A signalizer functor theorem for groups of finite Morley rank

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1. Introduction

There is a longstanding conjecture, due to Gregory Cherlin and Boris Zilber, that all simple groups of finite Morley rank are simple algebraic groups. Towards this end, the development of the theory of groups of finite Morley rank has achieved a good theory of Sylow 2-subgroups. It is now common practice to divide the Cherlin–Zilber conjecture into different cases depending on the nature of the connected component of the Sylow 2-subgroup, known as the Sylow° 2-subgroup.

We shall be working with groups whose Sylow° 2-subgroup is divisible, or *odd type* groups. To date, the main theorem in the area of odd type groups is Borovik's trichotomy theorem. The "trichotomy" here is a case division of the minimal counterexamples within odd type.

More technically, Borovik's result represents a major success at transferring signalizer functors and their applications from finite group theory to the finite Morley rank setting. The major difference between the two settings is the absence of a *solvable* signalizer functor theorem. This forced Borovik to work only with *nilpotent* signalizer functors, and the trichotomy theorem ends up depending on the assumption of tameness to assure that the necessary signalizer functors are nilpotent.

The present paper shows that one may obtain a connected nilpotent signalizer functor from any sufficiently non-trivial solvable signalizer functor. This result plugs seamlessly into Borovik's work to eliminate the assumption of tameness from his trichotomy theorem. In the meantime, a new approach to the trichotomy theorem has been developed by Borovik [7], based on the "generic identification theorem" of Berkman and Borovik [5].

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Borovik uses his original signalizer functor arguments, and incorporates the result of the present paper.

The paper is organized as follows. The first section will develop a limited characteristic zero notion of unipotence to complement the usual p-unipotence theory. The section on centralizers and generation which follows will establish some background needed in the rest of the paper. In Section 4 we prove our main result on signalizer functors, and in Section 5 we discuss some applications. With Borovik's kind permission, we include a proof of the nilpotent signalizer functor theorem as an appendix. The results of Section 3 are based in part on a section of an unpublished version of [3].

2. Unipotence

We say a group of finite Morley rank is *connected* if it has no proper definable subgroup of finite index. We also define the connected component G° of a group G of finite Morley rank to be the intersection of all subgroups of finite index (see [6, §5.2]). We define the Fitting subgroup $F(G)$ of a group G of finite Morley rank to be the maximal normal nilpotent subgroup of G (see [6, §7.2]). As it turns out, this naive notion of unipotence is not sufficiently robust for many purposes. For example, it lacks an analog of Fact 2.3 below.

For p prime, we say that a subgroup of a connected solvable group H of finite Morley rank is *p-unipotent* if it is a definable connected p-group of bounded exponent. This definition works amazingly well when one does not need to worry about fields of characteristic zero. This section is dedicated to providing a characteristic zero notion of unipotence, with analogs of the following three facts about p-unipotent groups:

Fact 2.1 (Fact 2.15 of [9] and Fact 2.36 of [2]). *Let H be a connected solvable group of finite Morley rank. Then there is a unique maximal p-unipotent subgroup $U_p(H)$ of H, and $U_p(H) \leqslant F^\circ(H)$.*

Fact 2.2. *The image of a p-unipotent group under a definable homomorphism is p-unipotent.*

Fact 2.3 (Lemma 1 of [4]). *Let H be a connected solvable group of finite Morley rank with $U_p(H) = 1$. Then no definable section of H is p-unipotent.*

The definition of the 0-unipotent radical U_0 will be covered in Section 2.1. Next, Section 2.2 contains analogs of Facts 2.2 and 2.3. In Section 2.3 we will show that our new notion of 0-unipotence, together with the usual notion of p-unipotence, offers a kind of completeness which had no analog in the pure p-unipotence theory. Lastly, Section 2.4 will prove that U_0 is indeed contained in the Fitting subgroup, finishing off our analog of Fact 2.1.

2.1. The characteristic zero notion

We seek here to define a characteristic zero notion of unipotence. Our approach will be to identify special torsion-free "root groups." The point is to pick up groups which appear to play the role of additive groups, while avoiding those that may act like pieces of the multiplicative group of a field.

Let A be an abelian group of finite Morley rank. We say a pair $A_1, A_2 < A$ of proper subgroups is *supplemental* if $A_1 + A_2 = A$. We may call A_2 a *supplement* to A_1 in A. We will use the term *indecomposable* to mean a definable connected abelian group without a supplemental pair of proper definable subgroups.

Lemma 2.4. *Every connected abelian group of finite Morley rank can be written as a finite sum of indecomposable subgroups.*

Proof. Induction on Morley rank. □

Lemma 2.5. *Let A be an indecomposable group. Then A is divisible or A has bounded exponent.*

Proof. Immediate from Theorem 6.8 of [6]. □

Lemma 2.6. *Let A be an abelian group of finite Morley rank, and let A_1 and A_2 be definable subgroups without definable supplement in A, i.e., there is no definable $B_i < A$ such that $A = A_i + B_i$. Then $A_1 + A_2$ has no definable supplement in A.*

Proof. Immediate from definitions. □

The *radical* $J(A)$ of a definable abelian group is defined to be the maximal proper definable connected subgroup without a definable supplement ($J(A)$ exists and is unique by Lemma 2.6 for $A \neq 1$). In particular, the radical $J(A)$ of an indecomposable group A is its unique maximal proper definable connected subgroup.

