Note on the uniqueness of subsonic Euler flows in an axisymmetric nozzle

Lili Du\textsuperscript{a,c}, Ben Duan\textsuperscript{b,c,}\textsuperscript{*}

\textsuperscript{a} Department of Mathematics, Sichuan University, Chengdu 610064, PR China
\textsuperscript{b} Department of Mathematics, Yeshiva University, 500 West 185th Street, Belfer Hall, New York, NY 10033, United States
\textsuperscript{c} The Institute of Mathematical Sciences, The Chinese University of Hong Kong, Shatin, NT, Hong Kong

A R T I C L E   I N F O

Article history:
Received 29 September 2010
Received in revised form 8 July 2011
Accepted 27 July 2011

Keywords:
Uniqueness
Subsonic flows
Steady Euler equations
Axisymmetric nozzle

A B S T R A C T

In this paper, we consider the uniqueness of globally subsonic compressible flows through an infinitely long axisymmetric nozzle. The flow is governed by the steady Euler equations and satisfies no-flow boundary conditions on the nozzle walls. We will show that for given mass flux and Bernoulli’s function in the upstream, the subsonic flow is unique in the class of all axisymmetric solutions, which possess the asymptotic behaviors at the far fields. This result extends the uniqueness of solutions in the previous paper Du and Duan (2011) [1].

1. Introduction

This is a continuation of our study the subsonic problem of the compressible flows in an axisymmetric nozzle. In the previous work [1], we established the existence of globally subsonic flows throughout an axisymmetric nozzle, provided that the variation of the given Bernoulli’s function \( B(r) \) is sufficiently small and the mass flux \( m \) is less than the critical value. More precisely, the authors constructed a special subsonic flow which is axisymmetric and possesses a zero-swirl component. Furthermore, we showed that such an axisymmetric flow is unique in the sense of the flow possessing a zero-swirl component. In this short paper, we will consider the uniqueness of the subsonic flows in a more general sense.

We investigate the steady compressible flow of ideal isentropic gas. The flow is governed by the conservation laws of mass and momentum, which provide the following system for velocity field \( u = (u_1, u_2, u_3) \), the density \( \rho \) and the pressure \( p \),

\[
\begin{align*}
(\rho u_1)_x &+ (\rho u_2)_y + (\rho u_3)_z = 0, \\
(\rho u_1^2 + p)_x &+ (\rho u_1 u_2)_y + (\rho u_1 u_3)_z = 0, \\
(\rho u_1 u_2)_x &+ (\rho u_2^2 + p)_y + (\rho u_2 u_3)_z = 0, \\
(\rho u_1 u_3)_x &+ (\rho u_3 u_2)_y + (\rho u_3^2 + p)_z = 0.
\end{align*}
\]

In general, we assume \( p = p(\rho) \) is a smooth function with \( p'(\rho) > 0 \) and \( p''(\rho) \geq 0 \) for \( \rho > 0 \). For example, the state equation for the ideal polytropic gas is given by

\[
p(\rho) = A\rho^\gamma.
\]
where $A$ is a positive constant depending on the entropy and $\gamma$ is the adiabatic exponent between 1 and 5/3. The quantity $c(\rho) = \sqrt{\rho / (\rho)}$ is called the sound speed and the ratio $M = \frac{|u|}{c(\rho)}$ is called the Mach number. The flow is subsonic for $M < 1$, sonic for $M = 1$ and supersonic for $M > 1$.

Suppose that the lengths of the nozzles to be considered are usually much larger than their cross-sections in the practical application, then the problem can be formulated mathematically into an infinitely long nozzle problem. In this paper, we always assume that the nozzle is infinitely long and axisymmetric as

$$
\Omega = \left\{(x_1, x_2, x_3) \in \mathbb{R}^3 | 0 \leq \sqrt{x_2^2 + x_3^2} < f(x_1), -\infty < x_1 < +\infty, \right\},
$$

where $f(x_1)$ satisfies

$$
f(x_1) \to 1, \quad \text{as} \quad x_1 \to -\infty, \quad f(x_1) \to r_0 > 0, \quad \text{as} \quad x_1 \to +\infty,
$$

$$
\|f\|_{C^2(\mathbb{R})} \leq C \quad \text{for some} \quad \alpha > 0, \quad C > 0 \quad \text{and} \quad \inf \limits_{\mathbb{R}} f(x_1) = b > 0. \quad (1.2)
$$

