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Cyclotomic double affine Hecke algebras and affine parabolic category $\mathcal{O}$

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Abstract

Using the orbifold KZ connection we construct a functor from an affine parabolic category $\mathcal{O}$ of type A to the category $\mathcal{O}$ of a cyclotomic rational double affine Hecke algebra $\mathbf{H}$. Then we give several results concerning this functor.

Keywords: Cyclotomic double affine Hecke algebras; Affine Lie algebras; Affine parabolic category $\mathcal{O}$

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Introduction

Fix positive integers n, ℓ. Let W be the wreath product of $\mathfrak{S}_n$ and $\mathbb{Z}/\ell\mathbb{Z}$. The cyclotomic rational double affine Hecke algebra $\mathbf{H}$ is a deformation of the semi-direct product $\mathbb{C}[\mathbb{C}^{2n}] \rtimes W$. Its category $\mathcal{O}$ is a quasi-hereditary cover of the Ariki–Koike algebra, see [22]. One important problem is to compute the dimension of the simple modules of the category $\mathcal{O}$ of $\mathbf{H}$, or, equivalently, the Jordan–Hölder multiplicities of the standard modules. So far the main approach to the representation theory of double affine Hecke algebras associated with complex reflection groups is geometric and uses D-modules on quiver varieties. However a dimension formula for simple modules seems out of reach yet by these techniques. It is expected that the Jordan–Hölder multiplicities of the standard modules are values at one of some affine parabolic Kazhdan–Lusztig polynomial, see [22, Sec. 6.5]. We'll call this the dimension conjecture. See Section 8 below for details. These multiplicities are encoded in a combinatorial object called the level ℓ Fock space.

If $\ell = 1$ the dimension conjecture is proved. It follows from [22] and [28], or from [24] and [29]. In this case there is another algebraic approach to the algebra $\mathbf{H}$ due to Suzuki. He constructed a functor from Kazhdan–Lusztig's category of modules over the type $A^{(1)}$ affine Lie algebra to the category $\mathcal{O}$ of $\mathbf{H}$. We give a proof that this functor is an equivalence in Section A.5 below. This functor is constructed via affine coinvariants over the configuration space of $\mathbb{P}^1$ and the Knizhnik–Zamolodchikov connection.

In this paper we construct a similar functor for any ℓ. The new ingredient is the space of orbifold affine coinvariants over the configuration space of the stack $[\mathbb{P}^1/(\mathbb{Z}/\ell\mathbb{Z})]$ and the corresponding Knizhnik–Zamolodchikov connection. A priori this space of coinvariants involves a choice of a twisted affine Lie algebra. Choosing an inner twist of the type $A^{(1)}$ affine Lie algebra, we get a functor $\mathfrak{E}$ from an affine parabolic category $\mathcal{O}$ to the category $\mathcal{O}$ of $\mathbf{H}$. Then we study $\mathfrak{E}$, in particular its behavior on standardly filtered modules. We do not prove the dimension conjecture. Contrarily to the case $\ell = 1$ mentioned above, the functor $\mathfrak{E}$ is not an equivalence of quasi-hereditary categories in general. However, we expect the functor $\mathfrak{E}$ to be an important tool to prove it. In particular $\mathfrak{E}$ should behave nicely on indecomposable projective modules, as explained in this paper. We'll come back to this elsewhere.

Let us now describe the structure of the paper. The first and second sections are reminders on DAHA's (= double affine Hecke algebras) and Suzuki's functor. In the third one we consider the case $\ell \neq 1$. Using the orbifold Knizhnik–Zamolodchikov connection we define a functor taking a smooth module over a twisted affine Lie algebra to a $\mathbf{H}$-module and we compute the image of parabolic Verma modules. In the fourth section we compare the space of twisted affine coinvariants and the space of non-twisted ones when the affine Lie algebra is equipped with an inner twist. In the sixth section we define the functor $\mathfrak{E}$. It goes from the affine parabolic category $\mathcal{O}$ of type $A^{(1)}$ to the category $\mathcal{O}$ of $\mathbf{H}$. We prove that $\mathfrak{E}$ preserves the posets of standard modules and is exact on standardly filtered modules in Section 7. We conjecture that it preserves the set of indecomposable projective modules. In the last section we compare $\mathfrak{E}$ with what one expects from the dimension conjecture.

0. Notations

0.1. First, let us gather a few basic notations on categories. The categories we'll consider are all $\mathbb{C}$-linear, i.e., they are additive and the Hom sets are $\mathbb{C}$-vector spaces. A category is *Artinian* if the Hom sets are finite dimensional $\mathbb{C}$-vector spaces and every object has a finite length. Write $\mathcal{A}^{fg}$ for the full subcategory of an Abelian category $\mathcal{A}$ consisting of the objects of finite length and $\mathcal{A}^{\mathrm{proj}} \subset \mathcal{A}^{fg}$ for the full exact subcategory of projective objects. Given a set I of objects of the Abelian category $\mathcal{A}$, we denote by $\mathcal{A}^I$ the exact full subcategory of I-filtered objects, i.e., of objects M with a finite filtration such that each successive quotient is isomorphic to an object of I.

Write $[M : S]$ for the Jordan–Hölder multiplicity of a simple module S in an object M of finite length.

By a *quasi-hereditary category* we mean a highest weight category in the sense of [3] which is equivalent, as a highest weight category, to the category of finitely generated modules of a quasi-hereditary (finite dimensional) $\mathbb{C}$-algebra. In other words, it is an Artinian Abelian category with a finite poset $\Delta_{\mathcal{A}}$ of standard modules satisfying the following axioms:

(a) we have $\mathrm{End}_{\mathcal{A}}(M) = \mathbb{C}$ for each $M \in \Delta_{\mathcal{A}}$,
(b) if $\mathrm{Hom}_{\mathcal{A}}(M_1, M_2) \neq 0$ and $M_1, M_2 \in \Delta_{\mathcal{A}}$ then $M_1 \leqslant M_2$,
(c) if $\mathrm{Hom}_{\mathcal{A}}(M, N) = 0$ for each $M \in \Delta_{\mathcal{A}}$ then $N = 0$,
(d) if $M \in \Delta_{\mathcal{A}}$ there is a projective object $P \in \mathcal{A}$ and an epimorphism $P \to M$ whose kernel is $\Delta_{\mathcal{A}}$-filtered with subquotients $> M$.

See [3, Thm. 3.6], [6, Appendix] for more details. An equivalence of highest weight categories is an equivalence of categories which restricts to a bijection between both sets of standard modules. We'll abbreviate $\mathcal{A}^{\Delta} = \mathcal{A}^{\Delta_{\mathcal{A}}}$.

We'll write $[\mathcal{B}]$ for the Grothendieck group of an Abelian or an exact category $\mathcal{B}$. Let $[M]$ denote the class in $[\mathcal{B}]$ of an object M. Note that the obvious embedding $\mathcal{A}^{\Delta} \subset \mathcal{A}^{fg}$ yields a group isomorphism $[\mathcal{A}^{\Delta}] = [\mathcal{A}^{fg}]$.

0.2. Let R be a commutative Noetherian ring with 1, and let $\mathbf{A}$ be an R-algebra. Write **A-mod** for the module category of $\mathbf{A}$, **A-proj** for $\mathbf{A}\text{-}\mathbf{mod}^{\mathrm{proj}}$ and $\mathbf{A}\text{-}\mathbf{mod}^{fg}$ for $(\mathbf{A}\text{-}\mathbf{mod})^{fg}$. Let $\mathrm{Irr}(\mathbf{A})$ be the set of isomorphism classes of simple objects of $\mathbf{A}\text{-}\mathbf{mod}^{fg}$. To any R-algebra homomorphism $\phi : \mathbf{A} \to \mathbf{B}$ we associate the functor

$$\phi : \mathbf{B}\text{-}\mathbf{mod} \to \mathbf{A}\text{-}\mathbf{mod}, \qquad M \mapsto {}^{\phi}M,$$

where ${}^{\phi}M$ is the twist of M by ϕ.

0.3. Let M be a $\mathbb{C}$-vector space and R be a commutative $\mathbb{C}$-algebra. Assume that R is the functions algebra of a $\mathbb{C}$-variety R. We write

$$M[X] = M_R = M \otimes R.$$

Given an automorphism F of a set M, let M^F be the fixed points set. If F is an R-linear automorphism F of M_R we may abbreviate $M_R^F = (M_R)^F$.

1. The cyclotomic rational DAHA and the Dunkl operators

1.1. The complex reflection group $G(\ell, 1, n)$. Let $D_\ell \subset \mathbb{C}^\times$ be the group consisting of the ℓ-th roots of unity. Fix a generator of D_ℓ once for all. We'll denote it by ε. Let $\mathfrak{S}_n$ be the symmetric group on n letters and W be the semi-direct product $\mathfrak{S}_n \ltimes (D_\ell)^n$, where $(D_\ell)^n$ is the Cartesian product of n copies of D_ℓ. To avoid any confusion we may write

$$W_A = W, \quad A = \{1, 2, \dots, n\}.$$

Let $\varepsilon_i \in (D_\ell)^n$ be the element with ε at the i-th place and to 1 at the other ones. If $a \in \mathbb{Z}$ we may identify $\mathbb{Z}/a\mathbb{Z}$ with the set $\{1, 2, \dots, a\}$ in the obvious way, hoping it will not create any confusion. Set

$$\Lambda = \{1, 2, \dots, \ell\} \simeq \mathbb{Z}/\ell\mathbb{Z}.$$

For each $p \in \Lambda$ and each $i \neq j$, we write $s_{i,j}^{(p)}$ for $s_{i,j}\varepsilon_i^p \varepsilon_j^{-p}$.

1.2. The cyclotomic rational DAHA. Fix a basis (x, y) of $\mathbb{C}^2$. Let x_i, y_i denote the elements x, y respectively in the i-th summand of $(\mathbb{C}^2)^{\oplus n}$. The group W acts on $(\mathbb{C}^2)^{\oplus n}$ such that for distinct i, j, k we have

$$\varepsilon_i(x_i) = \varepsilon^{-1}x_i, \qquad \varepsilon_i(x_j) = x_j, \qquad \varepsilon_i(y_i) = \varepsilon y_i, \qquad \varepsilon_i(y_j) = y_j,$$
$$s_{i,j}(x_i) = x_j, \qquad s_{i,j}(y_i) = y_j, \qquad s_{i,j}(x_k) = x_k, \qquad s_{i,j}(y_k) = y_k.$$

Fix $k \in \mathbb{C}$ and $\gamma_p \in \mathbb{C}$ for $0 \neq p \in \Lambda$. We'll write γ for the ℓ-tuple (γ_p). The CRDAHA (= cyclotomic rational DAHA) is the quotient $\mathbf{H}_{k,\gamma}$ of the smash product of $\mathbb{C}W$ and the tensor algebra of $(\mathbb{C}^2)^{\oplus n}$ by the relations

$$[y_i, x_i] = 1 - k\sum_{j\neq i}\sum_p s_{i,j}^{(p)} - \sum_{p\neq 0}\gamma_p\varepsilon_i^p,$$
$$[y_i, x_j] = k\sum_p \varepsilon^p s_{i,j}^{(p)} \quad \text{if } i \neq j,$$
$$[x_i, x_j] = [y_i, y_j] = 0.$$

This presentation is the same as in [7]. We'll use another presentation where the parameters are h, h_p with $p \in \Lambda$ and $\sum_p h_p = 0$. Set $H = (h_1, h_2, \dots, h_{\ell-1})$. The corresponding algebra is denoted by the symbol $\mathbf{H}_{h,H}$. It is isomorphic to the algebra $\mathbf{H}_{k,\gamma}$ with $k = -h$ and $\gamma_p = -\sum_{p'\in\Lambda}\varepsilon^{-pp'}h_{p'}$. Our parameter h_p is the same as the parameter H_p in [11] and it is equal to $h_{H_0,p} - h_{H_0,p-1}$ with respect to the parameters $h_{H_0,p}$, $p \in \Lambda$, in [22].

1.3. The Dunkl operators. The subalgebras of $\mathbf{H}_{h,H}$ generated by $\{x_1, \dots, x_n\}$ and $\{y_1, \dots, y_n\}$ are free commutative. We'll write $\mathbf{R}$ for the first one and $\mathbf{R}^*$ for the second one. Note that

$\mathbf{R} = \mathbb{C}[\mathbb{C}^n]$. There is a $\mathbb{C}$-linear representation of $\mathbf{H}_{h,H}$ on $\mathbf{R}$ such that $x_i \mapsto x_i$, $w \mapsto w$ and $y_i \mapsto \bar{y}_i$ with

$$\bar{y}_i = \partial_{x_i} + k \sum_{j \neq i} \sum_p \frac{1}{x_i - \varepsilon^{-p} x_j} \big(s_{i,j}^{(p)} - 1 \big) + \sum_{p \neq 0} \frac{\gamma_p}{x_i - \varepsilon^{-p} x_i} \big(\varepsilon_i^p - 1 \big).$$

The operators $\bar{y}_i$ are called the *Dunkl operators*.

1.4. Combinatorics. Let $\mathcal{C}_{m,\ell}$ be the set of compositions of m with ℓ parts, i.e., the set of tuples $\nu = (\nu_1, \nu_2, \ldots, \nu_\ell) \in \mathbb{N}^\ell$ with sum $|\nu| = m$. Let $\mathcal{C}_{m,\ell,n}$ be the subset of compositions whose parts are all $\geqslant n$. Set

$$J = \{1, 2, \ldots, m\}.$$

To each $\nu \in \mathcal{C}_{m,\ell}$ we associate the following partition

$$J = J_{\nu,1} \sqcup J_{\nu,2} \sqcup \cdots \sqcup J_{\nu,\ell}, \qquad J_{\nu,p} = \{i_p, i_p + 1, \ldots, j_p\},$$
$$i_p = 1 + \nu_1 + \cdots + \nu_{p-1}, \qquad j_p = i_{p+1} - 1, \quad p \in \Lambda.$$

We may write $J_p = J_{\nu,p}$ if there is no risk of confusion. Next, we write

$$\mathbb{C}^\nu_{\geqslant 0} = \big\{ \lambda \in \mathbb{C}^m;\ \lambda_i - \lambda_{i+1} \in \mathbb{N},\ \forall i \neq j_1, j_2, \ldots \big\},$$
$$\mathbb{Z}^\nu_{\geqslant 0} = \mathbb{Z}^m \cap \mathbb{C}^\nu_{\geqslant 0}, \qquad \mathbb{Z}^\nu_{>0} = \rho + \mathbb{Z}^\nu_{\geqslant 0}.$$

The elements of $\mathbb{C}^\nu_{\geqslant 0}$ are called the *ν-dominant weights*. The elements of $\mathbb{Z}^\nu_{\geqslant 0}$ are called the *ν-dominant integral weights*. Let $\mathcal{P}_n$ be the set of partitions of n, i.e., the set of non-increasing sequences λ of integers $\lambda_1, \lambda_2, \ldots > 0$ with sum n. We write ${}^t\lambda$ for the transposed partition, $|\lambda|$ for the weight of λ, $n(\lambda)$ for the integer $\sum_i \lambda_i (i-1)$ and $l(\lambda)$ for its length, i.e., for the number of parts in λ. Let $\mathcal{P} = \bigsqcup_n \mathcal{P}_n$ be the set of all partitions. Let $\mathcal{P}^\ell_n$ be the set of ℓ-partitions of n. It is the set of Λ-tuples $\lambda = (\lambda_p)$ of partitions with $\sum_p |\lambda_p| = n$. Let $\mathcal{P}^\ell = \bigsqcup_n \mathcal{P}^\ell_n$ be the set of all ℓ-partitions. Given any ℓ-tuple $\nu = (\nu_p)$ in $\mathbb{N}^\ell$ we set

$$\mathcal{P}^\ell_{n,\nu} = \big\{ \lambda \in \mathcal{P}^\ell_n;\ l(\lambda_p) \leqslant \nu_p \big\}.$$

The transpose of an ℓ-partition λ is given by

$${}^t\lambda = \big({}^t\lambda_\ell, \ldots, {}^t\lambda_2, {}^t\lambda_1 \big).$$

Any ℓ-partition $\lambda \in \mathcal{P}^\ell_{n,\nu}$ may be viewed as an element in $\mathbb{N}^\nu_{\geqslant 0}$ by adding zeroes on the right of each partition λ_p such that $l(\lambda_p) < \nu_p$. This yields a bijection

$$\mathbb{N}^\nu_{\geqslant 0} = \mathbb{Z}^\nu_{\geqslant 0} \cap \mathbb{N}^m = \bigsqcup_n \mathcal{P}^\ell_{n,\nu}. \tag{1.1}$$

Finally, for any ℓ-tuple $\nu = (\nu_1, \nu_2, \ldots, \nu_\ell)$ we'll write $\nu^\circ_p = (\nu^\circ)_p$ and $\nu^\bullet_p = (\nu^\bullet)_p$ where

$$\nu^\circ = (\nu_\ell, \nu_{\ell-1}, \ldots, \nu_1), \qquad \nu^\bullet = (\nu_{\ell-1}, \nu_{\ell-2}, \ldots, \nu_1, \nu_\ell).$$

1.5. Representations of W. For $p \in \Lambda$ there is a unique character

$$\chi_p : D_\ell \to \mathbb{C}^\times, \qquad \varepsilon \mapsto \varepsilon^p.$$

We set

$$\mathrm{Irr}(\mathbb{C}\mathfrak{S}_n) = \{\mathfrak{X}_\lambda; \lambda \in \mathcal{P}_n\}, \qquad \mathrm{Irr}(\mathbb{C}W) = \{\mathfrak{X}_\lambda; \lambda \in \mathcal{P}_n^\ell\}. \tag{1.2}$$

If $\lambda \in \mathcal{P}_n$ then $\mathfrak{X}_\lambda$ is defined as in (2.1) below. If $\lambda \in \mathcal{P}_n^\ell$ then $\mathfrak{X}_\lambda$ defined as follows. Any composition $\mu \in \mathcal{C}_{n,\ell}$ can be regarded as a partition $A = A_{\mu,1} \sqcup \cdots \sqcup A_{\mu,\ell}$ as above. We consider the subgroups

$$\mathfrak{S}_\mu = \prod_p \mathfrak{S}_{A_{\mu,p}} \subset \mathfrak{S}_n, \qquad W_\mu = \prod_p W_{A_{\mu,p}} \subset W = W_A.$$

Let w_μ be the longest element in $\mathfrak{S}_\mu$. Write w_0 for $w_{(n)}$. Fix an ℓ-partition $\lambda = (\lambda_p)$ in $\mathcal{P}_n^\ell$. The tuple $\mu = (\mu_p)$, with $\mu_p = |\lambda_p|$ for each p, belongs to $\mathcal{C}_{n,\ell}$. Let $\mathfrak{X}_{\lambda_p}\chi_{p-1}^{\otimes\mu_p}$ be the representation of $W_{A_{\mu,p}}$ which is the tensor product of the $\mathfrak{S}_{A_{\mu,p}}$-module $\mathfrak{X}_{\lambda_p}$ and the one-dimensional $(D_\ell)^{\mu_p}$-module $\chi_{p-1}^{\otimes\mu_p}$. Then the W-module $\mathfrak{X}_\lambda$ is given by

$$\mathfrak{X}_\lambda = \Gamma_{W_\mu}^W\big(\mathfrak{X}_{\lambda_1}\chi_\ell^{\otimes\mu_1} \otimes \mathfrak{X}_{\lambda_2}\chi_1^{\otimes\mu_2} \otimes \cdots \otimes \mathfrak{X}_{\lambda_\ell}\chi_{\ell-1}^{\otimes\mu_\ell}\big), \tag{1.3}$$

where the symbol Γ denotes the induction.

1.6. The highest weight category of $\mathbf{H}_{h,H}$. Let $\mathcal{H}_{h,H}$ be the category of $\mathbf{H}_{h,H}$-modules which are locally nilpotent over $\mathbf{R}^*$. The category $\mathcal{H}_{h,H}$ is quasi-hereditary by [10]. The standard modules of $\mathcal{H}_{h,H}$ are the induced modules

$$\Delta_{\lambda,h,H} = \Gamma_{W\ltimes\mathbf{R}^*}^{\mathbf{H}_{h,H}}(\mathfrak{X}_\lambda), \quad \lambda \in \mathcal{P}_n^\ell.$$

Here $\mathfrak{X}_\lambda$ is viewed as a $W \ltimes \mathbf{R}^*$-module such that $y_1, \ldots, y_n$ act trivially. Let $S_{\lambda,h,H}$, $P_{\lambda,h,H}$ denote the top and the projective cover of $\Delta_{\lambda,h,H}$.

The category $\mathcal{H}_{h,H}^{fg}$ consists of the $\mathbf{H}_{h,H}$-modules which are locally nilpotent over $\mathbf{R}^*$ and finitely generated over $\mathbf{R}$. The Grothendieck group of $\mathcal{H}_{h,H}^{fg}$ is spanned by the set $\{[S_{\lambda,h,H}]; \lambda \in \mathcal{P}_n^\ell\}$ and by the set $\{[\Delta_{\lambda,h,H}]; \lambda \in \mathcal{P}_n^\ell\}$.

The algebra $\mathbf{H}_{h,H}$ is given the inner $\mathbb{Z}$-grading such that x_i, y_i, w have the degrees 1, −1, 0 respectively. The sum $\mathrm{eu} = \sum_i x_i y_i + \mathrm{eu}_0$, with

$$\mathrm{eu}_0 = h\sum_{i<j}\sum_{p\in\Lambda}\big(1 - s_{i,j}^{(p)}\big) + \sum_{i=1}^n\sum_{p,p'=1}^{\ell-1}\varepsilon^{-pp'}(h_1 + \cdots + h_{p'})\varepsilon_i^p,$$

is an Euler element for the grading, i.e., an element $x \in \mathbf{H}_{h,H}$ is of degree i iff we have $[\mathrm{eu}, x] = ix$.

For each $\lambda \in \mathcal{P}_n^\ell$ we denote by θ_λ the scalar by which eu_0 acts on $\mathfrak{X}_\lambda$. We have the following formula [22, Prop. 6.2]

$$\theta_\lambda = \ell \sum_{p=2}^{\ell} |\lambda_p|(h_1 + \cdots + h_{p-1}) - h\ell \sum_p \big(n(\lambda_p) - n({}^t\lambda_p)\big) + \theta_0,$$

where θ_0 is a constant independent of λ. The partial order $\succcurlyeq$ on the set of standard modules is the unique order relation such that [10, Sec. 2.5]

$$\Delta_{\mu,h,H} \succ \Delta_{\lambda,h,H} \quad \Longleftrightarrow \quad \theta_\lambda - \theta_\mu \in \mathbb{Z}_{>0}. \tag{1.4}$$

1.7. From local systems to $\mathbf{H}_{h,H}$-modules. Let $\mathbf{R}_{n,\ell} = \mathbb{C}[C_{n,\ell}]$, where $C_{n,\ell} \subset \mathbb{C}^n$ is the complement of the hypersurface

$$x_1 x_2 \cdots x_n \prod_p \prod_{i \neq j} \big(x_i - \varepsilon^p x_j\big) = 0.$$

Note that $C_{1,\ell} = \mathbb{C}^\times$. For each $\mathbf{R}$-module M we write

$$M_{n,\ell} = M \otimes_{\mathbf{R}} \mathbf{R}_{n,\ell}.$$

In particular we have the $\mathbb{C}$-algebra

$$\mathbf{B} = \mathbb{C}W \ltimes \mathbf{R}, \qquad \mathbf{B}_{n,\ell} = \mathbb{C}W \ltimes \mathbf{R}_{n,\ell}. \tag{1.5}$$

We'll abbreviate $\mathbf{H}_{h,H,n,\ell} = (\mathbf{H}_{h,H})_{n,\ell}$. The algebra $\mathbf{H}_{h,H,n,\ell}$ does not depend on the choice of the parameters h, H, up to canonical isomorphisms. See [10, Thm. 5.6] for details. We'll need the following basic result.

1.8. Proposition.

(a) *Let $M_{n,\ell}$ be a $\mathbf{B}_{n,\ell}$-module with an integrable W-equivariant connection $\nabla = \sum_i \nabla_i \, dx_i$. Set*

$$\bar{y}_i = \nabla_i + k \sum_{j \neq i} \sum_p \frac{1}{x_i - \varepsilon^{-p} x_j} s_{i,j}^{(p)} + \sum_{p \neq 0} \frac{\gamma_p}{x_i - \varepsilon^{-p} x_i} \varepsilon_i^p.$$

The assignment $y_i \mapsto \bar{y}_i$ yields a $\mathbb{C}$-linear representation of $\mathbf{H}_{h,H}$ on $M_{n,\ell}$.

(b) *Let M be a $\mathbf{B}$-module with an integrable W-equivariant connection. Assume that M is torsion free as an $\mathbf{R}$-module and that the operators $\bar{y}_1, \bar{y}_2, \ldots, \bar{y}_n$ on $M_{n,\ell}$ preserve the subset M. Then M is an $\mathbf{H}_{h,H}$-submodule of $M_{n,\ell}$.*

Proof. Part (b) is obvious. Let us concentrate on part (a). Set ∇' equal to $\sum_i \nabla'_i \, dx_i$, where

$$\nabla'_i = \nabla_i + k \sum_{j \neq i} \sum_p \frac{1}{x_i - \varepsilon^{-p} x_j} + \sum_{p \neq 0} \frac{\gamma_p}{x_i - \varepsilon^{-p} x_i}.$$

We have

$$\bar{y}_i = \nabla'_i + k\sum_{j\neq i}\sum_p \frac{1}{x_i - \varepsilon^{-p}x_j}\big(s_{i,j}^{(p)} - 1\big) + \sum_{p\neq 0}\frac{\gamma_p}{x_i - \varepsilon^{-p}x_i}\big(\varepsilon_i^p - 1\big).$$

Thus it is enough to check that ∇' is an integrable W-equivariant connection and to apply the same argument as for the Dunkl operators with ∇'_i instead of ∂_{x_i}. The W-equivariance of ∇' is obvious. If $i \neq j$ a direct computation yields

$$\begin{aligned}
\big[\nabla'_i, \nabla'_j\big] &= \left[\partial_{x_i}, k\sum_{r\neq j}\sum_p \frac{1}{x_j - \varepsilon^{-p}x_r} + \sum_{p\neq 0}\frac{\gamma_p}{x_j - \varepsilon^{-p}x_j}\right] \\
&\quad + \left[k\sum_{r\neq i}\sum_p \frac{1}{x_i - \varepsilon^{-p}x_r} + \sum_{p\neq 0}\frac{\gamma_p}{x_i - \varepsilon^{-p}x_i}, \partial_{x_j}\right] \\
&= k\left[\partial_{x_i}, \sum_p \frac{1}{x_j - \varepsilon^{p}x_i}\right] + k\left[\sum_p \frac{1}{x_i - \varepsilon^{-p}x_j}, \partial_{x_j}\right] \\
&= 0. \qquad \square
\end{aligned}$$

2. The affine category $\mathcal{O}$

According to Suzuki [25] the Knizhnik–Zamolodchikov connection gives a functor from Kazhdan–Lusztig's category of modules over the affine Lie algebra to the category $\mathcal{O}$ of the rational DAHA (for $\ell = 1$). This functor uses affine coinvariants. In this section we briefly review this construction. Since the results here are not new, we do not give proofs.

