NOTE

Some New Inequalities Involving Elementary Mean Values

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A formula is derived from which one can obtain a family of two-sided inequalities involving the elementary mean values \( \sum_k w_k x_k^{1/r} \). In particular, one member of this family provides a new refinement of the arithmetic mean-geometric mean inequality.

Let \( A(x) \) and \( G(x) \) be the arithmetic and geometric means of the positive numbers \( x_k \) formed with the positive weights \( w_k \), where \( \sum w_k = 1 \) and \( k = 1, 2, \ldots, n \). We shall write \( M_r(x) \) for the elementary mean \( (\sum w_k x_k^r)^{1/r} \), and we note that \( M_1(x) = A(x) \) and \( M_0(x) = G(x) \), where \( M_0(x) \) denotes the limit of \( M_r(x) \) as \( r \to 0^+ \). We shall write \( A \) for \( A(x) \) and similarly for the other means when there is no risk of confusion. Furthermore, \( \Sigma \) will always mean \( \Sigma^r_x \), and the operators max and min will always be taken over all subscripts from 1 to \( n \).

In 1978, D. I. Cartwright and M. J. Field [2] proved the following sharpened form of the arithmetic mean-geometric mean inequality:

\[
A - G \geq \frac{1}{2} \frac{1}{\max(x_k)} \sum w_k (x_k - A)^2. \tag{1}
\]
Recently H. Alzer [1] improved this to be the following:

\[ A - G \geq \frac{1}{2} \frac{1}{\max(x_k)} \sum w_k (x_k - G)^2. \]  

(2)

The proof of (2) followed lines similar to that of (1). These proofs involved induction on \( n \) combined with a novel application of the Lagrange multiplier method.

Our purpose here is to derive a formula (see (5)) that, for example, gives immediately the pair of inequalities:

\[
\left| \frac{r(r - s)}{l(l - s)} \right| \frac{1}{\left[ \min(x_k) \right]^{1-r}} \geq \frac{\left| M'_r(x) - M'_s(x) \right|}{\left| M'_l(x) - M'_l(x) \right|} \geq \left| \frac{r(r - s)}{l(l - s)} \right| \frac{1}{\left[ \max(x_k) \right]^{1-r}} \quad (l > r, s \neq 0). 
\]  

(3)

If we take \( l = 2, r = 1 \), and let \( s \to 0 \) in this we get

\[
\frac{1}{4} \frac{1}{\min(x_k)} \left( \sum w_k x_k^2 - G^2 \right) \geq A - G \geq \frac{1}{4} \frac{1}{\max(x_k)} \left( \sum w_k x_k^2 - G^2 \right).
\]  

(4)

The right-hand inequality here closely resembles (1) and (2). However, as we shall see, the last member in (4) is not comparable to either of the last members of (1) or (2). We now derive the formula that provides inequalities such as (3).

**Lemma.** Let \( x_k \) and \( w_k \) \((k = 1, 2, \ldots, n)\) be as above and denote by \( J(x) \) the smallest closed interval that contains all of the \( x_k \). Now let \( f \) and \( g \) be two real-valued functions defined on \( J(x) \) and suppose that each of them possesses a continuous second derivative there. Then we have

\[
\frac{\sum w_k f(x_k) - f(A)}{\sum w_k g(x_k) - g(A)} = \frac{f''(\xi)}{g''(\xi)} \quad \text{for some } \xi \in J(x),
\]  

(5)

provided that the denominator of the left-hand side is nonzero.
Proof. Write
\[(Qf)(t) = \sum w_k f(t x_k + (1 - t) A) - f(A),\]
so that
\[(Qf)'(t) = \sum w_k (x_k - A) f'(t x_k + (1 - t) A)\]
and
\[(Qf)''(t) = \sum w_k (x_k - A)^2 f''(t x_k + (1 - t) A).\]
Now consider
\[W(t) = (Qf)(t) - K(Qg)(t),\]
where
\[K = \frac{(Qf)(1)}{(Qg)(1)} = \frac{\sum w_k f(x_k) - f(A)}{\sum w_k g(x_k) - g(A)}\]
and recall that
\[\sum w_k = 1 \quad \text{and} \quad \sum w_k x_k = A.\]
We get
\[W(0) = W'(0) = 0 \quad \text{and} \quad W(1) = 0,\]
and so two successive applications of the mean value theorem give \(W''(\eta) = 0\) for some \(\eta \in (0, 1).\) That is,
\[\sum w_k (x_k - A)^2 \{f''(\eta x_k + (1 - \eta) A) - K g''(\eta x_k + (1 - \eta) A)\} = 0.\]
(6)
Since all of the arguments of the function \(f'' - K g''\) here lie in the interval \(J(x)\) and since this function is continuous, (6) then reads
\[\{f''(\xi) - K g''(\xi)\} \sum w_k (x_k - A)^2 = 0 \quad \text{for some} \ \xi \in J(x),\]
which gives \(f''(\xi) - K g''(\xi) = 0.\) This concludes the proof of the lemma.