We define the *reduced rank* $\bar{r}(A)$ of a definable abelian group A to be the Morley rank of the quotient $A/J(A)$, i.e., $\bar{r}(A) = \mathrm{rk}(A/J(A))$. We define the 0-rank of any group G of finite Morley rank to be

$$\bar{r}_0(G) = \max\big\{\bar{r}(A)\colon A \leqslant G \text{ is indecomposable and } A/J(A) \text{ is torsion-free}\big\}.$$

This gives us the necessary terminology to define 0-unipotence:

Definition 2.7. Let G be a group of finite Morley rank. We define $U_0(G) = U_{0,\bar{r}_0(G)}(G)$ where

$$U_{0,r}(G) = \big\langle A \leqslant G\colon A \text{ is indecomposable}, \bar{r}(A) = r, A/J(A) \text{ is torsion free}\big\rangle.$$

We shall usually preserve the $U_{0,r}$ notation for those results where we wish to emphasize the fact that r need not be maximal. We say G is a $U_{0,r}$-*group* (alternatively $(0, r)$-*unipotent*) or a U_0-*group* (alternatively 0-*unipotent*) if G is a group of finite Morley rank and $U_{0,r}(G) = G$ or $U_0(G) = G$, respectively.

Remark 2.8. Let G be a group of finite Morley rank. Then $U_{0,r}(U_{0,r}(G)) = U_{0,r}(G)$ and $U_{0,r}(G)$ is connected. Also $U_0(G) \neq 1$ iff $\bar{r}_0(G) > 0$.

We should mention that this is not the first notion of 0-unipotence to be developed. Altseimer and Berkman [1] have worked with various interesting notions. Our current notion mixes well with the signalizer functor theory.

2.2. Homomorphisms

Since U_0 is defined from indecomposable abelian groups, we first investigate how indecomposable groups behave under homomorphisms.

Lemma 2.9 (Push-forward of indecomposables). *Let A be an indecomposable abelian subgroup of a group G of finite Morley rank and let $f : A \to G$ be a definable homomorphism. Then $f(A)$ is indecomposable and $f(J(A)) = J(f(A))$. If $f(A) \neq 1$ then the induced map $\hat{f} : A/J(A) \to f(A)/J(f(A))$ has finite kernel. Furthermore, if $A/J(A)$ is a $\pi^{\perp}$-group (i.e., a group with no non-trivial π-elements) then $f(A)/J(f(A))$ is a $\pi^{\perp}$-group too.*

Proof. The inverse image of a proper subgroup of the image is a proper subgroup, so the image of an indecomposable is indecomposable. Suppose $\ker(f) < A$. Then $\ker(f)^{\circ} \leqslant J(A)$ and $f(J(A)) < f(A)$. Since the image of the connected group $J(A)$ is connected, $f(J(A)) \leqslant J(f(A))$.

Since $J(f(A)) < f(A)$, $C := f^{-1}(J(f(A)))^{\circ} \leqslant J(A)$. Since $f(C)$ has finite index in $J(f(A))$, $J(f(A)) = f(C) \leqslant f(J(A))$. Thus $f(J(A)) = J(f(A))$ and the induced map $\hat{f} : A/J(A) \to f(A)/J(f(A))$ has finite kernel. By Exercise 13b on page 72 of [6], a non-trivial p-element of $f(A)/J(f(A))$ lifts, via $\hat{f}$, to a non-trivial p-element of $A/J(A)$.   □

Lemma 2.10 (Pull-back of indecomposables). *Let $f : G \to H$ be a definable homomorphism between definable groups in a structure of finite Morley rank. Let $B \leqslant f(G)$ be an indecomposable abelian subgroup such that $B/J(B)$ contains an element of infinite order. Then f sends some indecomposable group $A \leqslant G$ onto B. Furthermore, if $B/J(B)$ is torsion-free then $A/J(A)$ is torsion-free.*

Proof. Fix $b \in B$ which has infinite order modulo $J(B)$. We use $d(b)$ to denote the intersection of all definable subgroups of H containing b. For some n we have $b^n \in d(b)^{\circ}$; as $b^n \notin J(B)$ we have $d(b)^{\circ} = B$.

There is an $a \in G$ such that $f(a) = b$. Then $b \in f(d(a))$ and $B = d(b) \leqslant f(d(a))$. As $f(d(a)^{\circ})$ has finite index in $f(d(a)) = B$ and B is connected, we have $f(d(a)^{\circ}) = B$. By Lemma 2.4, there is a decomposition $d(a)^{\circ} = A_1 + \cdots + A_n$ of $d(a)^{\circ}$ into indecomposable

groups A_i; hence there is an indecomposable group $A \leqslant d(a)^\circ$ such that $f(A)$ is not contained in $J(B)$. Since $f(A)$ is also connected and B is indecomposable, $f(A) = B$.

Suppose $B/J(B)$ is torsion-free and $A/J(A)$ has an element of order p. Since $A/J(A)$ must have an element of infinite order and is indecomposable, it is divisible by Lemma 2.5. Thus $A/J(A)$ must have an element of order p^n for every n, contradicting the fact that the kernel of the induced map $A/J(A) \to B/J(B)$ is finite. $\quad\square$

We can restate the last two results in the U_0 language as follows:

Lemma 2.11 (Push-forward and Pull-back). *Let $f : G \to H$ be a definable homomorphism between two groups of finite Morley rank. Then*

(1) *(Push-forward)* $f(U_{0,r}(G)) \leqslant U_{0,r}(H)$ *is a $U_{0,r}$-group.*
(2) *(Pull-back) If $U_{0,r}(H) \leqslant f(G)$ then $f(U_{0,r}(G)) = U_{0,r}(H)$.*

In particular, an extension of a $U_{0,r}$-group by a $U_{0,r}$-group is a $U_{0,r}$-group.