2.1. Lie algebras. Fix once for all an integer $m > 0$. We set $\mathfrak{g} = \mathfrak{gl}_m(\mathbb{C})$ and $G = \mathrm{GL}_m(\mathbb{C})$. To avoid any confusion we may write

$$\mathfrak{g}_J = \mathfrak{g}_m = \mathfrak{g}, \qquad G_J = G_m = G.$$

Let $U(\mathfrak{g})$ be the enveloping algebra of $\mathfrak{g}$. For any $g \in G$, $\xi \in \mathfrak{g}$ let $g\xi$ be the adjoint action of g on ξ and let ${}^t\xi$ be the transpose of ξ. Let

$$\mathfrak{b} \subset \mathfrak{g}, \qquad \mathfrak{t} \subset \mathfrak{g}, \qquad T \subset G$$

be the Borel Lie subalgebra consisting of upper triangular matrices and the maximal tori consisting of the diagonal matrices. Let (ϵ_i), $(\check{\epsilon}_i)$, $i \in J$, be the canonical bases of $\mathfrak{t}^*$, $\mathfrak{t}$. There is a unique G-invariant pairing on $\mathfrak{g}$ such that $\langle \check{\epsilon}_i : \check{\epsilon}_j\rangle = \delta_{i,j}$ for each $i, j \in J$. We have canonical isomorphisms $\mathfrak{t} = \mathfrak{t}^* = \mathbb{C}^J$ taking $\check{\lambda}$, λ to the tuples $(\check{\lambda}_i)$, (λ_i) with $\check{\lambda}_i = \check{\lambda}(\epsilon_i)$ and $\lambda_i = \lambda(\check{\epsilon}_i)$ respectively. The elements of $\mathfrak{t}$, $\mathfrak{t}^*$ are called coweights and weights respectively. The weights in $\mathbb{Z}^J$ are called *integral weights*. If the weight λ is given then the symbol $\check{\lambda}$ denote the coweight $\check{\lambda} = \sum_{j\in J}\lambda_j\check{\epsilon}_j$, and vice-versa. We put

$$\rho = (m, \ldots, 2, 1), \qquad \alpha_i = \epsilon_i - \epsilon_{i+1}, \quad i \in I,\ I = \{1, 2, \ldots, m-1\}.$$

Let $\Pi \subset \mathbb{Z}^J$ be the set of roots, $\Pi^+ \subset \Pi$ the set of positive roots containing $\{\alpha_i;\ i \in I\}$ and $\mathbb{Z}\Pi$ be the root lattice. Let $e_{k,l}$, $k, l \in J$, be the canonical basis vectors of $\mathfrak{g}$. For each simple root α_i we write $e_i = e_{i,i+1}$, $f_i = e_{i+1,i}$ for the corresponding root vectors in $\mathfrak{g}$. Write

$$L(\lambda) = L(\mathfrak{g}, \lambda)$$

for the simple $\mathfrak{g}$-module with highest weight λ (relative to the Borel Lie subalgebra $\mathfrak{b}$). If λ is integral and dominant we define the $\mathfrak{S}_n$-module

$$\mathfrak{X}_\lambda = H_0\big(\mathfrak{g}, \big(\mathbf{V}^*\big)^{\otimes n} \otimes L(\lambda)\big), \tag{2.1}$$

where H_0 denotes the space of coinvariants. It vanishes unless $\lambda \in \mathcal{P}_n$. Compare (1.2). Let $\mathbf{V}$ be the vectorial representation of G and let $\mathbf{V}^*$ be the dual module. We can identify $\mathbf{V}$ with $\mathfrak{t}$ as $\mathbb{C}$-vector spaces. Hence we may view (ϵ_j) as a basis of $\mathbf{V}^*$. For each $\nu \in \mathcal{C}_{m,\ell}$ and $p \in \Lambda$, let

$$\mathbf{V}_p^* \subset \mathbf{V}^*$$

be the subspace spanned by the vectors ϵ_j, $j \in J_{\nu,p}$.

2.2. Affine Lie algebras. Let t be a formal variable. For each integer r we set

$$\mathbf{g} = \mathfrak{g} \otimes \mathbb{C}\big[t, t^{-1}\big], \qquad \mathbf{g}_{\geqslant r} = \mathfrak{g} \otimes t^r\mathbb{C}[t], \qquad \mathbf{g}_{\leqslant r} = \mathfrak{g} \otimes t^r\mathbb{C}\big[t^{-1}\big], \qquad \mathbf{b} = \mathfrak{b} \oplus \mathbf{g}_{>0}.$$

Let $\hat{\mathbf{g}}$ be the central extension of $\mathbf{g}$ by $\mathbb{C}$ associated with the cocycle $(\xi \otimes f, \zeta \otimes g) \mapsto \langle \xi : \zeta \rangle \operatorname{Res}_{t=0}(g df)$. Write $\mathbf{1}$ for the canonical central element of $\hat{\mathbf{g}}$. We abbreviate $\hat{\mathbf{g}}_{\geqslant 0} = \mathbf{g}_{\geqslant 0} \oplus \mathbb{C}\mathbf{1}$ and $\hat{\mathbf{b}} = \mathbf{b} \oplus \mathbb{C}\mathbf{1}$, the trivial central extensions. The element $\partial = t\partial_t$ acts on $\mathbf{g}$ in the obvious way, yielding a derivation of the Lie algebra $\hat{\mathbf{g}}$ such that $\partial(\mathbf{1}) = 0$. We put $\mathbf{t} = \mathbb{C}\partial \oplus \mathfrak{t} \oplus \mathbb{C}\mathbf{1}$, and $\tilde{\mathbf{g}} = \mathbb{C}\partial \oplus \hat{\mathbf{g}}$, $\tilde{\mathbf{b}} = \mathbb{C}\partial \oplus \hat{\mathbf{b}}$. For any commutative $\mathbb{C}$-algebra R with 1 we set $\mathfrak{g}_R = \mathfrak{g} \otimes R$, $\hat{\mathbf{g}}_R = \hat{\mathbf{g}} \otimes R$, etc. We regard $\mathfrak{g}_R$, $\hat{\mathbf{g}}_R$ and $\tilde{\mathbf{g}}_R$ as R-Lie algebras.

The adjoint $\mathbf{t}$-action on $\hat{\mathbf{g}}$ is diagonalizable. An element of $\mathbf{t}^*$ is called an *affine weight*. Let $\hat{\Pi} \subset \mathbf{t}^*$ be the set of roots of $\hat{\mathbf{g}}$, let $\hat{\Pi}^+ \subset \hat{\Pi}$ be the set of roots of the pro-nilpotent radical $\mathbf{n}$ of $\hat{\mathbf{b}}$, and let $\hat{\Pi}_{\mathrm{re}} \subset \hat{\Pi}$ be the set of real roots. Let $\delta, \omega_0 \in \mathbf{t}^*$ be the linear forms given by

$$\delta(\partial) = \omega_0(\mathbf{1}) = 1, \qquad \omega_0(\mathbb{C}\partial \oplus \mathfrak{t}) = \delta(\mathfrak{t} \oplus \mathbb{C}\mathbf{1}) = 0.$$

The simple roots in $\hat{\Pi}^+$ are

$$\hat{\alpha}_i, \quad i \in \hat{I}, \quad \hat{I} = \{0, 1, \ldots, m-1\}.$$

From now on we'll use the canonical isomorphism $\mathbf{t}^* = \mathbb{C}^{m+2} = \mathbb{C} \times \mathbb{C}^m \times \mathbb{C}$ such that $\hat{\alpha}_i \mapsto (0, \alpha_i, 0)$ if $i \neq 0$, $\omega_0 \mapsto (0, 0, 1)$, and $\delta \mapsto (1, 0, 0)$. When there is no risk of confusion we'll abbreviate $\alpha_i = \hat{\alpha}_i$. Recall that $\alpha_0 = \delta - \epsilon_1 + \epsilon_m$. Let $\langle : \rangle$ denote also the symmetric bilinear form on $\mathbf{t}^*$ such that

$$\langle \omega_0 : \delta \rangle = 1, \quad \langle \epsilon_i : \delta \rangle = \langle \epsilon_i : \omega_0 \rangle = 0, \quad \langle \epsilon_i : \epsilon_j \rangle = \delta_{i,j}.$$

An affine weight λ such that $\lambda(\partial) = 0$ is called a *classical affine weight*. Let $\mathbf{t}' \subset \mathbf{t}^*$ be the set of classical affine weights.

2.3. Enveloping algebras. For any commutative $\mathbb{C}$-algebra R with 1 let $U(\mathfrak{g}_R)$, $U(\hat{\mathbf{g}}_R)$ be the enveloping algebras of $\mathfrak{g}_R$, $\hat{\mathbf{g}}_R$ over R. Given an element $\kappa \in R$ let $U(\hat{\mathbf{g}}_R) \to \hat{\mathbf{g}}_{R,\kappa}$ be the quotient by the two-sided ideal generated by the element $\mathbf{1} - c$ where

$$c = \kappa - m.$$

We call c the *level*. A $\hat{\mathbf{g}}_{R,\kappa}$-module is the same as a $\hat{\mathbf{g}}_R$-module such that the element $\mathbf{1}$ acts as the multiplication by c. If $R = \mathbb{C}$ we'll abbreviate $\hat{\mathbf{g}}_\kappa = \hat{\mathbf{g}}_{\mathbb{C},\kappa}$, etc.

2.4. Smooth $\hat{\mathbf{g}}_R$-modules. A $\hat{\mathbf{g}}_{R,\kappa}$-module M is *almost smooth* if each element of M is annihilated by $\mathbf{g}_{R,\geqslant r}$ for a large enough integer r. Let $\mathcal{C}(\hat{\mathbf{g}}_{R,\kappa})$ for the category of almost smooth $\hat{\mathbf{g}}_{R,\kappa}$-modules. We may abbreviate

$$\mathcal{C}_{R,\kappa} = \mathcal{C}(\hat{\mathbf{g}}_{R,\kappa}).$$

For each integer $r \geqslant 1$ let $Q_{R,r} \subset \hat{\mathbf{g}}_{R,\kappa}$ be the subspace generated by the products of r elements of $\mathbf{g}_{R,\geqslant 1}$. Let also $Q_{R,0} = R$. Given a $\hat{\mathbf{g}}_{R,\kappa}$-module M let $M(r) \subset M$ be the annihilator of $Q_{R,r}$ and let $M(-r) \subset M$ be the annihilator of $Q_{R,-r} = \sharp(Q_{R,r})$. Set

$$M(\infty) = \bigcup_{r \geqslant 0} M(r).$$

Note that $M(r)$ is a $\hat{\mathbf{g}}_{R,\geqslant 0}$-submodule of M and that $M(\infty)$ is a $\hat{\mathbf{g}}_{R,\kappa}$-submodules of M. The $\hat{\mathbf{g}}_{R,\kappa}$-module M is called *smooth* if $M = M(\infty)$. All smooth modules are almost smooth. See [18, Lem. 1.10] for details.

2.5. The Sugawara operator. Put $\xi^{(a)} = \xi \otimes t^a$ for $\xi \in \mathfrak{g}$. Let $R^\times \subset R$ be the set of units. From now on we'll always assume that $\kappa \in R^\times$. For each integer $b \in \mathbb{Z}$ the formal sum

$$\mathbf{L}_b = \frac{1}{2\kappa} \sum_{a \geqslant -b/2} \sum_{k,l \in J} e_{k,l}^{(-a)} e_{l,k}^{(a+b)} + \frac{1}{2\kappa} \sum_{a < -b/2} \sum_{k,l \in J} e_{k,l}^{(a+b)} e_{l,k}^{(-a)}$$

is called the Sugawara operator. It lies in a completion of $\hat{\mathbf{g}}_{R,\kappa}$. For any $\hat{\mathbf{g}}_{R,\kappa}$-module M and any integer b the Sugawara operator $\mathbf{L}_b$ acts on $M(\infty)$ and we have

$$\left[\mathbf{L}_b, \xi^{(a)}\right] = -a\xi^{(a+b)}. \tag{2.2}$$

Let $\Omega = \sum_{k,l \in J} e_{k,l} e_{l,k}$ be the Casimir element in $U(\mathfrak{g}_R)$. Then $\mathbf{L}_0 - \Omega/2\kappa$ acts trivially on the subset $M(1)$.

2.6. Duality for $\hat{\mathbf{g}}_R$-modules. For each $\hat{\mathbf{g}}_R$-module M we define the $\hat{\mathbf{g}}_R$-modules ${}^\sharp M$, ${}^\dagger M$, M^*, and M^d as follows

- ${}^\sharp M$ is the $\hat{\mathbf{g}}_R$-module equal to the twist of M by the automorphism $\sharp : \hat{\mathbf{g}}_R \to \hat{\mathbf{g}}_R$ such that $\xi^{(r)} \mapsto (-1)^r \xi^{(-r)}$ and $\mathbf{1} \mapsto -\mathbf{1}$,
- ${}^\dagger M$ is the $\hat{\mathbf{g}}_R$-module equal to the twist of M by the automorphism $\dagger : \hat{\mathbf{g}}_R \to \hat{\mathbf{g}}_R$ such that $\xi^{(r)} \mapsto -{}^t\xi^{(r)}$ and $\mathbf{1} \mapsto \mathbf{1}$,

- $M^* = \mathrm{Hom}_R(M, R)$ with the $\hat{\mathbf{g}}_R$-action such that $(\xi^{(r)}\varphi, m) = -(\varphi, \xi^{(r)}m)$ and $(\mathbf{1}\varphi, m) = -(\varphi, \mathbf{1}m)$,
- M^d is the set of $\mathfrak{g}_{R,\nu}$-finite elements of M^*, i.e., it is the sum of all $\mathfrak{g}_{R,\nu}$-submodules of M^* which are finitely generated as R-modules.

The functors $M \mapsto {}^\dagger M, {}^\sharp M, M^*, M^d$ commute to each other. We set $\ddagger = \dagger \circ \sharp$, $DM = ({}^\sharp M^*)(\infty)$, and ${}^\dagger DM = ({}^\ddagger M^*)(\infty)$. As an R-module ${}^\dagger DM$ consists of those R-linear forms $M \to R$ which are zero on $Q_{R,-r}M$ for some $r \geqslant 1$.

For each $\mathfrak{g}$-module M we define the $\mathfrak{g}$-modules ${}^\dagger M$, M^*, M^d and the duality functor ${}^\dagger D$ as in the affine case.

2.7. The (parabolic) category $\mathcal{O}$ of $\mathfrak{g}$ and $\hat{\mathfrak{g}}$. In this section we set $R = \mathbb{C}$. The adjoint $\mathbf{t}$-action on $\tilde{\mathbf{g}}$ preserves the Lie subalgebra $\hat{\mathbf{g}}$. By a parabolic Lie subalgebra of $\hat{\mathbf{g}}$ we'll mean a $\mathbf{t}$-diagonalizable Lie subalgebra $\hat{\mathbf{q}} \subset \hat{\mathbf{g}}_{\geqslant 0}$ containing a conjugate of $\hat{\mathbf{b}}$. Fix a Levi subalgebra $\hat{\mathbf{l}} \subset \hat{\mathbf{q}}$. Let $\mathcal{O}(\hat{\mathbf{g}}_\kappa, \hat{\mathbf{q}})$ be the category of the $\hat{\mathbf{g}}_\kappa$-modules which are $\hat{\mathbf{l}}$-semisimple and $\hat{\mathbf{q}}$-locally finite. We abbreviate

$$\hat{\mathcal{O}}_\kappa = \mathcal{O}(\hat{\mathbf{g}}_\kappa) = \mathcal{O}(\hat{\mathbf{g}}_\kappa, \hat{\mathbf{b}})$$

with $\hat{\mathbf{l}} = \mathfrak{t} \oplus \mathbb{C}\mathbf{1}$. Fix a composition $\nu \in \mathcal{C}_{m,\ell}$. Let $\mathfrak{q}_\nu \subset \mathfrak{g}$ be the standard parabolic Lie subalgebra with block diagonal Levi subalgebra

$$\mathfrak{h}_\nu = \mathfrak{g}_{\nu_1} \oplus \cdots \oplus \mathfrak{g}_{\nu_\ell}.$$

Let $\mathfrak{u}_\nu$ be the nilpotent radical of $\mathfrak{q}_\nu$. Let $\hat{\mathbf{q}}_\nu$ be the parabolic Lie subalgebra of $\hat{\mathbf{g}}$ with Levi subalgebra $\mathfrak{h}_\nu \oplus \mathbb{C}\mathbf{1}$, and let $\mathbf{u}_\nu$ be the pronilpotent radical of $\hat{\mathbf{q}}_\nu$. We abbreviate

$$\hat{\mathcal{O}}_{\nu,\kappa} = \mathcal{O}(\hat{\mathbf{g}}_\kappa, \hat{\mathbf{q}}_\nu), \qquad \hat{\mathcal{O}}_{\geqslant 0,\kappa} = \mathcal{O}(\hat{\mathbf{g}}_\kappa, \hat{\mathbf{g}}_{\geqslant 0}).$$

In the same way let $\mathcal{O}(\mathfrak{g}, \mathfrak{q}_\nu)$ be the category of the $\mathfrak{g}$-modules which are $\mathfrak{h}_\nu$-semisimple and $\mathfrak{q}_\nu$-locally finite. We abbreviate

$$\mathcal{O} = \mathcal{O}(\mathfrak{g}, \mathfrak{b}), \qquad \mathcal{O}_\nu = \mathcal{O}(\mathfrak{g}, \mathfrak{q}_\nu), \qquad \mathcal{O}_{\geqslant 0} = \mathcal{O}(\mathfrak{g}, \mathfrak{g}).$$

So $\mathcal{O}_{\geqslant 0}$ is the category of all semisimple $\mathfrak{g}$-modules. Let $\mathfrak{q}'_\nu = w_0(\mathfrak{q}_\nu)$, $\hat{\mathbf{q}}'_\nu = w_0(\hat{\mathbf{q}}_\nu)$, and $\hat{\mathbf{b}}' = w_0(\hat{\mathbf{b}})$.

2.8. Induction and generalized Weyl modules. Set $R = \mathbb{C}$. For any $\mathfrak{q}_\nu$-module M we set

$$M_\nu = \Gamma_{\mathfrak{q}_\nu}^{\mathfrak{g}}(M),$$

the induced $\mathfrak{g}$-module. An $\mathfrak{h}_\nu$-module M may be viewed as a $\mathfrak{q}_\nu$-module such that $\mathfrak{u}_\nu$ acts as zero, yielding again a $\mathfrak{g}$-module M_ν. Similarly, for any $\hat{\mathbf{q}}_\nu$-module M of level c we set

$$M_{\nu,\kappa} = \Gamma_{\hat{\mathbf{q}}_\nu}^{\hat{\mathbf{g}}}(M).$$

If M is an $\mathfrak{h}_\nu$-module we define the $\hat{\mathbf{g}}_\kappa$-module $M_{\nu,\kappa}$ in the obvious way.

If $\ell = 1$ then we have $\mathfrak{h}_\nu = \mathfrak{g}$, $\hat{\mathbf{q}}_\nu = \hat{\mathbf{g}}_{\geqslant 0}$ and we write

$$M_\kappa = M_{\nu,\kappa}.$$

If there is an integer $r > 0$ such that Q_r acts by zero on the $\hat{\mathbf{g}}_{\geqslant 0,\kappa}$-module M, then M_κ is called a *generalized Weyl module*. More precisely, if M is a $\hat{\mathbf{g}}_{\geqslant 0,\kappa}$-module with a finite filtration by $\hat{\mathbf{g}}_{\geqslant 0,\kappa}$-modules such that the successive quotients are annihilated by Q_1 and lie in $\mathcal{O}_\nu^{fg}$ (as $\mathfrak{g}$-modules) then M_κ belongs to the category $\hat{\mathcal{O}}_{\nu,\kappa}^{fg}$, and it is called a *generalized Weyl module of type* ν. Further, if M is a simple $\mathfrak{g}$-module in $\mathcal{O}_\nu$ which is regarded as a $\hat{\mathbf{g}}_{\geqslant 0,\kappa}$-module such that $\mathbf{g}_{>0}$ acts as zero, then M_κ is called a *Weyl module of type* ν.

For each weight $\lambda \in \mathfrak{t}^*$ let $L(\mathfrak{h}_\nu, \lambda)$ be the simple $\mathfrak{h}_\nu$-module with highest weight λ (relative to the Borel Lie subalgebra $\mathfrak{b} \cap \mathfrak{h}_\nu$ of $\mathfrak{h}_\nu$). We have the induced $\mathfrak{g}$-module

$$M(\lambda)_\nu = L(\mathfrak{h}_\nu, \lambda)_\nu.$$

The top of $M(\lambda)_\nu$ is $L(\lambda)$. Consider the classical affine weight

$$\hat{\lambda} = \lambda + c\omega_0. \tag{2.3}$$

We define a $\hat{\mathbf{g}}_\kappa$-module by setting

$$M(\hat{\lambda})_\nu = L(\mathfrak{h}_\nu, \lambda)_{\nu,\kappa}.$$

Let $L(\hat{\lambda})$ be the top of $M(\hat{\lambda})_\nu$. It is the simple $\hat{\mathbf{g}}_\kappa$-module with highest weight $\hat{\lambda}$. Recall that $\mathfrak{t}^*$ is identified with $\mathbb{C}^m$ and that the elements of $\mathbb{C}^\nu_{\geqslant 0} \subset \mathbb{C}^m$ are called ν-dominant weights. If λ is ν-dominant then $M(\hat{\lambda})_\nu$ is a generalized Weyl module of type ν. More precisely, if $\lambda \in \mathbb{C}^\nu_{\geqslant 0}$ then $M(\lambda)_\nu \in \mathcal{O}_\nu$, $M(\hat{\lambda})_\nu \in \hat{\mathcal{O}}_{\nu,\kappa}$ and they are both called *parabolic Verma modules*. If $\ell = 1$ we'll abbreviate

$$M(\hat{\lambda}) = M(\hat{\lambda})_\nu \in \hat{\mathcal{O}}_{\geqslant 0,\kappa}.$$

Although $\hat{\mathcal{O}}_{\nu,\kappa}$ is not a highest weight category, we'll adopt the following abuse of language: an object M is said to be *standardly filtered* if it is equipped with a finite filtration by submodules such that the successive quotients are parabolic Verma modules. Let $\hat{\mathcal{O}}_{\nu,\kappa}^\Delta \subset \hat{\mathcal{O}}_{\nu,\kappa}^{fg}$ be the full subcategory of the standardly filtered modules. Let $\Delta_{\hat{\mathcal{O}}_{\nu,\kappa}}$ be the set of parabolic Verma modules in $\hat{\mathcal{O}}_{\nu,\kappa}$.

2.9. Proposition. *Assume that* $\kappa \notin \mathbb{Q}_{\geqslant 0}$.

(a) *All modules in* $\hat{\mathcal{O}}_{\nu,\kappa}$ *are smooth. A* $\hat{\mathbf{g}}_\kappa$*-module lies in* $\hat{\mathcal{O}}_{\nu,\kappa}^{fg}$ *iff it is a quotient of a generalized Weyl module of type* $\mathcal{O}_\nu^{fg}$.
(b) *The category* $\hat{\mathcal{O}}_{\nu,\kappa}^{fg}$ *consists of the finitely generated smooth* $\hat{\mathbf{g}}_\kappa$*-modules* M *such that* $M(r) \in \mathcal{O}_\nu^{fg}$ *for each* $r > 0$.
(c) *The category* $\hat{\mathcal{O}}_{\nu,\kappa}^{fg}$ *is Abelian, any object has a finite length, and Hom sets are finite dimensional.*

(d) *If M lies in $\hat{\mathcal{O}}^{fg}_{\nu,\kappa}$ then we have ${}^\dagger DM = {}^\ddagger M^d$. The functor ${}^\dagger D$ takes $\hat{\mathcal{O}}^{fg}_{\nu,\kappa}$ to itself, and it yields an involutive anti-auto-equivalence which fixes the simple modules.*

Proof. Clearly, any module in $\hat{\mathcal{O}}_{\nu,\kappa}$ is smooth, and $M(r)$ lies in $\mathcal{O}_{\nu,\kappa}$ for each M in $\hat{\mathcal{O}}_{\nu,\kappa}$ and each $r > 0$. Now, assume that M lies in $\hat{\mathcal{O}}^{fg}_{\nu,\kappa}$. There is an integer $r > 0$ and a $\hat{\mathbf{g}}_{\geqslant 0,\kappa}$-submodule $V \subset M(r)$ such that M is generated by V as a $\hat{\mathbf{g}}_\kappa$-module and $V \in \mathcal{O}^{fg}_\nu$ as a $\mathfrak{g}$-module. Then M is a quotient of the generalized Weyl module V_κ. Recall that $\mathcal{O}^{fg}_\nu$ is an Abelian category, and that a subquotient of a $\mathfrak{g}$-module which lies in $\mathcal{O}^{fg}_\nu$ lies again in $\mathcal{O}^{fg}_\nu$. Thus V has a finite filtration by $\hat{\mathbf{g}}_{\geqslant 0,\kappa}$-submodules such that the successive quotients are annihilated by Q_1 and lies in $\mathcal{O}^{fg}_\nu$. So V_κ is a generalized Weyl module of type ν. Next, observe that a subquotient of a $\hat{\mathbf{g}}$-module which lies in $\hat{\mathcal{O}}_{\nu,\kappa}$ lies again in $\hat{\mathcal{O}}_{\nu,\kappa}$. Thus a quotient of a generalized Weyl module of type $\mathcal{O}^{fg}_\nu$ lies in $\hat{\mathcal{O}}^{fg}_{\nu,\kappa}$. This proves (a). Part (b) follows from (a) and [30, Thm. 3.5(1)].

Part (c) follows from parts (a), (b). Indeed, since $\hat{\mathcal{O}}_{\nu,\kappa}$ is obviously Abelian, to prove that $\hat{\mathcal{O}}^{fg}_{\nu,\kappa}$ is Abelian it is enough to prove that any submodule of a $\hat{\mathbf{g}}$-module in $\hat{\mathcal{O}}^{fg}_{\nu,\kappa}$ is finitely generated. This follows from [30, Thm. 3.5(3)]. Next, we must check that any object has a finite length and that the Hom sets are finite dimensional. The first claim follows from [30, Thm. 3.5(3)], because $\mathcal{O}^{fg}_\nu$ is Artinian. The second claim is proved as in [18, Prop. I.2.29]. It follows from Frobenius reciprocity and the following three facts: any object in $\hat{\mathcal{O}}^{fg}_{\nu,\kappa}$ is a quotient of a generalized Weyl module of type ν, the Hom spaces in $\mathcal{O}^{fg}_\nu$ are finite dimensional and $M(r) \in \mathcal{O}^{fg}_\nu$ for each $M \in \hat{\mathcal{O}}^{fg}_{\nu,\kappa}$ and each integer $r > 0$.

Part (d) is left to the reader, compare [18, Sec. 1,2]. □

Set $\hat{\mathcal{O}}'_{\nu,\kappa} = \mathcal{O}(\hat{\mathbf{g}}_\kappa, \hat{\mathbf{q}}'_\nu)$. The functor $\dagger$ yields an involutive equivalence

$$\hat{\mathcal{O}}_{\nu,\kappa} \to \hat{\mathcal{O}}'_{\nu,\kappa}.$$

The functor D takes $\hat{\mathcal{O}}^{fg}_{\geqslant 0,\kappa}$ into itself, and it yields an involutive anti-auto-equivalence of $\hat{\mathcal{O}}^{fg}_{\geqslant 0,\kappa}$. Note that the modules $\mathbf{V}_\kappa$ and $\mathbf{V}^*_\kappa := (\mathbf{V}^*)_\kappa$ are parabolic Verma modules in $\hat{\mathcal{O}}_{\geqslant 0,\kappa}$ such that

$$D(\mathbf{V}_\kappa) = \mathbf{V}^*_\kappa, \qquad \mathbf{V}^*_\kappa = {}^\dagger\mathbf{V}_\kappa.$$

From now on we'll always assume that $\kappa \notin \mathbb{Q}_{\geqslant 0}$.

