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Super Height of an Ideal in a Noetherian Ring

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INTRODUCTION

This paper is essentially devoted to the proof of a generalization of the result of M. Hochster [K, Theorem (1.2)] which deals with certain ideals related to the Direct Summand Conjecture. The proof we will give is a modification of his proof, the main idea of which is to formulate the problem in terms of solving a system of polynomial equations and to apply the Artin approximation theorem. In this paper all rings are assumed to be commutative with identity.

DEFINITIONS. Let I be an ideal in a Noetherian ring R:

big ht $I = \sup\{\text{ht } P \mid P$ is a prime ideal of R minimal over $I\}$,
super ht $I = \sup\{\text{ht } IS \mid IS \neq S$, S is a Noetherian R-algebra$\}$,
finite super ht $I = \sup\{\text{ht } IS \mid IS \neq S$, S is a finitely generated R-algebra$\}$.

Remark. By the Krull altitude theorem, super ht I is at most the number of generators of I. In particular, super height is finite for all ideals in a Noetherian ring.

THEOREM 1. *Let R be a finitely generated algebra over a field k and let P be a prime ideal of R. Then super ht IR_p = finite super ht IR_p for all ideals I of R contained in P.*

Remark. In our proof we will keep track of the prime P. Even if we start with R without P, we have to keep track of some prime ideal in the case when k is not perfect (see Claim 1.).

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For a prime integer p, we denote by F_p the algebraic closure of $\mathbb{Z}/p\mathbb{Z}$ and by V_p the ring of Witt vectors over F_p.

THEOREM 2. *Let R be a finitely generated algebra over V_p. Then super ht $I=$ finite super ht I for all ideals I of R.*

Remark. The proof of Theorem 2 will show that if I is an ideal in a finitely generated $\mathbb{Z}$-algebra, then there is a finitely generated V_p-algebra for some p where the super height of I is obtained.

We recall the following easy consequence of Zariski's main theorem to use later in our example and to note that there is a fairly small class of algebras where the finite super height can be obtained.

Fact. Let I be a Noetherian ring R. Then finite super ht $I=$ sup$\{$*big ht $I(R/J)'$ | J is a prime ideal of R and $'$ denotes the normalization*$\}$.

Proof. [K, Fact (1.4)] and Going Down Theorem. Q.E.D.

We note that some of the corollaries of the big Cohen–Macaulay module conjecture are super height theorems, i.e., they bound a super height. For example, see [H_2, Section 7] for homological Krull altitude theorem and [K] for the discussion of the Direct Summand Conjecture in terms of super height.

The example included shows that the theorem cannot be generalized to the Noetherian case but one might still hope to generalize to the excellent case because all known examples are non-excellent.

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1. PROOF OF THE THEOREMS

We first prove Theorem 1. Let super ht $IR_p=n$. Let S be a Noetherian R_p-algebra and Q be a prime ideal of ht n which is minimal over IS. We may localize S at Q, complete, and kill a minimal prime of maximal coheight to assume that S is a complete local domain with maximal ideal Q. We may replace S by a suitable faithfully flat extension to assume that its residue field K is algebraically closed. Because we are free to replace R by a finitely generated R-subalgebras of S and IS is primary to Q, we may assume that $R=k[x_1, ..., x_n, y_1, ..., y_m] \subset S$, $x_1, ..., x_n$ forms a system of parameters for S, and I is generated by $x_1, ..., x_n$.

Notation. If T is a subalgebra of S containing $x_1, ..., x_n$, then IT will denote the ideal of T generated by $x_1, ..., x_n$.

We must show that finite super ht $IR_p = n$. Since $Q \cap R \subset P$, it suffices to show that finite super ht $IR_{(Q \cap R)} = n$.

Claim 1. *Let k_0 be the prime subfield of k. Then there is a finitely generated k_0-subalgebra B of R containing $x_1, ..., x_n$ such that finite super ht $IB_{(Q \cap B)} \leqslant$ finite super ht $IR_{(Q \cap R)}$.*

Proof. Write R as $k[X_1, ..., X_n, Y_1, ..., Y_m]/(F_1, ..., F_l)$ and let L be the subfield of k generated over k_0 by coefficients of $F_1, ..., F_l$. Let $R^* = L[X_1, ..., X_n, Y_1, ..., Y_m]/(F_1, ..., F_l)$. Then $R = R^* \otimes_L k$ is faithfully flat over R^* and hence $R_{(Q \cap R)}$ is faithfully flat over $R^*_{(Q \cap R^*)}$. One now checks easily from the Going Down Theorem that

$$\text{finite super ht } IR_{(Q \cap R)} \geqslant \text{finite super ht } IR^*_{(Q \cap R^*)}.$$

Because R^* is a localization of a finitely generated k_0-algebra, we may pick a suitable k_0-subalgebra B of R^* to complete a proof. Q.E.D.