Proposition 2.12. *Let H be a connected solvable group of finite Morley rank with $U_0(H) = 1$. Then no definable section of H is torsion-free.*

Proof. Suppose K is a definable torsion-free section of H. Let A be an infinite definable abelian subgroup of K, such as $d(a)$ for some $a \in K^*$. We may assume that A is indecomposable. By Lemma 2.11, $U_{0,\bar{r}(A)}(H) \neq 1$. Since $\bar{r}_0(H) \geqslant \bar{r}(A) > 0$, $U_0(H) \neq 1$. $\quad\square$

2.3. Good tori

We call a non-trivial divisible abelian group T of finite Morley rank a *torus*. By Remark 1 to Theorem 6.8 of [6], T has no connected subgroups of bounded exponent, so $U_p(T) = 1$ for any prime p. We call a torus T a *good torus* if every definable connected subgroup of T is the definable closure of its torsion. Obviously, a good torus T has no torsion-free sections, so $U_0(T) = 1$.

Lemma 2.13. *Every definable subgroup G (not necessarily connected) of a good torus is the definable closure of its torsion.*

Proof. Since G is abelian, $G = D \oplus B$ where $D \leqslant G$ is definable and divisible and $B \leqslant G$ has bounded exponent by Exercise 7 on page 78 of [6]. Since D is connected, D is the definable closure of its torsion. Since B is entirely torsion, G is the definable closure of its torsion. $\quad\square$

As a converse to our basic observations about tori and good tori, we find that some notion of unipotence must be non-trivial for groups which are not good tori.

Lemma 2.14. *Let G be a connected solvable non-nilpotent group of finite Morley rank. Then $U_p(G) \neq 1$ for some p prime or 0.*

Proof. By the proof of Corollary 9.10 from [6], G has a section which is the additive group of a field of characteristic p for some p prime or zero. The result follows from Fact 2.3 ($p > 0$) or Proposition 2.12 ($p = 0$). □

Theorem 2.15. *Let H be a connected solvable group of finite Morley rank. Suppose $U_p(H) = 1$ for all p prime or 0. Then H is a good torus.*

Proof. By Lemma 2.14, H is nilpotent. Let $G \leqslant H$ be definable and connected. By Theorem 6.8 of [6], $G = D * C$ where D and C are definable characteristic subgroups of G, D is divisible and C has bounded exponent. The Sylow$^\circ$ p-subgroup P of C is definable and connected by Theorem 9.29 of [6] so $P \leqslant U_p(H) = 1$ and $C = 1$. Let T be the torsion part of G. By Theorem 6.9 of [6], T is central in G and $G = T \oplus N$ for some torsion-free divisible nilpotent subgroup N. Since T is central, $G' = N' \subset N$ is torsion-free and definable. Suppose $a \in G'$ is non-trivial. Since G' is torsion-free, $d(a)$ is divisible and hence connected. There is now a non-trivial indecomposable subgroup A of $d(a)$. Since $A \subset G'$ is torsion-free and abelian and $U_0(H) = 1$, $G' \neq 1$ contradicts Proposition 2.12. Thus G is divisible abelian. By the structure of divisible abelian groups, $G/d(T)$ is torsion-free (or trivial). So $G \neq d(T)$ contradicts $U_0(H) = 1$ too. □

2.4. Nilpotence

We recall that, for any group G, $G^{k+1} = [G^k, G]$ with $G^0 = G$ and $G^{(k+1)} = [G^{(k)}, G^{(k)}]$ with $G^{(0)} = G$. These are connected if G is a connected group of finite Morley rank [6, Corollary 5.30].

Theorem 2.16. *Let H be a connected solvable group of finite Morley rank. Then $U_0(H) \leqslant F(H)$.*

Proof. Let A be an indecomposable abelian $U_{0,\bar{r}_0(H)}$-subgroup of H, i.e., $\bar{r}(A) = \bar{r}_0(H)$ and $A/J(A)$ is torsion-free. We will show that $A \leqslant F(H)$, and hence $U_0(H) \leqslant F(H)$.

We observe that $G^{(k)}$ gives a normal series whose quotients $G^{(k)}/G^{(k+1)}$ are connected and abelian. Let $\{V_i\}_{i=0}^n$ be a maximal series for H whose quotients V_i/V_{i-1} are connected and abelian. So $n \leqslant \mathrm{rk}(H)$. Then the quotients V_i/V_{i-1} are also A-minimal, i.e., V_i/V_{i-1} contains no proper definable infinite A-normal subgroup.

Let K_i be the kernel of the action $A \to \mathrm{Aut}(V_i/V_{i-1})$ given by conjugation. Suppose toward a contradiction that the action of A on V_i/V_{i-1} is non-trivial for some i. V_i/V_{i-1} is A/K_i-minimal. The action of A/K_i is faithful. By the Zilber field theorem [6, Theorem 9.1], there is a field k interpretable in $U_0(H)$ such that $A/K_i \hookrightarrow k^*$ and $V_i/V_{i-1} \cong k_+$ and the natural action of k^* on k_+ is our action. Since $K_i^\circ \leqslant J(A)$, $K_i J(A)/J(A)$ is finite. As $A/J(A)$ is torsion-free, $K_i \leqslant J(A)$ and $A/J(A)$ is a torsion-free section of k^*. By Corollary 9 of [13], a field of characteristic $p > 0$ has no definable torsion-free sections, so k must have characteristic zero. Let $b \in V_i - V_{i-1}$. Since k_+ is torsion-free, $d(b)^\circ$ is not contained in V_{i-1}. Let B be an indecomposable definable connected abelian subgroup of $d(b)^\circ$ which is not contained in V_{i-1}. By Corollary 3.3

