so $\mathcal{N}_0 <_K \mathcal{N}^3$ by finite character, since $\text{ran}(\bar{b}) = \mathcal{N}_0$. Thus $\mathcal{N}_0 <_K \mathcal{N}^3$ a.e. as desired. \hfill $\square$

The next lemma is the required improvement to Lemma 2.4.

**Lemma 3.3.** Assume that $(\mathcal{K}, <_K)$ has finite character. Let $\mathcal{M} \in \mathcal{K}$ and assume that $\mathcal{M} \equiv_{\omega, \omega} \mathcal{N}$. Then for every countable $\mathcal{N}_0 \subseteq \mathcal{N}$ there is some countable $\mathcal{N}_0 \subseteq \mathcal{N}$ such that $\mathcal{N}_0 \subseteq \mathcal{N}_0$ and $\mathcal{N}_0 <_K \mathcal{N}^3$ a.e.

**Proof.** We show how to find $\omega$-sequences $\bar{a}$ from $\mathcal{M}$ and $\bar{b}$ from $\mathcal{N}$ such that $\text{ran}(\bar{a}) = \mathcal{M}_0$ where $\mathcal{M}_0 <_K \mathcal{M}_0$, $(\mathcal{M}, \bar{a}) \equiv_{\omega, \omega} (\mathcal{N}, \bar{b})$, and $\mathcal{N}_0 <_K \mathcal{N}^3$. It then follows from Lemma 3.2 that $\mathcal{N}_0 = \text{ran}(\bar{b})$ is as desired.

Let $X = \{s \in \mathcal{P}_\omega(\mathcal{M}) : (\mathcal{M}, \bar{a}) \equiv_{\omega, \omega} (\mathcal{N}, \bar{b}) \text{ and have player I choose } a_0 \text{ in the game } G(X)\}$.

By Lemma 2.1. Enumerate $\mathcal{M}_0$ as $\{b_{2n} : n \in \omega\}$. Pick $a_0 \in \mathcal{M}$ such that $(\mathcal{M}, a_0) \equiv_{\omega, \omega} (\mathcal{N}, b_0)$. Then by finite character this formula is as required for the lemma. \hfill $\square$
Defining \( \varphi(\bar{x}) \) as \( \sqrt{\varphi(\cdot, \bar{a})} : M_0 \in K \) is countable, \( \text{ran}(\bar{a}) = M_0 \) the following is clear.

**Lemma 3.9.** Assume \((K, \prec_K)\) has finite character. Then for any \(N\) and any \(\omega\)-sequence \(b\) from \(N\), \(N \models \varphi(\bar{b}) \) iff \(\text{ran}(\bar{b}) = N_0\) for some \(\mathcal{N}_0\) such that \(\mathcal{N}_0 \prec_K \mathcal{N} \) a.e.

Finally we define \(\theta\) to be \(\forall x_0 \exists x_1 \ldots \exists x_{2n-1} \ldots \varphi(\bar{x})\). Then \(\theta \in L(\omega)\) and the proof that it axiomatizes \(K\) is similar to the proof of **Theorem 2.9**.

As in the preceding section, the obvious question is whether finite character implies \(K\) is \(L_{\omega, \omega}\) axiomatizable. We do not know the answer, but once again having ‘few’ models in some cardinality is a sufficient condition. See also Section 6 for examples which are not \(L_{\omega, \omega}\) axiomatizable.

**Theorem 3.10.** Assume \((K, \prec_K)\) has finite character. Assume that \(K\) contains at most \(\lambda\) models of cardinality \(\lambda\) for some infinite \(\lambda\). Then \(K = \text{Mod}(\theta)\) for some \(\theta \in L_{\omega, \omega}\). If \(K\) also contains at most \(\lambda\) models of cardinality \(\lambda\), then we can find \(\theta \in L_{\lambda^+, \omega}\).

**Proof.** Let \(\{M_i : i \in I\}\) list all models in \(K\) of cardinality \(\lambda\), up to isomorphism. For each \(i \in I\) let \(\sigma_i \in L_{\lambda^+, \omega}\) determine \(M_i\) up to \(L_{\omega, \omega}\)-elementary equivalence. Let \(\theta_1 = \bigvee_{i \in I} \sigma_i\). Then \(\theta_1 \in L_{\lambda^+, \omega}\) since \(|I| \leq \lambda\).

Let \(N \in K\) have cardinality \(> \lambda\). We claim that \(N \models \theta_1\). If not, then \(N \models \neg \theta_1\) hence \(N^i \models \neg \theta_1\) a.e. by **Theorem 1.10(a)** (since \((\neg \theta_1)^i = \neg \theta_1\) a.e.). \(N^i \in K\) \(\lambda\)-a.e. (by the remark following **Lemma 2.1**), and \(|N^i| = \lambda\) a.e. In particular some \(N^i\) will satisfy all three of these properties, contradicting the definition of \(\theta_1\).

We obtain the desired \(\theta\) from \(\theta_1\) exactly as in the proof of **Theorem 2.11**. \(\square\)

Note that the hypothesis of this theorem could be weakened to require just that \(K\) contains at most \(\lambda\) pairwise \(L_{\omega, \omega}\)-elementarily inequivalent structures of cardinality \(\lambda\).

4. Types and saturation

We continue to assume \((K, \prec_K)\) is an a.e.c. in a countable vocabulary with \(LS(K) = \omega\). In this section we study Galois types and the corresponding notions of saturation. Galois types are orbits in some large homogeneous model \(C\), the monster. We require that \((K, \prec_K)\) satisfies the amalgamation property (over models) and joint embedding and contains arbitrarily large models in order for the monster to exist. For tuples \(\bar{a}\) from \(C\) and ‘small’ subsets \(B\) of \(C\), the Galois type of \(\bar{a}\) over \(B\), \(tp^B(\bar{a}/B)\), is the set of all images of \(\bar{a}\) under automorphisms of \(C\) fixing \(B\) pointwise. We refer to [1, 4, 5, 13] for more detail. Also see the end of this section for an alternate way to define Galois types.

