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Derivations Mapping into the Radical, III

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We obtain commutativity-free characterizations of those derivations d on a unital complex Banach algebra A that map A into its radical: $dA \subseteq \mathrm{rad}(A)$ if and only if there exists a constant $M \geqslant 0$ such that $r(dx) \leqslant Mr(x)$ for all $x \in A$, which in turn is equivalent to $\sup\{r(z^{-1}dz) \mid z \in A \text{ invertible}\} < \infty$ (where $r(\cdot)$ is denoting the spectral radius). The second characterization answers positively a question raised by J. Zemánek. © 1995 Academic Press, Inc.

1. INTRODUCTION

The general or non-commutative Singer–Wermer conjecture states that a derivation d on a complex Banach algebra A (i.e., a linear mapping on A satisfying the Leibniz rule $d(xy) = x(dy) + (dx)\,y$ for all $x, y \in A$) which has the property that all commutators $x(dx) - (dx)\,x$, $x \in A$ belong to $\mathrm{rad}(A)$ (the Jacobson radical of A) has its image contained in $\mathrm{rad}(A)$. Equivalently, [6, p. 239], all primitive ideals of A are invariant under d. It is known to be true if d is bounded [3], [7] or if A is commutative [11], while the classical Singer–Wermer theorem [10] gave the affirmative answer if both hypotheses are satisfied. There is some evidence for the validity of the conjecture in general, cf. in particular [12], the strongest probably being that $dA \subseteq \mathrm{rad}(A)$ if, in the assumption, $\mathrm{rad}(A)$ is replaced by the smaller nil radical $\mathrm{nil}(A)$ of A [8]. A comprehensive account of the state-of-the-art and of how the general Singer–Wermer conjecture relates to other important open problems in Banach algebra theory is given in [6].

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The present paper is devoted to commutativity-free descriptions of derivations mapping into the radical. Let us introduce some terminology. Throughout, A will denote a complex Banach algebra which, without loss of generality, we assume to be unital. A linear mapping $T: A \to A$ is called *spectrally bounded* if there exists a constant $M \geqslant 0$ such that $r(Tx) \leqslant Mr(x)$ for all $x \in A$, where $r(.)$ stands for the spectral radius. We say that T is *spectrally infinitesimal* if, in the above estimate, we can take $M = 0$ or, equivalently, $TA \subseteq Q(A)$, the set of quasinilpotent elements of A. Clearly, one has the following implications none of which can be reversed in general

$$TA \subseteq \text{rad}(A) \Rightarrow T \text{ spectrally infinitesimal} \Rightarrow T \text{ spectrally bounded.}$$

Let d be a derivation on A. It is known [13], [7] that d spectrally infinitesimal implies that $dA \subseteq \text{rad}(A)$. Our main result, Theorem 2.5, establishes that d spectrally bounded already implies that $dA \subseteq \text{rad}(A)$. The proof merely uses the Jacobson density theorem and a weak version of continuity of spectrally bounded derivations (Lemma 2.4), bypassing the Turovskiĭ–Shul'man result. Building on work by Pták from the 1970s, the first-named author had previously been able to prove the result for inner derivations [2], while in [4] a simpler proof of a more general result avoiding subharmonicity properties of the spectral radius was found. That the result holds for non-inner derivations as well is explicitly surmised in [6] as it may be considered as an intermediate step towards the non-commutative Singer–Wermer conjecture.

In the 1970s, the interrelations between algebraic properties of the spectral radius and those of the Banach algebra A, in particular commutativity, were intensively studied by a number of authors. For example in [14], it was proved that an element $a \in A$ is central modulo $\text{rad}(A)$, i.e., the inner derivation d_a maps into $\text{rad}(A)$, if and only if $\sup\{r(z^{-1} d_a z) \mid z \text{ invertible}\} < \infty$. Zemánek [14, Question 1.4] raised the question whether this property takes over to arbitrary derivations, and using essentially the same methods as in Theorem 2.5, we will give an affirmative answer in Theorem 2.6.

Finally, in Theorem 2.8, we will characterize spectrally bounded generalized derivations extending the results of [4].

2. Results

We precede the proof of our main theorem by a series of lemmas.