2.10. The category $\mathcal{O}$ of $\hat{\mathbf{g}}$ versus $\tilde{\mathbf{g}}$. Set $R = \mathbb{C}$. Let $\hat{\mathbf{q}}$ be a parabolic Lie subalgebra of $\hat{\mathbf{g}}$. We write $\tilde{\mathbf{q}} = \hat{\mathbf{q}} \oplus \mathbb{C}\partial$ and $\tilde{\mathbf{l}} = \hat{\mathbf{l}} \oplus \mathbb{C}\partial$. Let $\mathcal{O}(\tilde{\mathbf{g}}_\kappa, \tilde{\mathbf{q}})$ be the category of the $\tilde{\mathbf{g}}_\kappa$-modules which are $\tilde{\mathbf{l}}$-semisimple, $\tilde{\mathbf{q}}$-locally finite. We abbreviate

$$\tilde{\mathcal{O}}_\kappa = \mathcal{O}(\tilde{\mathbf{g}}_\kappa) = \mathcal{O}(\tilde{\mathbf{g}}_\kappa, \tilde{\mathbf{b}}), \qquad \tilde{\mathcal{O}}_{\nu,\kappa} = \mathcal{O}(\tilde{\mathbf{g}}_\kappa, \tilde{\mathbf{q}}_\nu), \qquad \tilde{\mathcal{O}}_{\geqslant 0,\kappa} = \mathcal{O}(\tilde{\mathbf{g}}_\kappa, \tilde{\mathbf{g}}_{\geqslant 0}).$$

Let $\hat{\Omega} = \partial + \mathbf{L}_0$ be the generalized Casimir operator of $\tilde{\mathbf{g}}$, as in [15, Sec. 2.5]. Given a parabolic Lie subalgebra $\hat{\mathbf{q}} \subset \hat{\mathbf{g}}$, let

$$\mathcal{O}(\tilde{\mathbf{g}}_\kappa, \tilde{\mathbf{q}})^0 \subset \mathcal{O}(\tilde{\mathbf{g}}_\kappa, \tilde{\mathbf{q}})$$

be the full subcategory of the modules on which $\hat{\Omega}$ acts locally nilpotently. Forgetting the action of ∂ yields in equivalence of categories

$$\mathcal{O}(\tilde{\mathbf{g}}_\kappa, \tilde{\mathbf{q}})^0 \to \mathcal{O}(\hat{\mathbf{g}}_\kappa, \hat{\mathbf{q}}).$$

A quasi-inverse takes the $\hat{\mathbf{g}}$-module M to the unique $\tilde{\mathbf{g}}$-module $\tilde{M}$ which is equal to M as a $\hat{\mathbf{g}}$-module and such that ∂ acts as the semisimplification of the operator $-\mathbf{L}_0$. See [23, Prop. 8.1] for details. We identify M and $\tilde{M}$ if there is no ambiguity. For any affine weight $\lambda \in \mathbf{t}^*$ the λ-weight space of M (which is identified with $\tilde{M}$) is

$$M_\lambda = \{x \in \tilde{M};\ hx = \lambda(h)x,\ \forall h \in \mathbf{t}\}.$$

For each weight $\lambda \in \mathfrak{t}^*$ we write

$$\hat{\lambda} = \lambda + c\omega_0, \qquad \tilde{\lambda} = \hat{\lambda} + z_\lambda \delta, \qquad z_\lambda = -\langle \lambda : 2\rho + \lambda \rangle / 2\kappa.$$

We'll say that the affine weights $\hat{\lambda}$, $\tilde{\lambda}$ are ν-dominant if the weight λ is ν-dominant. The formula for $\mathbf{L}_0$ shows that the functor

$$\hat{\mathcal{O}}_{\nu,\kappa} \to \tilde{\mathcal{O}}_{\nu,\kappa}, \qquad M \mapsto \tilde{M}$$

takes $L(\hat{\lambda})$ to the simple module $L(\tilde{\lambda})$ with the (ν-dominant) highest weight $\tilde{\lambda}$ and $M(\hat{\lambda})_\nu$ to the parabolic Verma module $M(\tilde{\lambda})_\nu$ with the same highest weight. Here $\tilde{\lambda}$ is given by the formula above.

2.11. The formal loop Lie algebra. Fix a commutative $\mathbb{C}$-algebra R with 1. Given a finite set S we fix formal variables t_i, $i \in S$, and we set

$$R((t_S)) = \bigoplus_{i \in S} R((t_i)).$$

Each $f(t) \in R((t))$ yields an element in the i-th factor of $R((t_S))$ denoted by

$$f(t)_{[i]} = f(t_i).$$

We set

$$\mathcal{G}_R = \mathfrak{g} \otimes R((t)), \qquad \mathcal{G}_{R,S} = \mathfrak{g} \otimes R((t_S)).$$

As above, if $R = \mathbb{C}$ we simply forget the subscript R everywhere. Let $\hat{\mathcal{G}}_{R,S}$ (resp. $\hat{\mathcal{G}}_R$) be the central extension of $\mathcal{G}_{R,S}$ (resp. $\mathcal{G}_R$) by R associated with the cocycle $(\xi \otimes f, \zeta \otimes g) \mapsto \langle \xi : \zeta \rangle \sum_i \mathrm{Res}_{t_i=0}(g\,df)$. Write $\mathbf{1}$ for the canonical central element of $\hat{\mathcal{G}}_{R,S}$. Let

$$U(\hat{\mathcal{G}}_{R,S}) \to \hat{\mathcal{G}}_{R,S,\kappa}$$

be the quotient by the two-sided ideal generated by $\mathbf{1} - c$. Now, let M_i, $i \in S$, be almost smooth $\hat{\mathbf{g}}_{R,\kappa}$-modules. We'll use the following notation: if γ is an operator on M_i then $\gamma_{(i)}$ is the operator on

$$_S M = \bigotimes_{i \in S} M_i$$

given by the action of γ on the i-th factor. We'll abbreviate

$$e_{k,l,(i)} = (e_{k,l})_{(i)}, \qquad \mathbf{L}_{b,(i)} = (\mathbf{L}_b)_{(i)}. \tag{2.4}$$

The assignment

$$\xi \otimes f(t)_{[i]} \mapsto \xi \otimes f(t)_{(i)} \tag{2.5}$$

yields a representation of $\hat{\mathcal{G}}_{R,S,\kappa}$ on $_S M$.

2.12. The space of affine coinvariants. Fix a point $\infty \in \mathbb{P}^1$ and a coordinate z on $\mathbb{P}^1 - \{\infty\}$. We'll identify $\mathbb{P}^1 - \{\infty\}$ with $\mathbb{C}$ and $\mathbb{C}[\mathbb{C}]$ with $\mathbb{C}[z]$. Let $0 \in \mathbb{P}^1$ be the point such that $z = 0$ and $\mathbb{C}^\times = \{z \in \mathbb{C};\ z \neq 0\}$. Fix an S-tuple $x = (x_i;\ i \in S)$ of distinct points in $\mathbb{P}^1$. We set $z_i = z - x_i$ if $x_i \neq \infty$ and $z_i = -z^{-1}$ else. So z_i is a coordinate on $\mathbb{P}^1$ centered on x_i. Write

$$\mathbb{P}^1_x = \mathbb{P}^1 - \{x_i;\ i \in S\}.$$

Composing the expansion of rational functions on $\mathbb{P}^1$ at x_i with the assignment $z_i \mapsto t_i$ we get the inclusion

$$\iota_x : \mathbb{C}\big[\mathbb{P}^1_x\big] \to \mathbb{C}((t_S)).$$

By the residue theorem the embedding

$$\mathfrak{g}\big[\mathbb{P}^1_x\big] \to \mathcal{G}_S, \qquad \xi \otimes f \mapsto \xi \otimes \iota_x(f)$$

lifts to a Lie algebra embedding

$$\mathfrak{g}\big[\mathbb{P}^1_x\big] \to \hat{\mathcal{G}}_S. \tag{2.6}$$

If M_i, $i \in S$, are almost smooth $\hat{\mathbf{g}}_\kappa$-modules then $\hat{\mathcal{G}}_S$ acts on $_S M$. Thus $\mathfrak{g}[\mathbb{P}^1_x]$ acts also on $_S M$ through (2.6).

2.13. Definition. The space of affine coinvariants of the M_i's is $\langle M_i;\ i \in S\rangle_x = H_0(\mathfrak{g}[\mathbb{P}^1_x],\ _S M)$.

Given a point x_0 in $\mathbb{P}^1_x$ we set

$$\hat{S} = \{0\} \cup S, \qquad \hat{x} = (x_0, x).$$

The following properties are well-known, see e.g., [18, Props. 9.15, 9.16, 9.18].

2.14. Proposition. *Assume that $M_i \in \mathcal{C}_\kappa$ for each $i \in S$.*

(a) *If $M_i = N_{i,\kappa}$ is a generalized Weyl module for each i then there is a natural isomorphism $\langle M_i;\ i \in S\rangle_x = H_0(\mathfrak{g},\ {}_S N)$.*
(b) *If $M_i \in \hat{\mathcal{O}}^{fg}_{\geqslant 0,\kappa}$ for each i then $\langle M_i;\ i \in S\rangle_x$ is finite dimensional.*
(c) *If $M_0 = M(c\omega_0)$ there is a natural isomorphism $\langle M_i;\ i \in \hat{S}\rangle_{\hat{x}} = \langle M_i;\ i \in S\rangle_x$.*
(d) *The canonical vector space isomorphisms $M_i \to {}^\dagger M_i$, $i \in S$, yield an isomorphism $\langle M_i;\ i \in S\rangle_x \to \langle {}^\dagger M_i;\ i \in S\rangle_x$.*

2.15. Remark. Note that $M(c\omega_0)$ is simple, i.e., it is equal to $L(c\omega_0)$, if $\kappa \in \mathbb{Q}_{<0}$ by [18, Prop. 2.12(b)].

In the rest of Section 2 we'll assume that $S = A \cup \{n+1\}$ and $x_{n+1} = \infty$. Now, we allow the tuple $(x_1, \dots, x_n)$ to vary in the set C_n, where

$$C_n \subset \mathbb{C}^n$$

is the complement of the big diagonal. So we may view the x_i's as regular functions on C_n, i.e., as elements of the algebra $\mathbf{R}_n = \mathbb{C}[C_n]$. Let $R \subset \mathbf{R}_n(z)$ be the $\mathbf{R}_n$-submodule spanned by the rational functions

$$z_i^{-a},\ z^b \quad \text{with } i \in A,\ a > 0,\ b \geqslant 0.$$

It is an $\mathbf{R}_n$-algebra. The assignment

$$x_1, x_2, \dots, x_n, z \mapsto x_1, x_2, \dots, x_n, x_{n+1}$$

yields an $\mathbf{R}_n$-algebra isomorphism $R \to \mathbf{R}_{n+1}$. The expansion at $\{x_i;\ i \in S\}$ of a rational function in R yields $\mathbf{R}_n$-linear maps

$$R \to \mathbf{R}_n((t_S)), \qquad \mathfrak{g}_R \to \hat{\mathcal{G}}_{\mathbf{R}_n,S}.$$

For almost smooth $\hat{\mathbf{g}}_\kappa$-modules M_i, $i \in S$, the $\mathbf{R}_n$-module ${}_S M_{\mathbf{R}_n} = ({}_S M)_{\mathbf{R}_n}$ is equipped with a $\hat{\mathcal{G}}_{\mathbf{R}_n,S}$-action. So we get a $\mathfrak{g}_R$-action on ${}_S M_{\mathbf{R}_n}$ such that the element $\xi \otimes z^a$ acts as the sum

$$\sum_{i \in A} \bigl(\xi \otimes (t + x_i)^a\bigr)_{(i)} + \bigl(\xi \otimes \bigl(-t^{-1}\bigr)^a\bigr)_{(n+1)}. \tag{2.7}$$

Consider the $\mathbf{R}_n$-module

$$\langle M_i;\ i \in S\rangle = H_0(\mathfrak{g}_R,\ {}_S M_{\mathbf{R}_n}).$$

The following is well known, see e.g., [18, Sec. 9.13, Prop. 12.12].

2.16. Proposition. *If $M_i \in \hat{\mathcal{O}}^{fg}_{\geqslant 0,\kappa}$ for each $i \in S$, then $\langle M_i;\ i \in S\rangle$ is a projective $\mathbf{R}_n$-module of finite rank whose specialization at the point $x \in C_n$ is equal to $\langle M_i;\ i \in S\rangle_x$.*

2.17. The space of affine coinvariants of $T(M)_{\mathbf{R}_n}$. Now we fix a module $M \in \mathcal{C}_\kappa$. Let $R \subset \mathbf{R}_n(z)$ be as above. We'll apply the construction of affine coinvariants to the $\hat{\mathbf{g}}_\kappa$-modules $M_1 = \cdots = M_n = \mathbf{V}_\kappa^*$ and $M_{n+1} = M$. Write

$$\mathbf{B}_n = \mathbb{C}\mathfrak{S}_n \ltimes \mathbf{R}_n.$$

There is an obvious representation of $\mathbf{B}_n$ on ${}_S M_{\mathbf{R}_n}$, such that the group $\mathfrak{S}_n$ switches the modules $M_1, M_2, \dots, M_n$ and acts on $\mathbf{R}_n$, and $\mathbf{R}_n$ acts by multiplication. This action centralizes the $\mathfrak{g}_R$-action, see formula (2.7). Thus $\langle M_i;\ i \in S\rangle$ is equipped with a representation of $\mathbf{B}_n$. To unburden notation we'll set

$$T(M) = \left(\mathbf{V}^*\right)^{\otimes n} \otimes M.$$

The canonical inclusion $\mathbf{V}^* \subset \mathbf{V}_\kappa^*$ yields an embedding

$$T(M) \subset {}_S M.$$

Equip the $\mathbf{R}_n$-module $T(M)_{\mathbf{R}_n} = T(M) \otimes \mathbf{R}_n$ with the representation of $\mathfrak{g}[\mathbb{C}]$ such that

$$\xi \otimes z^a \mapsto \sum_{i \neq n+1} \xi_{(i)} \otimes x_i^a + \left(\sharp\xi^{(a)}\right)_{(n+1)} \otimes 1, \quad a \geqslant 0. \tag{2.8}$$

2.18. Proposition. *The inclusion $T(M) \subset {}_S M$ yields an $\mathbf{R}_n$-module isomorphism*

$$H_0\left(\mathfrak{g}[\mathbb{C}], T(M)_{\mathbf{R}_n}\right) \simeq \left\langle \mathbf{V}_\kappa^*, \dots, \mathbf{V}_\kappa^*, M\right\rangle.$$

If λ is a dominant weight there is an isomorphism of $\mathbf{B}_n$-modules

$$H_0\left(\mathfrak{g}[\mathbb{C}], T\left(M(\hat{\lambda})\right)_{\mathbf{R}_n}\right) \simeq \Gamma_{\mathfrak{S}_n}^{\mathbf{B}_n}(\mathfrak{X}_\lambda).$$

Proof. We do not give a proof here, since it is rather standard. Note that part (a) is a particular case of formula (3.15) below (set $\ell = 1$), and part (b) is a particular case of Proposition 3.8(b) below (set again $\ell = 1$). So the proof can be recovered from its twisted version given below. □

2.19. The local system of affine coinvariants of $T(M)_{\mathbf{R}_n}$. Let M be as above. Recall that $S = A \cup \{n+1\}$. We define differential operators on the $\mathbf{R}_n$-module ${}_S M_{\mathbf{R}_n}$ by the formula

$$\nabla_i = \partial_{x_i} + \mathbf{L}_{-1,(i)}, \quad \forall i \in A.$$

The operators ∇_i commute to each other and give commuting operators on the $\mathbf{R}_n$-module $T(M)_{\mathbf{R}_n}$, see [1, Lem. 13.3.7] for instance. The connection $\nabla = \sum_i \nabla_i \, dx_i$ is called the KZ connection. Let us compute it.

For each $i \in S$ we have the $\mathbb{C}$-linear operator of $T(M)$ given by

$$e_{k,l,(i)}^{(a)} = \left(e_{k,l}^{(a)}\right)_{(i)},$$

where $e_{k,l}^{(a)}$ acts on $\mathbf{V}^*_{\mathbf{R}_n} = \mathbf{V}^* \otimes \mathbf{R}_n$ as the operator $e_{k,l} \otimes x_i^a$. Next, for $i, j \in A$ the Casimir tensor σ yields the $\mathbb{C}$-linear operator on $T(M)$ given by

$$\sigma_{i,j} = \sum_{k,l} e_{k,l,(j)} e_{l,k,(i)}.$$

We define the $\mathbf{R}_n$-linear operators γ_i on $T(M)_{\mathbf{R}_n}$ by the formula

$$\gamma_i = \sum_{j \neq i} \gamma_{i,j} + \gamma_{i,n+1},$$

$$\gamma_{i,j} = \frac{1}{\kappa} \frac{\sigma_{i,j}}{x_i - x_j}, \qquad \gamma_{i,n+1} = \frac{1}{\kappa} \sum_{a>0} \sum_{k,l} (-1)^{a-1} e_{k,l,(n+1)}^{(a)} e_{l,k,(i)}^{(a-1)}.$$

The following is standard, see [1, Sec. 13.3.8] for instance.

2.20. Proposition. *Under the identification in Proposition* 2.18 *we have* $\partial_{x_i} + \gamma_i = \nabla_i$. *These operators normalize* $\mathfrak{g}[\mathbb{C}]$ *and yield an integrable* $\mathfrak{S}_n$*-equivariant connection on the* $\mathbf{R}_n$*-module* $H_0(\mathfrak{g}[\mathbb{C}], T(M)_{\mathbf{R}_n})$.

Note that Propositions 1.8, 2.20 yield a representation of $\mathbf{H}_{h,H}$ on $H_0(\mathfrak{g}[\mathbb{C}], T(M)_{\mathbf{R}_n})$.

3. Twisted affine coinvariants

In this section ℓ is any integer > 0. We'll use the orbifold Knizhnik–Zamolodchikov connection over the configuration space of the stack $[\mathbb{P}^1/D_\ell]$. It yields a functor taking modules over a twisted affine Lie algebra to $\mathbf{H}_{h,H}$-modules. This functor is a generalization of Suzuki's functor for any ℓ. As in the case $\ell = 1$ we can easily compute the image of parabolic Verma modules.

3.1. The twisted affine Lie algebra. Equip the G-module $\mathbf{V}^*$ with a representation of the group D_ℓ. Since $\mathbf{V}^*$ is the dual of the vectorial representation of G, there is a unique element $g \in G$ which acts on $\mathbf{V}^*$ as the generator of D_ℓ does. Let $H \subset G$ be the centralizer of g. We set

$$\mathfrak{g} = \bigoplus_{p \in \Lambda} \mathfrak{g}_p, \quad \mathfrak{g}_p = \{\xi \in \mathfrak{g};\ g\xi = \varepsilon^p \xi\}, \qquad \mathfrak{h} = \mathfrak{g}_0. \tag{3.1}$$

Let F be the automorphism of $\hat{\mathbf{g}}$ given by

$$F : \hat{\mathbf{g}} \to \hat{\mathbf{g}}, \qquad \xi^{(a)} \mapsto \varepsilon^a (g\xi)^{(a)}, \qquad \mathbf{1} \mapsto \mathbf{1}.$$

The twisted affine Lie algebra is the fixed points set $\hat{\mathbf{g}}^F \subset \hat{\mathbf{g}}$. We'll say that a $\hat{\mathbf{g}}^F$-module is of level c if the element $\mathbf{1}$ acts as $(c/\ell)\,\mathrm{id}$. Let $\hat{\mathbf{g}}^F_\kappa$ be the quotient of the enveloping algebra $U(\hat{\mathbf{g}}^F)$ by the two-sided ideal generated by the element $\mathbf{1} - c/\ell$.

3.2. Modules over the twisted affine Lie algebra. We have the triangular decomposition

$$\hat{\mathbf{g}}^F = \mathbf{g}^F_{<0} \oplus (\mathfrak{h} \oplus \mathbb{C}\mathbf{1}) \oplus \mathbf{g}^F_{>0} = \mathbf{g}^F_{<0} \oplus \hat{\mathbf{g}}^F_{\geqslant 0}.$$

The middle term is a reductive Lie algebra. So the categories

$$\hat{\mathcal{O}}^F_\kappa = \mathcal{O}(\hat{\mathbf{g}}^F_\kappa), \qquad \hat{\mathcal{O}}^F_{\geqslant 0,\kappa} = \mathcal{O}(\hat{\mathbf{g}}^F_\kappa, \hat{\mathbf{g}}^F_{\geqslant 0}), \qquad \mathcal{C}^F_\kappa = \mathcal{C}(\hat{\mathbf{g}}^F_\kappa)$$

are defined as in the untwisted case. The definitions above still make sense by replacing F by the automorphism

$$F' : \hat{\mathbf{g}} \to \hat{\mathbf{g}}, \qquad \xi^{(a)} \mapsto \varepsilon^{-a}(g\xi)^{(a)}, \qquad \mathbf{1} \mapsto \mathbf{1}.$$

They are indicated by the upperscript F' instead of F. Note that

$$\sharp(\hat{\mathbf{g}}^{F'}_{-\kappa+2m}) = \hat{\mathbf{g}}^F_\kappa, \qquad \dagger(\hat{\mathbf{g}}^{F'}_\kappa) = \hat{\mathbf{g}}^F_\kappa.$$

In particular the functor ${}^\dagger D$ yields an involutive anti-auto-equivalence of $\hat{\mathcal{O}}^{F,fg}_{\geqslant 0,\kappa}$. For each $\hat{\mathbf{g}}^F_{\geqslant 0}$-module M of level c let M^F_κ be the induced $\hat{\mathbf{g}}^F_\kappa$-module. The generalized Weyl $\hat{\mathbf{g}}^F_\kappa$-modules are defined as in the untwisted case. If M is an $\mathfrak{h}_\nu$-module then it can be equipped with the unique representation of $\hat{\mathbf{g}}^F_{\geqslant 0}$ of level c such that $\mathbf{g}^F_{>0}$ acts trivially, yielding an induced $\hat{\mathbf{g}}^F_\kappa$-module. We denote it again by M^F_κ. For each $\lambda \in \mathfrak{t}^*$ we put

$$\hat{\lambda} = \lambda + (c/\ell)\omega_0. \tag{3.2}$$

Compare (2.3). Consider the $\hat{\mathbf{g}}^F_\kappa$-module

$$M(\hat{\lambda})^F = L(\mathfrak{h}_\nu, \lambda)^F_\kappa.$$

Let $L(\hat{\lambda})^F$ be the top of $M(\hat{\lambda})^F$. Finally, consider the $\mathbb{C}$-algebra automorphism

$$\flat : \mathbb{C}[\mathbb{C}] \to \mathbb{C}[\mathbb{C}], \qquad f(z) \mapsto f(\varepsilon^{-1}z),$$

and let F denote also the automorphism $g \otimes \flat$ of $\mathfrak{g}[\mathbb{C}]$.

3.3. The space of twisted affine coinvariants of $T(M)$. Recall that for any vector space M we have

$$T(M) = (\mathbf{V}^*)^{\otimes n} \otimes M.$$

Consider the following endomorphisms of $T(M)$

$$g_i = g_{(i)}, \qquad \sigma^{(p)}_{i,j} = \sum_{k,l} (g^p e_{k,l})_{(j)} e_{l,k,(i)}.$$

Since $\mathbf{V}$ is a D_ℓ-module, the group W acts on the vector space $(\mathbf{V}^*)^{\otimes n}$ in such a way that $s^{(p)}_{i,j}$, ε_i act as $\sigma^{(p)}_{i,j}$, g_i respectively. This action gives rise to a representation of W on $T(M)$. The Lie algebra $\mathfrak{h}$ acts on $\mathbf{V}^*$ in the obvious way. Given an $\mathfrak{h}$-module M let $\mathfrak{h}$ act diagonally on $T(M)$. The representation of W on $T(M)$ factors to a representation of W on

$$\mathfrak{X}(M) = H_0(\mathfrak{h}, T(M)). \tag{3.3}$$

The diagonal $\mathfrak{h}$-action and the W-action on $(\mathbf{V}^*)^{\otimes n}$ satisfy the double centralizer property, see [19, Sec. 6] for instance. So $\mathfrak{X}$ yields a map

$$\mathfrak{X} : \mathrm{Irr}\big(U(\mathfrak{h})\big) \to \mathrm{Irr}(\mathbb{C}W),$$

where the symbol Irr denotes the set of isomorphism classes of finite dimensional modules. Now, recall the algebras $\mathbf{B}$, $\mathbf{B}_{n,\ell}$ in (1.5). Given $w \in W$ we'll use the symbol w for the w-action on the $\mathbf{B}_{n,\ell}$-module $\Gamma_W^{\mathbf{B}_{n,\ell}}(T(M))$ and the symbol $\underline{\omega}$ for the operator $\mathrm{id} \otimes w$ on

$$T(M)_{\mathbf{R}_{n,\ell}} = T(M) \otimes \mathbf{R}_{n,\ell}$$

such that w acts on $\mathbf{R}_{n,\ell}$ as in Section 1.2. We'll identify

$$T(M)_{\mathbf{R}} = \Gamma_W^{\mathbf{B}}\big(T(M)\big), \qquad T(M)_{\mathbf{R}_{n,\ell}} = \Gamma_W^{\mathbf{B}_{n,\ell}}\big(T(M)\big) \tag{3.4}$$

in the obvious way. Next, if M is a $\hat{\mathbf{g}}_\kappa^F$-module we equip $T(M)_{\mathbf{R}_{n,\ell}}$ with the unique $\mathbf{R}_{n,\ell}$-linear representation of the Lie algebra $\mathfrak{g}[\mathbb{C}]^F$ such that

$$\xi \otimes z^a \mapsto \sum_{i \neq n+1} \xi_{(i)} \otimes x_i^a + \big(\sharp\xi^{(a)}\big)_{(n+1)} \otimes 1. \tag{3.5}$$

Note that $\sharp\xi^{(a)} \in \hat{\mathbf{g}}^F$ because $\xi \otimes z^a \in \mathfrak{g}[\mathbb{C}]^F$. This action preserves the subspace $T(M)_{\mathbf{R}}$.

3.4. Definition. For each $\hat{\mathbf{g}}_\kappa^F$-module M we define the $\mathbf{R}$-module of twisted affine coinvariants of $T(M)$ by the following formula

$$\mathfrak{C}(M) = H_0\big(\mathfrak{g}[\mathbb{C}]^F, T(M)_{\mathbf{R}}\big). \tag{3.6}$$

By formula (3.5) it is quite clear that the $\mathfrak{g}[\mathbb{C}]^F$-action on $T(M)_{\mathbf{R}}$ centralizes the $\mathbf{B}$-action. Thus the representation of $\mathbf{B}$ factors to $\mathfrak{C}(M)$, yielding a functor

$$\mathfrak{C} : \hat{\mathbf{g}}_\kappa^F\textbf{-mod} \to \mathbf{B}\textbf{-mod}. \tag{3.7}$$

3.5. The $\mathbf{H}_{h,H}$-action on the space of twisted affine coinvariants of $T(M)$. Now, let M be an almost smooth $\hat{\mathbf{g}}_\kappa^F$-module. Set $S = A \cup \{n+1\}$. Let us define new operators on $T(M)_{\mathbf{R}_{n,\ell}}$. For each $i \in S$ we have the $\mathbb{C}$-linear operator of $T(M)$ given by

$$e_{k,l,(i)}^{(a)} = \big(e_{k,l}^{(a)}\big)_{(i)},$$

where $e_{k,l}^{(a)}$ acts on $\mathbf{V}_{\mathbf{R}_n}^*$ as the operator $e_{k,l} \otimes x_i^a$. See Section 2.19. Assume that

$$p \in \Lambda,\ i \in A,\ j \in S, \quad (j,p) \neq (i,0).$$

If $j \neq n+1$ we set

$$\gamma_{i,j}^{(p)} = \sigma_{i,j}^{(p)}/\kappa\big(x_i - \varepsilon^{-p} x_j\big),$$

$$\gamma_{i,n+1}^{(p)} = \frac{\varepsilon^p}{\kappa} \sum_{a>0} \sum_r (-1)^{a-1} \big(F'\big)^{-p} \big(e_{k,l,(i)}^{(a-1)}\big) e_{l,k,(n+1)}^{(a)}.$$

The sum converges because M is almost smooth. Note that the action of

$$e_{l,k,(n+1)}^{(a)}$$

here does not involves any twist by the automorphism $\sharp$, contrarily to the $\mathfrak{g}[\mathbb{C}]^F$-action on $T(M)_{\mathbf{R}_{n,\ell}}$ in (3.5). Next, if $i \neq j$ we set

$$\gamma_{i,j}^F = \sum_p \gamma_{i,j}^{(p)}, \qquad \gamma_{i,i}^F = \sum_{p \neq 0} \gamma_{i,i}^{(p)}.$$

Finally we consider the $\mathbb{C}$-linear operator $\bar{y}_i$ on $T(M)_{\mathbf{R}_{n,\ell}}$ given by

$$\bar{y}_i = \partial_{x_i} - \sum_{j \neq i, n+1} \sum_p \gamma_{i,j}^{(p)} \big(\underline{s_{i,j}^{(p)}} - 1\big) - \sum_{p \neq 0} \gamma_{i,i}^{(p)} \big(\underline{\varepsilon_i^p} - 1\big) + \gamma_{i,n+1}^F. \tag{3.8}$$

By (3.4) we have a representation of $\mathbf{B}_{n,\ell}$ on $T(M)_{\mathbf{R}_{n,\ell}}$. This representation can be extended to a representation of $\mathbf{H}_{h,H,n,\ell}$ as follows.