We prove (**Theorem 4.1**) that the Galois type of a finite tuple over the empty set is preserved under \(L_{\omega, \omega}\)-elementary equivalence, and that if \(K\) contains ‘few’ models in some infinite cardinality then Galois types of finite tuples over the empty set are \(L_{\omega, \omega}\) definable. These results are the analogues of **Theorems 3.4 and 3.10**. The class of \(\omega\)-Galois saturated models turns out to be well behaved (**Theorems 4.4 and 4.5**), especially when finite character is also assumed (**Theorem 4.6**).

We abbreviate the assumptions of amalgamation, joint embedding and arbitrarily large models to (AP etc.). In this section, \(\bar{x}, \bar{a}\), etc. refer to finite tuples unless explicitly noted otherwise.

**Theorem 4.1** (**AP etc.**). Let \(M, N \in K\) and let \(\bar{a}, \bar{b}\) be tuples of the same (finite) length from \(M, N\) respectively. Assume that \((\bar{a}, \bar{b}) \equiv_{\infty, \omega} (\bar{N}, \bar{b})\). Then \(tp^B(\bar{a}/\bar{b}) = tp^B(\bar{a}/\bar{b})\).

**Proof.** We have \(M, N \prec_K C\). By hypothesis and **Theorem 1.7(b)** we know that \((M^i, \bar{a}) \equiv_{N^i, \bar{b}}\) a.e. Further, by **Lemma 2.1**, we know that \(M^i \prec_{\infty, \omega} N^i\) a.e. and \(N^i \prec_K N\) a.e. In particular, then, there are countable \(M_0 \prec_K M\) and \(N_0 \prec_K N\) such that \((M_0, \bar{a}) \equiv_K (N_0, \bar{b})\). This isomorphism extends to an automorphism of the monster, establishing the equality of the Galois types. \(\square\)

Now that it follows that for every \(M \in K\) and every tuple \(\bar{a} \in M\) there is some \(\varphi_0^M(\bar{x}) \in L_{\infty, \omega}\) such that whenever \(N \in K\) and \(M \equiv_{\infty, \omega} N\) and \(\bar{b} \in N\) then \(N \models \varphi_0^M(\bar{b})\) iff \(tp^B(\bar{a}/\bar{b}) = tp^B(\bar{a}/\bar{b})\).

Are Galois types uniformly definable, that is, is there a formula \(\varphi_0(\bar{x}) \in L_{\infty, \omega}\) which has this property for every \(N \in K\)? We do not know the answer, but having ‘few’ models in some cardinality is a sufficient condition. As with **Theorem 3.10** the hypothesis could be weakened to refer to the number of \(L_{\omega, \omega}\)-elementarily inequivalent models.

**Corollary 4.2** (**AP etc.**). Assume that \(K\) contains at most \(\lambda\) models of cardinality \(\lambda\) for some \(\lambda \geq \omega\). Then for every \(\bar{a} \in C\) there is \(\varphi_0(\bar{x}) \in L_{\infty, \omega}\) such that for every \(N \in K\), \(\bar{b} \in N\), \(N \models \varphi_0(\bar{b})\) iff \(tp^B(\bar{a}/\bar{b}) = tp^B(\bar{a}/\bar{b})\).

**Proof.** As shown in the proof of **Theorem 3.10** the hypothesis implies that there is a set \(\{M_i : i \in I\}\) of models in \(K\) such that for every \(N \in K\) is \(L_{\infty, \omega}\)-elementary equivalent to \(M_i\) for some \(i \in I\) for each \(i \in I\) let \(\sigma_i \in L_{\omega, \omega}\) determine \(M_i\) up to \(L_{\omega, \omega}\)-elementary equivalence and let \(\varphi_i(\bar{x}) \in L_{\infty, \omega}\) define \(tp^B(\bar{a}/\bar{b})\) in all models in \(K\) of \(\sigma_i\). Then \(\varphi_i(\bar{x}) = \bigvee_{i \in I} [\sigma_i \land \varphi_i(\bar{x})]\) will define \(tp^B(\bar{a}/\bar{b})\) in every model in \(K\). \(\square\)

\(\lambda\)-Galois saturated models may be defined in the expected way. We are mostly concerned with \(\omega\)-Galois saturated models, which may be characterized as follows: \(M \in K\) is \(\omega\)-Galois saturated iff for every \(\bar{a} \in K\), every \(\bar{b}_0, \ldots, \bar{b}_{n-1} \in M\), every \(N \in K\) with \(M \prec_K N\) and every \(b_n \in N\) there is \(a_n \in M\) such that \(tp^B(a_0, \ldots, a_n/\bar{b}) = tp^B(a_0, \ldots, a_{n-1}, b_n/\bar{b})\).

The following lemma is useful in back-and-forth arguments.
Lemma 4.3 (AP etc.). Assume that \( M \in K \) is \( \omega \)-Galois saturated. Let \( N \in K \), let \( \bar{a} = a_0 \ldots a_{n-1} \in M, \bar{b} = b_0 \ldots b_{n-1} \in N \) and assume \( tp^{\infty}(\bar{a}/\emptyset) = tp^\omega(\bar{b}/\emptyset) \). Then for any \( b_n \in N \) there is \( a_n \in M \) such that \( tp^{\infty}(\bar{a}a_n/\emptyset) = tp^\omega(\bar{b}b_n/\emptyset) \).