Lemma 2.1. *Every spectrally bounded derivation leaves each primitive ideal invariant.*

Proof. Let P be a primitive ideal of A and let d be such that $r(dx) \leqslant Mr(x)$ for all $x \in A$ and some $M \geqslant 0$. A standard application of the iterated Leibniz rule yields that

$$d^n(x^n) - n!\,(dx)^n \in P$$

for all $x \in P$ and $n \in \mathbf{N}$, cf. [9, Lemma 2.1]. In the semisimple quotient Banach algebra A/P we therefore obtain that

$$\begin{aligned} r(dx + P) &= r((dx)^n + P)^{1/n} \\ &= (n!)^{-1/n}\, r(d^n(x^n) + P)^{1/n} \\ &\leqslant (n!)^{-1/n}\, r\,(d^n(x^n))^{1/n} \\ &\leqslant (n!)^{-1/n}\, Mr(x) \qquad (x \in P, n \in \mathbf{N}) \end{aligned}$$

whence $dx + P \in Q(A/P)$ for each $x \in P$. Consequently, the ideal $(dP + P)/P$ is contained in $Q(A/P)$, hence in $\operatorname{rad}(A/P) = \{0\}$, and $dP \subseteq P$ follows. ∎

The following elementary observation is merely recorded for completeness.

LEMMA 2.2. *Let E be a vector space and a be a linear mapping on E. If $\{1, a\}$ is linearly independent, then there is $\xi \in E$ such that $\{\xi, a\xi\}$ is linearly independent.*

The technical key result to our theorems is the following algebraic lemma. It is interesting to note that we have to treat some of the inner derivations separately. This is due to the following observation of Sinclair. Following [9], we denote by E_ξ the linear space

$$E_\xi = \{\eta \in E \mid \pi(x)\,\eta = 0 \text{ for all } x \in A \text{ such that } \pi(x)\,\xi = d_\pi \pi(x)\,\xi = 0\},$$

whenever (π, E) is an irreducible representation of A, $\xi \in E$, and d_π is a non-zero derivation on $\pi(A)$. By [9], $\dim E_\xi \leqslant 2$ and equality holds for some ξ if and only if $d_\pi = [\,\cdot, b]$ for some linear, not necessarily bounded mapping b on E. Let $\dim E = 2$, so that $E = E_\xi$ will occur. Then, $\pi(A) = M_2(\mathbf{C})$ and $d_\pi = [\,\cdot, b]$ with $b \notin \mathbf{C}1$. Take $\xi \in E \backslash \{0\}$. If $\{\xi, b\xi\}$ is linearly independent, then $\pi(x)\,\xi = d_\pi \pi(x)\,\xi = 0$ implies that $\pi(x)\,b\xi = 0$ whence $\pi(x) = 0$. If $b\xi = \lambda\xi$ for some $\lambda \in \mathbf{C}$, then $d_\pi \pi(y)\,\xi = 0$ for every $y \in A$ with $\pi(y)\,\xi = 0$. In any case, not both conditions (1) and (2) in Lemma 2.3 below can be fulfilled.

LEMMA 2.3. *Let (π, E) be an irreducible representation of A with $\dim E \geqslant 3$ and d_π be a non-zero derivation on $\pi(A)$. There exist $\xi, \eta \in E \backslash \{0\}$ and $x, y \in A$ such that*

$$\pi(x)\,\xi = d_\pi \pi(x)\,\xi = 0 \qquad \textit{and} \qquad \pi(x)\,\eta = \xi \tag{1}$$

and

$$\pi(y)\,\xi = \pi(y)\,\eta = 0 \qquad \textit{and} \qquad d_\pi \pi(y)\,\xi = \eta. \tag{2}$$

Proof. Take $\xi \in E \setminus \{0\}$. Suppose that, for all $y \in A$,

$$\pi(y)\,\xi = 0 \Rightarrow d_\pi \pi(y)\,\xi = 0.$$

Then, $b(\pi(y)\,\xi) = -d_\pi \pi(y)\,\xi$, $y \in A$ defines a linear mapping b on E, as $\pi(A)\,\xi = E$. Note that

$$\begin{aligned} d_\pi \pi(x)\,\pi(y)\,\xi &= d_\pi \pi(xy)\,\xi - \pi(x)\,d_\pi \pi(y)\,\xi \\ &= (\pi(x)\,b\pi(y) - b\pi(xy))\,\xi \\ &= [\pi(x), b]\,\pi(y)\,\xi \end{aligned}$$

for all $x, y \in A$. Hence, $d_\pi = [\cdot, b]$ is an inner derivation on $\pi(A)$. As $d_\pi \neq 0$, $b \notin \mathbf{C}1$ so that there is $\xi' \in E$ such that $\{\xi', b\xi'\}$ is linearly independent (Lemma 2.2). Thus, by the Jacobson density theorem, there is $y \in A$ satisfying $\pi(y)\,\xi' = 0$ and

$$d_\pi \pi(y)\,\xi' = \pi(y)\,b\xi' - b\pi(y)\,\xi' = \pi(y)\,b\xi' \neq 0.$$

Taking either ξ or ξ' we can therefore assume that there is $y_0 \in A$ with

$$\pi(y_0)\,\xi = 0 \qquad \text{and} \qquad d_\pi \pi(y_0)\,\xi \neq 0. \tag{3}$$