of [10], k has no proper definable additive subgroup, so $B/(B \cap V_{i-1}) \cong V_i/V_{i-1}$ is minimal and $J(B) \leqslant V_{i-1}$. So $\mathrm{rk}(k_+) = \bar{r}(B)$. By choice of A, $\bar{r}(B) \leqslant \bar{r}(A)$. Thus

$$\mathrm{rk}(k_+) \leqslant \bar{r}(A) \leqslant \mathrm{rk}(A/K_i) \leqslant \mathrm{rk}(k^*) \leqslant \mathrm{rk}(k_+).$$

So $J(A) = K_i$ and $k^* \cong A/J(A)$ is torsion-free, a contradiction.

Hence A acts trivially on V_i/V_{i-1} and $[V_i, A] \subset V_{i-1}$ for each $i = 1, \ldots, n$. This means A satisfies the *left n-Engel condition*, i.e., for all $x \in H$ and all $a \in A$, the nth left commutator $[\cdots[x, a], \cdots], a]$ is trivial [12, Definition 1.4.1]. By Lemma 1.4.1 of [12], $A \leqslant \bar{L}(H) \leqslant F(H)$. $\quad\square$

Theorem 2.16 is one of the main reasons for restricting our attention to indecomposable subgroups with maximal reduced rank. In particular, we will often find that lemmas can be proved using the relativized $U_{0,r}$ notation, but that we must restrict to the U_0 notation to get our final results. For example, our homomorphism lemma alone provides us with the tools necessary to show that the central series of a nilpotent $U_{0,r}$-group consists of $U_{0,r}$-groups, but we will still need Theorem 2.16 to know that our groups are nilpotent in the first place.

Lemma 2.17. *Let G be a nilpotent $U_{0,r}$-group. Then the derived subgroups G^k and their quotients G^k/G^{k+1} are $U_{0,r}$-groups for all k.*

Proof. We may assume that G^{k+1} is a $U_{0,r}$-group (or trivial) by downward induction on k. By Lemma 2.11, G/G' is a $U_{0,r}$-group. The bilinear map $f : G/G' \times G^{k-1}/G^k \to G^k/G^{k+1}$ induced by $(x, y) \mapsto [x, y]$ is surjective. By Lemma 2.11, $f(G/G', g) \leqslant G^k/G^{k+1}$ is a $U_{0,r}$-group. Since these groups generate G^k/G^{k+1}, the quotient G^k/G^{k+1} is a $U_{0,r}$-group too. By Lemma 2.11 (and induction), G^k is a $U_{0,r}$-group. $\quad\square$

3. Centralizers and generation

This section develops the basic background necessary for our main result. The results of this section are based in part on an unpublished version of [3]. They were originally intended to be used in the proof of Borovik's nilpotent signalizer functor theorem for characteristic p.

Fact 3.1 (Theorem 9.35 of [6]). *Any two maximal π-subgroups, known as Hall π-subgroups, of a solvable group of finite Morley rank are conjugate.*

Fact 3.2 [3]. *Let $G = H \rtimes T$ be a group of finite Morley rank. Suppose T is a solvable π-group of bounded exponent and $Q \lhd H$ is a definable solvable T-invariant $\pi^\perp$-subgroup. Then*

$$C_H(T)Q/Q = C_{H/Q}(T).$$

Proof. Clearly, it is enough to show that $C_{H/Q}(T) \leqslant C_H(T)Q/Q$. Let $L = C_H(T \bmod Q)$, i.e., $L = \{h \in H \colon [h,t] \in Q \text{ for all } t \in T\}$. Since $[L,T] \leqslant Q$, L normalizes QT. Since Q and T are solvable, QT is solvable. For any $x \in L$, $T^x \leqslant QT$ is a Hall π-subgroup of QT and $T^x = T^a$ for some $a \in Q$ by Fact 3.1. Thus $xa^{-1} \in N_L(T)$. But $N_L(T) = C_L(T)$, so $x \in QC_L(T) \leqslant QC_H(T)$. $\square$

Fact 3.3 [3]. *Let $G = H \rtimes T$ be a group of finite Morley rank. Suppose that T is a solvable π-group of bounded exponent and that H is a definable abelian $\pi^\perp$-group. Then $H = [H,T] \oplus C_H(T)$.*

Proof. Since $[H,T]$ is T-invariant and normal in H, Fact 3.2 yields

$$H = [H,T]C_H(T).$$

Suppose $x = [h_1,t_1] + \cdots + [h_n,t_n] \in C_H(T)$ for some $h_i \in H$ and $t_i \in T$. An abelian group of bounded exponent is locally finite and an extension of locally finite groups is locally finite by Theorem 1.45 of [11], so the solvable group T is locally finite; and hence $T_0 = \langle t_1, \ldots, t_n \rangle$ is finite. Consider the endomorphism $E = \sum_{t \in T_0} t$. Now

$$E\big([h,s]\big) = \sum_{t \in T_0} \big(h - h^s\big)^t = \sum_{t \in T_0} h^t - \sum_{t \in T_0} h^t = 0$$

for $h \in H$ and $s \in T_0$. So $E(x) = 0$. But $E(x) = |T_0|x$ since $x \in C_H(T)$, so $x = 0$. Thus $C_H(T) \cap [H,T] = 0$. $\square$