Proof. We have \( M, N \prec K \). Then there is an automorphism \( h \) of the monster such that \( h(\bar{a}) = \bar{b} \). Let \( a'_n \) be such that \( h(a'_n) = b_n \), so \( tp^\omega(\bar{a}a'_n/\emptyset) = tp^\omega(\bar{b}b_n/\emptyset) \). Since \( M \) is \( \omega \)-Galois saturated there is \( a_n \in M \) such that \( tp^\infty(\bar{a}a_n/\emptyset) = tp^\infty(\bar{a}a'_n/\emptyset) \), so \( a_n \) is as desired. \( \square \)

The basic uniqueness and existence facts about saturated models in first order logic extend easily to Galois saturation. In particular, assuming (AP etc.), every model in \( K \) has an \( \omega \)-Galois saturated \( K \)-extension, and \( K \) contains a countable \( \omega \)-Galois saturated model iff there are just countably many Galois types of (finite) tuples over the empty set.

Some earlier results become biconditional on \( \omega \)-Galois saturated models.

Theorem 4.4 (AP etc.). Let \( M, N \in K \) and assume that \( M \) is \( \omega \)-Galois saturated. Then \( N \) is \( \omega \)-Galois saturated iff \( M \equiv_{\infty,\omega} N \).

Proof. The proof from left to right is a familiar back-and-forth argument using Lemma 4.3.

For the other direction assume that \( M \equiv_{\infty,\omega} N \), let \( \bar{b} = b_0 \ldots b_{n-1} \in N \) and let \( b'_n \in N ' \) where \( N ' \prec K \). Let \( \bar{a} = a_0 \ldots a_{n-1} \in M \) be such that \( (M, \bar{a}) \equiv_{\infty,\omega} (N, \bar{b}) \). Then \( tp^\omega(\bar{a}/\emptyset) = tp^\omega(\bar{b}/\emptyset) \) by Theorem 4.4.1, and so by Lemma 4.3 (applied to \( M \) and \( N ' \)) there is \( a_n \in M \) such that \( tp^\omega(\bar{a}a_n/\emptyset) = tp^\omega(\bar{b}b'_n/\emptyset) \). Now let \( b_n \in N \) be such that \( (M, \bar{a}a_n) \equiv_{\infty,\omega} (N, \bar{b}b'_n) \). Then, by Theorem 4.4.1 again, \( tp^\omega(\bar{a}a_n/\emptyset) = tp^\omega(\bar{a}a'_n/\emptyset) = tp^\omega(\bar{b}b'_n/\emptyset) \) as desired. \( \square \)

The next result is a consequence of Theorem 4.4.1 and an easy back-and-forth argument which we omit.

Theorem 4.5 (AP etc.). Let \( M, N \subseteq K \) both be \( \omega \)-Galois saturated. Let \( \bar{a} \in M, \bar{b} \in N \). Then \( tp^\omega(\bar{a}/\emptyset) = tp^\omega(\bar{b}/\emptyset) \) iff \( (M, \bar{a}) \equiv_{\infty,\omega} (N, \bar{b}) \).

Hyttenin and Keselà define an a.e.c. (in a countable vocabulary with countable Löwenheim–Skolem number) to be finitary iff it has finite character and satisfies (AP etc.) (this is the definition in [5] replacing the stronger property considered in [4]). The class of \( \omega \)-Galois saturated models of a finitary a.e.c. is extremely well behaved. Following Hyttenin and Keselà we use \( K^{\omega} \) for the class of \( \omega \)-Galois saturated models in \( K \).

Theorem 4.6. Assume \( (K, \prec_{K}) \) is finitary.

(a) \( K^{\omega} = Mod(\sigma) \) for a complete sentence \( \sigma \in L_{\infty,\omega} \), if \( K^{\omega} \) contains a countable model then \( \sigma \in L_{\omega,1}^{\omega} \).

(b) For \( M, N \in K^{\omega} \) we have \( M \prec_{K} N \) iff \( M \prec_{\infty,\omega} N \); if \( K^{\omega} \) contains a countable model then \( M \prec_{K} N \) iff \( M \prec_{L^{\omega}} \) for a countable fragment \( L^{\omega} \) of \( L_{\omega,1}^{\omega} \).

Proof. (a) is immediate from Theorems 3.4 and 4.4. (b) is immediate from Corollary 3.6 and Theorem 4.5. \( \square \)

We similarly obtain a biconditional strengthening of Theorem 3.5 for \( \omega \)-Galois saturated models.

Theorem 4.7. Assume \( (K, \prec_{K}) \) is finitary. Let \( M, N \subseteq K^{\omega} \), let \( M_0 \prec_{K} M \) and let \( h : M_0 \rightarrow N \). Then \( h \) is a \( K \)-embedding iff \( h \) is \( L_{\infty,\omega} \)-elementary as a map of \( M \) to \( N \).

We briefly discuss types over models.

Definition 4.8. Let \( \bar{a} \in C, M \prec_{K} C \). Then:

\[ tp_{\infty,\omega}(\bar{a}/M) = \{ \psi(\bar{x}, \bar{m}) : \psi(\bar{x}, \bar{y}) \in L_{\infty,\omega}, \bar{m} \in M, C \models \psi(\bar{a}, \bar{m}) \} \]

Note that we would get the same result if \( C \) is replaced by any \( \omega \)-Galois saturated model.

We refer to [4] for the definition of weak types, notation \( tp^{\omega}(\bar{a}/M) \). The following is an immediate consequence of the definitions and Theorem 4.5.

Corollary 4.9. Assume \( (K, \prec_{K}) \) is finitary. Then \( tp^{\omega}(\bar{a}/M) = tp^{\omega}(\bar{b}/M) \) iff \( tp^{\infty}(\bar{a}/M) = tp^{\infty}(\bar{b}/M) \).

Recall from [4] that if \( (K, \prec_{K}) \) is finitary and \( \omega \)-Galois stable then \( tp^{\infty}(\bar{a}/M) = tp^{\infty}(\bar{b}/M) \) iff \( tp^{\omega}(\bar{a}/M) = tp^{\omega}(\bar{b}/M) \) for any countable \( M \subseteq K \). Adding the hypothesis of tameness removes the cardinality restriction on \( M \). Thus, under the same hypotheses, Galois types coincide with \( L_{\infty,\omega} \)-types.