Our next claim is that there is $x_0 \in A$ such that $\pi(x_0)\,\xi = d_\pi \pi(x_0)\,\xi = 0$ and $\pi(x_0) \neq 0$. Clearly, $\xi \in E_\xi$ and since $\dim E_\xi \leqslant 2$ [9, Lemma 3.2] while $\dim E \geqslant 3$, there exists $\eta \in E \setminus E_\xi$. Consequently, there has to be $x_0 \in A$ satisfying $\pi(x_0)\,\xi = d_\pi \pi(x_0)\,\xi = 0$, but $\pi(x_0)\,\eta \neq 0$. Take $x_1 \in A$ such that $\pi(x_1)\,\pi(x_0)\,\eta = \xi$. Then $x = x_1 x_0$ satisfies (1).

We next proceed to find $y \in A$ satisfying (2). Using (3) there is $y_1 \in A$ such that $\pi(y_1)\,d_\pi \pi(y_0)\,\xi = \eta$. Then $y_2 = y_1 y_0$ satisfies

$$\pi(y_2)\,\xi = 0 \qquad \text{and} \qquad d_\pi \pi(y_2)\,\xi = \eta.$$

If $\{\eta, \pi(y_2)\,\eta\}$ is linearly independent, again the Jacobson density theorem allows us to pick $y_3 \in A$ such that

$$\pi(y_3)\,\eta = \eta \qquad \text{and} \qquad \pi(y_3)\,\pi(y_2)\,\eta = 0.$$

Putting $y = y_3 y_2$ we then arrive at the assertion.

Otherwise, $\pi(y_2)\eta = \mu\eta$ for some $\mu \in \mathbf{C}$. By (1), the element $y_2' = y_2 + x$ fulfills

$$\pi(y_2')\xi = 0, \qquad d_\pi \pi(y_2')\xi = \eta \qquad \text{and} \qquad \pi(y_2')\eta = \mu\eta + \xi.$$

Thus, proceeding as above with y_2' in place of y_2, we find $y = y_3 y_2'$ fulfilling (2). ∎

The following continuity property, sufficient for our purposes, is immediate from Lemma 2.1 and [5].

LEMMA 2.4. *Let d be a spectrally bounded derivation on A and (π, E) be an irreducible representation (with a topology on E such that the action by A is continuous). For each $\eta \in E$, the mapping $x \mapsto \pi(dx)\eta$ from A to E is continuous.*

THEOREM 2.5. *Every spectrally bounded derivation d on a unital Banach algebra A maps into the radical.*

Proof. Suppose that $dA \not\subseteq \operatorname{rad}(A)$ so that there is an irreducible representation (π, E) of A with $d_\pi \neq 0$, where d_π denotes the induced derivation on $\pi(A)$ (Lemma 2.1). Our aim is to produce a sequence $(z_n)_{n \in \mathbf{N}}$ of invertible elements in A, $x \in A$ and $\xi \in E \backslash \{0\}$ such that

$$d_\pi \pi(z_n^{-1} x z_n)\xi = n\xi \qquad \text{for all} \quad n \in \mathbf{N}. \tag{4}$$

Suppose that (4) holds. Then, $r(d_\pi \pi(z_n^{-1} x z_n)) \geqslant n$ and thus

$$n \leqslant r(d(z_n^{-1} x z_n)) \leqslant M r(z_n^{-1} x z_n) = M r(x)$$

for some $M \geqslant 0$ and all $n \in \mathbf{N}$. This contradiction will show that $dA \subseteq \operatorname{rad}(A)$.

If $\dim E \leqslant 2$, there is $b \in A$ such that $d_\pi = [\,\cdot\,, \pi(b)]$. Since $d_\pi \neq 0$, $\pi(b) \notin \mathbf{C}1$ wherefore there exists $\xi \in E$ such that $\{\xi, \pi(b)\xi\}$ is linearly independent (Lemma 2.2). By [1, Corollary 4.2.6], we can take $x \in A$ and $z_n \in A$ invertible with the property that

$$\pi(x)\xi = 0, \qquad \pi(x)\pi(b)\xi = \xi, \qquad \pi(z_n)\xi = \xi$$

and

$$\pi(z_n)\pi(b)\xi = n\pi(b)\xi.$$

Then,

$$d_\pi \pi(z_n^{-1} x z_n)\xi = \pi(z_n^{-1})\pi(x)\pi(z_n)\pi(b)\xi - \pi(b)\pi(z_n^{-1})\pi(x)\pi(z_n)\xi = n\xi$$

as required.