3.6. Proposition. *Let $M \in \mathcal{C}_\kappa^F$, $k = -1/\kappa$ and $\gamma_p = -\mathrm{tr}(g^p)/\kappa$.*

(a) *The assignment $y_i \mapsto \bar{y}_i$ defines a $\mathbf{H}_{h,H,n,\ell}$-action on the $\mathbf{B}_{n,\ell}$-module $T(M)_{\mathbf{R}_{n,\ell}}$ which normalizes the $\mathfrak{g}[\mathbb{C}]^F$-action.*
(b) *The operator $\bar{y}_i - \gamma_{i,n+1}^F$ vanishes on the subspace $T(M) \subset T(M)_{\mathbf{R}_{n,\ell}}$. The representation of $\mathbf{H}_{h,H}$ on $T(M)_{\mathbf{R}_{n,\ell}}$ preserves the subspace $T(M)_{\mathbf{R}} \subset T(M)_{\mathbf{R}_{n,\ell}}$. It factors to a representation of $\mathbf{H}_{h,H}$ on $\mathfrak{C}(M)$.*

Note that the $\mathbf{H}_{h,H,n,\ell}$-action on $T(M)_{\mathbf{R}_{n,\ell}}$ in (a) yields a representation of $\mathbf{H}_{h,H,n,\ell}$ for any parameter h, H, since the algebra $\mathbf{H}_{h,H,n,\ell}$ does not depend on the choice of h, H. However the formula (3.8) for the action of y_i does not hold for arbitrary h, H. Hence (b) is false for arbitrary h, H. We'll prove Proposition 3.6 from Section 3.9 onwards.

3.7. The image by $\mathfrak{C}$ of the parabolic Verma modules. Conjugacy classes of elements $g \in G$ such that $g^\ell = 1$ are labeled by integral weights of level ℓ in the dominant alcove. More precisely, since $g^\ell = 1$ there is a cocharacter $\mathbb{C}^\times \to G$ such that $\varepsilon \mapsto g^{-1}$. We may assume that this cocharacter maps into the torus T, so it is identified with a coweight. We may also assume that its coordinates in the basis $(\check{\epsilon}_i)$ are $(-1)^{\nu_1}, (-2)^{\nu_2}, \dots, (-\ell)^{\nu_\ell}$ for some $\nu \in \mathcal{C}_{m,\ell}$ because $\varepsilon^\ell = 1$. Then the Lie algebra $\mathfrak{h}$ in (3.1) is equal to $\mathfrak{h}_\nu$ and we have

$$g = \bigoplus_p \varepsilon^p \, \mathrm{id}_{\mathbf{V}_p^*}.$$

From now on we'll always assume that $\mathfrak{h}$, ν, g are as above. Now we compute the image by $\mathfrak{C}$ of the parabolic Verma modules.

3.8. Proposition.

(a) *Given a ν-dominant weight λ we have $\mathfrak{X}(L(\mathfrak{h}_\nu, \lambda)) = 0$ if $|\lambda| \notin \mathcal{P}^\ell_{n,\nu}$ and $\mathfrak{X}(L(\mathfrak{h}_\nu, \lambda)) \simeq \mathfrak{X}_{\lambda^\circ}$ else. If $\nu \in \mathcal{C}_{m,\ell,n}$ then $\mathfrak{X}(\mathrm{Irr}(U(\mathfrak{h}_\nu))) = \mathrm{Irr}(\mathbb{C}W)$.*
(b) *For each $\hat{\mathbf{g}}^F_{\geqslant 0}$-module M the canonical inclusion $M \subset M^F_\kappa$ yields a $\mathbf{B}$-module isomorphism $\Gamma^{\mathbf{B}}_W(\mathfrak{X}(M)) \to \mathfrak{C}(M^F_\kappa)$.*

Note that in (b) the symbol $\mathfrak{X}(M)$ means the functor $\mathfrak{X}$ in (3.3) applied to the restriction of M to the Lie subalgebra $\mathfrak{h}_\nu \subset \hat{\mathbf{g}}^F_{\geqslant 0}$ and that in (a) the W-module $\mathfrak{X}_{\lambda^\circ}$ is as in (1.3).

Proof. For each $\mu \in \mathcal{C}_{n,\ell}$ the subspace

$$\bigotimes_p \left(\mathbf{V}^*_p\right)^{\otimes \mu_p} \subset \left(\mathbf{V}^*\right)^{\otimes n}$$

is preserved by the action of the parabolic subgroup $W_\mu \subset W$ in Section 1.5. Fix a $W \times U(\mathfrak{h}_\nu)$-module isomorphism

$$\left(\mathbf{V}^*\right)^{\otimes n} \simeq \bigoplus_{\mu \in \mathcal{C}_{n,\ell}} \Gamma^W_{W_\mu}\left(\bigotimes_p \left(\mathbf{V}^*_p\right)^{\otimes \mu_p}\right).$$

It yields a W-module isomorphism

$$\mathfrak{X}\big(L(\mathfrak{h}_\nu, \lambda)\big) \simeq \bigoplus_\mu \Gamma^W_{W_\mu}\left(\bigotimes_p H_0\big(\mathfrak{g}_{J_p}, \left(\mathbf{V}^*_p\right)^{\otimes \mu_p} \otimes L(\lambda_p)\big)\right).$$

If $\lambda_p \in \mathcal{P}_{\mu_p}$ then the W_{μ_p}-module $H_0(\mathfrak{g}_{J_p}, (\mathbf{V}^*_p)^{\otimes \mu_p} \otimes L(\lambda_p))$ is the tensor product of the $\mathfrak{S}_{\mu_p}$-module $\mathfrak{X}_{\lambda_p}$ in (2.1) and the one-dimensional $(D_\ell)^{\mu_p}$-module $\chi_{-p}^{\otimes \mu_p}$. Else it is zero. Now fix $\mu = (|\lambda_p|)$. Then we have

$$\mathfrak{X}\big(L\big(\mathfrak{h}_\nu, \lambda^\circ\big)\big) = \Gamma^W_{W_{\mu^\circ}}\big(\mathfrak{X}_{\lambda_\ell} \chi_{\ell-1}^{\otimes|\lambda_\ell|} \otimes \mathfrak{X}_{\lambda_{\ell-1}} \chi_{\ell-2}^{\otimes|\lambda_{\ell-1}|} \otimes \cdots \otimes \mathfrak{X}_{\lambda_1} \chi_\ell^{\otimes|\lambda_1|}\big).$$

By (1.3) we have also

$$\mathfrak{X}_\lambda = \Gamma^W_{W_\mu}\big(\mathfrak{X}_{\lambda_1} \chi_\ell^{\otimes|\lambda_1|} \otimes \mathfrak{X}_{\lambda_2} \chi_1^{\otimes|\lambda_2|} \otimes \cdots \otimes \mathfrak{X}_{\lambda_\ell} \chi_{\ell-1}^{\otimes|\lambda_\ell|}\big).$$

Thus we have $\mathfrak{X}(L(\mathfrak{h}_\nu, \lambda^\circ)) \simeq \mathfrak{X}_\lambda$ as W-modules. Part (a) is proved. Now, let us prove (b). Set

$$N = M^F_\kappa, \qquad \mathfrak{a} = \mathfrak{g}[\mathbb{C}]^F, \qquad \mathfrak{a}' = \big(z\mathfrak{g}[\mathbb{C}]\big)^F, \qquad \mathfrak{b} = \mathfrak{h}_\nu.$$

The linear map

$$\mathfrak{g}[\mathbb{C}] \to \hat{\mathbf{g}}, \qquad \xi \otimes z^b \mapsto \sharp\xi^{(b)}$$

yields a representation of $\mathfrak{a}_{\mathbf{R}}$ on $N_{\mathbf{R}}$. View $M_{\mathbf{R}}$ as a $\mathfrak{b}_{\mathbf{R}}$-module via the restriction to the Lie subalgebra $\mathfrak{b}_{\mathbf{R}} \subset (\hat{\mathbf{g}}_{\mathbf{R},\geqslant 0})^F$. We have an isomorphism of $\mathfrak{a}_{\mathbf{R}}$-modules

$$N_{\mathbf{R}} \simeq \Gamma_{\mathfrak{b}_{\mathbf{R}}}^{\mathfrak{a}_{\mathbf{R}}}(M_{\mathbf{R}}).$$

The assignment $\xi \otimes z^b \mapsto \sum_i x_i^b \xi_{(i)}$ yields a representation of $\mathfrak{a}_{\mathbf{R}}$ on $(\mathbf{V}^*)_{\mathbf{R}}^{\otimes n}$. Since the induction datum $(\mathfrak{a}_{\mathbf{R}}, \mathfrak{b}_{\mathbf{R}}, \mathfrak{a}'_{\mathbf{R}}, M_{\mathbf{R}})$ is $\mathbf{R}$-split, the tensor identity yields an isomorphism of $\mathfrak{a}_{\mathbf{R}}$-modules

$$T(N)_{\mathbf{R}} \simeq \left(\mathbf{V}^*\right)_{\mathbf{R}}^{\otimes n} \otimes_{\mathbf{R}} N_{\mathbf{R}} \simeq \left(\mathbf{V}^*\right)_{\mathbf{R}}^{\otimes n} \otimes_{\mathbf{R}} \Gamma_{\mathfrak{b}_{\mathbf{R}}}^{\mathfrak{a}_{\mathbf{R}}}(M_{\mathbf{R}}) \simeq \Gamma_{\mathfrak{b}_{\mathbf{R}}}^{\mathfrak{a}_{\mathbf{R}}}\left(T(M)_{\mathbf{R}}\right),$$

see Section A.3. Therefore we have an isomorphism of $\mathbf{B}$-modules

$$\mathfrak{C}(N) = H_0\left(\mathfrak{a}_{\mathbf{R}}, T(N)_{\mathbf{R}}\right) \simeq H_0\left(\mathfrak{b}_{\mathbf{R}}, T(M)_{\mathbf{R}}\right) \simeq \Gamma_W^{\mathbf{B}}\left(\mathfrak{X}(M)\right). \qquad \square$$

3.9. Twisted affine coinvariants. In the rest of Section 3 we'll prove Proposition 3.6. First we recall what twisted affine coinvariants are, following [8]. The group D_ℓ acts faithfully on $\mathbb{P}^1$ by multiplication, yielding a cyclic cover $\pi : \mathbb{P}^1 \to \mathbb{P}^1/D_\ell$ which is ramified at $0, \infty$. Fix an S-tuple of distinct points $y_i \in \mathbb{P}^1/D_\ell$ and pick points x_i of $\pi^{-1}(y_i)$ for each $i \in S$. Let ℓ_i be the number of points in the D_ℓ-orbit of x_i. Next, fix $S = A \cup \{n+1\}$ and $x_{n+1} = y_{n+1} = \infty$. Put

$$\mathbb{P}_y^1 = \mathbb{P}^1 - \pi^{-1}\left(\{y_i;\ i \in S\}\right).$$

Let $\flat \in \operatorname{Aut}(\mathbb{C}[\mathbb{P}_y^1])$ be the comorphism of the action of the generator of D_ℓ. We set

$$z_i = z - x_i, \qquad z_{n+1} = -z^{-1}, \qquad z_{i,p} = z - \varepsilon^p x_i, \quad i \neq n+1.$$

We'll abbreviate $z_{i,p}^a = (z_{i,p})^a$ for each a. Note that $\flat(z) = \varepsilon^{-1} z$, $\flat(z_{n+1}) = \varepsilon z_{n+1}$ and $\flat(z_{i,p}) = \varepsilon^{-1} z_{i,p+1}$. We get the following automorphism of $\mathfrak{g}[\mathbb{P}_y^1]$

$$F = g \otimes \flat : \mathfrak{g}\left[\mathbb{P}_y^1\right] \to \mathfrak{g}\left[\mathbb{P}_y^1\right].$$

Let $\mathcal{G}_S^F$ be the Lie subalgebra of $\mathcal{G}_S$ spanned by $\mathcal{G}_{[n+1]}^F$ and $\mathcal{G}_{[i]}$ for $i \neq n+1$. Let $\hat{\mathcal{G}}_S^F$ be the central extension of $\mathcal{G}_S^F$ by $\mathbb{C}$ associated with the cocycle $(\xi \otimes f, \zeta \otimes g) \mapsto \langle \xi : \zeta \rangle \sum_{i \in S} \ell_i \operatorname{Res}_{t_i=0}(g df)$. Consider the algebra homomorphism

$$\iota_x : \mathbb{C}\left[\mathbb{P}_y^1\right] \to \mathbb{C}((t_S)), \qquad f(z) \mapsto \sum_{i \neq n+1} f(t + x_i)_{[i]} + f\left(-t^{-1}\right)_{[n+1]}.$$

Here $f(t + x_i)$, $f(-t^{-1})$ are identified with the corresponding formal series in t. If $\xi \otimes f$, $\zeta \otimes g \in \mathfrak{g}[\mathbb{P}_y^1]^F$ then we have

$$\sum_{i \in S} \ell_i \operatorname{Res}_{z_i=0}\left(\langle \xi : \zeta \rangle g df\right) = 0, \tag{3.9}$$

because the residue of the one-form $\langle \xi : \zeta \rangle g df$ is the same at each point of the D_ℓ-orbit of x_i. Thus the linear map

$$\mathfrak{g}\big[\mathbb{P}^1_y\big]^F \to \mathcal{G}^F_S, \qquad \xi \otimes f \mapsto \xi \otimes \iota_x(f)$$

lifts to a Lie algebra embedding

$$\mathfrak{g}\big[\mathbb{P}^1_y\big]^F \to \hat{\mathcal{G}}^F_S. \tag{3.10}$$

Now, assume that $M_1, M_2, \dots, M_n \in \mathcal{C}_\kappa$ and $M_{n+1} \in \mathcal{C}^F_\kappa$. The assignment

$$\xi \otimes f(t)_{[i]} \mapsto \xi \otimes f(t)_{(i)} \tag{3.11}$$

yields a representation of $\hat{\mathcal{G}}^F_S$ on the tensor product

$${}_S M = \bigotimes_{i \in S} M_i.$$

So ${}_S M$ is also a $\mathfrak{g}[\mathbb{P}^1_y]^F$-module via the map (3.10).

3.10. Definition. The space of the twisted affine coinvariants of the M_i's is

$$\langle M_i;\ i \in S \rangle_x = H_0\big(\mathfrak{g}\big[\mathbb{P}^1_y\big]^F,\ {}_S M\big).$$

Now, let the point x vary in $C_{n,\ell}$. View the x_i's as coordinates on $C_{n,\ell}$. Let $R \subset \mathbf{R}_{n,\ell}(z)$ be the $\mathbf{R}_{n,\ell}$-submodule spanned by

$$z_{i,p}^{-a},\ z^b \quad \text{with } a > 0,\ i \in A,\ p \in \Lambda,\ b \geqslant 0.$$

It is closed under multiplication. Let $R' \subset R$ be the $\mathbf{R}_{n,\ell}$-subalgebra consisting of the functions which vanish on $\{z = \infty\}$. As an $\mathbf{R}_{n,\ell}$-module R' is spanned by

$$z_{i,p}^{-a} \quad \text{with } a > 0,\ i \in A,\ p \in \Lambda.$$

There is a unique $\mathbf{R}_{n,\ell}$-algebra automorphism $\flat$ of R, R' such that $f(z) \mapsto f(\varepsilon^{-1} z)$. It yields the automorphism $F = g \otimes \flat$ of the $\mathbf{R}_{n,\ell}$-Lie algebras $\mathfrak{g}_R$, $\mathfrak{g}_{R'}$. The construction above yields the $\mathbf{R}_{n,\ell}$-Lie algebra $\hat{\mathcal{G}}^F_{\mathbf{R}_{n,\ell},S}$ and the morphism of $\mathbf{R}_{n,\ell}$-Lie algebras

$$(\mathfrak{g}_R)^F \to \hat{\mathcal{G}}^F_{\mathbf{R}_{n,\ell},S} \tag{3.12}$$

which is analogous to (3.10). Assume that $M_1, M_2, \dots, M_n \in \mathcal{C}_\kappa$ and $M_{n+1} \in \mathcal{C}^F_\kappa$. The assignment (3.11) yields a representation of $\mathcal{G}^F_{\mathbf{R}_{n,\ell},S}$ on ${}_S M_{\mathbf{R}_{n,\ell}}$. Thus (3.12) yields a representation of $(\mathfrak{g}_R)^F$ on ${}_S M_{\mathbf{R}_{n,\ell}}$. Taking the coinvariants with respect to the Lie algebras $(\mathfrak{g}_R)^F$, $(\mathfrak{g}_{R'})^F$ we get the following two $\mathbf{R}_{n,\ell}$-modules:

$$\langle M_i;\ i \in S \rangle, \qquad \langle M_i;\ i \in S \rangle'.$$

We'll also use a more general version of twisted affine coinvariants obtained by inserting a new module $M_0 \in \mathcal{C}_\kappa^{F'}$ at the point 0. Set

$$\hat{S} = \{0\} \cup S = A \cup \{0, n+1\}.$$

Let $\hat{x} = (0, x)$ and $\hat{y} = (0, y)$. The Lie algebra $\mathfrak{g}[\mathbb{P}^1_{\hat{y}}]^F$ acts on the tensor product ${}_{\hat{S}}M$ in the obvious way. We define

$$\langle M_i;\ i \in \hat{S}\rangle_{\hat{x}} = H_0\big(\mathfrak{g}\big[\mathbb{P}^1_{\hat{y}}\big]^F,\ {}_{\hat{S}}M\big).$$

3.11. Proposition.

(a) *If $M_1, \ldots, M_n \in \hat{\mathcal{O}}^{fg}_{\geqslant 0,\kappa}$ and $M_{n+1} \in \hat{\mathcal{O}}^{F,fg}_{\geqslant 0,\kappa}$ the $\mathbf{R}_{n,\ell}$-module $\langle M_i;\ i \in S\rangle$ is projective of finite rank. Its fiber at the point $x \in C_{n,\ell}$ is equal to the $\mathbb{C}$-vector space $\langle M_i;\ i \in S\rangle_x$.*

(b) *Let $\hat{x} = (0, x) \in C_{n+1,\ell}$, $M_1, \ldots, M_n \in \mathcal{C}_\kappa$ and $M_{n+1} \in \mathcal{C}_\kappa^F$. Set $M_0 = M(c\omega_0)^{F'}$. The obvious inclusion $L(\mathfrak{h}_\nu, 0) \subset M_0$ yields an isomorphism $\langle M_i;\ i \in \hat{S}\rangle_{\hat{x}} = \langle M_i;\ i \in S\rangle_x$.*

Proof. The proof is quite standard, so we'll be very brief. Part (a) is the twisted version of Proposition 2.16. The proof that the $\mathbf{R}_{n,\ell}$-module $\langle M_i;\ i \in S\rangle$ is finitely generated is the same as in the untwisted case, see e.g., [18, Prop. 9.12]. Indeed, observe first that for each $i \in A$ there is an integer $r_i > 0$ such that M_i is generated by $M_i(r_i)$ as a $\hat{\mathbf{g}}$-module. Similarly there is an integer $r_{n+1} > 0$ such that M_{n+1} is generated by $M_{n+1}(r_{n+1})$ as a $\hat{\mathbf{g}}^F$-module, where $M_{n+1}(r_{n+1})$ is the set of elements of M_{n+1} killed by any product of r_{n+1} elements of $\mathbf{g}^F_{>0}$. Next, set

$$f_i = z_{i,0}^{-1}, \qquad f_{n+1} = -z, \quad i \in A.$$

Then $f_1, f_2, \ldots, f_{n+1} \in R$ and for each $i, j \in S$ the expansion of f_i at x_j is t^{-1} if $i = j$ and it belongs to $\mathbf{R}_{n,\ell}[\![t]\!]$ else. The rest of the proof is by induction, using the spaces $M_i(r_i)$ and the functions f_i as in [18]. Next, the $\mathbf{R}_{n,\ell}$-module $\langle M_i;\ i \in S\rangle$ is projective, because it is finitely generated and admits a connection, compare [18, Prop. 12.12]. This connection is called the *orbifold KZ connection*. It is constructed in a quite general setting in [26]. We have given the construction of the orbifold KZ connection in the second step of the proof of Proposition 3.6, see Section 3.13 below. Finally, the third claim in (a) is also proved as in the untwisted case, see e.g., [18, Sec. 9.13]. It is a twisted analogue of the commutation of affine coinvariants with base change.

Part (b) is the twisted analogue of the *propagation of vacua* recalled in Proposition 2.14(c). It is proved as in the untwisted case in [18, Prop. 9.18]. More precisely, the expansion at $\hat{y}$ yields a Lie algebra embedding

$$\mathfrak{g}\big[\mathbb{P}^1_{\hat{y}}\big]^F \to \hat{\mathcal{G}}^F_{\hat{S}},$$

where $\hat{\mathcal{G}}^F_{\hat{S}}$ is a central extension of the Lie subalgebra $\mathcal{G}^F_{\hat{S}} \subset \mathcal{G}_{\hat{S}}$ spanned by $\mathcal{G}^{F'}_{[0]}$, $\mathcal{G}_{[i]}$, $i \in A$, and $\mathcal{G}^F_{[n+1]}$. Then $\langle M_i;\ i \in \hat{S}\rangle_{\hat{x}}$ is the space of coinvariants of the representation of $\mathfrak{g}[\mathbb{P}^1_{\hat{y}}]^F$ on the space

$${}_{\hat{S}}M = M(c\omega_0)^{F'} \otimes {}_S M.$$

Note that the $\hat{\mathcal{G}}^F_{\hat{S}}$-module ${}_{\hat{S}}M$ is induced from the $\hat{\mathcal{G}}^{F,+}_{\hat{S}}$-module $\mathbb{C} \otimes {}_S M$, where $\hat{\mathcal{G}}^{F,+}_{\hat{S}} \subset \hat{\mathcal{G}}^F_{\hat{S}}$ is the central extension of the Lie subalgebra $\mathcal{G}^{F,+}_{\hat{S}} \subset \mathcal{G}^F_{\hat{S}}$ spanned by $(\mathfrak{g} \otimes \mathbb{C}[\![t_0]\!])^{F'}$, $\mathcal{G}_{[i]}$, $i \in A$, and $\mathcal{G}^F_{[n+1]}$. Therefore the claim follows from the tensor identity as in [18], and from the following equality

$$\hat{\mathcal{G}}^F_{\hat{S}} = \hat{\mathcal{G}}^{F,+}_{\hat{S}} + \mathfrak{g}\big[\mathbb{P}^1_{\hat{y}}\big]^F,$$

which is left to the reader. Compare Proposition 4.5 below. □

3.12. Remark. The $\mathbf{R}_{n,\ell}$-module of twisted affine coinvariants $\langle M_i;\ i \in S\rangle$ and the $\mathbf{R}$-module $\mathfrak{C}(M)$ of twisted affine coinvariants of $T(M)$ are related by the following $\mathbf{R}_{n,\ell}$-module isomorphism, see (3.15) below

$$M_1 = \cdots = M_n = \mathbf{V}^*_\kappa, \qquad M_{n+1} = M \in \mathcal{C}^F_\kappa \quad \Rightarrow \quad \langle M_i;\ i \in S\rangle \simeq \mathfrak{C}(M)_{n,\ell}.$$

Proof of Proposition 3.6. Recall that $M \in \mathcal{C}^F_\kappa$. Let $R' \subset R \subset \mathbf{R}_{n,\ell}(z)$ be as in Section 3.9. To avoid cumbersome notation we write

$$\mathfrak{a} = (\mathfrak{g}_R)^F, \qquad \mathfrak{a}' = (\mathfrak{g}_{R'})^F, \qquad \mathfrak{b} = (\mathfrak{g}_{\mathbf{R}_{n,\ell}[z]})^F.$$

We have $\mathfrak{a} = \mathfrak{a}' \oplus \mathfrak{b}$, and the $\mathbf{R}_{n,\ell}$-modules $\mathfrak{a}'$, $\mathfrak{b}$ are respectively spanned by

- $\xi \otimes z^p(z^\ell - x_i^\ell)^{-a}$ with $\xi \in \mathfrak{g}_p$, $a > 0$, $p = 0, \ldots, \ell - 1$, $i \in A$,
- $\xi \otimes z^b$ with $\xi \in \mathfrak{g}_b$, $b \geqslant 0$.

The proof is long and technical. We'll split it into several steps.

Step 1: First, we prove the formula (3.15) below, which identifies $T(M)_{\mathbf{R}_{n,\ell}}$ with a set of twisted affine coinvariants. To do so we put

$$M_1 = \cdots = M_n = \mathbf{V}^*_\kappa, \qquad M_{n+1} = M.$$

To simplify the notation we'll set $n = 1$ in this part of the proof. The case $n > 1$ is identical. Therefore we have

$$T(M) = \mathbf{V}^* \otimes M, \qquad {}_S M = \mathbf{V}^*_\kappa \otimes M$$

and the inclusion $\mathbf{V}^* \subset \mathbf{V}^*_\kappa$ yields an $\mathbf{R}_{1,\ell}$-linear embedding

$$T(M)_{\mathbf{R}_{1,\ell}} \subset {}_S M_{\mathbf{R}_{1,\ell}}. \tag{3.13}$$

Recall that $\mathbf{R}_{1,\ell} = \mathbb{C}[x_1^{\pm 1}]$ and that $\mathfrak{a}$ acts on ${}_S M_{\mathbf{R}_{1,\ell}}$ via the map (3.12). Thus, by definition of the map ι_x the element $\xi \otimes z^b \in \mathfrak{b}$ acts as the sum

$$\xi \otimes (t + x_1)^b_{[1]} + \xi \otimes \big(-t^{-1}\big)^b_{[2]}.$$

Since the subspace $\mathbf{V}^* \subset \mathbf{V}_\kappa^*$ is killed by $\mathbf{g}_{>0}$, we get that $\xi \otimes z^b$ acts on the subspace $T(M)_{\mathbf{R}_{1,\ell}}$ as the sum

$$\big(x_1^b \xi\big)_{(1)} + \big(\sharp\xi^{(b)}\big)_{(2)}.$$

This is precisely the $\mathfrak{b}$-action on $T(M)_{\mathbf{R}_{1,\ell}}$ given in (3.5). Therefore, the inclusion (3.13) is an embedding of $\mathfrak{b}$-modules.