We next note the generalizations of Theorems 4.1 and 4.4.4–4.6 to \( \omega \)-sequences and \( \omega_{1} \)-Galois saturated models. In these results, \( \bar{a} \) and \( \bar{b} \) are sequences of the same length \( \leq \omega \).

Theorem 4.10 (AP etc.). Let \( M, N \subseteq K \) and let \( \bar{a}, \bar{b} \) be from \( M, N \) respectively. Assume that \( (M, \bar{a}) \equiv_{\infty,\omega} (N, \bar{b}) \). Then \( tp^{\infty}(\bar{a}/\emptyset) = tp^{\infty}(\bar{b}/\emptyset) \).

Proof Outline. The hypothesis implies that \( (M, \bar{a}) \equiv_{\infty,\omega} (N, \bar{b}) \) a.e. – note that \( L_{\infty,\omega} \)-elementary equivalence would not suffice for this implication when the sequences are infinite. The conclusion now follows exactly as in the proof of Theorem 4.1. \( \square \)

Let \( K^{\omega_{1}} \) be the class of all \( \omega_{1} \)-Galois saturated models in \( K \).

Theorem 4.11 (AP etc.). Let \( M, N \subseteq K \) and assume that \( M \in K^{\omega_{1}} \). Then \( N \subseteq K^{\omega_{1}} \) iff \( M \equiv_{\infty,\omega_{1}} N \).
Theorem 4.11 follows from Theorem 4.10 and the generalization of Lemma 4.3 to this context just as Theorem 4.4 follows from Theorem 4.1 and Lemma 4.3. The proofs of the next two Theorems are also similar to the proofs of Theorems 4.5 and 4.6. We omit the details.

Theorem 4.12 (AP etc.). Assume that $\mathcal{M}, \mathcal{N} \in \mathbb{K}^{\omega_1}$ and that $\bar{a}, \bar{b}$ are from $\mathcal{M}, \mathcal{N}$ respectively. Then the following are equivalent:

$$(\mathcal{M}, \bar{a}) \equiv_{\omega_1, \omega} (\mathcal{N}, \bar{b}), \quad tp^\mathcal{M}(\bar{a}/\emptyset) = tp^\mathcal{N}(\bar{b}/\emptyset), \quad (\mathcal{M}, \bar{a}) \equiv_{\omega_1, \omega} (\mathcal{N}, \bar{b}).$$

Theorem 4.13 (AP etc.). (a) $\mathbb{K}^{\omega_1} = \text{Mod}(\sigma)$ for some complete $\sigma \in \mathbb{L}_{\omega_1, \omega}$. (b) Assume that $\mathcal{M}, \mathcal{N} \in \mathbb{K}^{\omega_1}$. Then the following are equivalent:

$$\mathcal{M} \not\equiv_{\mathbb{K}^{\omega_1}, \omega_1} \mathcal{N}, \quad \mathcal{M} \not\equiv_\mathbb{K} \mathcal{N}, \quad \mathcal{M} \not\equiv_{\omega_1, \omega_1} \mathcal{N}.$$

Use of the monster is a standard convenience but it is not essential. It should be noted that many of the results proved using the monster do not require all three of the hypotheses needed to obtain the monster. For example, no result in this section requires $\mathbb{K}$ to contain arbitrarily large models, and many do not require joint embedding either.

Without the monster, Galois types may be defined as follows (see [1,4,13] for more detail). We will use this material in Section 6.

Definition 4.14. Let $\mathcal{M}, \mathcal{N} \in \mathbb{K}$, let $\bar{a}, \bar{b}$ be tuples of the same length from $\mathcal{M}, \mathcal{N}$ respectively. Then $tp^\mathcal{M}(\bar{a}/\emptyset, \mathcal{M}) = tp^\mathcal{N}(\bar{b}/\emptyset, \mathcal{N})$ if there is some $\mathcal{M}' \in \mathbb{K}$ and $\mathbb{K}$-embeddings $g$ and $h$ of $\mathcal{M}$ and $\mathcal{N}$ into $\mathcal{M}'$ such that $g(\bar{a}) = h(\bar{b})$.

Alternate Proof of Theorem 4.1. As in the original proof we obtain countable $\mathcal{M}_0 \not\equiv_\mathbb{K} \mathcal{N}_0 \not\equiv_\mathbb{K} \mathcal{N}_0'$ such that $(\mathcal{M}_0, \bar{a}) \equiv (\mathcal{N}_0, \bar{b})$, hence $tp^\mathcal{M}(\bar{a}/\emptyset, \mathcal{M}_0) = tp^\mathcal{N}(\bar{b}/\emptyset, \mathcal{N}_0)$. But it is clear from the definition that $tp^\mathcal{M}(\bar{a}/\emptyset, \mathcal{M}_0) = tp^\mathcal{N}(\bar{a}/\emptyset, \mathcal{N}_0)$ and $tp^\mathcal{N}(\bar{b}/\emptyset, \mathcal{N}_0)$, and so the result follows.

Note that this argument shows that if $\mathbb{K}$ satisfies amalgamation and if $\mathcal{M}, \mathcal{N} \in \mathbb{K}$ are such that $\mathcal{M} \equiv_{\omega_1, \omega} \mathcal{N}$ then $\mathcal{M}$ and $\mathcal{N}$ can both be $\mathbb{K}$-embedded into some $\mathcal{N}' \in \mathbb{K}$.

Assuming just amalgamation we still have that every model in $\mathbb{K}$ has an $\omega$-Galois saturated $\mathbb{K}$-extension. If joint embedding fails there will be $\omega$-Galois saturated models in $\mathbb{K}$ which are not $\mathbb{L}_{\omega_1, \omega}$-elementarily equivalent. But if $\mathcal{M}, \mathcal{N} \in \mathbb{K}$ are both $\omega$-Galois saturated and $\mathcal{M} \equiv_{\omega_1, \omega} \mathcal{N}$ then the equivalence in Theorem 4.5 holds. In particular if $\mathcal{M}$ is $\omega$-Galois saturated and $\bar{a}, \bar{b}$ are both from $\mathcal{M}$ then $tp^\mathcal{M}(\bar{a}/\emptyset, \mathcal{N}) = tp^\mathcal{M}(\bar{b}/\emptyset, \mathcal{M})$ if $(\mathcal{M}, \bar{a}) \equiv_{\omega_1, \omega}(\mathcal{M}, \bar{b})$.