Note. Throughout this note we assume that all of the \(x_k\) are positive, but in fact, for the truth of the lemma, this assumption is unnecessary. We should also mention that the special case of the lemma in which \(g\) is the function \(e^x\) (where \(e^x(x) = x^2\)) was proved in [3]. The proof of the general case above is quite different from the earlier one, which appears to be limited to that special choice of \(g.\)
We now use the lemma to deduce (3). Taking \( f(x) = x^{r/s} \) and \( g(x) = x^{i/s} \) in (5), we get

\[
\frac{\sum w_k x_k^{r/s} - (\sum w_k x_k)^{r/s}}{\sum w_k x_k^{i/s} - (\sum w_k x_k)^{i/s}} = \frac{r(r-s)}{l(l-s)} \xi^{(r-s)/(r-s)} \quad \xi \in J(x)
\]

Now for each \( k \) write \( x_k = u_k^r \) and write \( \xi = \gamma^r \) when this reads

\[
\frac{M'(u) - M'(u)}{M'(u) - M'(u)} = \frac{r(r-s)}{l(l-s)} \gamma^{r-s} \quad \gamma \in J(u) \tag{7}
\]

There is clearly no loss of generality if we suppose that \( l > r \), and since \( \gamma \) is positive, the quotients on each side here have the same sign. So to cover the different cases arising from different values of \( s \), we shall write (7) as

\[
\left| \frac{M'(u) - M'(u)}{M'(u) - M'(u)} \right| = \frac{r(r-s)}{l(l-s)} \gamma^{r-s} \quad \gamma \in J(u) \tag{8}
\]

To conform to the lettering in (3), we replace the \( u_k \) with \( x_k \). And, having done this, since \( l > r \) and \( \min(x_k) \leq \gamma \leq \max(x_k) \), we obtain (3).

It is now a simple matter to see that the two inequalities in (3) are the best possible in the sense that the constant \( |r(r-s)/l(l-s)| \) cannot be replaced by a larger number on the right or by a smaller one on the left. To show this, take \( n \) positive numbers \( a \), and write \( x_k = t \alpha + (1-t)A \) (0 < \( t < 1 \)), in which \( A = A(x) = A(x) \). Clearly \( A \) is independent of \( t \). Inserting this in (3) and letting \( t \to 0 \), we find that

\[
\left| \frac{M'(x) - M'(x)}{M'(x) - M'(x)} \right| \to \frac{r(r-s)}{l(l-s)} \frac{1}{A^r},
\]

and this implies the above assertion.

As mentioned in 1, the inequality (2) is sharper than (1) because

\[
\sum w_k (x_k - G)^2 \geq \sum w_k (x_k - A)^2,
\]

but neither (1) nor (2) can be compared to the right-hand inequality in (4) in this way. This can be seen from the following numerical examples. With \( n = 4 \) we take the \( x_k \) as (23, 23, 24, 86) and we find that

\[
\frac{\sum w_k x_k^2 - G^2}{2 \sum w_k (x_k - G)^2} = 0.776 \ldots \quad \text{and} \quad \frac{\sum w_k x_k^2 - G^2}{2 \sum w_k (x_k - A)^2} = 0.823 \ldots,
\]
but when the $x_k$ are (6, 23, 25, 30), the values of these ratios are, respectively, 1.104… and 1.232….

We conclude this note by stating four more inequalities that can be obtained from (3) by giving various values to $l$, $r$, and $s$:

$$\frac{1}{[\min(x_k)]^2}(A - G)$$

$$\geq H^{-1} - G^{-1} \geq \frac{1}{[\max(x_k)]^2}(A - G) \quad (l = 1, r = -1, s = 0)$$

$$\frac{\max(x_k) - A}{[\min(x_k)]^2} \geq H^{-1} - [\max(x_k)]^{-1}$$

$$\geq \frac{\max(x_k) - A}{[\max(x_k)]^2} \quad (l = 1, r = -1, s \to +\infty)$$

$$\frac{(M_2^2 - H^2)}{3 \min(x_k)} \geq A - H \geq \frac{(M_2^2 - H^2)}{3 \max(x_k)} \quad (l = 2, r = 1, s = -1)$$

$$\frac{1}{[\min(x_k)]^3}(M_2^2 - A^2)$$

$$\geq H^{-1} - A^{-1}$$

$$\geq \frac{1}{[\max(x_k)]^3}(M_2^2 - A^2) \quad (l = 2, r = -1, s = 1).$$

In these, of course, $H = H(x) = M_{-1}(x)$ is the harmonic mean of the $x_k$.

REFERENCES