If $\dim E \geqslant 3$, we may apply Lemma 2.3 to obtain $\xi, \eta \in E$ and $x, y \in A$ fulfilling (1) and (2). For each $n \in \mathbf{N}$ put $z_n = e^{ny}$. By (2), we have $\pi(z_n)\xi = \xi$, $\pi(z_n)\eta = \eta$, and by induction, $d_\pi \pi(y^k)\xi = 0$ for all $k \geqslant 2$. By Lemma 2.4 and (2) it follows that

$$d_\pi \pi(z_n)\xi = \sum_{k=0}^{\infty} \frac{n^k}{k!} d_\pi \pi(y^k)\xi = n\eta.$$

Using (1) we conclude that

$$\begin{aligned} d_\pi \pi(z_n^{-1} x z_n)\xi &= d_\pi \pi(z_n^{-1})\pi(x)\pi(z_n)\xi + \pi(z_n^{-1}) d_\pi \pi(x)\pi(z_n)\xi \\ &\quad + \pi(z_n^{-1})\pi(x) d_\pi \pi(z_n)\xi \\ &= n\xi. \end{aligned}$$

The proof is complete. ∎

Note that there exist unbounded spectrally bounded derivations, but that the separating space of each such derivation is contained in the radical. Non-zero inner derivations on semisimple Banach algebras provide examples of bounded spectrally unbounded derivations.

A variant on the proof of Theorem 2.5 enables us to affirm a conjecture by Zemánek first stated in 1977, cf. [14, Question 1.4].

THEOREM 2.6. *Let d be a derivation on a unital Banach algebra A. Then $dA \subseteq \operatorname{rad}(A)$ if and only if* $\sup\{r(z^{-1}dz) \mid z \in A \text{ invertible}\} < \infty$.

Proof. Let $s = \sup\{r(z^{-1}dz) \mid z \in A \text{ invertible}\}$. If $dA \subseteq \operatorname{rad}(A)$, then $s = 0$. Supposing conversely that $s < \infty$ we first show that $d\operatorname{rad}(A) \subseteq \operatorname{rad}(A)$. Given $a \in \operatorname{rad}(A)$ we have $(1+a)^{-1} = 1 - a(1+a)^{-1} \in 1 + \operatorname{rad}(A)$ and therefore

$$r((1+a)^{-1} d(1+a)) = r((1 - a(1+a)^{-1})\, da) = r(da).$$

By assumption, it follows that $r(da) \leqslant s < \infty$ for all $a \in \operatorname{rad}(A)$ whence $d\operatorname{rad}(A) \subseteq Q(A)$. Consequently,

$$r(x\,da) = r(d(xa) - (dx)a) = r(d(xa)) = 0 \qquad (x \in A,\ a \in \operatorname{rad}(A))$$

wherefore $d\operatorname{rad}(A) \subseteq \operatorname{rad}(A)$.

Since the induced derivation $\hat{d}$ on the semisimple Banach algebra $\hat{A} = A/\operatorname{rad}(A)$ is continuous [5, Remark 4.3], it leaves each primitive ideal $\hat{P}$ of $\hat{A}$ invariant [9, Theorem 2.2]. If P is a primitive ideal of A, then $\hat{P} = P/\operatorname{rad}(A)$ is a primitive ideal of $\hat{A}$ whence $\hat{d}\hat{P} \subseteq \hat{P}$ implies that $dP \subseteq P$. We thus conclude that d fixes each primitive ideal of A.

Now suppose that $dA \not\subseteq \mathrm{rad}(A)$. Take an irreducible representation (π, E) of A such that the induced derivation d_π is non-zero. By the first part of the proof of Lemma 2.3, i.e. (3), which does not depend on the dimension of E, there are $y_0 \in A$ and $\xi \in E\backslash\{0\}$ such that $\pi(y_0)\,\xi = 0$ and $d_\pi \pi(y_0)\,\xi \neq 0$. Take $y_1 \in A$ with $\pi(y_1)\, d_\pi \pi(y_0)\,\xi = \xi$, then $y = y_1 y_0$ satisfies

$$\pi(y)\,\xi = 0 \qquad \text{and} \qquad d_\pi \pi(y)\,\xi = \xi. \tag{5}$$

Putting $z_n = e^{ny}$ $(n \in \mathbf{N})$ we obtain invertible elements in A with the property that $\pi(z_n)\,\xi = \xi$ and