5. Categoricity

We continue to assume that $\mathbb{K}$ is a.e.c. in a countable vocabulary and that $L^\mathbb{K}(\mathcal{K}) = \omega$. In this brief section we consider such a.e.c.'s which are also categorical in some infinite power. Our main result (Theorem 5.2) states that if $\mathbb{K}$ is also finitary then $\mathbb{K}$ is 'almost' $L_{\omega_1, \omega}$ axiomatizable.

Lemma 5.1. Assume that $\mathbb{K}$ is $\lambda$-categorical for some $\lambda \geq \omega$.

(a) If $\mathcal{M}, \mathcal{N} \in \mathbb{K}$ and $|\mathcal{M}|, |\mathcal{N}| \geq \lambda$ then $\mathcal{M} \equiv_{\lambda, \omega} \mathcal{N}$.

(b) (AP etc.) All models in $\mathbb{K}$ of cardinality at least $\lambda$ are $\omega$-Galois saturated and are $\mathbb{L}_{\omega_1, \omega}$-elementarily equivalent to countable models.

Proof. (a) This is immediate from the proof of Theorem 3.10.

(b) The first assertion is immediate from part (a) and Theorem 4.4, since $\mathbb{K}$ contains models of cardinality at least $\lambda$ which are $\omega$-Galois saturated. For the second, it suffices to show there is a countable $\omega$-Galois saturated model in $\mathbb{K}$. If $\lambda = \omega$ this follows since there will be just countably many Galois types of finite tuples over the empty set. If $\lambda > \omega$ this follows since $\lambda$-categoricity for $\lambda > \omega$ implies $\omega$-Galois stability.

Part (b) of this lemma for $\lambda > \omega$ requires the full strength of (AP etc.) – in particular there are $\omega_1$-categorical a.e.c.'s which have finite character and satisfy amalgamation and joint embedding but which do not have countable $\omega$-Galois saturated models – we give an example in the next section following Example 6.3.

Theorem 5.2. Assume that $\mathbb{K}$ is finitary and is $\lambda$-categorical for some $\lambda \geq \omega$. Then there is some complete $\sigma \in L_{\omega_1, \omega}$ such that:

(i) $\mathcal{M} \in \mathbb{K}$ iff $\mathcal{M} \models \sigma$ for every $\mathcal{M}$ with $|\mathcal{M}| \geq \lambda$.

(ii) there is a countable fragment $L^*$ of $L_{\omega_1, \omega}$ such that $\mathcal{K}_{\lambda, \omega}, \mathcal{K}_{\omega, \omega}$ and $\mathcal{L}_{\omega}^*$ are equivalent on $\text{Mod}(\sigma)$, and

(iii) $\text{Mod}(\sigma)$ is a finitary a.e.c. with respect to any of the three substructure relations in (ii).

Proof. Immediate from Theorem 4.6 and Lemma 5.1(b) where $\text{Mod}(\sigma) = \mathbb{K}_{\omega}$. □
Under the hypotheses of Theorem 5.2, \( K \) is certainly \( L_{\infty, \omega} \)-axiomatizable. It is an open question whether or not \( K \) must be \( L_{\omega_1, \omega} \)-axiomatizable when \( \lambda > \omega \) (when \( \lambda = \omega \) this is clear). Note that by Theorem 3.10 it would suffice to show that a \( \lambda \)-categorical finitary a.e.c. contains just countably many models of cardinality \( \omega \) (since joint embedding will then imply that \( K \) contains only countably many finite models as well).

D. Marker has an example of a sentence of \( L_{\omega_1, \omega} \) (which therefore defines an a.e.c. with finite character) which is \( \kappa \)-categorical for all \( \kappa > \omega \) but has \( 2^\omega \) countable models. It satisfies amalgamation but is not finitary since joint embedding fails.

Finally, we also have the following result, which does not use finite character.

Theorem 5.3. Assume that \( (K, <_K) \) is \( \lambda \)-categorical for some \( \lambda \) such that \( \lambda^{\omega_1} = \lambda \).

(a) If \( M, N \in K \) and \( |M|, |N| \geq \lambda \) then \( M \cong_{\lambda, \omega_1} N \).

(b) (AP etc.) All models in \( K \) of cardinality at least \( \lambda \) are \( \omega_1 \)-Galois saturated.

(c) (AP etc.) There is a complete \( \sigma \in L(2^{\omega_1}+, \omega_1) \) such that \( K \) and \( \text{Mod}(\sigma) \) coincide on all models of cardinality at least \( \lambda \).

Proof. (a) is immediate from the proof of Theorem 2.11. (b) follows from the existence of \( \omega_1 \)-Galois saturated models and Theorem 4.11. (c) is immediate from part (b) and Theorem 4.13(a). \( \square \)

In parts (b) and (c) the hypothesis on \( \lambda \) can be weakened to \( \text{cf}(\lambda) > \omega \), since the model in \( K \) of cardinality \( \lambda \) will be \( \omega_1 \)-Galois saturated and this immediately implies that all larger models are also \( \omega_1 \)-Galois saturated. Note also that the sentence \( \sigma \) in part (c) will have a model of cardinality \( \omega_1 \) since \( (K, <_K) \) is \( \omega \)-Galois stable (see [1]).