Fact 3.4 [3]. *Let G be a connected solvable $p^\perp$-group of finite Morley rank and let P be a finite p-group of definable automorphisms of G. Then $C_G(P)$ is connected.*

Proof. Let A be a non-trivial definable characteristic connected abelian subgroup of G, say $G^{(n)}$ for some n. Inductively, we assume that $C_{G/A}(P)$ is connected, so $H := C_G(P \bmod A)$ is connected. By Fact 3.2, $H = AC_G(P)$. Since H is connected, $H = AC_G^\circ(P)$ so

$$C_G(P) = C_H(P) = C_A(P)C_G^\circ(P).$$

By Fact 3.3, $A = [A,P] \oplus C_A(P)$ so $C_A(P)$ is connected. Hence $C_G(P)$ is connected. $\square$

Corollary 3.5. *Let G be a solvable p-unipotent group of finite Morley rank and let P be a finite q-group of definable automorphisms of G for some $q \neq p$. Then $C_G(P)$ is p-unipotent.*

There is a "characteristic zero" (recall Definition 2.7) analog to the forgoing.

Lemma 3.6. *Let G be a nilpotent $(0,r)$-unipotent $p^\perp$-group of finite Morley rank and let P be a finite p-group of definable automorphisms of G. Then $C_G(P)$ is $(0,r)$-unipotent.*

Proof. Let A be a non-trivial definable characteristic abelian $U_{0,r}$-subgroup of G, say G^n for some n (see Lemma 2.17). By Fact 3.3, $A = [A, P] \oplus C_A(P)$. By Lemma 2.11, $C_A(P)$ is $(0, r)$-unipotent. Inductively, we assume that $C_{G/A}(P)$ is $(0, r)$-unipotent. By Fact 3.2, $C_G(P)/C_A(P) \cong C_G(P)A/A = C_{G/A}(P)$ so $C_G(P)$ is an extension of a $U_{0,r}$-group by a $U_{0,r}$-group. By Lemma 2.11, $C_G(P)$ is a $U_{0,r}$-group. $\square$

The last two results of this section are not used until the proof of the nilpotent signalizer functor theorem in the appendix. They are provided here to consolidate our facts about centralizers.

Fact 3.7. *Let H be a solvable $p^\perp$-group of finite Morley rank. Let E be a finite elementary abelian p-group acting definably on H. Then*

$$H = \langle C_H(E_0) \colon E_0 \leqslant E, [E : E_0] = p \rangle.$$

Proof. We may assume E has rank at least 2. We proceed by induction on the rank and degree of H. Let A be a non-trivial E-invariant abelian normal subgroup of H such that H/A has smaller rank or degree, say $Z(F(H))$ or its connected component. By induction, $H/A = \langle C_{H/A}(E_0) \colon E_0 \leqslant E, [E : E_0] = p \rangle$. By Fact 3.2,

$$H = A \langle C_H(E_0 \bmod A) \colon E_0 \leqslant E, [E : E_0] = p \rangle$$
$$= A \langle C_H(E_0) \colon E_0 \leqslant E, [E : E_0] = p \rangle.$$

Thus we may assume that $H = A$ is abelian E-invariant and either infinite, or finite and non-trivial. In either case, we may also assume that A contains no proper non-trivial E-invariant subgroup with the same properties.

Let R be the subring of $\text{End}(H)$ generated by E. First, suppose H is connected. For $r \in R^*$, $\ker r$ is E-invariant (since E is abelian), so $\ker r$ is finite if H is connected and trivial if H is finite. By Exercise 8 on page 78 of [6] if H is connected (and by counting otherwise), $rH = H$. Thus R is an integral domain. The image of E in R is therefore cyclic. Since E has rank at least 2, there is some $E_0 \leqslant E$ with $[E : E_0] = p$ which acts trivially on H, i.e., $H = C_H(E_0)$. $\square$

Fact 3.8. *Let G be a connected solvable $p^\perp$-group of finite Morley rank. Let E be a finite elementary abelian p-group of rank at least 3 acting on G. Suppose $C_G(s)$ is nilpotent for every $s \in E^*$. Then G is nilpotent.*

Proof. Let A be an E-minimal abelian normal subgroup of G. By induction on Morley rank, we assume that G/A is nilpotent. Since $A \lhd G$, $[G, A] \leqslant A$ is E-invariant, so $[G, A] = A$ or 1. By Theorem 9.8 of [6], $[G', A] = 1$. Consider $H := A \rtimes (G/G')$. Since G is nilpotent if $[G, A] = 1$, it suffices to show that $[H, A] \neq A$.

Let $E_0 \leqslant E$ have rank 2. For $v \in E_0^*$, let $H_v = C_H(v \bmod A)$. By Fact 3.7, $H = \langle H_v \colon v \in E_0^* \rangle$. Since $A \leqslant H_v$ and H/A is abelian, H_v is normal in H. By Exercise 8 on page 88 of [6] (existence of Fitting subgroup), H is nilpotent if the H_v are all

nilpotent. This follows by induction when $H_v < H$, so we may assume $H_v = H$. By Fact 3.2, $H = AC_H(v)$. By Fact 3.3, $A = C_A(v) \oplus [A, v]$. If both factors are non-trivial then $H/C_A(v)$ and $H/[A, v]$ are nilpotent, so $H \hookrightarrow H/C_A(v) \times H/[A, v]$ is nilpotent. If $C_A(v) = A$ then $H = C(v)$ is nilpotent by hypothesis, so we may assume $C_A(v) = 1$.