Now we prove that there is an isomorphism of $\mathfrak{a}$-modules

$$_SM_{\mathbf{R}_{1,\ell}} \simeq \Gamma_{\mathfrak{b}}^{\mathfrak{a}}\big(T(M)_{\mathbf{R}_{1,\ell}}\big). \tag{3.14}$$

Let $\hat{\mathfrak{b}} = \mathfrak{b} \oplus \mathbf{R}_{1,\ell}$ (the trivial central extension) and let $\hat{\mathfrak{a}}$ be the central extension of $\mathfrak{a}$ by $\mathbf{R}_{1,\ell}$ associated with the cocycle $(\xi \otimes f, \zeta \otimes g) \mapsto \langle \xi : \zeta \rangle \operatorname{Res}_{z=x_1}(g\,df)$. By the residue theorem there is an $\mathbf{R}_{1,\ell}$-Lie algebra homomorphism

$$\hat{\mathfrak{a}} \to \hat{\mathbf{g}}^F_{\mathbf{R}_{1,\ell}}, \qquad \xi \otimes f(z) \mapsto \xi \otimes f\big(-t^{-1}\big), \qquad \mathbf{1} \mapsto -\ell\mathbf{1},$$

compare (3.9). Thus, since M is a $\hat{\mathbf{g}}^F$-module of level c, the assignment

$$\xi \otimes f(z) \mapsto \xi \otimes f\big(-t^{-1}\big)$$

yields a representation of $\hat{\mathfrak{a}}$ on $M_{\mathbf{R}_{1,\ell}}$ of level $-c$. Similarly, since $\mathbf{V}_\kappa^*$ is a $\hat{\mathbf{g}}^F$-module of level c, the assignment

$$\xi \otimes f(z) \mapsto \xi \otimes f(t + x_1)$$

yields a representation of $\hat{\mathfrak{a}}$ of level c on $(\mathbf{V}_\kappa^*)_{\mathbf{R}_{1,\ell}}$. The representation of $\mathfrak{a}$ on $_SM_{\mathbf{R}_{1,\ell}}$ is the restriction of the tensor product of the $\hat{\mathfrak{a}}$-modules $M_{\mathbf{R}_{1,\ell}}$ and $(\mathbf{V}_\kappa^*)_{\mathbf{R}_{1,\ell}}$. The $\mathbf{R}_{1,\ell}$-submodule $\mathbf{V}^*_{\mathbf{R}_{1,\ell}} \subset (\mathbf{V}_\kappa^*)_{\mathbf{R}_{1,\ell}}$ is preserved by the $\hat{\mathfrak{b}}$-action. So the quadruple $(\hat{\mathfrak{a}}, \hat{\mathfrak{b}}, \mathfrak{a}', \mathbf{V}^*_{\mathbf{R}_{1,\ell}})$ is an $\mathbf{R}_{1,\ell}$-split induction datum. See Section A.3 for details.

We claim that the representation of $\hat{\mathfrak{a}}$ on $(\mathbf{V}_\kappa^*)_{\mathbf{R}_{1,\ell}}$ is isomorphic to the induced module $\Gamma_{\hat{\mathfrak{b}}}^{\hat{\mathfrak{a}}}(\mathbf{V}^*_{\mathbf{R}_{1,\ell}})$. Then (3.14) follows from the tensor identity, because we have

$$_SM_{\mathbf{R}_{1,\ell}} = \Gamma_{\hat{\mathfrak{b}}}^{\hat{\mathfrak{a}}}\big(\mathbf{V}^*_{\mathbf{R}_{1,\ell}}\big) \otimes_{\mathbf{R}_{1,\ell}} M_{\mathbf{R}_{1,\ell}} = \Gamma_{\hat{\mathfrak{b}}}^{\hat{\mathfrak{a}}}\big(\mathbf{V}^*_{\mathbf{R}_{1,\ell}} \otimes_{\mathbf{R}_{1,\ell}} M_{\mathbf{R}_{1,\ell}}\big) = \Gamma_{\mathfrak{b}}^{\mathfrak{a}}\big(T(M)_{\mathbf{R}_{1,\ell}}\big).$$

Now we prove the claim. The $\hat{\mathfrak{a}}$-action on $(\mathbf{V}_\kappa^*)_{\mathbf{R}_{1,\ell}}$ yields a map

$$\Gamma_{\hat{\mathfrak{b}}}^{\hat{\mathfrak{a}}}\big(\mathbf{V}^*_{\mathbf{R}_{1,\ell}}\big) \to \big(\mathbf{V}_\kappa^*\big)_{\mathbf{R}_{1,\ell}}, \qquad \big(\xi \otimes f(z)\big) \otimes v \mapsto \big(\xi \otimes f(t + x_1)\big)v.$$

We must prove that it is invertible. Recall that

$$\Gamma_{\hat{\mathfrak{b}}}^{\hat{\mathfrak{a}}}\big(\mathbf{V}^*_{\mathbf{R}_{1,\ell}}\big) = U\big(\mathfrak{a}'\big) \otimes_{\mathbf{R}_{1,\ell}} \mathbf{V}^*_{\mathbf{R}_{1,\ell}},$$

$$\big(\mathbf{V}_\kappa^*\big)_{\mathbf{R}_{1,\ell}} = \Gamma_{\hat{\mathbf{g}}_{\geqslant 0}}^{\hat{\mathbf{g}}}\big(\mathbf{V}^*_{\mathbf{R}_{1,\ell}}\big) = U(\mathbf{g}_{\mathbf{R}_{1,\ell},<0}) \otimes_{\mathbf{R}_{1,\ell}} \mathbf{V}^*_{\mathbf{R}_{1,\ell}}.$$

Further, for each $a > 0$ the assignment $z \mapsto t + x_1$ takes $z^p(z^\ell - x_1^\ell)^{-a}$ to a formal series in

$$(\mathbf{R}_{1,\ell})^\times t^{-a} + \mathbf{R}_{1,\ell}[\![t]\!]t^{1-a}.$$

Therefore the claim is obvious.

Note that (3.14) yields the second of the two isomorphisms below

$$\langle M_i;\ i \in S\rangle' \simeq T(M)_{\mathbf{R}_{n,\ell}}, \qquad \langle M_i;\ i \in S\rangle \simeq \mathfrak{C}(M)_{n,\ell}. \tag{3.15}$$

Indeed, with the notation used in this proof the definition (3.6) yields

$$\mathfrak{C}(M)_{n,\ell} = H_0\big(\mathfrak{b}, T(M)_{\mathbf{R}_{n,\ell}}\big).$$

The proof of the first isomorphism is identical because the quadruple $(\hat{\mathfrak{a}}, \hat{\mathfrak{b}}, \mathfrak{a}', T(M)_{\mathbf{R}_{1,\ell}})$ is also an $\mathbf{R}_{1,\ell}$-split induction datum, hence (3.14) and Section A.3 imply that

$${}_S M_{\mathbf{R}_{1,\ell}} \simeq \Gamma^{\mathfrak{a}'}\big(T(M)_{\mathbf{R}_{1,\ell}}\big).$$

Step 2: Now we define an integrable connection on the $\mathbf{R}_{n,\ell}$-modules $T(M)_{\mathbf{R}_{n,\ell}}$ and $\mathfrak{C}(M)_{n,\ell}$. For each i we consider the differential operator on ${}_S M_{\mathbf{R}_{n,\ell}}$ given by

$$\nabla_i = \partial_{x_i} + \mathbf{L}_{-1,(i)}.$$

Next we set $\nabla = \sum_i \nabla_i\, dx_i$. It is an integrable connection on the $\mathbf{R}_{n,\ell}$-module ${}_S M_{\mathbf{R}_{n,\ell}}$. By (2.2), for each $\xi \in \mathfrak{g}$ and $f \in \mathbf{R}_{n,\ell}((t_S))$ we have

$$[\nabla_i, \xi \otimes f] = \xi \otimes (\partial_{x_i} f - \partial_{t_i} f). \tag{3.16}$$

We claim that ∇ normalizes the $\mathfrak{a}$-action given by (3.12). Hence the isomorphisms (3.15) yield an integrable connection on the $\mathbf{R}_{n,\ell}$-module $T(M)_{\mathbf{R}_{n,\ell}}$ which factors to a connection on $\mathfrak{C}(M)_{n,\ell}$. Write ∇ again for these connections.

Now we prove this claim. For each integers a, b with $a > 0$ we define the constant c_a^b by the following formula

$$\partial_t^{a-1}\big(t^b\big)/(a-1)! = c_a^b t^{b-a+1}.$$

A direct computation yields the following relations in $\mathbf{R}_{n,\ell}((t_S))$

$$\iota_x\big(z_{j,p}^{-a}\big) = \delta_{p,0} t_{[j]}^{-a} - \sum_k \sum_{b \geqslant 0} c_a^{-b-1}\big(\varepsilon^p x_j - x_k\big)^{-b-a} t_{[k]}^b - \sum_{b \geqslant 0} (-1)^b c_a^b \big(\varepsilon^p x_j\big)^{b-a+1} t_{[n+1]}^{b+1},$$

$$\iota_x\big(z^b\big) = \sum_k (x_k + t)_{[k]}^b + (-1)^b t_{[n+1]}^{-b},$$

where $k = 1, 2, \ldots, n$ and $(k, p) \neq (j, 0)$. Thus the derivation $\partial_{x_i} - \partial_{t_i}$ annihilates $\iota_x(z^b)$ and takes $\iota_x(z_{j,p}^{-a})$ to $\delta_{i,j}\varepsilon^p a \iota_x(z_{j,p}^{-a-1})$. So $(\partial_{x_i} - \partial_{t_i}) \circ \iota_x = \iota_x \circ \partial_{x_i}$ on R. Hence (3.16) yields the following relations

$$[\nabla_i, \xi \otimes \iota_x(z^b)] = 0,$$
$$[\nabla_i, \xi \otimes \iota_x(z^p(z^\ell - x_j^\ell)^{-a})] = \delta_{i,j} a\ell x_i^{\ell-1} \xi \otimes \iota_x(z^p(z^\ell - x_j^\ell)^{-a-1}).$$

The claim is proved.

Step 3: Now we claim that the connection ∇ on $T(M)_{\mathbf{R}_{n,\ell}}$ is W-equivariant and

$$\bar{y}_i = \nabla_i + k \sum_{j \neq i} \sum_p \frac{1}{x_i - \varepsilon^{-p} x_j} s_{i,j}^{(p)} + \sum_{p \neq 0} \frac{\gamma_p}{x_i - \varepsilon^{-p} x_i} \varepsilon_i^p. \tag{3.17}$$

Thus part (a) of the proposition follows from Proposition 1.8(a). Note that the $\mathbf{H}_{h,H}$-action on $T(M)_{\mathbf{R}_{n,\ell}}$ we have just constructed normalizes the $\mathfrak{b}$-action given by (3.5), because the connection ∇ normalizes the $\mathfrak{a}$-action on ${}_SM_{\mathbf{R}_{n,\ell}}$ given by (3.12) by the previous claim.

Now, let us prove the latest claim. We'll compute the connection ∇ on $T(M)_{\mathbf{R}_{n,\ell}}$. Since ∇_i is a derivation, it is enough to consider its restriction to the subspace $T(M) \subset T(M)_{\mathbf{R}_{n,\ell}}$. Under the quotient map ${}_SM_{\mathbf{R}_{n,\ell}} \to T(M)_{\mathbf{R}_{n,\ell}}$ in (3.15), the obvious inclusion $T(M) \subset {}_SM_{\mathbf{R}_{n,\ell}}$ is taken to the obvious inclusion $T(M) \subset T(M)_{\mathbf{R}_{n,\ell}}$. So it is enough to compute the action of the operator $\mathbf{L}_{-1,(i)}$ on the subspace $T(M) \subset_S M_{\mathbf{R}_{n,\ell}}$.

For $\xi \in \mathfrak{g}$ we consider the following elements

$$\sigma(\xi, i) = \sum_p \varepsilon^p g^p \xi \otimes z_{i,p}^{-1} \in \mathfrak{a}', \qquad \xi_{[j]}^{(b)} = \xi \otimes t_{[j]}^b \in \hat{\mathcal{G}}_{S,\mathbf{R}_{n,\ell}}.$$

Note that

$$\iota_x(\sigma(\xi, i)) = \xi_{[i]}^{(-1)} - \sum_{j,p,b} \varepsilon^p (\varepsilon^p x_i - x_j)^{-b-1} (g^p \xi)_{[j]}^{(b)} - \sum_{b,p} (-1)^b \varepsilon^p (\varepsilon^p x_i)^b (g^p \xi)_{[n+1]}^{(b+1)},$$

where $j \in A$, $b \geqslant 0$, $p \in \Lambda$ and $(j,p) \neq (i,0)$ in the first sum. Let $v \in T(M)$, viewed as a subspace of ${}_SM_{\mathbf{R}_{n,\ell}}$. Then, we have

$$\xi_{[i]}^{(-1)} v = \sum_{j,p} \frac{g^p \xi_{(j)}}{x_i - \varepsilon^{-p} x_j} v + \sum_{b,p} (-1)^b x_i^b F^p(\xi^{(b+1)})_{(n+1)} v \quad \operatorname{mod} \mathfrak{a}'({}_SM_{\mathbf{R}_{n,\ell}}).$$

Note that $\xi_{(i)} v = 0$ for each $\xi \in \hat{\mathbf{g}}_{>0}$, $i \in A$. We have

$$\mathbf{L}_{-1,(i)} v = \frac{1}{\kappa} \sum_{k,l} e_{k,l,(i)}^{(-1)} e_{l,k,(i)} v, \quad i \in A,$$

$$\gamma_{i,j}^{(p)} = \begin{cases} \frac{1}{\kappa} \sum_{a>0} \sum_{k,l} (-1)^{a-1} x_i^{a-1} F^p(e_{k,l}^{(a)})_{(n+1)} e_{l,k,(i)} & \text{if } j = n+1, \\ \sum_{k,l} (g^p e_{k,l})_{(j)} e_{l,k,(i)} / \kappa(x_i - \varepsilon^{-p} x_j) & \text{else.} \end{cases}$$

Setting $\xi = e_{k,l}$ in the formula above, we get

$$\mathbf{L}_{-1,(i)} v = \gamma_{i,1}^F v + \cdots + \gamma_{i,n+1}^F v \quad \operatorname{mod} \mathfrak{a}'({}_SM_{\mathbf{R}_{n,\ell}}).$$

In other words, the connection ∇ on $T(M)_{\mathbf{R}_{n,\ell}}$ is given by

$$\nabla_i = \partial_{x_i} + \frac{1}{\kappa}\sum_{j\neq i}\sum_{p}\frac{\sigma_{i,j}^{(p)}}{x_i-\varepsilon^{-p}x_j} + \frac{1}{\kappa}\sum_{p\neq 0}\frac{\sigma_{i,i}^{(p)}}{x_i-\varepsilon^{-p}x_i} + \gamma_{i,n+1}^F,$$

with $j \in A$. Now, for $p \neq 0$ the operator $\sigma_{i,i}^{(p)}$ acts on $T(M)_{\mathbf{R}_{n,\ell}}$ as the operator $\mathrm{tr}(g^p)g_i^p = -\kappa\gamma_p g_i^p$. Thus we have

$$\nabla_i = \partial_{x_i} + \frac{1}{\kappa}\sum_{j\neq i}\sum_{p}\frac{\sigma_{i,j}^{(p)}}{x_i-\varepsilon^{-p}x_j} - \sum_{p\neq 0}\frac{\gamma_p\, g_i^p}{x_i-\varepsilon^{-p}x_i} + \gamma_{i,n+1}^F.$$

Since $k = -1/\kappa$ the right-hand side of (3.17) is

$$\begin{aligned}&\partial_{x_i} - \frac{1}{\kappa}\sum_{j\neq i}\sum_{p}\frac{1}{x_i-\varepsilon^{-p}x_j}\big(s_{i,j}^{(p)} - \sigma_{i,j}^{(p)}\big) + \sum_{p\neq 0}\frac{\gamma_p}{x_i-\varepsilon^{-p}x_i}\big(\varepsilon_i^p - g_i^p\big) + \gamma_{i,n+1}^F\\ &\quad= \partial_{x_i} - \sum_{j\neq i}\sum_{p}\gamma_{i,j}^{(p)}\big(\underline{s_{i,j}^{(p)}} - 1\big) - \sum_{p\neq 0}\gamma_{i,i}^{(p)}\big(\underline{\varepsilon_i^p} - 1\big) + \gamma_{i,n+1}^F = \bar{y}_i.\end{aligned}$$

So (3.17) is proved. The W-equivariance of ∇ is obvious from the formula above.

Let us prove (b). If $v \in T(M)$ then (3.8) gives $\bar{y}_i v = \gamma_{i,n+1}^F v$. The subspace $T(M)_{\mathbf{R}} \subset T(M)_{\mathbf{R}_{n,\ell}}$ is preserved by the $\mathbf{B}$-action in (3.4). By (3.8) it is also preserved by the operators $\bar{y}_1, \bar{y}_2, \ldots, \bar{y}_n$, i.e., the denominator in $\gamma_{i,j}^{(p)}$ simplifies. Thus $T(M)_{\mathbf{R}}$ is a $\mathbf{H}_{h,H}$-submodule. The representation of $\mathbf{H}_{h,H}$ on $T(M)_{\mathbf{R}}$ normalizes the $\mathfrak{g}[\mathbb{C}]^F$-action by part (a). Hence $\mathbf{H}_{h,H}$ acts on $\mathfrak{C}(M)$. □

3.13. Remarks. (a) The connection ∇ is a particular case of the orbifold KZ connection in [26].

(b) In Section 4.4 we'll also use a more general $\mathbf{H}_{h,H}$-module $\mathfrak{C}(M', M)_{n,\ell}$ obtained by inserting also an almost smooth $\hat{\mathbf{g}}_\kappa^{F'}$-module M' at the point $0 \in \mathbb{P}^1$ before taking twisted affine coinvariants. See also Section 3.9. Then Proposition 3.8 generalizes in the following way: if $M' = (E')_\kappa^{F'}$ and $M = E_\kappa^F$ then we have an isomorphism of $\mathbf{B}_{n,\ell}$-modules

$$\mathfrak{C}\big(M', M\big)_{n,\ell} \simeq \Gamma_W^{\mathbf{B}_{n,\ell}}\big(\mathfrak{X}\big(E' \otimes E\big)\big).$$

4. Untwisting the space of twisted affine coinvariants

Any twisted affine Lie algebra associated with an inner automorphism is isomorphic to a non-twisted one. In this section we prove that, in a similar way, the space of twisted affine coinvariants can expressed in terms of non-twisted affine coinvariants. Recall, however, that twisted affine coinvariants yield a local system over the configuration space of the stack $[\mathbb{P}^1/D_\ell]$ while affine coinvariants yield a local system over the configuration space of $\mathbb{P}^1$.

4.1. Notation. Given a weight π we may shift the origin in $\mathfrak{t}^*$ and write

$$\lambda_\pi = \lambda + \pi, \qquad \hat{\lambda}_\pi = \lambda_\pi + c\omega_0, \quad \forall \lambda \in \mathfrak{t}^*.$$

We'll choose a new origin π as follows

$$\pi = c\gamma/\ell, \quad \gamma = (-1, -1, \ldots, -1, -2, \ldots, -\ell),$$

where the integer $-p$ has multiplicity ν_p. The coweight $\check{\gamma}$ associated with γ can be viewed as a cocharacter of the torus T. So it yields the element $\check{\gamma}(z) \in T$ for each $z \in \mathbb{C}^\times$. Note that $\check{\gamma}(\varepsilon) = g^{-1}$. Note also that accordingly to (3.2) we have $\hat{\lambda} = \lambda + (c/\ell)\omega_0$.

4.2. Proposition.

(a) *There is an algebra isomorphism* $\varkappa : \hat{\mathbf{g}}_\kappa^F \to \hat{\mathbf{g}}_\kappa$ *yielding equivalences of categories* $\mathcal{C}_\kappa^F \to \mathcal{C}_\kappa$, $\hat{\mathcal{O}}_{\geqslant 0,\kappa}^F \to \hat{\mathcal{O}}_{\nu,\kappa}$ *such that* $M(\hat{\lambda})^F \mapsto M(\hat{\lambda}_\pi)_\nu$, $L(\hat{\lambda})^F \mapsto L(\hat{\lambda}_\pi)$.

(b) *There is an algebra isomorphism* $\varkappa : \hat{\mathbf{g}}_\kappa^{F'} \to \hat{\mathbf{g}}_\kappa$ *yielding equivalences of categories* $\mathcal{C}_\kappa^{F'} \to \mathcal{C}_\kappa$, $\hat{\mathcal{O}}_{\geqslant 0,\kappa}^{F'} \to \hat{\mathcal{O}}_{\nu,\kappa}$ *such that* $M(\hat{\lambda})^{F'} \mapsto M(\hat{\mu}_\pi)_\nu$, $L(\hat{\lambda})^{F'} \mapsto L(\hat{\mu}_\pi)$ *with* $\mu = -w_\nu(\lambda)$.

(c) *If* $M \in \mathcal{O}_\kappa^{F,fg}$ *then* ${}^\dagger D({}^\varkappa M) = {}^\varkappa({}^\dagger DM)$. *If* $M \in \mathcal{C}_\kappa^F$ *then* ${}^\varkappa M = {}^\varkappa({}^\dagger M)$.

Proof. (a) We have

$$\mathbf{g}^F = \bigoplus_{a \equiv b \bmod \ell} \mathfrak{g}(b) \otimes t^a, \qquad \mathfrak{g}(b) = \{\xi \in \mathfrak{g};\ [\check{\gamma}, \xi] = b\xi\}. \tag{4.1}$$

Conjugating by $\check{\gamma}(-t)^{-1}$ takes $\mathfrak{g}(b) \otimes t^a$ onto $\mathfrak{g}(b) \otimes t^{a-b}$. Composing this map with the assignment $(-t)^\ell \mapsto -t$ yields a Lie algebra isomorphism $\mathbf{g}^F \to \mathbf{g}$. It lifts to a Lie algebra isomorphism

$$\varkappa : \hat{\mathbf{g}}^F \to \hat{\mathbf{g}}, \qquad \mathbf{1} \mapsto \mathbf{1}/\ell, \qquad \xi^{(a)} \mapsto \check{\gamma}(-t)^{-1/\ell}\big(\xi^{(a/\ell)}\big) - \delta_{a,0}\langle\check{\gamma} : \xi\rangle\mathbf{1}/\ell, \tag{4.2}$$

see [15, Thm. 8.5]. The Lie algebra $\varkappa(\hat{\mathbf{g}}_{\geqslant 0}^F)$ is spanned by $\mathbf{1}$ and the elements $\xi^{(a)}$ with $\xi \in \mathfrak{g}(b)$ and $a\ell + b \geqslant 0$. Therefore we have

$$\varkappa\big(\hat{\mathbf{g}}_{\geqslant 0}^F\big) = \hat{\mathbf{q}}_\nu,$$
$$\varkappa(\xi + z\mathbf{1}) = \xi + \big(z - \langle\check{\gamma} : \xi\rangle\big)\mathbf{1}/\ell, \quad \forall \xi \in \mathfrak{h}_\nu.$$

Part (b) is the same, using the conjugation by $\check{\gamma}(t)$ instead of $\check{\gamma}(-t)^{-1}$. Indeed, we get the Lie algebra isomorphism

$$\varkappa' : \hat{\mathbf{g}}^{F'} \to \hat{\mathbf{g}}, \qquad \mathbf{1} \mapsto \mathbf{1}/\ell, \qquad \xi^{(a)} \mapsto \check{\gamma}(t)^{1/\ell}\big(\xi^{(a/\ell)}\big) + \delta_{a,0}\langle\check{\gamma} : \xi\rangle\mathbf{1}/\ell. \tag{4.3}$$

Set $\varkappa = \dagger \circ \varkappa'$. Note that ${}^\dagger L(\hat{\lambda}) = L(\widehat{-\lambda})$ where, in the right-hand side, the highest weight is relative to $\hat{\mathbf{b}}'$. Note also that twisting by $\varkappa'$ takes $L(\hat{\lambda})^{F'}$ to the simple module $L(w_\nu(\hat{\lambda}_{-\pi}))$ in $\hat{\mathcal{O}}'_{\nu,\kappa}$. Part (c) is obvious. $\square$

To unburden notation, we'll write $M \mapsto {}^\varkappa M$ both for the equivalence $\mathcal{C}_\kappa^F \to \mathcal{C}_\kappa$ and its inverse, hoping it will not create any confusion.

4.3. Notation. Let $S = A \cup \{n+1\}$. Let x_i, y_i, $i \in S$, be as in Section 3.9. Recall that $x_{n+1} = y_{n+1} = \infty$. Recall n-tuples $(x_i;\ i \in A)$, $(y_i;\ i \in A)$ belong to $C_{n,\ell}$, $C_{n,1}$ respectively. View x_i, y_i as coordinates on $C_{n,\ell}$, $C_{n,1}$. Then the assignment $y_i \mapsto x_i^\ell$ yields an inclusion

$$\mathbf{R}_{n,1} \subset \mathbf{R}_{n,\ell}.$$

The group D_ℓ acts on $\mathbb{P}^1_y$ by multiplication. We have

$$\mathbb{C}\big[\mathbb{P}^1_y/D_\ell\big] = \mathbb{C}\big[\mathbb{P}^1_y\big]^\flat.$$

Next, there is an obvious isomorphism $\mathbb{P}^1/D_\ell \to \mathbb{P}^1$ which gives rise to an isomorphism $\mathbb{P}^1_y/D_\ell \to \mathbb{P}^1 - \{x_i^\ell;\ i \in S\}$. So we may identify

$$\mathbb{C}\big[\mathbb{P}^1_y\big]^\flat = \mathbb{C}\big[z, (z-y_i)^{-1}\big].$$

4.4. Twisted affine coinvariants versus untwisted ones. We want to compare twisted affine coinvariants with untwisted ones. To do that we first generalize the construction of the functor $\mathfrak{C}$ in (3.6) by inserting an almost smooth $\hat{\mathbf{g}}_\kappa^{F'}$-module at the point $x_0 = 0$. More precisely, given $M \in \mathcal{C}_\kappa^F$ and $M' \in \mathcal{C}_\kappa^{F'}$ we consider the vector space

$$T\big(M', M\big) = M' \otimes T(M) = M' \otimes \big(\mathbf{V}^*\big)^{\otimes n} \otimes M.$$

Equip the $\mathbf{R}_{n,\ell}$-module $T(M', M)_{\mathbf{R}_{n,\ell}}$ with the unique $\mathbf{R}_{n,\ell}$-linear representation of the Lie algebra $\mathfrak{g}[\mathbb{C}^\times]^F$ such that

$$\xi \otimes z^a \mapsto \big(\xi^{(a)}\big)_{(0)} + \sum_{i=1}^n x_i^a \xi_{(i)} + \big(\sharp\xi^{(a)}\big)_{(n+1)}.$$

Then we set

$$\mathfrak{C}\big(M', M\big)_{n,\ell} = H_0\big(\mathfrak{g}\big[\mathbb{C}^\times\big]^F, T\big(M', M\big)_{\mathbf{R}_{n,\ell}}\big).$$

The $\mathfrak{g}[\mathbb{C}^\times]^F$-action on $T(M', M)_{\mathbf{R}_{n,\ell}}$ centralizes the $\mathbf{B}_{n,\ell}$-action. We get a bifunctor

$$\hat{\mathbf{g}}_\kappa^{F'}\text{-}\mathbf{mod} \times \hat{\mathbf{g}}_\kappa^{F}\text{-}\mathbf{mod} \to \mathbf{B}_{n,\ell}\text{-}\mathbf{mod}, \qquad \big(M', M\big) \mapsto \mathfrak{C}\big(M', M\big)_{n,\ell}.$$

The $\mathbf{B}_{n,\ell}$-modules $\mathfrak{C}(M)_{n,\ell}$ and $\mathfrak{C}(M', M)_{n,\ell}$ are related in the following way.