6. Finite character revisited

In this section we continue to assume that \( (K, <_K) \) is an a.e.c in a countable vocabulary with Löwenheim–Skolem number \( \omega \) and examine finite character more closely. We first give a characterization of finite character (Theorem 6.1) in terms of \( L_{\infty, \omega} \)-reducts. We next (Theorem 6.2) show that there are exactly \( 2^{2^\omega} \) finitary a.e.c.’s. We then give an example of a finitary a.e.c. which is not closed under \( L_{\omega_1, \omega} \)-elementary equivalence. We finally formulate a closure property and collect some questions about finitary a.e.c.’s.

Theorem 6.1. Assume that \( (K, <_K) \) satisfies the amalgamation property. Then the following are equivalent:

(i) \( (K, <_K) \) has finite character.

(ii) For every \( M_0, M \in K \) with \( M_0 <_K M \) and every \( h : M_0 \to N \), if \( h \) is \( L_{\infty, \omega} \)-elementary as a map of \( M \) to \( N \) then \( h \) is a \( K \)-embedding.

Proof. Theorem 3.5 established that (i) implies (ii) without assuming amalgamation. We will show the other implication by deriving finite character from the special case of (ii) in which \( M = N \) is \( \omega \)-Galois saturated (recall from the end of Section 4 that amalgamation suffices for the existence of \( \omega \)-Galois saturated models).

Let \( N_0, N_1 \in K \) with \( N_0 \subseteq N_1 \) and assume that for every tuple \( b \) from \( N_0 \) there is a \( K \)-embedding of \( N_0 \) into \( N_1 \) fixing \( b \). We must show \( N_0 <_K N_1 \).

As noted above, we know that there is some \( \omega \)-Galois saturated \( M \in K \) such that \( N_1 <_K M \). By assumption there is some \( K \)-embedding \( g \) of \( N_0 \) into \( N_1 \). Let \( M_0 = g[N_0] \). Then \( M_0 <_K M \) and \( g \) is an embedding of \( N_0 \) onto \( M_0 \). Let \( b \) be from \( N_0 \). Then \( tp^K(b/\emptyset, M) = tp^K(g(b)/\emptyset, N_1) \) since \( N_1 <_K M \). By assumption, \( tp^K(b/\emptyset, N_1) = tp^K(b/\emptyset, N_0) \), and \( tp^K(b/\emptyset, N_0) = tp^K(g(b)/\emptyset, M) \) since \( g \) is a \( K \)-embedding of \( N_0 \) into \( N_1 \) and hence into \( M \).

Let \( h = g^{-1} \). Then \( h \) is an isomorphism of \( M_0 \) onto \( N_0 \) and by the previous paragraph we see that for every \( a \) from \( M_0 \) we have \( tp^K(\widehat{a}/\emptyset, M) = tp^K(h(\widehat{a})/\emptyset, M) \). Since \( M \) is \( \omega \)-Galois saturated we conclude, by the remark at the end of Section 4, that \( (\widehat{M}, \widehat{a}) \cong_{\infty, \omega} (M, h(\widehat{a})) \) for all \( a \) from \( M_0 \). Thus \( h \) is \( L_{\infty, \omega} \)-elementary as a map of \( M \) to \( M \). By (ii) we conclude that \( h \) is a \( K \)-embedding so \( N_0 <_K N_1 \) and hence \( N_0 <_K N_1 \) by coherence. \( \square \)

Since an a.e.c. \( (K, <_K) \) (in a countable vocabulary and with Löwenheim–Skolem number \( \omega \)) is completely determined by the class \( K_0 \) of countable structures in \( K \) and the restriction of \( <_K \) to \( K_0 \) (see Lemma 1.2), it is clear that there are at most \( 2^{2^\omega} \) such a.e.c.’s. In fact there are that many finitary a.e.c.’s.

Theorem 6.2. There are \( 2^{2^\omega} \) finitary a.e.c.’s.

Proof. Fix a countable relational vocabulary \( L_0 \) such that there are \( 2^\omega \) different complete first-order theories of \( L_0 \). Enumerate the complete \( L_0 \)-theories as \( \{T_i : i \in 2^\omega\} \). Let \( E \) be the vocabulary obtained by adding a new binary predicate symbol \( E \) to \( L_0 \). For every non-empty \( S \subseteq 2^\omega \) we define a finitary a.e.c. \( (K_S, <_S) \) so that different choices of \( S \) define different classes \( K_S \).

Let \( K_S \) be the class of all \( L \)-structures \( M \) such that \( E^M \) is an equivalence relation and every \( E^M \)-class is a model of \( T_i \) for some \( i \in S \). We do not require that models of every such \( T_i \) occur in \( M \) and we allow the same theory to occur any number of times. If \( M \in K_S \) and \( a \in M \) we define \( M_a \) to be the \( L_0 \)-reduct of the substructure of \( M \) whose universe is the \( E^M \)-class of \( a \) (thus \( M_a \models T_i \) for some \( i \in S \)). For \( M, N \in K_S \) we define \( M <_S N \) to hold if \( M \subseteq N \) and \( M_a < N_a \) for all \( a \in M \). It is easy to check that \( (K_S, <_S) \) is a finitary a.e.c. \( \square \)
Since there are only $2^\omega$ sentences of $L_{\omega_1,\omega}$ it follows that ‘most’ classes $K_\omega$ are not axiomatizable by a sentence of $L_{\omega_1,\omega}$.

However each $K_\omega$ is axiomatizable by a set of sentences of $L_{\omega_1,\omega}$ and is thus closed under $L_{\omega_1,\omega}$-elementary equivalence.

We next give an explicit example of a finitary a.e.c. which is not closed under $L_{\omega_1,\omega}$-elementary equivalence. It also shows that Vaught’s Conjecture fails for finitary a.e.c.’s.

**Example 6.3.** There is a finitary a.e.c. $(K, <_K)$ such that $K$ is not closed under $L_{\omega_1,\omega}$-elementary equivalence. $K = \text{Mod}(\sigma)$ for some $\sigma \in L_{\omega_2,\omega}$, $K$ contains exactly $\omega_1$ countable models, $(K, <_K)$ is not $\omega$-Galois stable (in fact there is no countable $\omega$-Galois saturated model) but it is $\kappa$-Galois stable for all $\kappa > \omega$.