Let $E_1 \leqslant E$ be a rank 2 subgroup not containing v. By the first half of the preceding argument, we may suppose that there is a $u \in E_1^*$ centralizing H/A; hence $E_2 = \langle u, v \rangle$ centralizes H/A. By the preceding argument, $C_A(x) = 1$ for $x \in E_2^*$. By Fact 3.7, $A = \langle C_A(x): x \in E_2^* \rangle$, a contradiction. $\quad \square$

4. Signalizer functors

The theory of signalizer functors plays an important role in the classification of the finite simple groups, and was transfered to the context of groups of finite Morley rank by Borovik. Signalizer functors are used in both the finite and finite Morley rank cases to control $O(C(i))$ for i an involution (see Section 5).

Let G be a group of finite Morley rank, let p be a prime, and let $E \leqslant G$ be an elementary abelian p-group. An E-signalizer functor on G is a family $\{\theta(s)\}_{s \in E^*}$ of definable $p^\perp$-subgroups of G satisfying:

(1) $\theta(s)^g = \theta(s^g)$ for all $s \in E^*$ and $g \in G$.
(2) $\theta(s) \cap C_G(t) \leqslant \theta(t)$ for any $s, t \in E^*$.

We observe that the first condition implies that $\theta(s)$ is E-invariant and $\theta(s) \lhd C_G(s)$ for each $s \in E^*$. We should also note that the second condition is equivalent to

$$\theta(s) \cap C_G(t) = \theta(t) \cap C_G(s)$$

for any $s, t \in E^*$.

As one would expect, we say θ is a *finite, connected, solvable, nilpotent, $(0, r)$-unipotent*, or *p-unipotent* signalizer functor if the groups $\theta(s)$ are finite, connected, solvable, nilpotent, $(0, r)$-unipotent, or p-unipotent, respectively, for all $s \in E^*$. Similarly, we say θ is a *non-finite* signalizer functor if $\theta(s)$ is infinite for some $s \in E^*$. By Fact 2.1 or Theorem 2.16, p-unipotent or 0-unipotent solvable signalizer functors are nilpotent; they are also connected.

Lemma 4.1. *Let G be a group of finite Morley rank and let $E \leqslant G$ be an elementary abelian p-group. Let θ be an E-signalizer functor on G, let $r := \max_{t \in E^*} \bar{r}_0(\theta(t))$ be the largest available reduced rank and set $\theta^\circ(\cdot) := \theta(\cdot)^\circ$. Then*

(0) *θ° is a connected E-signalizer functor,*
(1) *$\theta_0 := U_{0,r}(\theta(\cdot))$ is a 0-unipotent E-signalizer functor,*
(2) *$\theta_q := U_q(\theta(\cdot))$ is a q-unipotent E-signalizer functor for every prime q.*

Proof. First, let $R(H)$ be H°, $U_{0,r}(H)$, or $U_q(H)$ for some prime q and let $\tilde{\theta}(\cdot) = R(\theta(\cdot))$. For any $s, t \in E^*$, $C_{R(\theta(s))}(t) = R(C_{R(\theta(s))}(t))$ by either Lemma 3.6 when $R \equiv U_{0,r}$ or by Fact 3.4 when $R \equiv U_q$ or $R \equiv \cdot^\circ$.

Since θ is an E-signalizer functor,

$$\tilde{\theta}(s) \cap C_G(t) = C_{R(\theta(s))}(t) = R\big(C_{R(\theta(s))}(t)\big) \leqslant R\big(C_{\theta(s)}(t)\big) \leqslant R\big(\theta(t)\big) = \tilde{\theta}(t).$$

Since composition with R also preserves the conjugacy condition, the result follows. $\quad\square$

Our main result is the following:

Theorem 4.2. *Let G be a group of finite Morley rank and let $E \leqslant G$ be an elementary abelian p-group. Suppose G admits a non-finite solvable E-signalizer functor θ. Then G admits a non-trivial connected nilpotent E-signalizer functor, which is a normal subfunctor of θ.*

Proof. Since $\theta(s)$ is assumed infinite for some $s \in I(S)$, θ° is non-trivial. For q prime or 0, θ_q is a nilpotent signalizer functor by Lemma 4.1. So we may assume θ_q is trivial for all q prime or 0. In particular,

$$r := \max_{t \in E^*} \bar{r}_0\big(\theta(t)\big) = 0$$

and $U_0(\theta(s))$ is trivial for all $s \in E^*$. Now $\theta^\circ(t)$ is nilpotent for all $s \in E^*$ by Theorem 2.15. $\quad\square$

5. Applications

We should begin by discussing Borovik's "old" trichotomy theorem. Borovik's theorem is identical to Theorem 5.1 below, except that it requires the additional assumption of tameness.

Theorem 5.1. *Let G be a simple K^*-group of finite Morley rank and odd type. Then one of the following statements is true*:

(1) $n(G) \leqslant 2$.
(2) *G has a proper 2-generated core.*
(3) *G satisfies the B-conjecture and contains a classical involution.*

We will not define the terms appearing above; the first two are notions of "smallness" for groups, while the third represents a point of departure for the identification of the "generic" algebraic group. The "B-conjecture" states that $O(C_G(i)) = 1$ for any involution $i \in G$.

In any case, Borovik makes use of tameness at only one point in his argument, in connection with the B-conjecture. He shows that $\theta(i) := O(C_G(i))$ is a signalizer functor, observes that under the tameness assumption it is nilpotent, and applies his nilpotent

signalizer functor theorem, discussed further in the appendix, to prove that θ is trivial when clauses 1 and 2 do not apply.