4.5. Proposition. *There is a natural isomorphism of $\mathbf{B}_{n,\ell}$-modules $\mathfrak{C}(M)_{n,\ell} \to \mathfrak{C}(M(c\omega_0)^{F'}, M)_{n,\ell}$.*

Proof. This is a consequence of Proposition 3.11(b). To unburden the notation we give a proof for $n = 1$ and we set $R = \mathbf{R}_{n,\ell}$. The case $n > 1$ is the same. Set

$$\mathfrak{a} = \mathfrak{g}\big[\mathbb{C}^\times\big]^F, \qquad \mathfrak{b} = \mathfrak{g}[\mathbb{C}]^F, \qquad \mathfrak{a}' = \big(z^{-1}\mathfrak{g}\big[z^{-1}\big]\big)^F.$$

Let $\hat{\mathfrak{b}} = \mathfrak{b} \oplus \mathbb{C}$, the trivial central extension of $\mathfrak{b}$ by $\mathbb{C}$, and let $\hat{\mathfrak{a}}$ be the central extension of $\mathfrak{a}$ by $\mathbb{C}$ associated with the cocycle $(\xi \otimes f, \zeta \otimes g) \mapsto \langle \xi : \zeta \rangle \operatorname{Res}_{z=0}(g df)$. The assignments

$$\xi \otimes z^b \mapsto \xi^{(b)}, \qquad \xi \otimes z^b \mapsto x_1^b \xi_{(1)} + \big(\sharp\xi^{(b)}\big)_{(2)}$$

yield representations of $\hat{\mathfrak{a}}_R$ on M'_R and $\mathbf{V}_R^* \otimes_R M_R$ of level c and $-c$ respectively. We have an isomorphism of $\hat{\mathfrak{a}}_R$-modules

$$M'_R \simeq \Gamma_{\hat{\mathfrak{b}}_R}^{\hat{\mathfrak{a}}_R}(R).$$

Here $\mathfrak{b}_R$ acts trivially on R. The induction datum $(\hat{\mathfrak{a}}_R, \hat{\mathfrak{b}}_R, \mathfrak{a}'_R, R)$ is R-split. Thus the tensor identity yields an isomorphism of $\mathfrak{a}_R$-modules

$$T\big(M', M\big)_R \simeq \Gamma_{\mathfrak{b}_R}^{\mathfrak{a}_R}\big(T(M)_R\big).$$

Thus we have

$$\mathfrak{C}\big(M', M\big)_{n,\ell} \simeq H_0\big(\mathfrak{b}, T(M)_R\big) = \mathfrak{C}(M)_{n,\ell}. \qquad \square$$

Next, assume that $M \in \mathcal{C}_\kappa^F$ and $M' \in \mathcal{C}_\kappa^{F'}$. Set $N = {}^{\varkappa}M$ and $N' = {}^{\varkappa}M'$. The $\hat{\mathbf{g}}_\kappa$-modules N, N' are almost smooth by Proposition 4.2. They are canonically isomorphic to M, M' respectively as vector spaces. Let $\mathbf{B}_{n,\ell}$ act on $(\mathbf{V}^*)^{\otimes n}_{\mathbf{R}_{n,\ell}}$ as in (3.4). Setting $\ell = 1$ we get a $\mathbf{B}_{n,1}$-action on $(\mathbf{V}^*)^{\otimes n}_{\mathbf{R}_{n,1}}$. The inclusion $\mathbf{R}_{n,1} \subset \mathbf{R}_{n,\ell}$ in Section 4.3 yields an embedding $W \ltimes \mathbf{R}_{n,1} \subset \mathbf{B}_{n,\ell}$, where the ε_i's act trivially on $\mathbf{R}_{n,1}$ in the left-hand side. Thus, by induction, the algebra $\mathbf{B}_{n,\ell}$ acts on

$$\big(\mathbf{V}^*\big)^{\otimes n}_{\mathbf{R}_{n,1}} \otimes_{\mathbf{R}_{n,1}} \mathbf{R}_{n,\ell}.$$

Now, consider the $\mathbf{R}_{n,\ell}$-linear map

$$\begin{gathered}\big(\mathbf{V}^*\big)^{\otimes n}_{\mathbf{R}_{n,\ell}} \to \big(\mathbf{V}^*\big)^{\otimes n}_{\mathbf{R}_{n,1}} \otimes_{\mathbf{R}_{n,1}} \mathbf{R}_{n,\ell},\\ v = v_1 \otimes v_2 \otimes \cdots \otimes v_n \mapsto \check{\gamma}(x_1)v_1 \otimes \check{\gamma}(x_2)v_2 \otimes \cdots \otimes \check{\gamma}(x_n)v_n,\\ \check{\gamma}(x_i) = \big(x_i^{-1}, x_i^{-1}, \ldots, x_i^{-1}, x_i^{-2}, \ldots, x_i^{-\ell}\big).\end{gathered}$$

It is indeed a $\mathbf{B}_{n,\ell}$-linear map (the $\mathfrak{S}_n$-linearity is obvious, and to prove the W-linearity it is enough to observe that $\check{\gamma}(\varepsilon) = g^{-1}$, see Section 4.1, and that $\varepsilon_i v = g_{(i)} v$, see Section 3.1). This yields also a $\mathbf{B}_{n,\ell}$-linear map

$$\varkappa : T\big(M', M\big)_{\mathbf{R}_{n,\ell}} \to T\big({}^{\dagger}N', N\big)_{\mathbf{R}_{n,1}} \otimes_{\mathbf{R}_{n,1}} \mathbf{R}_{n,\ell}. \tag{4.4}$$

4.6. Proposition. *Let $M \in \mathcal{C}_\kappa^F$, $M' \in \mathcal{C}_\kappa^{F'}$, $N = {}^{\varkappa}M$ and $N' = {}^{\varkappa}M'$. The map $\varkappa$ gives a $\mathbf{B}_{n,\ell}$-module isomorphism $\mathfrak{C}(M', M)_{n,\ell} \to \mathfrak{C}({}^{\dagger}N', N)_{n,1} \otimes_{\mathbf{R}_{n,1}} \mathbf{R}_{n,\ell}$.*

Proof. We must prove that the $\mathbf{B}_{n,\ell}$-linear map $\varkappa$ in (4.4) factors to an isomorphism of $\mathbf{B}_{n,\ell}$-modules

$$H_0\big(\mathfrak{g}\big[\mathbb{C}^\times\big]^F, T\big(M', M\big)_{\mathbf{R}_{n,\ell}}\big) \to H_0\big(\mathfrak{g}\big[\mathbb{C}^\times\big], T\big({}^\dagger N', N\big)_{\mathbf{R}_{n,1}}\big) \otimes_{\mathbf{R}_{n,1}} \mathbf{R}_{n,\ell}.$$

The vector space $\mathfrak{g}[\mathbb{C}^\times]^F$ is spanned by the elements

$$\xi \otimes z^a \quad \text{with } \xi \in \mathfrak{g}(b), \quad a \equiv -b \bmod \ell.$$

Here $\mathfrak{g}(b)$ is as in (4.1). The conjugation by $\check{\gamma}(z)$ takes $\xi \otimes z^a$ to

$$\check{\gamma}(z)\big(\xi \otimes z^a\big) = \xi \otimes z^{a+b}.$$

Composing it with the linear map $\xi \otimes z^{a+b} \mapsto \xi \otimes z^{(a+b)/\ell}$ yields a Lie algebra isomorphism

$$\mathfrak{g}\big[\mathbb{C}^\times\big]^F \to \mathfrak{g}\big[\mathbb{C}^\times\big]. \tag{4.5}$$

We'll regard $\mathbf{R}_{n,1}$ as the subalgebra of $\mathbf{R}_{n,\ell}$ generated by the elements $y_i = x_i^\ell$. Under the map (4.5) the element $\xi \otimes z^a \in \mathfrak{g}[\mathbb{C}^\times]^F$ above acts on $T({}^\dagger N', N)_{\mathbf{R}_{n,1}}$ as

$$\begin{aligned} &\varkappa'\big(\xi^{(a)}\big)_{(0)} + \sum_{i=1}^n y_i^{(a+b)/\ell} \xi_{(i)} + \varkappa\big(\sharp\xi^{(a)}\big)_{(n+1)} \\ &\quad = \varkappa'\big(\xi^{(a)}\big)_{(0)} + \sum_{i=1}^n x_i^a \big(\check{\gamma}(x_i)\xi\big)_{(i)} + \varkappa\big(\sharp\xi^{(a)}\big)_{(n+1)}. \end{aligned}$$

Here the maps $\varkappa' : \hat{\mathbf{g}}^{F'} \to \hat{\mathbf{g}}$ and $\varkappa : \hat{\mathbf{g}}^F \to \hat{\mathbf{g}}$ are as in (4.2), (4.3). Note that

$$T\big({}^\dagger N', N\big)_{\mathbf{R}_{n,\ell}} = T\big({}^\dagger N', N\big)_{\mathbf{R}_{n,1}} \otimes_{\mathbf{R}_{n,1}} \mathbf{R}_{n,\ell}.$$

Under the canonical isomorphisms

$${}^\dagger N' = {}^{\varkappa'} M' \simeq M', \qquad N = {}^{\varkappa} M \simeq M$$

the actions of the elements $\varkappa'(\xi^{(a)})$, $\varkappa(\sharp\xi^{(a)})$ on ${}^\dagger N'$, N are the same as the actions of the elements $\xi^{(a)}$, $\sharp\xi^{(a)}$ on M', M respectively. Therefore, we have

$$\xi \otimes z^a \cdot \varkappa(v) = \varkappa\big(\big(\xi \otimes z^a\big) \cdot v\big), \quad \forall v \in T\big(M', M\big)_{\mathbf{R}_{n,\ell}}. \qquad \square$$

5. Complements on the category $\hat{\mathcal{O}}_{\nu,\kappa}$

This section is a reminder on the structure of the category $\mathcal{O}$ for (untwisted) affine Lie algebras. This section does not contain new results. Proposition 5.8 is proved in Appendix A. It is standard, but we have not found any proof in the literature. Recall that we'll always assume that $\kappa \in \mathbb{C}^\times$.

5.1. Complements on the affine Weyl group. When it creates no confusion we'll abbreviate $\mathfrak{S} = \mathfrak{S}_m$. Let $\hat{\mathfrak{S}}$ be the affine Weyl group of $\mathfrak{S}$, i.e., the semi-direct product $\mathfrak{S} \ltimes \mathbb{Z}\Pi$. For any real affine root a let $s_\alpha \in \hat{\mathfrak{S}}$ be the reflection associated with α. There is a unique linear representation of $\hat{\mathfrak{S}}$ on $\mathbf{t}^*$ such that $w \in \mathfrak{S}$, $\tau \in \mathbb{Z}\Pi$ act in the following way:

$$w(\epsilon_i) = \epsilon_{v(i)}, \qquad w(\omega_0) = \omega_0, \qquad w(\delta) = \delta, \qquad \tau(\delta) = \delta,$$
$$\tau(\epsilon_i) = \epsilon_i - \langle \tau : \epsilon_i \rangle \delta, \qquad \tau(\omega_0) = \tau + \omega_0 - \langle \tau : \tau \rangle \delta/2.$$

We abbreviate

$$\hat{\rho} = \rho + m\omega_0.$$

The dot-action of $\hat{\mathfrak{S}}$ on $\mathbf{t}^*$ is given by

$$w \bullet \lambda = w(\lambda + \hat{\rho}) - \hat{\rho}.$$

Both actions factor to representations of $\hat{\mathfrak{S}}$ on the vector space $\mathbf{t}'$. Recall the notation

$$\hat{\lambda} = \lambda + c\omega_0, \qquad \tilde{\lambda} = \hat{\lambda} + z_\lambda \delta, \qquad z_\lambda = -\langle \lambda : 2\rho + \lambda \rangle / 2\kappa, \quad \forall \lambda \in \mathrm{t}^*. \tag{5.1}$$

We have $\hat{\lambda} = w \bullet \hat{\mu}$ iff $\tilde{\lambda} = w \bullet \tilde{\mu}$.

For each $\lambda \in \mathbf{t}^*$ we set

$$\hat{\Pi}(\lambda) = \{\alpha \in \hat{\Pi};\ 2\langle \lambda + \hat{\rho} : \alpha \rangle \in \mathbb{Z}\langle \alpha : \alpha \rangle\}.$$

Let $\mathbf{t}_0^* = \{\lambda \in \mathbf{t}^*;\ \langle \lambda + \hat{\rho} : \delta \rangle \neq 0\}$. Note that if $\hat{\lambda}$ is as in (5.1) then it lies in $\mathbf{t}_0^*$ iff $\kappa \neq 0$. For each $\lambda \in \mathbf{t}_0^*$ we have

$$\hat{\Pi}(\lambda) = \{\alpha \in \hat{\Pi}_{\mathrm{re}};\ \langle \lambda + \hat{\rho} : \alpha \rangle \in \mathbb{Z}\},$$

a root system with the set of positive roots $\hat{\Pi}(\lambda)^+ = \hat{\Pi}^+ \cap \hat{\Pi}(\lambda)$. Let $\hat{\mathfrak{S}}(\lambda)$ be its Weyl group. We call $\hat{\mathfrak{S}}(\lambda)$ the *integral Weyl group* associated with λ.

For each $\lambda \in \mathbf{t}_0^*$ we consider also the group

$$\hat{\mathfrak{S}}_\lambda = \{w \in \hat{\mathfrak{S}};\ w \bullet \lambda = \lambda\}.$$

It is a (finite) subgroup isomorphic to the Weyl group of the root system

$$\hat{\Pi}_\lambda = \{\alpha \in \hat{\Pi}_{\mathrm{re}};\ \langle \lambda + \hat{\rho} : \alpha \rangle = 0\}.$$

See [17, Sec. 2] for details.

Finally, let Π_ν, Π_ν^+, $\mathfrak{S}_\nu$ be the root system, the set of positive roots and the Weyl group of $\mathfrak{h}_\nu$. Note that $\Pi_\nu \subset \Pi$ and $\Pi_\nu^+ = \Pi^+ \cap \Pi_\nu$.

5.2. The partial order on $\mathbf{\Delta}_{\hat{\mathcal{O}}_{\nu,\kappa}}$. We'll say that an affine weight $\lambda \in \mathbf{t}^*$ is ν-*regular* (resp. ν-*integral*) if we have $\langle \lambda : \alpha \rangle \neq 0$ (resp. $\langle \lambda : \alpha \rangle \in \mathbb{Z}$) for each $\alpha \in \Pi_\nu$. Note that if $\lambda + \hat{\rho}$ is ν-regular and integral then there exists a unique element $w \in \mathfrak{S}_\nu$ such that

$$\lambda_+ := w \bullet \lambda$$

is ν-dominant, and we set

$$sn(\lambda) = (-1)^{l(w)}.$$

We equip the set $\mathbf{t}^*$ with the partial order given by

$$\lambda \geqslant \mu \quad \Longleftrightarrow \quad \lambda - \mu \in \mathbb{N}\hat{\Pi}^+.$$

Now, we define the following partial orders on the set of ν-dominant affine weights.

(a) Let $\preccurlyeq$ be the transitive closure of the binary relation such that $\lambda \Uparrow \mu$ iff the simple $\tilde{\mathbf{g}}$-module $L(\lambda)$ is a Jordan–Hölder factor of the parabolic Verma module $M(\mu)_\nu$.
(b) Let $\trianglelefteqslant$ be the transitive and reflexive closure of the binary relation such that $\lambda \uparrow \mu$ iff there is a root α in $\hat{\Pi}^+_{\mathrm{re}} \setminus \Pi^+_\nu$ such that $\langle \mu + \hat{\rho} : \alpha \rangle \in \mathbb{Z}_{>0}$, $s_\alpha \bullet \mu + \hat{\rho}$ is ν-regular, and we have

$$\lambda = (s_\alpha \bullet \mu)_+ < \mu.$$

5.3. Proposition.

(a) *The partial order $\trianglelefteqslant$ refines the partial order $\preccurlyeq$.*
(b) *If* $\mathrm{Hom}_{\hat{\mathbf{g}}}(M(\hat{\lambda}_1)_\nu, M(\hat{\lambda}_2)_\nu) \neq 0$ *then* $\hat{\lambda}_1 \trianglelefteqslant \hat{\lambda}_2$.

Proof. Part (b) follows from (a), because if $\phi \in \mathrm{Hom}_{\hat{\mathbf{g}}}(M(\hat{\lambda}_1)_\nu, M(\hat{\lambda}_2)_\nu)$ is non-zero then $\phi(M(\hat{\lambda}_1)_\nu)$ is a submodule of $M(\hat{\lambda}_2)_\nu$ whose top contains $L(\hat{\lambda}_1)$. Hence $L(\hat{\lambda}_1)$ is a Jordan–Hölder factor of $M(\hat{\lambda}_2)_\nu$. Now, we prove (a). We must prove that if $\mu \preccurlyeq \lambda$ then $\mu \trianglelefteqslant \lambda$. Given a ν-dominant affine weight $\lambda \in \mathbf{t}^*$, the Kac–Kazhdan formula for the Shapovalov determinant of the contravariant form on $M(\lambda)_\nu$ restricted to its weight μ-subspace is given, up to a non-zero scalar, by the expression

$$\det(\lambda)_{\nu,\mu} = \prod_{n>0} \prod_{\alpha \in \hat{\Pi}^+ \setminus \Pi^+_\nu} \big(2\langle \lambda + \hat{\rho} : \alpha \rangle - n\langle \alpha : \alpha \rangle\big)^{\chi(\lambda - n\alpha)_{\nu,\mu}},$$

$$\chi(\lambda)_{\nu,\mu} = \sum_{w \in \mathfrak{S}_\nu} (-1)^{l(w)} \dim M(w \bullet \lambda)_{\nu,\mu},$$

see [16, p. 107]. Let us consider the sets

$$\begin{aligned} \mathcal{S}_\lambda &= \big\{\alpha \in \hat{\Pi}^+ \setminus \Pi^+_\nu;\ \exists n \in \mathbb{Z}_{>0},\ 2\langle \lambda + \hat{\rho} : \alpha \rangle = n\langle \alpha : \alpha \rangle\big\} \\ &= \big\{\alpha \in \hat{\Pi}^+_{\mathrm{re}} \setminus \Pi^+_\nu;\ \langle \lambda + \hat{\rho} : \alpha \rangle \in \mathbb{Z}_{>0}\big\}, \\ \mathcal{S}^0_\lambda &= \{\alpha \in \mathcal{S}_\lambda;\ s_\alpha \bullet \lambda + \hat{\rho} \text{ is } \nu\text{-regular}\}. \end{aligned}$$

Now, suppose that $L(\mu)$ is a subquotient of $M(\lambda)_\nu$ with λ, μ both ν-dominant affine weights and $\mu < \lambda$. Then $L(\mu)$ must be a subquotient of the maximal submodule $M(\lambda)^1_\nu$ of $M(\lambda)_\nu$. The Jantzen filtration yields a decreasing sequence of submodules $M(\lambda)^k_\nu$, $k > 0$, of $M(\lambda)_\nu$ such that

$$\sum_{k>0} \operatorname{ch}\bigl(M(\lambda)^k_\nu\bigr) = \sum_{\alpha \in \mathcal{S}^0_\lambda} sn(s_\alpha \bullet \lambda) \operatorname{ch}\bigl(M\bigl((s_\alpha \bullet \lambda)_+\bigr)_\nu\bigr),$$

where the symbol ch denotes the formal character, see e.g., [13, Sec. 4.1]. Thus there is a real affine root $\alpha \in \mathcal{S}^0_\lambda$ such that $L(\mu)$ is a subquotient of $M((s_\alpha \bullet \lambda)_+)_\nu$. Since $(s_\alpha \bullet \lambda)_+ \uparrow \lambda$, an obvious induction implies the proposition. Compare the discussion in [14, Sec. 9.4–10.6] for instance. □

If $\tilde{\lambda}, \tilde{\mu}$ are as in (5.1) we may write $\mu \trianglelefteq \lambda$ for $\tilde{\mu} \trianglelefteq \tilde{\lambda}$. To define the order relation we embed $\hat{\mathcal{O}}_{\nu,\kappa}$ in $\tilde{\mathcal{O}}_{\nu,\kappa}$ and we equip the set $\Delta_{\hat{\mathcal{O}}_{\nu,\kappa}}$ with the order $\trianglelefteq$ of the highest weights of the parabolic Verma modules.

5.4. Remark. Note that if $\lambda \trianglelefteq \mu$ then $\mu - \lambda \in \mathbb{N}\Pi^+$ and $\lambda \in \hat{\mathfrak{S}} \bullet \mu$.

5.5. The truncated category. Recall that $\mathbf{u}_\nu$ denotes the pronilpotent radical of $\hat{\mathbf{q}}_\nu$. Let $\hat{\Pi}^\nu \subset \hat{\Pi}^+$ be the set of roots of $\mathbf{u}_\nu$. Set $\mathbf{z} = \mathbb{C}\partial \oplus \mathfrak{z} \oplus \mathbb{C}\mathbf{1}$ where $\mathfrak{z} \subset \mathfrak{h}_\nu$ is the central Lie subalgebra such that $\mathfrak{h}_\nu = [\mathfrak{h}_\nu, \mathfrak{h}_\nu] \oplus \mathfrak{z}$. Note that $\mathbf{z} \subset \mathbf{t}$. Let $z : \mathbf{t}^* \to \mathbf{z}^*$ be the restriction of linear forms. We equip the set $\mathbf{z}^*$ with the partial order such that

$$z_2 \leqslant z_1 \quad \Longleftrightarrow \quad z_1 - z_2 \in z\bigl(\mathbb{N}\hat{\Pi}^\nu\bigr).$$

Given a finite subset $B \subset \mathbf{z}^*$ we put

$${}^B\Lambda = \bigl\{\lambda \in \mathbf{t}^*;\ \exists \beta \in B,\ z(\lambda) \leqslant \beta\bigr\}.$$

Let ${}^B\hat{\mathcal{O}}_{\nu,\kappa} \subset \hat{\mathcal{O}}_{\nu,\kappa}$ be the Serre subcategory consisting of the modules whose λ-weight space vanishes if $\lambda \notin {}^B\Lambda$. Note that any object of $\hat{\mathcal{O}}^{fg}_{\nu,\kappa}$ lies in ${}^B\hat{\mathcal{O}}_{\nu,\kappa}$ for some B. The following is well known.

5.6. Proposition. *The category ${}^B\hat{\mathcal{O}}^{fg}_{\nu,\kappa}$ is a highest weight category. The poset of standard modules is the set of parabolic Verma modules which belong to ${}^B\hat{\mathcal{O}}^{fg}_{\nu,\kappa}$ with the order relation $\trianglelefteq$.*

Proof. By Proposition 2.9(c) the category $\hat{\mathcal{O}}^{fg}_{\nu,\kappa}$ is Abelian, any object has a finite length, and Hom sets are finite dimensional. Thus the category ${}^B\hat{\mathcal{O}}^{fg}_{\nu,\kappa}$ is Abelian and Artinian. The axioms (a) and (c) of quasi-hereditary categories, see Section 0.1, are obvious. The axiom (b) follows from Proposition 5.3(b). Thus it is enough to check that any parabolic Verma module M has a projective cover in ${}^B\hat{\mathcal{O}}^{fg}_{\nu,\kappa}$ whose kernel is standardly filtered with subquotients $> M$. This is well known, and proved in [21, Cor. 10, Cor. 13, Thm. 4]. See also [9,23] for a more recent exposition. □

5.7. Kazhdan–Lusztig polynomials. Set

$$\mathfrak{A} = \bigl\{\lambda \in \mathbf{t}^*_0;\ \langle \lambda + \hat{\rho} : \alpha\rangle \leqslant 0,\ \forall \alpha \in \hat{\Pi}(\lambda)^+\bigr\}.$$

Recall that for each $\lambda \in \mathbf{t}_0^*$ such that $\hat{\Pi}(\lambda) \neq \emptyset$ and $\langle \lambda + \hat{\rho} : \delta \rangle \notin \mathbb{Q}_{>0}$ the set $(\hat{\mathfrak{S}}(\lambda) \bullet \lambda) \cap \mathfrak{A}$ consists exactly of one element, see [17, Lem. 2.10]. Fix $\lambda \in \mathbf{t}_0^*$. Let $\hat{\mathfrak{S}}^\lambda \subset \hat{\mathfrak{S}}$ be the set of minimal length representatives in the left cosets relative to $\hat{\mathfrak{S}}_\lambda$. Since $\hat{\mathfrak{S}}_\lambda$ is also a parabolic subgroup of the integral Weyl group $\hat{\mathfrak{S}}(\lambda)$, the set $\hat{\mathfrak{S}}(\lambda)^\lambda$ is well defined. We'll use the symbol $P_{v,w}^{\lambda,-1}$ for Deodhar's parabolic Kazhdan–Lusztig polynomials of type $\hat{\mathfrak{S}}(\lambda)/\hat{\mathfrak{S}}_\lambda$, see [4].

5.8. Proposition. *Let $\lambda \in \mathfrak{A}$ be such that $\hat{\Pi}(\lambda) \neq \emptyset$. If $w \in \hat{\mathfrak{S}}(\lambda)^\lambda$ is such that $w \bullet \lambda$ is ν-dominant, then we have the following formula in $[\hat{\mathcal{O}}_{\nu,\kappa}^{fg}]$*

$$\big[L(w \bullet \lambda)\big] = \sum_{v \leqslant w} (-1)^{l(w)-l(v)} P_{v,w}^{\lambda,-1}(1)\big[M(v \bullet \lambda)_\nu\big].$$

The sum is over all $v \in \hat{\mathfrak{S}}(\lambda)^\lambda$ such that $v \bullet \lambda$ is ν-dominant. The symbol $\leqslant$ denotes the Bruhat order on $\hat{\mathfrak{S}}(\lambda)$.

5.9. Remark. Note that if $\lambda \in \mathfrak{A}$ is such that $\hat{\Pi}(\lambda) \neq \emptyset$ then we have $\kappa := \langle \lambda + \hat{\rho} : \delta \rangle \notin \mathbb{Q}_{\geqslant 0}$, see e.g., [17, Lem. 2.10].

6. Definition of the functor $\mathfrak{E}$

We can now construct our main functor $\mathfrak{E}$. It takes a module from the affine parabolic category $\mathcal{O}$ to a module in $\mathcal{H}_{h,H}$. We'll prove that the functor $\mathfrak{E}$ preserves the posets of standard modules. In this section we'll make the following assumption

$$\nu \in \mathcal{C}_{m,\ell}, \qquad \kappa \notin \mathbb{Q}_{\geqslant 0}, \qquad h = 1/\kappa,$$
$$h_p = \nu_p^\bullet/\kappa - m/\ell\kappa, \quad \forall p \in \Lambda.$$

6.1. Notation and definition of $\mathfrak{E}$. Recall that by Proposition 3.6(b) we have a functor

$$\mathfrak{E} : \mathcal{C}_\kappa^F \to \mathbf{H}_{h,H}\text{-}\mathbf{mod}.$$

Composing it with the functor $\varkappa$ in Proposition 4.2(a) we get the functor

$$\mathfrak{E} : \mathcal{C}_\kappa \to \mathbf{H}_{h,H}\text{-}\mathbf{mod}, \qquad M \mapsto H_0\big(\mathfrak{g}[\mathbb{C}]^F, T\big({}^\varkappa M\big)_{\mathbf{R}}\big). \tag{6.1}$$

Recall that

$$\pi = c(-1, -1, \ldots, -1, -2, \ldots, -\ell)/\ell,$$

where the integer $-p$ has multiplicity ν_p. Set

$$\lambda_\pi = \lambda + \pi, \qquad \hat{\lambda}_\pi = \lambda_\pi + c\omega_0, \qquad \tilde{\lambda}_\pi = \hat{\lambda}_\pi + z_{\lambda_\pi}\delta, \quad \forall \lambda \in \mathfrak{t}^*.$$

If $\lambda \in \mathfrak{t}^*$ is a ν-dominant weight we'll abbreviate

$$\Delta_{\lambda,\nu,\kappa} = M(\hat{\lambda}_\pi)_\nu, \qquad S_{\lambda,\nu,\kappa} = L(\hat{\lambda}_\pi).$$

Given a finite subset $B \subset \mathbf{z}^*$ let ${}^B P_{\lambda,\nu,\kappa}$ denote the projective cover of $S_{\lambda,\nu,\kappa}$ in ${}^B\hat{\mathcal{O}}_{\nu,\kappa}$.

6.2. The functor $\mathfrak{E}$ and the standard modules. First, let us compute the image by $\mathfrak{E}$ of the standard modules.