$$d_\pi \pi(z_n)\,\xi = \sum_{k=0}^{\infty} \frac{n^k}{k!}\, d_\pi \pi(y^k)\,\xi = n\, d_\pi \pi(y)\,\xi = n\xi$$

as $d_\pi \pi(y^k)\,\xi = 0$ for all $k \geqslant 2$ and the same continuity argument as in Lemma 2.4 applies. It follows that

$$n \leqslant r(\pi(z_n^{-1})\, d_\pi \pi(z_n)) \leqslant r(z_n^{-1}\, dz_n) \leqslant s \qquad (n \in \mathbf{N})$$

which is impossible. ∎

A linear mapping δ on A is said to be a *generalized derivation* if

$$\delta(xyz) = \delta(xy)\, z - x\delta(y)\, z + x\delta(yz) \qquad (x, y, z \in A). \tag{6}$$

In the applications such operators correspond to irreversible dynamics while derivations generate reversible ones. Put $a = \delta(1)$. Using (6) it is easily computed that $dx = \delta x - ax$, $x \in A$ defines a derivation on A. Hence, every generalized derivation δ is of the form

$$\delta = L_a + d$$

with $a = \delta(1)$ and d a derivation, and every generalized *inner* derivation is given by $L_a + d_b = L_{a-b} + R_b$ (here, L_a and R_b denote left and right multiplication by a and b, respectively). A spectrally bounded generalized derivation need not map into the radical, but if it is inner, both its constituents L_a and d_b have to be spectrally bounded as is proved in [4]. Adapting the proof of Theorem 2.5 we finally extend this result to arbitrary spectrally bounded generalized derivations.

LEMMA 2.7. *Every spectrally bounded generalized derivation leaves each primitive ideal invariant.*

Proof. Let $\delta = L_a + d$ be spectrally bounded. As above, it suffices to show that d fixes the radical. Let $M \geqslant 0$ be such that $r(\delta x) \leqslant M r(x)$ for all $x \in A$. For each $y \in \mathrm{rad}(A)$, we have that

$$r((dy)x) = r(d(yx) - ydx) = r(\delta(yx) - ayx - ydx)$$
$$= r(\delta(yx)) \leqslant Mr(yx) = 0$$

for all $x \in A$ wherefore $dy \in \text{rad}(A)$. ∎

Remark. Of course, the arguments in the above proof can also be used to prove Lemma 2.1; however, we preferred to give a more elementary proof not using automatic continuity theory.

THEOREM 2.8. *Let $\delta = L_a + d$ with $a = \delta(1)$ be a generalized derivation on a unital Banach algebra A. The following conditions are equivalent.*

(a) *δ is spectrally bounded.*

(b) *Both L_a and d are spectrally bounded.*

Proof. (b) ⇒ (a) By Theorem 2.5, $dA \subseteq \text{rad}(A)$ wherefore

$$r(\delta x) = r(ax + dx) = r(ax) \leqslant Mr(x)$$

for some $M \geqslant 0$ and all $x \in A$. Hence, δ is spectrally bounded.

(a) ⇒ (b) By Theorem 2.5, it suffices to show that d is spectrally bounded. For then, $dA \subseteq \text{rad}(A)$ and $r(ax) = r(\delta x)$ for all $x \in A$ yield that L_a is spectrally bounded with the same constant as δ.

Suppose that $dA \nsubseteq \text{rad}(A)$ and let (π, E) be an irreducible representation of A such that $\pi(dA) \neq \{0\}$. By Lemma 2.7, d induces a non-zero derivation d_π on $\pi(A)$. As in the proof of Theorem 2.5 there are $z_n \in A$ invertible, $x \in A$, and $\xi \in E \setminus \{0\}$ such that, for all $n \in \mathbf{N}$, $d_\pi \pi(z_n^{-1} x z_n)\, \xi = n\xi$ as well as $\pi(x)\, \pi(z_n)\, \xi = 0$. Therefore,

$$\delta_\pi \pi(z_n^{-1} x z_n)\, \xi = \pi(a)\, \pi(z_n^{-1} x z_n)\, \xi + d_\pi \pi(z_n^{-1} x z_n)\, \xi = n\xi$$

which shows that

$$n \leqslant r(\delta_\pi \pi(z_n^{-1} x z_n)) \leqslant r(\delta(z_n^{-1} x z_n)) \leqslant Mr(x)$$

for some $M \geqslant 0$ and all $n \in \mathbf{N}$ contradicting the spectral boundedness of δ. ∎

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