The vocabulary $L$ of the example consists of a unary predicate symbol $P$ and a binary predicate symbol $. K$ is the class of all $L$-structures $M$ such that $<^M$ holds only between elements of $P^M$ and $(P^M, <^M) \equiv (\alpha, <)$ for some ordinal $\alpha \leq \omega_1$. Thus $(M \setminus P^M)$ is an arbitrary set with no structure. For $M, N \in K$ we define $M <_K N$ to hold if $M \subseteq N$ and $(P^M, <^M)$ is an initial segment of $(P^N, <^N)$.

It is easy to check that this is an a.e.c. with Löwenheim–Skolem number $\omega$ satisfying (AP etc.). We verify that it has finite character. Let $M, N \in K$ with $M \subseteq N$ and assume that for all $a \in M$ there is a $K$-embedding of $M$ into $N$ fixing $a$. To prove that $M <_K N$ it suffices to show that $(P^M, <^M)$ is an initial segment of $(P^N, <^N)$. If not, let $b_0 \in N$ be the least element of $(P^N \setminus P^M)$. Necessarily $b_0 <^N a$ for some $a \in P^M$; let $a_0$ be the least such $a$. Note that

$$\{a \in P^M : a < ^M a_0\}, <^M) = \{b \in P^N : b < ^N b_0, <^N).$$

But by hypothesis there is a $K$-embedding of $M$ into $N$ fixing $a_0$. In particular, then,

$$\{a \in P^M : a < ^M a_0\}, <^M) \equiv \{b \in P^N : b < ^N a_0, <^N).$$

But this is impossible since $(b \in P^N : b < ^N b_0, <^N)$ is a proper initial segment of $(\{b \in P^N : b < ^N a_0, <^N)$. Since $(\omega_1, <) \equiv_{\omega_1, \omega} (\omega_2, <)$ (see [10]) we conclude that $K$ is not closed under $L_{\omega_1,\omega}$-elementary equivalence. The other properties of the example are easily verified.

If we change the preceding Example by dropping $P$, so that $K$ is the class of all $\{<\}$-structures isomorphic to $(\alpha, <)$ for some $\alpha \leq \omega_1$ and $<_K$ is still initial segment, then the result is an a.e.c. with finite character satisfying amalgamation and joint embedding which is $\omega_1$-categorical but contains no countable $\omega$-Galois saturated model. In particular this shows the necessity of assuming the existence of arbitrarily large models in Lemma 5.1(b).

We briefly discuss a.e.c.’s which are closed in the sense of the following definition (see [9,11]).

**Definition 6.4.** An a.e.c. $(K, <_K)$ in a vocabulary $L$ is closed iff for every $L$-structure $M, M \in K$ iff $M^* \in K$ a.e.

The implication from left to right holds for any a.e.c. (with Löwenheim–Skolem number $\omega$) by Lemma 2.1. The interest of this notion is that if $(K, <_K)$ is closed then $K$ is completely determined by the class of countable models in $K$ independent of $<_K$.

Certainly if $K = \text{Mod}(\sigma)$ for some $\sigma \in L_{\omega_1,\omega}$ then $(K, <_K)$ is closed, and if $(K, <_K)$ is closed then $K$ is closed under $L_{\omega_1,\omega}$-elementary equivalence. **Example 6.3** is a finitary a.e.c. which is not closed. All of the finitary a.e.c.’s constructed in the proof of Theorem 6.2 are closed, so a closed finitary a.e.c. need not be $L_{\omega_1,\omega}$-axiomatizable. The following lemma gives sufficient conditions for an a.e.c. $(K, <_K)$ to be closed.

**Lemma 6.5.** Let $L^*$ be a countable fragment of $L_{\omega_1,\omega}$.

(a) Assume that $K$ is closed under $L^*$-elementary equivalence. Then $(K, <_K)$ is closed.

(b) Assume that whenever $M, N \in K$ and $M <_L N$ then $M <_K N$. Then $(K, <_K)$ is closed.

**Proof.** We just prove (b). Assume that $M^* \in K$ a.e. We show that $M \in K$. We know that $M^* <_L M$ a.e. (cf. Lemma 2.1). Let $S$ be (countable) $M_0 < L > M_1$. Then $S$ contains almost all countable substructures of $M$, and if $M_0, M_1 \in S$ and $M_0 \subseteq M_1$ then $M_0 <_L M_1$ and so $M_0 <_K M_1$ by hypothesis. Thus $M = \bigcup S \in K$ by Lemma 1.2.

Note that **Example 6.3** shows that the result in (b) fails if the countable fragment $L^*$ is replaced by $L_{\omega_1,\omega}$.

We end this section with some questions. Let $(K, <_K)$ be a finitary a.e.c.

(1) Must $K$ be axiomatizable by a sentence of $L_{\omega_1,\omega}$? As we showed in Section 3 any counterexample would have to contain more than $\lambda$ pairwise $L_{\omega_1,\omega}$-elementarily inequivalent structures of cardinality $\lambda$ for every infinite $\lambda$.

(2) Under what conditions will $K$ be axiomatizable by a sentence of $L_{\omega_1,\omega}$? In particular, as discussed in Section 5, does $\kappa$-categoricity for some $\kappa > \omega$ suffice?

(3) How many $\omega_1$-categorical finitary a.e.c.’s are there?

(4) Under what conditions will $(K, <_K)$ be closed? Specifically, will this happen if it is $\omega$-Galois stable? Also, what other consequences does being closed have?
7. Uncountable Löwenheim-Skolem number

In this final section we assume that \((\aleph_\kappa, <_\kappa)\) is an a.e.c. in a vocabulary of cardinality at most \(\kappa\) where \(LS(\kappa) = \kappa > \omega\). We briefly survey the results which hold in this case.