6.3. Proposition.

(a) *The functor $\mathfrak{E}$ is right exact and takes $\hat{\mathcal{O}}_{\nu,\kappa}$ to $\mathcal{H}_{h,H}$.*
(b) *We have $\mathfrak{E}(\Delta_{\lambda,\nu,\kappa}) = \Delta_{\lambda^\circ,h,H}$ if $\lambda \in \mathcal{P}^\ell_{n,\nu}$, and 0 else.*
(c) *The functor $\mathfrak{E}$ takes $\hat{\mathcal{O}}^{fg}_{\nu,\kappa}$ to $\mathcal{H}^{fg}_{h,H}$.*

Proof. First, we prove (a). It is enough to check that $T(M)_{\mathbf{R}}$ is a locally nilpotent $\mathbf{R}^*$-module for $M \in \hat{\mathcal{O}}_{\nu,\kappa}$. We'll identify M with the $\hat{\mathbf{g}}^F$-module ${}^\varkappa M$. By Proposition 3.6(b) the operator $\bar{y}_i - \gamma^F_{i,n+1}$ vanishes on the vector space $T(M)$. Fix a finite dimensional $\hat{\mathbf{g}}^F_{\geqslant 0}$-submodule $E \subset M$. Formula (3.8) implies that

$$\big(\bar{y}_i - \gamma^F_{i,n+1}\big)\big(T(E)_{\mathbf{R}_{\leqslant a+1}}\big) \subset T(E)_{\mathbf{R}_{\leqslant a}},$$

where $\mathbf{R}_{\leqslant a} \subset \mathbf{R}$ is the subspace of the polynomials of degree $\leqslant a$. We have also

$$\gamma^F_{i,n+1}\big(T(E)_{\mathbf{R}}\big) \subset T\big(\mathbf{g}^F_{>0}E\big)_{\mathbf{R}}. \tag{6.2}$$

Thus, since the action of $\mathbf{g}^F_{>0}$ on E is nilpotent and E is finite dimensional, there is an integer $d > 0$ such that $(\gamma^F_{i,n+1})^d(T(E)_{\mathbf{R}}) = 0$. Therefore, if b is large enough then $\bar{y}_i^b(T(E)_{\mathbf{R}_{\leqslant a}}) = 0$. The right exactness of $\mathfrak{E}$ is obvious, because taking coinvariants is a right exact functor.

Now, we prove (b). Given a $\hat{\mathbf{g}}^F_{\geqslant 0}$-module M we consider the induced $\hat{\mathbf{g}}^F$-module M^F_κ. We have an isomorphism of $\mathbf{B}$-modules $\mathfrak{E}(M^F_\kappa) = \Gamma^{\mathbf{B}}_W(\mathfrak{X}(M))$ by Proposition 3.8(b). The y_i-action on an element $v \in \mathfrak{X}(M)$ is given by $y_i v = \gamma^F_{i,n+1} v$ by Proposition 3.6(b). If $\mathbf{g}^F_{>0}$ annihilates M then (6.2) yields $y_i v = 0$. Thus the subalgebra $\mathbf{B}^* \subset \mathbf{H}_{h,H}$ acts trivially on the subspace $\mathfrak{X}(M) \subset \mathfrak{E}(M^F_\kappa)$. This yields a $\mathbf{B}^*$-module isomorphism

$$\mathfrak{E}\big(M^F_\kappa\big) \simeq \Gamma^{\mathbf{H}_{h,H}}_{\mathbf{B}^*}\big(\mathfrak{X}(M)\big).$$

Now, Proposition 4.2(a) yields

$$\mathfrak{E}(\Delta_{\lambda,\nu,\kappa}) = \mathfrak{E}\big(M(\hat{\lambda}_\pi)_\nu\big) = \mathfrak{E}\big(L(\mathfrak{h}_\nu,\lambda)^F_\kappa\big).$$

We are done, because $\mathfrak{X}(L(\mathfrak{h}_\nu,\lambda)) = \mathfrak{X}_{\lambda^\circ}$ by Proposition 3.8(a).

Finally, we prove (c). Given $M \in \hat{\mathcal{O}}^{fg}_{\nu,\kappa}$ we must check that $\mathfrak{E}(M)$ has a finite length. This follows from an easy induction on the length of M. First, if M is simple then it is a quotient of a module $\Delta_{\lambda,\nu,\kappa}$. Thus, since $\mathfrak{E}$ is right exact, $\mathfrak{E}(M)$ is a quotient of $\Delta_{\lambda^\circ,h,H}$. Thus $\mathfrak{E}(M)$ lies in $\mathcal{H}^{fg}_{h,H}$. Next, there is an exact sequence

$$\Delta_{\lambda,\nu,\kappa} \to M \to N \to 0$$

such that the length of N is strictly less than the length of M. Since $\mathfrak{E}$ is right exact this yields the exact sequence

$$\Delta_{\lambda^\circ,h,H} \to \mathfrak{E}(M) \to \mathfrak{E}(N) \to 0.$$

Thus $\mathfrak{E}(M)$ has a finite length by the induction hypothesis. □

Now, we can compare the partial orders on the set of standard modules in $\hat{\mathcal{O}}_{\nu,\kappa}$ and in $\mathcal{H}_{h,H}$.

6.4. Proposition. *If* $\lambda, \mu \in \mathcal{P}^{\ell}_{n,\nu}$ *and* $\Delta_{\mu,\nu,\kappa} \trianglelefteq \Delta_{\lambda,\nu,\kappa}$ *then* $\Delta_{\mu^\circ,h,H} \preccurlyeq \Delta_{\lambda^\circ,h,H}$.

Proof. Recall the decomposition $J = J_1 \sqcup J_2 \sqcup \cdots \sqcup J_\ell$ in Section 1.4. Let γ, τ be the weights given by

$$\gamma_j = h_1 + h_2 + \cdots + h_{\ell-p}, \qquad \tau_j = m + \nu_\ell, \quad \forall j \in J_p.$$

For each weights λ, μ we set

$$a(\lambda,\mu) = \langle \lambda : \lambda + 2\tau + 2\rho\rangle/2\kappa - \langle \mu : \mu + 2\tau + 2\rho\rangle/2\kappa.$$

A computation yields

$$\begin{aligned}\langle \tilde\lambda_\pi - \tilde\mu_\pi : \pi + c\omega_0\rangle &= cz_{\lambda_\pi} - cz_{\mu_\pi} + \langle \lambda - \mu : \pi\rangle \\ &= m\langle \lambda - \mu : \pi\rangle/\kappa - c\langle \lambda : \lambda + 2\rho\rangle/2\kappa + c\langle \mu : \mu + 2\rho\rangle/2\kappa \\ &= \langle \lambda - \mu : c\tau + m\pi\rangle/\kappa - ca(\lambda,\mu).\end{aligned}$$

On the other hand, we have

$$\kappa a(\lambda,\mu) = \sum_p \bigl(n\bigl({}^t\lambda_p\bigr) - n\bigl({}^t\mu_p\bigr) - n(\lambda_p) + n(\mu_p)\bigr).$$

So we get

$$\begin{aligned}\theta_{\mu^\circ} - \theta_{\lambda^\circ} &= -\ell \sum_p (h_1 + \cdots + h_{p-1})\bigl(\bigl|\lambda^\circ_p\bigr| - \bigl|\mu^\circ_p\bigr|\bigr) - \ell a(\lambda,\mu) \\ &= -\ell\langle \lambda - \mu : \gamma\rangle - \ell a(\lambda,\mu).\end{aligned}$$

We claim that we have

$$c(\theta_{\mu^\circ} - \theta_{\lambda^\circ})/\ell = \langle \tilde\lambda_\pi - \tilde\mu_\pi : \pi + c\omega_0\rangle. \tag{6.3}$$

It is enough to check that $c\kappa\gamma + c\tau + m\pi = 0$. Since this tuple has the same entries on each segment J_p, it is enough to prove that its i_p-th entry is zero. The latter is

$$\begin{aligned}&\sum_{r=1}^{\ell-p} c\kappa h_r + c(\nu_\ell + \nu_1 + \nu_2 + \cdots + \nu_{p-1}) - cmp/\ell \\ &\quad = \sum_{r=1}^{\ell-p} c\bigl(\kappa h_r - \nu^\bullet_r\bigr) + cm(\ell - p)/\ell = \sum_{r=1}^{\ell-p} c\bigl(\kappa h_r - \nu^\bullet_r + m/\ell\bigr) = 0.\end{aligned}$$

Now we prove the proposition using formula (6.3). Assume that $\mu_\pi \triangleleft \lambda_\pi$. By definition of the partial order $\trianglelefteq$, a simple induction allows us to assume that

$$\tilde\mu_\pi = (s_\alpha \bullet \tilde\lambda_\pi)_+ < \tilde\lambda_\pi, \quad \alpha \in \hat\Pi^+_{\mathrm{re}} \setminus \Pi^+_\nu.$$

For each simple affine root α_i we have $\langle\alpha_i : \pi + c\omega_0\rangle = c/\ell$ or 0. Therefore

$$\alpha \in \hat{\Pi}^+_{\mathrm{re}} \setminus \Pi^+_\nu \quad\Rightarrow\quad \langle\alpha : \pi + c\omega_0\rangle \in \mathbb{Z}_{>0}c/\ell.$$

This implies that

$$\langle\tilde{\lambda}_\pi - \tilde{\mu}_\pi : \pi + c\omega_0\rangle = \langle\tilde{\lambda}_\pi - s_\alpha \bullet \tilde{\lambda}_\pi : \pi + c\omega_0\rangle \in \mathbb{Z}_{>0}c/\ell.$$

Using (6.3) this yields

$$\theta_{\mu^\circ} - \theta_{\lambda^\circ} \in \mathbb{Z}_{>0}.$$

Thus, the definition of the order in (1.4) implies that

$$\Delta_{\mu^\circ,h,H} \prec \Delta_{\lambda^\circ,h,H}. \qquad \square$$

6.5. Remarks. (a) We have $\mathfrak{E}(M) \in \mathcal{H}_{h,H}$ for any $M \in \hat{\mathcal{O}}_\kappa$, but $\mathfrak{E}(M) = 0$ if M is a simple module which does not belong to $\hat{\mathcal{O}}_{\nu,\kappa}$.

(b) The Lie algebra $\hat{\mathbf{g}}^F$ is $\mathbb{Z}$-graded by letting $\xi^{(a)}$ be of degree $-a$. We can consider the category of $\mathbb{Z}$-graded modules which belong to $\hat{\mathcal{O}}^F_{\geqslant 0,\kappa}$. The algebra $\mathbf{H}_{h,H}$ is $\mathbb{Z}$-graded as in Section 1.6 and we can also consider the category of $\mathbb{Z}$-graded modules which belong to $\mathcal{H}_{h,H}$. The functor $\mathfrak{E}$ lifts to a functor between these categories of graded modules.

6.6. Examples. Write 1_p for the ℓ-partition whose p-th partition is (1) and all other are zero.

(a) Set $\kappa = -2$, $n = 1$, $m = 10$, $\ell = 4$, $\nu = (2, 1, 6, 1)$. We have $\Delta_{1^\circ_3,h,H} \succ \Delta_{1^\circ_2,h,H} \succ \Delta_{1^\circ_1,h,H} \succ \Delta_{1^\circ_4,h,H}$ and $\Delta_{1_3,\nu,\kappa} \rhd \Delta_{1_2,\nu,\kappa} \not\rhd \Delta_{1_1,\nu,\kappa} \rhd \Delta_{1_4,\nu,\kappa}$. Thus the implication in Proposition 6.4 is not an equivalence.

(b) A direct computation shows that the module $\Delta_{0,\nu,\kappa}$ may not be simple (f.i., set $\kappa = -1$, $m = 7$, $\ell = 4$, and $\nu = (1, 1, 4, 1)$).

7. The functor $\mathfrak{E}$ is exact on standardly filtered modules

If $\ell = 1$ the functor $\mathfrak{E}$ is an equivalence of quasi-hereditary categories. Since there is no proof in the literature we have given one in Section A.5. For an arbitrary positive integer ℓ this is not true anymore. However we expect $\mathfrak{E}$ to be an important tool to prove the dimension conjecture. We'll prove that $\mathfrak{E}$ is exact on standardly filtered modules. We conjecture that it preserves the set of indecomposable projective modules. In this section we'll assume once again that

$$\nu \in \mathcal{C}_{m,\ell}, \qquad \kappa \notin \mathbb{Q}_{\geqslant 0}, \qquad h = 1/\kappa,$$
$$h_p = \nu^\bullet_p/\kappa - m/\ell\kappa, \quad \forall p \in \Lambda.$$

7.1. Reminder on the Kazhdan–Lusztig tensor product. A *monoidal category* is a tuple $(\mathcal{A}, \otimes, a, \mathbf{1})$ consisting of a category $\mathcal{A}$, a functor $\otimes : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$, a natural isomorphism

$$a_{L,M,N} : (L \otimes M) \otimes N \to L \otimes (M \otimes N), \quad L, M, N \in \mathcal{A},$$

and a unit object $\mathbf{1}$ satisfying the triangle and pentagon axioms, see e.g., [20, Sec. 2.2]. The isomorphism a is called the *associativity isomorphism*. A category $\mathcal{M}$ is a *left module category* over the monoidal category $(\mathcal{A}, \otimes, a, \mathbf{1})$ iff there exists a functor $\otimes : \mathcal{A} \times \mathcal{M} \to \mathcal{M}$ together with natural associativity and unit isomorphisms

$$(M \otimes N) \otimes X \to M \otimes (N \otimes X), \qquad \mathbf{1} \otimes X \to X, \quad X \in \mathcal{M},\ M, N \in \mathcal{A},$$

which satisfy appropriate pentagon and triangle axioms. In other words, a left module category is the same as the datum of a monoidal functor $\mathcal{A} \to \mathrm{Fun}(\mathcal{M}, \mathcal{M})$ to the monoidal category of endofunctors of $\mathcal{M}$, see [20, Prop. 2.2]. Similarly one defines the structure of a *right module category*. Finally, a category $\mathcal{M}$ is a *bimodule category* over $(\mathcal{A}, \otimes, a, \mathbf{1})$ iff $\mathcal{M}$ has both left and right $(\mathcal{A}, \otimes, a, \mathbf{1})$-module category structures and a natural family of isomorphisms

$$(X \otimes M) \otimes Y \to X \otimes (M \otimes Y), \quad X, Y \in \mathcal{A},\ M \in \mathcal{M},$$

satisfying the obvious pentagon axioms, see e.g., [12, Prop. 2.10].

Now, recall that the Kazhdan–Lusztig tensor product

$$\dot{\otimes} : \hat{\mathcal{O}}^{fg}_{\geqslant 0,\kappa} \times \hat{\mathcal{O}}^{fg}_{\geqslant 0,\kappa} \to \hat{\mathcal{O}}^{fg}_{\geqslant 0,\kappa}$$

equips $\hat{\mathcal{O}}^{fg}_{\geqslant 0,\kappa}$ with the structure of a monoidal category

$$\big(\hat{\mathcal{O}}^{fg}_{\geqslant 0,\kappa}, \dot{\otimes}, a, M(c\omega_0)\big),$$

see [18, Secs. 14.6, 18.2, 31]. The space of affine coinvariants is defined as in Remark A.2.2. It depends on the choice of a point x in the set $\mathcal{C}$ defined in Remark A.2.2. When no confusion is possible we'll abbreviate

$$\langle M_i;\ i \in S\rangle = \langle M_i;\ i \in S\rangle_x.$$

The following is folklore. See Section A.2 for details. Note that we'll note use Corollary 7.3 in this paper. It is given here for the sake of completeness.

7.2. Proposition.

(a) *There are right biexact bifunctors* $\dot{\otimes} : \hat{\mathcal{O}}^{fg}_{\geqslant 0,\kappa} \times \hat{\mathcal{O}}^{fg}_{\nu,\kappa} \to \hat{\mathcal{O}}^{fg}_{\nu,\kappa}$ *and* $\dot{\otimes} : \hat{\mathcal{O}}^{fg}_{\nu,\kappa} \times \hat{\mathcal{O}}^{fg}_{\geqslant 0,\kappa} \to \hat{\mathcal{O}}^{fg}_{\nu,\kappa}$ *yielding the structure of a bimodule category over* $(\hat{\mathcal{O}}^{fg}_{\geqslant 0,\kappa}, \dot{\otimes}, a, M(c\omega_0))$ *on* $\hat{\mathcal{O}}^{fg}_{\nu,\kappa}$.

(b) *For* $M_1, \dots, M_n \in \hat{\mathcal{O}}^{fg}_{\geqslant 0,\kappa}$ *and* $M_0, M_{n+1} \in \hat{\mathcal{O}}^{fg}_{\nu,\kappa}$ *there are natural isomorphisms of finite dimensional* $\mathbb{C}$*-vector spaces*

$$\begin{aligned}\big\langle M_0 \dot{\otimes} \cdots \dot{\otimes} M_n,\ {}^{\dagger}M_{n+1}\big\rangle &\simeq \big\langle M_0, M_1, \dots,\ {}^{\dagger}M_{n+1}\big\rangle \\ &\simeq \mathrm{Hom}_{\hat{\mathbf{g}}}\big(M_0 \dot{\otimes} \cdots \dot{\otimes} M_n,\ {}^{\dagger}DM_{n+1}\big)^*.\end{aligned}$$

(c) *The bifunctor* $\dot{\otimes}$ *takes* $\hat{\mathcal{O}}^{\Delta}_{\geqslant 0,\kappa} \times \hat{\mathcal{O}}^{\Delta}_{\nu,\kappa}$ *and* $\hat{\mathcal{O}}^{\Delta}_{\nu,\kappa} \times \hat{\mathcal{O}}^{\Delta}_{\geqslant 0,\kappa}$ *into* $\hat{\mathcal{O}}^{\Delta}_{\nu,\kappa}$.

7.3. Corollary. *For* $M \in \hat{\mathcal{O}}^{fg}_{\geqslant 0,\kappa}$ *the functor* $R_M : \hat{\mathcal{O}}^{fg}_{\nu,\kappa} \to \hat{\mathcal{O}}^{fg}_{\nu,\kappa}$, $N \mapsto N \dot{\otimes} M$ *is exact. If the module* M *is standardly filtered then* R_M *preserves the subcategory* $\hat{\mathcal{O}}^{\Delta}_{\nu,\kappa}$. *The functors* R_M, R_{DM} *are adjoint* (*left and right*) *to each other.*

In what follows we will omit from notation the associativity and unit isomorphisms, as is justified by the Mac Lane coherence theorem.

7.4. The KZ-functor. Fix $q \in \mathbb{C}^\times$ and a tuple $Q = (q_1, \dots, q_\ell)$ in $(\mathbb{C}^\times)^\ell$. The Ariki–Koike algebra $\mathbf{A}_{q,Q}$ is the $\mathbb{C}$-algebra with 1 generated by $T_0, T_1, \dots, T_{n-1}$ modulo the defining relations

$$(T_0 - q_1) \cdots (T_0 - q_\ell) = (T_i + 1)(T_i - q) = 0,$$
$$T_0 T_1 T_0 T_1 = T_1 T_0 T_1 T_0, \qquad T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1}, \qquad T_i T_j = T_j T_i.$$

Assume that

$$q = \exp(2i\pi h), \qquad q_p = \exp\bigl(2i\pi\bigl(h_1 + h_2 + \cdots + h_{p-1} + (p-1)/\ell\bigr)\bigr), \quad \forall p \in \Lambda. \tag{7.1}$$

Let $\heartsuit : \mathcal{H}_{h,H} \to \mathcal{H}_{h,H,\heartsuit}$ be the quotient by the Serre subcategory generated by the modules M such that $M_{n,\ell} = 0$. By [10, Secs. 5.1, 5.3] the Riemann–Hilbert correspondence yields an equivalence of categories

$$\mathrm{KZ} : \mathcal{H}_{h,H,\heartsuit} \to \mathbf{A}_{q,Q}\text{-}\mathbf{mod}.$$

Composing $\mathfrak{E}$, $\heartsuit$ and KZ yields the functor

$$\mathfrak{E}_{\mathrm{KZ}} : \hat{\mathcal{O}}_{\nu,\kappa} \to \mathbf{A}_{q,Q}\text{-}\mathbf{mod}.$$

7.5. Comparison of $\mathfrak{E}_{\mathbf{KZ}}$ with the Kazhdan–Lusztig tensor product. Recall that $\mathbf{V}$ is the dual of the vectorial representation of $\mathfrak{g}$ and that $\mathbf{V}^*_\kappa = D(\mathbf{V}_\kappa)$. Consider the following $\hat{\mathbf{g}}_\kappa$-module

$$\mathbf{V}_{n,\nu,\kappa} = \Delta_{0,\nu,\kappa} \dot{\otimes} (\mathbf{V}_\kappa)^{\dot{\otimes} n}.$$

It lies in the category $\hat{\mathcal{O}}^{fg}_{\nu,\kappa}$ by Proposition 7.2(a). So the $\mathbb{C}$-algebra

$$\mathbf{A}_{n,\nu,\kappa} = \mathrm{End}_{\hat{\mathbf{g}}}(\mathbf{V}_{n,\nu,\kappa})$$

is finite dimensional. The duality of $\mathbb{C}$-vector spaces $M \mapsto M^*$ yields a functor

$$\delta : \mathbf{A}^{\mathrm{op}}_{n,\nu,\kappa}\text{-}\mathbf{mod} \to \mathbf{A}_{n,\nu,\kappa}\text{-}\mathbf{mod}.$$

Composing the functor

$$\mathfrak{F}^{\mathrm{op}} : \hat{\mathcal{O}}_{\nu,\kappa} \to \mathbf{A}^{\mathrm{op}}_{n,\nu,\kappa}\text{-}\mathbf{mod}, \qquad M \mapsto \mathrm{Hom}_{\hat{\mathbf{g}}}(\mathbf{V}_{n,\nu,\kappa}, M)$$

with δ yields the functor

$$\mathfrak{F} : \hat{\mathcal{O}}_{\nu,\kappa} \to \mathbf{A}_{n,\nu,\kappa}\text{-}\mathbf{mod}.$$

7.6. Proposition. *There is an algebra homomorphism* $\phi : \mathbf{A}_{q,Q} \to \mathbf{A}_{n,\nu,\kappa}$ *such that* $\mathfrak{E}_{\mathrm{KZ}} = \phi \circ \mathfrak{F} \circ {}^{\dagger}D$.

Proof. Let $\mathfrak{E}_{\mathbb{C}}$ be the composition of $\mathfrak{E}_{\mathrm{KZ}}$ and the forgetful functor

$$\mathbf{A}_{q,Q}\text{-}\mathbf{mod} \to \mathbb{C}\text{-}\mathbf{mod}.$$

Let $\mathfrak{F}_{\mathbb{C}}$ be the composition of $\mathfrak{F}$ and the forget functor

$$\mathbf{A}_{n,\nu,\kappa}\text{-}\mathbf{mod} \to \mathbb{C}\text{-}\mathbf{mod}.$$

The proof consists of two steps: first we construct an isomorphism of functors

$$\psi : \mathfrak{E}_{\mathbb{C}} \circ {}^{\dagger}D \to \mathfrak{F}_{\mathbb{C}},$$

then we define an algebra homomorphism

$$\phi : \mathbf{A}_{q,Q} \to \mathbf{A}_{n,\nu,\kappa}$$

such that ψ lifts to an isomorphism of functors

$$\mathfrak{E}_{\mathrm{KZ}} \circ {}^{\dagger}D \to \phi \circ \mathfrak{F}.$$

Now fix a module N in $\hat{\mathcal{O}}^{fg}_{\nu,\kappa}$. Proposition 7.2(b) yields

$$\mathfrak{F}_{\mathbb{C}}(N) = \langle M(\pi + c\omega_0)_\nu, \mathbf{V}_\kappa, \dots, \mathbf{V}_\kappa, DN \rangle.$$

Consider the $\hat{\mathbf{g}}^F$-module $M = {}^{\varkappa}N$. By Proposition 4.2 we have

$$M(c\omega_0)^{F'} = {}^{\varkappa}\big({}^{\dagger}M(\pi + c\omega_0)_\nu\big).$$

Recall that we have

$$\mathbf{V}_\kappa^* = {}^{\dagger}\mathbf{V}_\kappa.$$

Fix tuples $x \in C_{n,\ell}$ and $y \in C_{n,1}$ as in Section 4.3. Assume that x, y belong to the set $\mathcal{C}$ defined in Remark A.2.2. Applying successively Proposition 2.14(d), Propositions 4.2(b) and 4.6, and Proposition 3.11(b), we get natural isomorphisms

$$\begin{aligned}
\mathfrak{F}_{\mathbb{C}}(N) &\simeq \langle {}^{\dagger}M(\pi + c\omega_0)_\nu, \mathbf{V}_\kappa^*, \dots, \mathbf{V}_\kappa^*, {}^{\dagger}DN \rangle_y \\
&\simeq \langle M(c\omega_0)^{F'}, \mathbf{V}_\kappa^*, \dots, \mathbf{V}_\kappa^*, {}^{\dagger}DM \rangle_x \\
&\simeq \langle \mathbf{V}_\kappa^*, \dots, \mathbf{V}_\kappa^*, {}^{\dagger}DM \rangle_x.
\end{aligned}$$

By Proposition 3.11(a) the vector space $\langle \mathbf{V}_\kappa^*, \dots, \mathbf{V}_\kappa^*, {}^\dagger DM\rangle_x$ is the fiber at the point x of the W-equivariant locally free sheaf over $C_{n,\ell}$ associated with the $\mathbf{R}_{n,\ell}$-module $\langle \mathbf{V}_\kappa^*, \dots, \mathbf{V}_\kappa^*, {}^\dagger DM\rangle$. Remark 3.12 yields

$$\big\langle \mathbf{V}_\kappa^*, \dots, \mathbf{V}_\kappa^*, {}^\dagger DM\big\rangle = \mathfrak{E}\big({}^\dagger DM\big)_{n,\ell}.$$

Further the $\mathbf{R}_{n,\ell}$-module $\mathfrak{E}({}^\dagger DM)_{n,\ell}$ is equipped with a flat W-equivariant connection ∇ which comes, via Proposition 1.8, from the representation of $\mathbf{H}_{h,H}$ on $\mathfrak{E}({}^\dagger DM)$ in Proposition 3.6(b).

Let $p : \tilde{C}_{n,\ell} \to C_{n,\ell}$ be the universal cover of the complex manifold associated with the $\mathbb{C}$-scheme $C_{n,\ell}$. By definition of the functor $\mathfrak{E}$ we have

$$\mathfrak{E}\big({}^\dagger DM\big) = \mathfrak{E}\big({}^\dagger DN\big).$$

Since the vector space $\mathfrak{E}_{\mathbb{C}}({}^\dagger DN)$ is obtained from the $\mathbf{A}_{q,Q}$-module $\mathfrak{E}_{\mathrm{KZ}}({}^\dagger DN)$ by forgetting the $\mathbf{A}_{q,Q}$-action, it is canonically identified with the vector space of holomorphic horizontal sections of the connection ∇ over $\tilde{C}_{n,\ell}$. Recall that $\mathcal{C}$ is a contractible subset of $C_{n,\ell}$ containing the point x. Fix once for all a contractible subset $\tilde{\mathcal{C}} \subset \tilde{C}_{n,\ell}$ such that p restricts to an isomorphism $\tilde{\mathcal{C}} \to \mathcal{C}$. Restricting functions on $\tilde{C}_{n,\ell}$ to $\tilde{\mathcal{C}}$ and taking the fiber at x, viewed as a point of $\tilde{\mathcal{C}}$ via the map p, yields a natural isomorphism of vector spaces

$$\psi(N) : \mathfrak{E}_{\mathbb{C}}\big({}^\dagger DN\big) \simeq \big\langle \mathbf{V}_\kappa^*, \dots, \mathbf{V}_\kappa^*, {}^\dagger DM\big\rangle_x \simeq \mathfrak{F}_{\mathbb{C}}(N).$$

Further, the $\mathbf{A}_{q,Q}$-action on the functor $\mathfrak{E}_{\mathbb{C}}$ given by $\mathfrak{E}_{\mathrm{KZ}}$ gives, via the isomorphism ψ, an $\mathbf{A}_{q,Q}$-action on the functor $\mathfrak{F}_{\mathbb{C}}$. This action comes from an algebra homomorphism ϕ as above, by the following consequence of Yoneda's lemma.

7.7. Lemma. *Let M be an object of an additive category $\mathcal{A}$. If a ring $\mathbf{A}$ acts on the functor* $\mathrm{Hom}_{\mathcal{A}}(M, -)$*, then there is a ring homomorphism $\phi : \mathbf{A} \to \mathrm{End}_{\mathcal{A}}(M)^{\mathrm{op}}$ such that the action of any element $a \in \mathbf{A}$ on F is the composition by $\phi(a)$.* □

7.8. Corollary. *The functor $\mathfrak{E}$ restricts to an exact functor $\hat{\mathcal{O}}^\Delta_{\nu,\kappa} \to \mathcal{H}^\Delta_{h,H}$.*

Proof. First, let us note the following basic fact whose proof is left to the reader.