Since the nozzle walls are impermeable, then the flow satisfies the following boundary condition

$$
(u_1, u_2, u_3) \cdot \vec{n} = 0 \quad \text{on} \quad \partial \Omega, \quad (1.3)
$$

where $\vec{n}$ is the unit outward normal to the nozzle walls. The conservation of mass and the no-flow boundary condition (1.3) imply that the mass flux $\int_\Sigma \rho \cdot \vec{l} ds$ remains for some positive constant $m_0$, where $\Sigma$ is any surface transversal to the $x_1$-axis direction, and $\vec{l}$ is the normal of $\Sigma$ in the positive $x_1$-axis direction.

The purpose of this paper is to show that there exists at most one axisymmetric solution to Eq. (1.1), which satisfies the same asymptotic behavior at the far fields and possesses the same mass flux. The result shows that the subsonic flows constructed in [1] are uniquely determined by the incoming mass flux and the asymptotic behaviors in the upstream, provided that the admissible flows are axisymmetric in the nozzle. This result provides an affirmation answer to an open question raised by the authors in the previous work [1] at least in the case of axisymmetric flows (see Remark 1.4 in [1]).

2. The main result and proof

Consider the problem in cylindrical coordinates, let the density and velocity field of the axisymmetric flows be $\rho(x, r)$ and $(U(x, r), V(x, r), W(x, r))$, where $U$, $V$, $W$ are axial velocity, radial velocity and swirl velocity respectively, $x = x_1$, $r = \sqrt{x_2^2 + x_3^2}$. Then, the cylindrical variables satisfy that

$$
u_1 = U(x, r), \quad \nu_2 = V(x, r) \frac{x_2}{r} - W(x, r) \frac{x_3}{r}, \quad \nu_3 = V(x, r) \frac{x_3}{r} + W(x, r) \frac{x_2}{r}.
$$

The axisymmetric flows satisfy

$$
\begin{align*}
(r \rho U)_x + (r \rho V)_r &= 0, \\
(r \rho U^2)_x + (r \rho UV)_r + r p_r &= 0, \\
(r \rho UV)_x + (r \rho V^2)_r + r p_r &= \rho W^2, \\
(r \rho UW)_x + (r \rho WV)_r + r V W &= 0.
\end{align*}
\quad (2.1)
$$

Define the mass flux of the axisymmetric flow in the cylindrical coordinates as

$$
\int_\Sigma (r \rho U, r \rho V, 0) \cdot l dS \equiv m = \frac{m_0}{2\pi},
$$

where $\Sigma$ is any curve transversal to the $x$-axis direction and $\vec{l}$ is unit normal of $\Sigma$. 


Assume that the hypotheses of Theorem 1 hold, it suffices to show that the swirl component of any smooth subsonic axisymmetric flow must be zero, provided that its asymptotic conditions through the nozzle which satisfies the properties (1.2) and Bernoulli's function $B(r)$ in the upstream satisfies

$$B > B_0, \quad B'(r) \in C^{0,1}((0, 1]), \quad B'(0) = B'(1) = 0, \quad B'(r) \geq 0 \text{ on } r \in [0, 1]$$

then there exists a critical mass flux $m_0$ such that for any $m \in (\delta_0^{1/4}, m_0)$, there exists a unique $C^{1,a}$-smooth axisymmetric flow $(\rho, U, V, 0)$ satisfying following conditions