All the results of Section 2, except for Theorem 2.13, hold with \(\omega\) replaced everywhere by \(\kappa\) (and with \(L^*_{\omega, \omega}\) replaced by \(L^*_{\kappa, \kappa}\)) with essentially the same proofs. As an example we state and outline the proof of the generalization of Lemma 2.3.

**Lemma 7.1.** Let \(M \in \aleph_\kappa, \mathcal{M}_0 <_{\aleph_\kappa} \mathcal{N}\) where \(|\mathcal{M}_0| \leq \kappa\), and let \(\bar{a}\) be a sequence with \(ran(\bar{a}) = \mathcal{M}_0\). Let \(\mathcal{N}\) be arbitrary and let \(\bar{b}\) be a sequence from \(\mathcal{N}\) such that \((\mathcal{M}, \bar{a}) \equiv_{\aleph_\kappa, \aleph_\kappa}^\kappa (\mathcal{N}, \bar{b})\). Then \(ran(\bar{b}) = \mathcal{N}_0\) where \(\mathcal{N}_0 <_{\aleph_\kappa} \mathcal{N}\) \(\kappa\)-a.e.

**Proof (Outline).** We may assume that \(a_i = b_i\) for all \(i < lh(\bar{a})\) so that \(\mathcal{M}_0 = \mathcal{N}_0\).

Let \(Y = \{s \in \mathcal{P}_+(\mathcal{N}) : \mathcal{N}_0 <_{\aleph_\kappa} \mathcal{N}^s\ \text{and}\ \mathcal{N}^s = s\}\). We wish to show that player II_Y has a winning strategy in the game \(G_{\kappa} + (Y)\).

Defining \(X = \{s \in \mathcal{P}_+(\mathcal{M}) : \mathcal{N}_0 <_{\aleph_\kappa} \mathcal{M}^s\ \text{and}\ \mathcal{M}^s = s\}\) we know that player II_X has a winning strategy in the game \(G_{\kappa} + (X)\).

We use the winning strategy of player II_Y and the back-and-forth properties of \(L^*_{\kappa, \kappa}\)-elementary equivalence to define a winning strategy for II_X. The details are just as in the proof of Lemma 2.3 except that the players choose \(\kappa\)-sequences instead of single elements. \(\Box\)

In particular we have the following theorem.

**Theorem 7.2.** (a) If \(M \in \aleph_\kappa, \mathcal{M} \equiv_{\aleph_\kappa, \aleph_\kappa}\mathcal{N}\) then \(\mathcal{N} \in \aleph_\kappa\).

(b) If \(M \in \aleph_\kappa, \mathcal{M} \not\equiv_{\aleph_\kappa, \aleph_\kappa}\mathcal{N}\).

The reason that Theorem 2.13 does not generalize is that \(L^*_{\kappa, \kappa}\)-elementarily equivalent structures of cardinality \(\kappa\) need not be isomorphic. For the same reason, Theorem 4.1 and the results depending on it do not generalize to uncountable \(\kappa\). But Theorems 4.10–4.13 generализe in the expected manner. We state and prove the generalization of Theorem 4.10.

**Theorem 7.3 (AP etc.).** Let \(\mathcal{M}, \mathcal{N} \in \aleph_\kappa, \mathcal{M} \not\equiv_{\aleph_\kappa, \aleph_\kappa}\mathcal{N}\), and let \(\bar{a}, \bar{b}\) be sequences of the same length \(\leq \kappa\) from \(\mathcal{M}, \mathcal{N}\) respectively. Assume that \((\mathcal{M}, \bar{a}) \equiv_{\aleph_\kappa, \aleph_\kappa}\mathcal{N}\), \(\bar{b}\). Then \(\text{tp}^\kappa(\bar{a}/\emptyset) = \text{tp}^\kappa(\bar{b}/\emptyset)\).

**Proof.** Choose \(\mathcal{M}_0 <_{\aleph_\kappa} \mathcal{M}\) such that \(\bar{a} \subseteq \mathcal{M}_0\) and \(|\mathcal{M}_0| \leq \kappa\). Let \(\bar{c}\) be such that \(\text{ran}(\bar{a}\bar{c}) = \mathcal{M}_0\). Next find \(\bar{d} \subseteq \mathcal{N}\) such that \((\mathcal{M}, \bar{c}\bar{d}) \equiv_{\aleph_\kappa, \aleph_\kappa}\mathcal{N}\). Then \(\text{ran}(\bar{b}\bar{d}) = \mathcal{N}_0\) where \(\mathcal{N}_0 <_{\aleph_\kappa} \mathcal{N}\) by Lemma 7.1. But \((\mathcal{M}_0, \bar{a}) \equiv_{\aleph_\kappa, \aleph_\kappa}(\mathcal{N}_0, \bar{b})\) so the equality of the Galois types follows. \(\Box\)

Finally, Theorem 5.3 also generalizes. In particular we have the following theorem.

**Theorem 7.4 (AP etc.).** Assume that \((\aleph_\kappa, <_{\aleph_\kappa})\) is \(\lambda\)-categorical for some \(\lambda\) such that \(\text{cf}(\lambda) > \kappa\). Then there is some complete \(\sigma \in L(\mathcal{M}, \bar{a})\) such that \(\mathcal{M}\) and \(\text{Mod}(\sigma)\) coincide on all models of cardinality at least \(\lambda\).

G. Johnson has been investigating finite character in a.e.c.'s with \(LS(\kappa) = \kappa > \omega\). He has shown, in particular, that if \(\text{cf}(\kappa) = \omega\) then \(\kappa\) is closed under \(L^*_{\kappa, \kappa}\)-elementary equivalence, but that this fails for all uncountable regular \(\kappa\). For full details see [7].

**Acknowledgements**

The author gratefully acknowledges many helpful conversations on abstract elementary classes with John Baldwin, Monica van Dieren, Meeri Kesälä and Alexei Kolesnikov.

**References**