By Claim 1, we may assume that k is perfect and we can pick a coefficient field K of S containing k. Since $x_1, ..., x_n$ is a system of parameters for S, S is module-finite over $K[[x_1, ..., x_n]]$. Enlarging S, if necessary, we may assume that S is the integral closure of $K[[x_1, ..., x_n]]$ in some finite Galois extension of the fraction field of $K[[x_1, ..., x_n]]$. (See [H_1, bottom of p. 23] for the case when char $k > 0$.) Let $\{\sigma_1, ..., \sigma_r\}$ be the Galois group of the extension. We may clearly enlarge R to assume that $R = k[x_1, ..., x_n, \sigma_i(y_j)]_{i \leqslant i \leqslant r, 1 \leqslant j \leqslant m} \subset S$.

Lemma 1. *Let A be a finitely generated k-subalgebra of $K[[x_1, ..., x_n]]$ containing $x_1, ..., x_n$. Then there exists a finitely generated A-algebra T and a prime ideal q of T such that*

(i) *q is minimal over IT,*

(ii) *ht $q = n$ ($=$super ht IA), and*

(iii) *$q \cap A = Q \cap A$.*

Proof. $A \subsetneq k[[x_1, ..., x_n]]$ means that a certain system of polynomial equations over k in variables $X_1, ..., X_n$ and others has a solution in $K[[x_1, ..., x_n]]$ in such a way that $X_i = x_i$ for $i \leqslant i \leqslant n$. Let (A^*, m^*) be the Henselization of $K[x_1, ..., x_n]_{(x_1, ..., x_n)}$. By the Artin approximation theorem, the system has a solution in A^* which is congruent to the original solution modulo the maxiamal ideal of $K[[x_1, ..., x_n]]$. This means that there is a ring homomorphism $\phi: A \to A^*$ such that $\phi(x_i) = x_i$ and $m^* \cap A = Q \cap A$. Since A^* is the union of pointed étale extensions of $K[x_1, ..., x_n]_{(x_1, ..., x_n)}$, $\phi(A)$ is contained in some A-subalgebra of A^* which is a localization of a finitely generated algebra over K. By the same argument used in the proof of Claim 1, we may assume that $\phi(A)$ is contained in a finitely generated A-subalgebra of A^* with the desired properties. Q.E.D.

The next claim will complete the proof of Theorem 1.

CLAIM 2. *There is a finitely generated k-subalgebra A of $K[[x_1, ..., x_n]]$ containing $x_1, ..., x_n$ such that*

$$\textit{finite super ht } IR_{(Q\cap R)} \geqslant \textit{finite super ht } IA_{(Q\cap A)}.$$

Notation. By $f_i(v_1, ..., v_r)$, $1 \leqslant i \leqslant r$, we denote the ith elementary symmetric function in $v_1, ..., v_r$, i.e.,

$$f_1(v_1, ..., v_r) = \sum_{1\leqslant i\leqslant r} v_i$$

$$f_2(v_1, ..., v_r) = \sum_{1\leqslant i<j\leqslant r} v_i v_j$$

$$\vdots$$

$$f_r(v_1, ..., v_r) = \prod_{1\leqslant i\leqslant r} v_i.$$

Proof of Claim 2. Let $C = k[x_1, ..., x_n, c_{ij}]_{1\leqslant i\leqslant r, 1\leqslant j\leqslant m} \subset S$, where $c_{ij} = f_i(\sigma_1(y_j), ..., \sigma_r(y_j))$. Then R is module-finite over C and $C \subset K[[x_1, ..., x_n]]$. Let $P_1, ..., P_t$ be all primes of R lying over $Q \cap C$ and distinct from $Q \cap R$. Since $P_l \neq Q \cap R$ for all $1 \leqslant l \leqslant t$, we can pick elements u_l of S which have their inverses in P_l.

Let $D = R[\sigma_i(u_l)]_{1\leqslant i\leqslant r, 1\leqslant l\leqslant t}$ and $A = C[a_{il}]_{1\leqslant i\leqslant r, 1\leqslant l\leqslant t}$, where

$$a_{il} = f_i(\sigma_1(u_l), ..., \sigma_r(u_l)).$$

Let T and q be as in Lemma 1. Let $T^* = T \otimes_A D$. Since T^* is integral over T, there is a prime ideal q^* of T^* lying over q such that $\text{ht } q^* = \text{ht } q$. Because q^* is automatically minimal over IT^*, we already have finite super ht $IR \geqslant \text{ht } q^*$. But to conclude that finite super $\text{ht } IR_{(Q\cap R)} \geqslant \text{ht } q^*$, it remains to check that $q^* \cap R \subset Q \cap R$. In fact, $q^* \cap R = Q \cap R$ because it is a prime lying over $Q \cap C$ and contracted from D(recall that $PD = D$ for all $1 \leqslant l \leqslant t$). Q.E.D.

Remark. The reason we modified C to A is to make sure that $q^* \cap R = Q \cap R$.