As this part of Borovik's argument can use any non-trivial nilpotent signalizer functor, Theorem 4.2 can be used instead of the tameness assumption; hence Theorem 5.1 holds. With the removal of the tameness assumption, one should also consider *degenerate type* groups, or groups with a finite Sylow 2-subgroup. One can check that the following version of Borovik's theorem applies in the degenerate case, where the B-conjecture leads to a contradiction rather than an identification.

Theorem 5.2. *Let G be a simple K^*-group of finite Morley rank and degenerate type. Then either $n(G) \leqslant 2$, or G has a proper 2-generated core.*

The reader familiar with finite group theory would expect us to eliminate tameness by proving a *solvable* signalizer functor theorem. This we do not do. However, we can prove the following weak version, obtained by combining Theorem 4.2 and the nilpotent signalizer functor theorem, Theorem A.2 below.

Theorem 5.3 (Weak solvable signalizer functor theorem). *Let G be a group of finite Morley rank, let p be a prime, and let $E \leqslant G$ be an elementary abelian p-group of rank at least 3. Let θ be a connected solvable non-finite E-signalizer functor. Then G admits a non-trivial complete (see Definition A.1 below) E-signalizer functor, which is a connected normal nilpotent subfunctor of θ.*

This theorem is weaker than a true solvable signalizer functor theorem in two respects: non-finiteness and the passage to the subfunctor. The assumption of non-finiteness does not really concern us, as we are generally working with connected groups anyway. To see that the passage to the subfunctor does not pose any problems, one must actually look at such applications in detail (see [7]).

In closing, we need to mention that the rest of the odd type story has evolved further. Berkman, Borovik, and Nesin have a new approach to the trichotomy theorem which produces stronger results and avoids the classical involution discussion entirely. The results of the present paper figure into the new version in a more or less identical fashion, however. The full picture is explained in [7,8], with essential references to [5]. Borovik and Nesin summarize the present state of affairs as follows:

Theorem 5.4 (Theorem 1 of [7]). *Let G be a simple K^*-group of finite Morley rank and odd or degenerate type. Then G is either a Chevalley group over an algebraically closed field of characteristic $\neq 2$, or has normal 2-rank $\leqslant 2$, or has a proper 2-generated core.*

Appendix A

This section contains a proof of Borovik's nilpotent signalizer functor theorem [6] for groups of finite Morley rank.

Definition A.1. Let G be a group of finite Morley rank and let $E \leqslant G$ be an elementary abelian p-group. Let θ be an E-signalizer functor. We define

$$\theta(E) = \langle \theta(s) \colon s \in E^* \rangle$$

and we say θ is *complete* (as an E-signalizer functor) if $\theta(E)$ is a $p^\perp$-group and

$$\theta(s) = C_{\theta(E)}(s)$$

for any $s \in E^*$.

We observe that the invariance condition in the definition of a signalizer functor implies that $\theta(s)$ is E-invariant and $\theta(s) \lhd C_G(s)$ for each $s \in E^*$. For this proof it will be convenient allow these two conditions to replace the full invariance condition in the definition of a signalizer functor. This allows us to both generalize the result and simplify the proof.

A special case of the following was proved in [6, Theorem B.30].

Theorem A.2. *Let G be a group of finite Morley rank, let p be a prime, and let $E \leqslant G$ be a finite elementary abelian p-group of rank at least* 3. *Let θ be a connected nilpotent E-signalizer functor. Then θ is complete and $\theta(E)$ is nilpotent.*

Proof. Let G be a counterexample with minimal rank. Let Θ be the collection of all definable connected solvable E-invariant $p^\perp$-subgroups Q of G such that $C_Q(s) = Q \cap \theta(s)$ for every $s \in E^*$. For any $Q \in \Theta$ and any $s \in E^*$, $C_Q(s) \leqslant \theta(s)$ is nilpotent. By Fact 3.8,

$$Q \text{ is nilpotent for any } Q \in \Theta.$$

The bulk of our argument will be directed at showing that

$$\Theta \text{ has a unique maximal element } Q^*. \qquad\qquad (\star)$$

Before proving this, however, we show that the theorem follows from the existence of Q^*.
By Fact 3.7,

$$Q^* = \langle C_{Q^*}(E_0) \colon E_0 \leqslant E, [E : E_0] = p \rangle \leqslant \langle C_{Q^*}(s) \colon s \in E^* \rangle \leqslant \langle \theta(s) \colon s \in E^* \rangle = \theta(E).$$

For every $s \in E^*$, $\theta(s)$ is a connected nilpotent E-invariant $p^\perp$-subgroup of $C_G(s)$, and

$$C_{\theta(s)}(t) = \theta(s) \cap \theta(t) \quad \text{for any } t \in E^*.$$

Thus $\theta(s) \in \Theta$. Since there must be some maximal element of Θ containing $\theta(s)$ for every $s \in E^*$, $\theta(E) \leqslant Q^*$; hence θ is complete, assuming $(\star)$.