1. The solution is globally uniformly subsonic and the axial velocity is always positive,
2. Asymptotic behaviors in the far fields

$$\rho \to \rho_0 > 0, \quad U \to U_0(r) > 0, \quad V \to 0 \text{ as } x \to -\infty,$$

$$\nabla \rho \to 0, \quad \nabla U \to (0, U'_0(r)) > 0, \quad \nabla V \to 0 \text{ as } x \to -\infty,$$

uniformly for $r \in K_1 \subset (0, 1)$, and

$$\rho \to \rho_1 > 0, \quad U \to U_1(r) > 0, \quad V \to 0 \text{ as } x \to +\infty,$$

$$\nabla \rho \to 0, \quad \nabla U \to (0, U'_1(r)) > 0, \quad \nabla V \to 0 \text{ as } x \to +\infty,$$

uniformly for $r \in K_2 \subset (0, r_0)$, where $\rho_0$ and $\rho_1$ are both positive constants, and $\rho_0, \rho_1, U_0(r)$ and $U_1(r)$ can be determined by $m$, $B(r)$ and $r_0$ uniquely.

In this short note, we will extend the uniqueness to general axisymmetric flow. The main result is stated as follows.

Theorem 2.1. Assume that the hypotheses of Theorem A hold, there exists at most one smooth axisymmetric subsonic flow through the nozzle which satisfies the properties (1), (2) in Theorem A, and $W = 0$ as $x \to -\infty$.

Proof. It suffices to show that the swirl component of any smooth subsonic axisymmetric flow must be zero, provided that it satisfies the asymptotic conditions.

The system of conservation laws (2.1) in the cylindrical coordinates can be written in a matrix form as

$$AF_r + BF_r + C = 0$$
where
\[
A = \begin{pmatrix}
\frac{U c^2(\rho)}{\rho} & c^2(\rho) & 0 & 0 \\
\frac{c^2(\rho)}{\rho U} & \frac{\rho U}{\rho} & 0 & 0 \\
0 & 0 & \rho U & 0 \\
0 & 0 & 0 & \rho U
\end{pmatrix}, \quad
B = \begin{pmatrix}
\frac{V c^2(\rho)}{\rho} & 0 & c^2(\rho) & 0 \\
0 & 0 & \rho V & 0 \\
0 & 0 & 0 & \rho V \\
0 & 0 & 0 & V
\end{pmatrix}, \quad
C = \begin{pmatrix}
\frac{V c^2(\rho)}{\rho} \\
0 \\
\rho W^2 \\
\frac{V W}{r}
\end{pmatrix},
\]
and \(F = (\rho, U, V, W)^T\) are the unknown variables. A direct computation yields that the eigenvalues of the symmetric system are
\[
\lambda_{1,2} = \frac{V}{U}, \quad \lambda_{3,4} = \frac{\rho U \pm c(\rho) \sqrt{U^2 + V^2 - c^2(\rho)}}{U^2 - c^2(\rho)},
\]
which are the solutions of
\[
\det(\lambda A - B) = 0.
\]
The symmetric system has at least two real eigenvalue \(\lambda_{1,2}\) which implies we have to deal with the hyperbolic modes.

**Case 1.** On the axis \(T\). It follows from the second equation in system (2.3) that the swirl velocity \(W\) must be zero for \(r = 0\).

**Case 2.** For \(r \neq 0\), we use the streamline argument to show the fact. It follows from the last equation in (2.3) that
\[
W_x + \frac{V}{U} W_r + \frac{V}{r U} W = 0 \quad \text{for } r \neq 0.
\]
Due to the positivity of \(U\), for any point in the inlet, there is one and only one streamline satisfying
\[
\frac{dr(x)}{dx} = \frac{V}{U}(x, r(x)), \quad r(x = -\infty) = a,
\]
for any \(a \in (0, 1]\). Obviously, it can be defined globally in the nozzle. Furthermore, any streamline cannot touch the axis for \(a \neq 0\). Thus,
\[
\frac{d}{dx} W(x, r(x)) + \frac{1}{r(x)} \frac{V W}{U}(x, r(x)) = 0, \quad W(x, r(x))|_{x=-\infty} = 0,
\]
which is a linear ordinary differential equation to \(W\). Hence, we have \(W = 0\). \(\square\)

**Remark 2.1.** The positivity of the axial velocity is crucial to the proof of the uniqueness, which guarantees a simple topological structure of streamline.

**Acknowledgments**

The authors would like to thank the referees for their valuable comments and suggestions.

**References**