We now prove Theorem 2. As in Theorem 1, we may assume that the super height is obtained in S which is the integral closure in some finite Galois extension of the fraction field of $W[[x_2, ..., x_n]]$, where W is the ring of Witt vectors over K, $R = V_p[x_2, ..., x_n, \sigma_i(y_j)]_{1\leqslant i\leqslant r, 1\leqslant j\leqslant m} \subset S$, and

I is the ideal generated by $(p, x_2, ..., x_n)$. Since there is a unique copy of F_p in K, $V_p \subset W$.

As before, Lemma 2 and Claim 2′ below prove Theorem 2.

LEMMA 2. *Let A be a finitely generated V_p-subalgebra of $W[[x_2, ..., x_n]]$ containing $x_2, ..., x_n$. Then finite super ht IA = super ht IA. (Recall that I is the ideal generated by $p, x_2, ..., x_n$.)*

Proof. This is again a question of solving a certain system of polynomial equations over V_p (henceforth V) in variables $X_1, ..., X_n$ and others. We claim that the system has a solution in $V[[x_2, ..., x_n]]$. By [PP, Theorem 2.4], it suffices to solve the system in $(V(p^l V)[[x_2, ..., x_n]]/(x_2, ..., x_n)^l$ for all $l \geq 1$. This ring is isomorphic to some affine space $F_p^{N(l)}$ such that the ring operations are polynomial functions in the coordinates. The equations in the system translate into equations over F_p in $N(l)$ vaiables. Since this new system has a solution in $K \supset F_p$, it has a solution in F_p because F_p is algebraically closed and our claim is proved. Hence the super height of IA is obtained in $V[[x_2, ..., x_n]]$ and the conclusion follows as in Lemma 1. Q.E.D.

Remark. When we solve the system in $V[[x_2, ..., x_n]]$, we get a ring homomorphism $\phi: A \to V[[x_2, ..., x_n]]$. The author does not know at present if $Q \cap A = \phi^{-1}$ (maximal ideal) $\cap A$ and it is for this reason that a localization is not allowed in Theorem 2.

CLAIM 2′. *There is a finitely generated V_p-subalgebra A of $W[[x_2, ..., x_n]]$ containing $x_2, ..., x_n$ such that finite super ht $IR \geq$ finite super ht IA.*

Proof. See proof of Claim 2. In this case, we can use C in Claim 2 because there is no prime to keep track of. Q.E.D.

2. EXAMPLE

By modifying Nagata's example [N, p. 209, Example 7] of 2-dimensional analytically reducible normal domain, we will get an example of a height 1 prime in a 2-dimensional non-excellent domain whose super height is 2 but finite super height is 1.

Let S be the ring $R[X]/(X^2 - z) R[X]$ described in [N, (E7.1)]. Then S is a 2-dimensional normal local domain dominated by $K[[x, y]]$ and its maximal ideal M_S is minimally generated by x, y, and w.

Let T = (the fraction field of S) $\cap K[[x, y]]$. Then (T, M_T) is a commutative ring with a unique maximal ideal (see Remark 1, below) which dominates (S, M_S) and is dominated by $K[[x,, y]]$. Since $M_S T = (x, y) T$ $(w = x(w/x)$ and $(w/x) \in T)$, $S \subsetneq T$. Pick $t \in T \backslash S$ and let $A = S[t] \subset T$.

Then $A \cong S[X]/J$, where J is the height 1 prime generated by $\{aX-b \mid a, b \in S, at-b=0\}$ (S is normal). Since $t \notin S$, $J \subset M_S(S[X])$ and $I = M_S A$ is a height 1 prime of a 2-dimensional domain A. Since I cntains x, y, and $A \subset K[[x, y]]$, super ht $I = 2$ (note that super ht $I \leqslant \dim A = 2$). Now suppose that finite super ht $I = 2$. By the fact in the Introduction, there is a height 2 prime in the normalization A' of A which is minimal over IA' ($\dim A = 2$). Since Q is a maximal ideal of A', Zariski's main theorem applied to $S \subset S[t] = A \subset A'$ and Q implies that $S = A'$ (S is local) which contradicts that $t \notin S$.

Hence finite super ht $I = 1$ and our example is complete (non-excellence of the ring A will be checked in Remark 2.).

Remark 1. W. Heinzer pointed out that there is a height 1 prime in S whose finite super ht is 1 and super height is 2. For his argument one needs to prove that T is a 2-dimensional local ($\Rightarrow$ Noetherian) ring.

Remark 2. G. Lyubeznik pointed out that Example 7 in [N] should have been the first place to look for this kind of example because of the following consequence of a result of Hartshorne [Ha, Theorem 3.1.]:

> If I is a height-1 ideal in a 2-dimensional analytically irreducible local ring, then super ht $I = 1$.

This implies that A cannot be excellent. If A were, then so is A' and a localization of A' is analytically irreducible. But some height-1 prime of A' lying over I is of super height 2. Hence A is not excellent.

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