We now prove $(\star)$. Suppose towards a contradiction that $Q, R \in \Theta$ are distinct and maximal. We may assume $D = (Q \cap R)^\circ$ has maximal possible rank. By Fact 3.7, $C_Q(E_1) \neq 1$ and $C_R(E_2) \neq 1$ for some $E_1, E_2 \leqslant E$ with $[E : E_i] \leqslant p$. Since E has rank at least 3, there is an $s \in E_1 \cap E_2$ such that $C_Q(s) \neq 1$ and $C_R(s) \neq 1$. By Fact 3.4, these two groups are connected. Since $\theta(s) \in \Theta$, there is a maximal $P \in \Theta$ containing $C_Q(s), C_R(s) \leqslant \theta(s)$. Thus $\mathrm{rk}((Q \cap P)^\circ) \geqslant \mathrm{rk}(C_Q(s)) > 0$ and $\mathrm{rk}((P \cap R)^\circ) \geqslant \mathrm{rk}(C_R(s)) > 0$, so $\mathrm{rk}(D) > 0$.

Let $H = N_G(D)$, $Q_1 = (H \cap Q)^\circ$, and $R_1 = (H \cap R)^\circ$. Consider the quotient $\overline{H} = H/D$. By the usual normalizer condition [6, Lemma 6.3], and nilpotence of Q_1 and R_1, $\overline{Q}_1$ and $\overline{R}_1$ are both infinite. Since D is E-invariant, $\overline{E} = ED/D$ is an elementary abelian p-subgroup of $\overline{H}$. Let $\theta_1(s) = (H \cap \theta(s))^\circ$ and let $\bar{\theta}_1(\bar{s}) = \theta_1(s)D/D$. So $\overline{Q}_1, \overline{R}_1$, and $\bar{\theta}_1(\cdot)$ are all nilpotent $\overline{E}$-invariant groups. By Exercise 13b on page 72 of [6], $\overline{Q}_1, \overline{R}_1$, and $\bar{\theta}_1(\cdot)$ are $p^\perp$-groups. Let $s, t \in E^*$. Since $D \lhd H$, $\bar{\theta}_1(\bar{s}) \cong \theta_1(s)/(\theta_1(s) \cap D)$ via the isomorphism $xD \mapsto x(\theta_1(s) \cap D)$. Since $\theta_1(s) \cap D \lhd \theta_1(s)$, Fact 3.2 yields,

$$C_{\bar{\theta}_1(\bar{s})}(\bar{t}) \cong C_{\theta_1(s)/(\theta_1(s) \cap D)}(t) = C_{\theta_1(s)}(t)(\theta_1(s) \cap D)/(\theta_1(s) \cap D).$$

The homomorphism $x(\theta_1(s) \cap D) \mapsto xD$ is the inverse to our first isomorphism on this group, so

$$C_{\bar{\theta}_1(\bar{s})}(\bar{t}) = C_{\theta_1(s)}(t)D/D.$$

By Fact 3.4, $C_{\theta_1(s)}(t)$ is connected, so $C_{\theta_1(s)}(t) \leqslant \theta_1(t)$. Thus $\bar{\theta}_1$ is a connected nilpotent signalizer functor on $\overline{H}$. Similarly,

$$
\begin{aligned}
C_{\overline{Q}_1}(\bar{t}) &= C_{Q_1}(t)D/D \quad \text{by Fact 3.2} \\
&= C_{Q_1}^\circ(t)D/D \quad \text{by Fact 3.4} \\
&\leqslant \left(H \cap C_Q(t)\right)^\circ D/D = \left(H \cap Q \cap \theta(t)\right)^\circ D/D \leqslant \overline{Q}_1 \cap \bar{\theta}_1(\bar{t}).
\end{aligned}
$$

Thus $\overline{Q}_1, \overline{R}_1$ are elements of $\overline{\Theta}_1$, the collection of all connected solvable E-invariant $p^\perp$-subgroups $\overline{Q}$ of $\overline{H}$ such that $C_{\overline{Q}}(\bar{s}) = \overline{Q} \cap \bar{\theta}_1(s)$ for every $s \in \overline{E}^*$.

Consider $\overline{S} \in \overline{\Theta}_1$ such that $\overline{Q}_1 \leqslant \overline{S}$. Let $S \leqslant H$ be the preimage of $\overline{S}$. Since D and $\overline{S}$ are connected, S is connected. As $\overline{S}$ and D are nilpotent $p^\perp$-groups, S is a solvable $p^\perp$-group. Let $t \in E^*$. Since $D \lhd H$, Fact 3.2 yields

$$C_{\overline{S}}(\bar{t}) = C_S(t)D/D \cong C_S(t)/C_D(t)$$

via the isomorphism $xD \mapsto xC_D(t)$. Since $Q, R \in \Theta$, $C_D(t) \leqslant C_{Q \cap R}(t) \leqslant \theta(t)$, so $C_D(t) = D \cap \theta(t)$. Hence

$$C_{\overline{S}}(\bar{t}) = \overline{S} \cap \bar{\theta}_1(\bar{t}) \cong S/C_D(t) \cap \theta_1(t)/C_D(t) = \left(S \cap \theta(t)\right)/C_D(t)$$

via the same isomorphism. Thus $C_S(t) = S \cap \theta(t)$ and $S \in \Theta$. Since $\overline{S} \geqslant \overline{Q}_1$, $(S \cap Q)^\circ \geqslant Q_1 > D$ and $S = Q$, so $\overline{Q}_1$ is maximal in $\overline{\Theta}_1$. Similarly, $\overline{R}_1$ is also maximal in $\overline{\Theta}_1$. Since $\mathrm{rk}(D) > 0$, $\mathrm{rk}(\overline{H}) < \mathrm{rk}(G)$; hence $\overline{\theta}_1$ is complete and $\overline{Q}_1 = \overline{R}_1$. Since $D = (Q \cap R)^\circ$, this is a contradiction. $\quad\square$

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Further reading

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