7.9. Lemma. *Let $\mathfrak{E} : \mathcal{A} \to \mathcal{B}$ be a right exact functor of quasi-hereditary categories such that $\mathfrak{E}(\Delta_{\mathcal{A}}) \subset \Delta_{\mathcal{B}} \cup \{0\}$. Let $\heartsuit : \mathcal{B} \to \mathcal{B}_\heartsuit$ be an exact functor such that* $\mathrm{Hom}_{\mathcal{B}}(M, N) = 0$ *for each* $M \in \mathrm{Ker}(\heartsuit)$*, $N \in \mathcal{B}^\Delta$. If $\heartsuit \circ \mathfrak{E}$ is exact on $\mathcal{A}^\Delta$ then $\mathfrak{E}$ restricts to an exact functor $\mathcal{A}^\Delta \to \mathcal{B}^\Delta$.*

There are no non-zero homomorphisms $M \to N$ for each $M \in \mathrm{Ker}(\heartsuit)$ and $N \in \mathcal{H}^\Delta_{h,H}$, because a standard $\mathbf{H}_{h,H}$-module is torsion free as an $\mathbf{R}$-module. To prove that $\heartsuit \circ \mathfrak{E}$ is exact on $\hat{\mathcal{O}}^\Delta_{\nu,\kappa}$, it is enough to check it for $\mathfrak{E}_{\mathrm{KZ}}$. Thus the claim follows from Proposition 7.6, because the module $\mathbf{V}_{n,\nu,\kappa}$ is standardly filtered by Proposition 7.2(c). □

7.10. Conjecture. *If ${}^B\Lambda$ is large enough then we have $\mathfrak{E}({}^B P_{\lambda,\nu,\kappa}) = P_{\lambda^\circ,h,H}$ for each $\lambda \in \mathcal{P}^\ell_{n,\nu}$.*

7.11. Remarks. (a) For $\ell > 1$ the functor $\mathfrak{E}$ may be not exact. It may also take a simple object to a non-simple non-zero $\mathbf{H}_{h,H}$-module. Indeed, set $\kappa = -1$, $m = 7$, $\ell = 4$, $\nu = (1, 1, 4, 1)$, $n = 1$, as in Example 6.5(b). We have $\mathfrak{E}(\Delta_{1_p,\nu,\kappa}) = \Delta_{1_p^\circ,h,H}$ for each p. We have also $\mathfrak{E}(\Delta_{\lambda_1,\nu,\kappa}) = 0$ and there is an exact sequence $\mathfrak{E}(\Delta_{\lambda_1,\nu,\kappa}) \to \mathfrak{E}(\Delta_{1_1,\nu,\kappa}) \to \mathfrak{E}(S_{1_1,\nu,\kappa}) \to 0$. Thus $\mathfrak{E}(S_{1_1,\nu,\kappa}) = \Delta_{1_1^\circ,h,H}$, which is not simple. Further we have $S_{1_4^\circ,h,H} = \Delta_{1_4^\circ,h,H}$ and there is an exact sequence $0 \to S_{1_4^\circ,h,H} \to \Delta_{1_1^\circ,h,H} \to S_{1_1^\circ,h,H} \to 0$. Thus the derived functor $L^{-1}\mathfrak{E}$ is non-zero and it takes $S_{\lambda_1,\nu,\kappa}$ to $S_{1_4^\circ,h,H}$.

(b) The morphism ϕ in Proposition 7.6 is injective. We do not know if it is invertible. To prove this we must check that the representation of $\mathbf{A}_{q,Q}$ on $\mathfrak{E}_{\mathrm{KZ}}(M)$ is faithful for some module M. Since KZ is exact and $\mathcal{H}_{h,H}$ has enough projective objects, there is a module $P_{\mathrm{KZ}} \in \mathcal{H}_{h,H}^{\mathrm{proj}}$ which represents KZ. We have $\mathbf{A}_{q,Q} = \mathrm{KZ}(P_{\mathrm{KZ}})$ by [10, Thm. 5.15]. We claim that if ${}^B\Lambda$ is large enough there is a projective module $P \in {}^B\hat{\mathcal{O}}_{\nu,\kappa}$ such that P_{KZ} is a direct summand of $\mathfrak{E}(P)$. Therefore the representation of $\mathbf{A}_{q,Q}$ on $\mathfrak{E}_{\mathrm{KZ}}(P)$ is faithful. The proof is omitted because we'll not use this result.

(c) Recall that we have $[\hat{\mathcal{O}}_{\nu,\kappa}^{fg}] = [\hat{\mathcal{O}}_{\nu,\kappa}^{\Delta}]$ and $[\mathcal{H}_{h,H}^{\Delta}] = [\mathcal{H}_{h,H}^{fg}]$. The derived functor $L^*\mathfrak{E}$ yields a group homomorphism $[\hat{\mathcal{O}}_{\nu,\kappa}^{fg}] \to [\mathcal{H}_{h,H}^{fg}]$ such that $[\Delta_{\lambda,\nu,\kappa}] \mapsto [\Delta_{\lambda^\circ,h,H}]$ if $\lambda \in \mathcal{P}_{n,\nu}^{\ell}$ and $[\Delta_{\lambda,\nu,\kappa}] \mapsto 0$ else. Conjecture 7.10 implies that $[{}^BP_{\mu,\nu,\kappa}]$ maps to $[P_{\mu^\circ,h,H}]$ for all $\mu \in \mathcal{P}_{n,\nu}^{\ell}$. So Brauer reciprocity implies that

$$\big[\nabla_{\lambda^\circ,h,H} : S_{\mu^\circ,h,H}\big] = [\nabla_{\lambda,\nu,\kappa} : S_{\mu,\nu,\kappa}], \tag{7.2}$$

where $\nabla_{\lambda^\circ,h,H}$, $\nabla_{\lambda,\nu}$ are the costandard modules with socles $S_{\lambda^\circ,h,H}$, $S_{\lambda,\nu}$ in $\mathcal{H}_{h,H}$, $\hat{\mathcal{O}}_{\nu,\kappa}$ respectively. This dimension formula is not the same as in the dimension conjecture. See Section 8 for a comparison of the two formulas. We do not know if they are equivalent.

(d) The module $\mathfrak{E}({}^BP_{\lambda,\nu,\kappa})$ does not depend on the set B if $\lambda \in \mathcal{P}_{n,\nu}^{\ell}$ and ${}^B\Lambda$ is large enough so that $\tilde{\mu}_\pi \in {}^B\Lambda$ for all $\mu \in \mathcal{P}_{n,\nu}^{\ell}$. Indeed, if $B \subset B'$ and $M \in {}^{B'}\hat{\mathcal{O}}_{\nu,\kappa}$ let ${}^BM$ be the maximal quotient of M which belongs to ${}^B\hat{\mathcal{O}}_{\nu,\kappa}$. The functor $M \to {}^BM$ maps ${}^{B'}P_{\lambda,\nu,\kappa}$ to ${}^BP_{\lambda,\nu,\kappa}$, see [6, Sec. A.1]. Brauer reciprocity implies that the kernel of the obvious projection ${}^{B'}P_{\lambda,\nu,\kappa} \to {}^BP_{\lambda,\nu,\kappa}$ is filtered by parabolic Verma modules whose highest weights belong to ${}^{B'}\Lambda \setminus {}^B\Lambda$. The claim follows.

8. The affine parabolic category $\mathcal{O}$ and the Fock space

Fix integers $m, \ell > 0$, $e > 1$ and fix a composition $s \in \mathcal{C}_{m,\ell}$. In this section we'll use freely the notation from Appendix A. In particular, we have defined there the integers $\Delta^+_{\lambda,\mu,e,-s}$, $\nabla^-_{\lambda,\mu,e,s^\circ}$, the $\widehat{\mathfrak{sl}}_e$-module Λ^s, and the basis elements $|\lambda, s, e, s^\circ\rangle$, $\mathcal{G}(\lambda, s, e, s^\circ)^-$ of Λ^s.

8.1. The affine parabolic category $\mathcal{O}$ and the Fock space. The level ℓ *Fock space* $Fock_{e,s^\circ}$ associated with the multicharge s° is a $\widehat{\mathfrak{sl}}_e$-module of level ℓ which is the limit of a filtered inductive system of vector spaces Λ^r, with r a positive integer. Each Λ^r is equipped with a level zero representation of $\widehat{\mathfrak{sl}}_e$. See Section A.4 for details.

Now, let $\kappa = -e$. For each weight $\lambda \in \mathfrak{t}^*$ we'll write $\hat{\lambda} = \lambda + (\kappa - m)\omega_0$. Let $\pi \in \mathfrak{t}^*$ be given by

$$\pi + \rho = (s_1, s_1 - 1, \ldots, 1, s_2, s_2 - 1, \ldots, s_\ell, s_\ell - 1, \ldots, 1).$$

Note that π is NOT the same as in Section 6.1. Given an s-dominant weight λ we abbreviate

$$\Delta_{\lambda,s,-e} = M(\hat{\lambda}_\pi)_s, \qquad S_{\lambda,s,-e} = L(\hat{\lambda}_\pi).$$

Let $\mathcal{A}_{s,-e} \subset \hat{\mathcal{O}}_\kappa$ be the Serre category generated by the modules $S_{\lambda,s,-e}$, $\lambda \in \mathbb{Z}^s_{\geqslant 0}$. The proposition below identifies the vector space $[\mathcal{A}^{fg}_{s,-e}] \otimes \mathbb{C}$ with a submodule Λ^s of the $\widehat{\mathfrak{sl}}_e$-submodule Λ^m. Thus we can regard $[\mathcal{A}^{fg}_{s,-e}] \otimes \mathbb{C}$ as a subspace of $Fock_{e,s^\circ}$.

8.2. Proposition. *There is a vector space isomorphism*

$$\big[\mathcal{A}^{fg}_{s,-e}\big] \otimes \mathbb{C} \to \Lambda^s, \qquad [\Delta_{\lambda,s,-e}] \mapsto \big|\lambda, s, e, s^\circ\big\rangle, \qquad [S_{\lambda,s,-e}] \mapsto \mathcal{G}\big(\lambda, s, e, s^\circ\big)^-.$$

We have $\nabla^-_{\lambda^\circ,\mu^\circ,e,s^\circ} = [\Delta_{\lambda,s,-e} : S_{\mu,s,-e}]$ *for each* $\lambda, \mu \in \mathcal{P}_{n,s}$.

Proof. We have $\lambda_\pi + \rho = \alpha(\lambda, s, s)$ by Remark A.4.4(a). Proposition 5.8 and Proposition A.4.3(b) yield

$$[S_{\mu,s,-e}] = \sum_\lambda (-1)^{l(v_\lambda)-l(v_\mu)} P^{\gamma,-1}_{v_\lambda,v_\mu}(1)[\Delta_{\lambda,s,-e}],$$

$$\mathcal{G}\big(\mu, s, e, s^\circ\big)^- = \sum_\lambda (-1)^{l(v_\lambda)-l(v_\mu)} P^{\gamma,-1}_{v_\lambda,v_\mu}(1)\big|\lambda, s, e, s^\circ\big\rangle,$$

where $\gamma \in \mathfrak{A}$, $v_\lambda, v_\mu \in \hat{\mathfrak{S}}^\gamma$ such that $v_\lambda \bullet \gamma = \hat{\lambda}_\pi$, $v_\mu \bullet \gamma = \hat{\mu}_\pi$ and $v_\mu \geqslant v_\lambda$. This proves the first claim. The second one follows from Proposition A.4.3(c). □

8.3. Yvonne's conjecture. Fix an integer $n > 0$, $q \in \mathbb{C}^\times$ and a tuple $Q = (q_1, \ldots, q_\ell)$ in $(\mathbb{C}^\times)^\ell$. Let $\mathbf{A}_{q,Q}$ be defined as in Section 7.4. The *cyclotomic q-Schur algebra* $\mathbf{S}_{q,Q}$ is the endomorphism algebra of a particular projective $\mathbf{A}_{q,Q}$-module. It is a quasi-hereditary algebra. In particular, for each $\lambda \in \mathcal{P}^\ell_n$ there is a standard module $\Delta_{\lambda,q,Q}$ with a simple top $S_{\lambda,q,Q}$. See [5, Def. 6.13] for details. Yvonne's conjecture [31, Conj. 2.13, Defs. 2.5, 4.4] is the following one (note that our hypothesis differs slightly from the ones in [31]).

8.4. Conjecture. *Assume that* $q = \exp(-2i\pi/e)$, $q_p = \exp(2i\pi s_p/e)$ *and* $s_{p+1} - s_p \geqslant n$ *for* $p \neq \ell$. *Then we have* $[\Delta_{\lambda,q,Q} : S_{\mu,q,Q}] = \Delta^+_{{}^t\mu,{}^t\lambda,e,-s}$ *for all* $\lambda, \mu \in \mathcal{P}^\ell_n$.

8.5. The dimension conjecture. Let q, Q be as in (7.1). Assume that

$$(q+1)\prod_{p' \neq p''}(q_{p'} - q_{p''}) \neq 0, \qquad h_p \geqslant (1-n)h, \qquad h < 0, \quad \forall p \neq \ell. \tag{8.1}$$

We have the following [22, Thm. 6.8].

8.6. Theorem. *If* (7.1), (8.1) *hold there is an equivalence of quasi-hereditary categories* $\mathcal{H}_{h,H} \to$ $\mathbf{S}_{q,Q}$-**mod** *taking* $\Delta_{\lambda,h,H}$ *to* $\Delta_{\lambda,q,Q}$.

Now, assume that (7.1), (8.1) hold and fix λ, μ in $\mathcal{P}_n^\ell$. Theorem 8.6 yields

$$[\Delta_{\lambda,h,H} : S_{\mu,h,H}] = [\Delta_{\lambda,q,Q} : S_{\mu,q,Q}].$$

Assume further that the following hold

$$h = -1/e, \qquad h_p = s_{p+1}/e - s_p/e - 1/\ell, \quad \forall p \neq \ell. \tag{8.2}$$

The *dimension conjecture* compares also $[\mathcal{H}_{h,H}^{fg}] \otimes \mathbb{C}$ with a subspace of $Fock_{e,s^\circ}$. We'll state a categorical analogue of this conjecture. It simply claims that $\mathcal{H}_{h,H}$ should be equivalent to a full subcategory $\mathcal{A}_{n,s,-e}$ of $\mathcal{A}_{s,-e}$ if $s \in \mathcal{C}_{m,\ell,n}$. Taking the Grothendieck groups we recover the usual dimension conjecture from its categorical version by Proposition 8.2. More precisely, let first note that (7.1), (8.1) and (8.2) imply that

$$q = \exp(-2i\pi/e), \qquad q_p = \exp(2i\pi s_p/e), \qquad s_{p+1} - s_p \geqslant n - 1 + e/\ell.$$

Assume also that the hypothesis in Conjecture 8.4 holds. Then we should have

$$[\Delta_{\lambda,h,H} : S_{\mu,h,H}] = \Delta^+_{{}^t\mu,{}^t\lambda,e,-s}. \tag{8.3}$$

The dimension conjecture says that this equality should hold without the lower bound in (8.1) on the parameters h_p, see [22, Sec. 6.5]. If $s \in \mathcal{C}_{m,\ell,n}$ then (A.6) and Proposition 8.2 yield the equality

$$\Delta^+_{{}^t\mu,{}^t\lambda,e,-s} = [\Delta_{\lambda^\circ,s,-e} : S_{\mu^\circ,s,-e}]. \tag{8.4}$$

Therefore, composing equalities (8.3) and (8.4) we get the following one

$$[\Delta_{\lambda,h,H} : S_{\mu,h,H}] = [\Delta_{\lambda^\circ,s,-e} : S_{\mu^\circ,s,-e}].$$

Before formulating the *categorical dimension conjecture* note the following easy fact, see Section A.6. Let $\mathcal{A}_{n,s,-e}$ be the Serre subcategory of $\mathcal{A}_{s,-e}$ generated by the modules $S_{\lambda,s,-e}$ with $\lambda \in \mathcal{P}_{n,s}$.

8.7. Proposition. *The category $\mathcal{A}_{n,s,-e}$ is quasi-hereditary with respect to the order $\trianglelefteq$. The standard modules are the modules $\Delta_{\lambda,s,-e}$ with $\lambda \in \mathcal{P}_{n,s}$.*

We conjecture the following.

8.8. Categorical dimension conjecture. *If $s \in \mathcal{C}_{m,\ell,n}$ and* (8.2) *holds there is an equivalence of categories $\mathcal{A}_{n,s,-e} \to \mathcal{H}_{h,H}$ taking $\Delta_{\lambda^\circ,s,-e}$ to $\Delta_{\lambda,h,H}$.*

8.9. Comparison of the dimension conjecture with the functor $\mathfrak{E}$. Fix an integer $m > 0$ and a composition $\nu \in \mathcal{C}_{m,\ell}$. Let $\mathcal{A}_{n,\nu,\kappa} \subset \hat{\mathcal{O}}_{\nu,\kappa}$ be the Serre subcategory generated by the simple modules $S_{\lambda,\nu,\kappa}$ with $\lambda \in \mathcal{P}_{n,\nu}^\ell$ (with the notation from Section 6.1). We define h, H in the following way

$$\kappa = -e, \qquad h = 1/\kappa, \qquad h_p = \nu_p^\bullet/\kappa - m\ell/\kappa, \quad \forall p \in \Lambda. \tag{8.5}$$

Then the functor $\mathfrak{E}$ restricts to a functor

$$\mathfrak{E} : \mathcal{A}_{n,\nu,\kappa} \to \mathcal{H}_{h,H}. \tag{8.6}$$

Note that the integer m in Conjecture 8.8 and the integer m in (8.6) are not the same. If $\ell = 1$ we may choose them to be equal. Then $\mathfrak{E}$ is Suzuki's functor and it yields the equivalence

$$\mathcal{A}_{n,s,-e} \to \mathcal{H}_{h,H}$$

which is conjectured in 8.8, by Theorem A.5.1. If $\ell > 1$ we do not know how to get a functor $\mathcal{A}_{n,s,-e} \to \mathcal{H}_{h,H}$ from $\mathfrak{E}$ because, given h, H, the composition s and the integer m in (8.2) are different from the composition ν and the integer m in (8.5).

8.10. Remarks. (a) Note that $\mathcal{A}_{n,\nu,\kappa}$ is not a quasi-hereditary category for the order $\trianglelefteq$ on the parabolic Verma modules, see Example 6.6(b).

(b) Let $\mathbf{S}'$ be the degenerate analogue of $\mathbf{S}_{q,Q}$ considered in [2]. Assume that $s \in \mathcal{C}_{m,\ell,n}$ and (8.2) hold. Here we allow e to be any non-zero complex number. Now, assume also that $e \notin \mathbb{Q}$. Then, it is not difficult to prove that the category $\mathcal{A}_{n,s,-e}$ is equivalent to a parabolic subcategory $\mathcal{A}_{n,s} \subset \mathcal{O}$. This is due to the fact that if $\kappa \notin \mathbb{Q}$ then the induction functor $\Gamma : \mathcal{O}_\nu^{fg} \to \hat{\mathcal{O}}_{\nu,\kappa}^{fg}$ is an equivalence of categories and also to the fact that there is a canonical isomorphism

$$\Gamma(M) \dot{\otimes} \Gamma(E) = \Gamma(M \otimes E)$$

for all $M \in \mathcal{O}_\nu^{fg}$ and all finite dimensional $\mathfrak{g}$-module E (the proof is standard and will be given elsewhere). It is probably not difficult to prove that, under this assumption, the categories $\mathcal{H}_{h,H}$ and $\mathbf{S}'$-**mod** are equivalent. We expect that, in this case, the equivalence in Conjecture 8.8 is precisely the equivalence of categories $\mathcal{A}_{n,s} \to \mathbf{S}'$-**mod** in [2, Thm. C].

8.11. Example. Assume that $m = 7$, $\ell = 4$, $\kappa = -1$, $\nu = (1, 1, 4, 1)$, $n = 1$ and $h = -1$, $H = (-9/4, 3/4, 3/4)$. Then we have $\nu \in \mathcal{C}_{m,\ell,n}$ and (8.5) holds. Now set $m' = 9$ and $e = 1$, $s = (3, 1, 2, 3)$. Then we have $s \in \mathcal{C}_{m',\ell,n}$ and (8.2) holds. Note that for this choice of m, ℓ and ν the equality (7.3) holds (a computation yields $[\nabla_{\lambda,\nu,\kappa} : S_{\mu,\nu,\kappa}] = 1$ if $\Delta_{\mu,\nu,\kappa} \trianglelefteq \Delta_{\lambda,\nu,\kappa}$ and 0 else).

Appendix A

A.1. Proof of Proposition 5.8. Fix $\lambda \in \mathfrak{A}_\rho$ and fix $w \in \hat{\mathfrak{S}}(\lambda)^\lambda$ such that $w \bullet \lambda$ is ν-dominant. Since $w \in \hat{\mathfrak{S}}(\lambda)^\lambda$ and $\langle \lambda + \hat{\rho} : \alpha \rangle \leqslant 0$ for all $\alpha \in \hat{\Pi}(\lambda)^+$, we have the following formula in $[\hat{\mathcal{O}}_\kappa^{fg}]$

$$\big[L(w \bullet \lambda)\big] = \sum_{v \in \hat{\mathfrak{S}}(\lambda)} (-1)^{l(w)-l(v)} P_{v,w}(1) \big[M(v \bullet \lambda)\big].$$

Here the sum is over all v's such that $w \geqslant v$ and $P_{v,w}$ is the Kazhdan–Lusztig polynomial relative to $\hat{\mathfrak{S}}(\lambda)$. See [17, Thm. 1.1] for details. Since $w \in \hat{\mathfrak{S}}(\lambda)^\lambda$, for each $u \in \hat{\mathfrak{S}}(\lambda)^\lambda$ we have also

$$\sum_{x \in \hat{\mathfrak{S}}_\lambda} (-1)^{l(x)} P_{ux,w}(1) = P_{u,w}^{\lambda,-1}(1)$$

by definition of the parabolic Kazhdan–Lusztig polynomial of type $\hat{\mathfrak{S}}(\lambda)/\hat{\mathfrak{S}}_\lambda$. This yields the following formula

$$\big[L(w\bullet\lambda)\big]=\sum_{u\in\hat{\mathfrak{S}}(\lambda)^\lambda}(-1)^{l(w)-l(u)}P^{\lambda,-1}_{u,w}(1)\big[M(u\bullet\lambda)\big]. \tag{A.1}$$

Next, observe that $\mathfrak{S}_\nu$ is a parabolic subgroup of $\hat{\mathfrak{S}}(\lambda)$. Indeed, given a simple root α_i which belongs to Π_ν we write

$$\langle\lambda+\hat{\rho}:\alpha_i\rangle=\big\langle\lambda+\hat{\rho}-w(\lambda+\hat{\rho}):\alpha_i\big\rangle+\langle w\bullet\lambda:\alpha_i\rangle+\langle\hat{\rho}:\alpha_i\rangle.$$

Then the first term is an integer because $w\in\hat{\mathfrak{S}}(\lambda)$, and the second one is also an integer because $w\bullet\lambda$ is ν-dominant. Thus we have

$$\mathfrak{S}_\nu\subset\hat{\mathfrak{S}}(\lambda).$$

We must check that if the simple root α_i lies in Π_ν then it belongs also to the basis of $\hat{\Pi}(\lambda)^+$. The latter consists of the real affine roots α such that $s_\alpha(\hat{\Pi}(\lambda)^+\setminus\{\alpha\})\subset\hat{\Pi}^+(\lambda)$. So the claim is obvious, because

$$s_{\alpha_i}\big(\hat{\Pi}^+\setminus\{\alpha_i\}\big)\subset\hat{\Pi}^+,\qquad s_{\alpha_i}\big(\hat{\Pi}(\lambda)\big)\subset\hat{\Pi}(\lambda).$$

Now, consider the set

$$S=\big\{u\in\hat{\mathfrak{S}}(\lambda)^\lambda;\ u>s_{\alpha_i}u,\ \forall s_{\alpha_i}\in\mathfrak{S}_\nu\big\}.$$

Since $\lambda\in\mathfrak{A}$, for each $u\in\hat{\mathfrak{S}}(\lambda)^\lambda$ we have

$$u\bullet\lambda\quad\text{is }\nu\text{-dominant}\quad\iff\quad u\in S.$$

Further, it is well known that if $u,v\in\hat{\mathfrak{S}}(\lambda)^\lambda$ are such that $u>s_{\alpha_i}u$ and $v>s_{\alpha_i}v$, then

$$s_{\alpha_i}u,s_{\alpha_i}v\in\hat{\mathfrak{S}}(\lambda)^\lambda,\qquad P^{\lambda,-1}_{s_{\alpha_i}u,v}(1)=P^{\lambda,-1}_{u,v}(1). \tag{A.2}$$

Therefore if $u,v\in\hat{\mathfrak{S}}(\lambda)^\lambda$ and $u\bullet\lambda$, $v\bullet\lambda$ are both ν-dominant then we have

$$P^{\lambda,-1}_{s_{\alpha_i}u,v}(1)=P^{\lambda,-1}_{u,v}(1),\quad\forall\alpha_i\in\Pi_\nu.$$

Note also that $w\in S$ and that if $u\bullet\lambda$ is ν-dominant then the BGG resolution yields

$$\big[M(u\bullet\lambda)_\nu\big]=\sum_{x\in\mathfrak{S}_\nu}(-1)^{l(x)}\big[M(xu\bullet\lambda)\big].$$

Thus the contribution of the elements $u\in\hat{\mathfrak{S}}(\lambda)^\lambda$ in the sum (A.1) which are of the form

$$u=xv,\quad x\in\mathfrak{S}_\nu,\ v\in S,$$

is equal to the sum

$$\sum_v (-1)^{l(w)-l(v)} P_{v,w}^{\lambda,-1}(1)\big[M(v\bullet\lambda)_\nu\big]$$

over all elements $v \in \hat{\mathfrak{S}}(\lambda)^\lambda$ such that $w \geqslant v$ and $v \bullet \lambda$ is ν-dominant. Using (A.2) once again one can easily check that the other u's do not contribute to the right-hand side of (A.1).

A.2. Proof of Proposition 7.2 and Corollary 7.3.

A.2.1. The Kazhdan–Lusztig tensor product. First, let us recall the definition of the Kazhdan–Lusztig tensor product. Let R be a commutative $\mathbb{C}$-algebra with 1. Assume that R is a Noetherian integral domain. Let F be its fraction field. Fix a unit $\kappa \in R^\times$, and set $S = \{1, 2, \dots, n\}$. We'll use the same notation as in Sections 2.11, 2.12. Recall that z is a local coordinate on $\mathbb{P}^1$ centered at 0 and that $x = (x_i;\ i \in S)$ is a family of distinct points of $\mathbb{P}^1$. Consider the Lie algebra

$$\Gamma_{R,x} = \mathfrak{g}\big[\mathbb{P}^1_x\big]_R.$$

Fix a family $z = (z_i;\ i \in S)$ of local coordinates on $\mathbb{P}^1$ such that the coordinate z_i is centered at x_i. We can regard z_i as an isomorphism $\mathbb{P}^1 \to \mathbb{P}^1$. To each rational function $f \in F(\mathbb{P}^1)$ and to each index i we associate the power series expansion in $F((z))$ of the rational function $f \circ z_i$. Composing this formal series with the assignment $z \mapsto t_i$ we get a field homomorphism $F(\mathbb{P}^1) \to F((t_i))$. Their sum gives an R-algebra homomorphism $R[\mathbb{P}^1_x] \to R((t_S))$ and an R-Lie algebra homomorphism

$$\Gamma_{R,x} \to \hat{\mathcal{G}}_{R,S}.$$

Now, we set $\hat{S} = \{0, 1, \dots, n\}$. Fix a family $\hat{x} = (x_i;\ i \in \hat{S})$ of distinct points of $\mathbb{P}^1$. Let $\hat{\Gamma}_{R,\hat{x}}$ be the central extension of the Lie algebra $\Gamma_{R,x}$ by R associated with the cocycle $(\xi_1 \otimes f_1, \xi_2 \otimes f_2) \mapsto \mathrm{Res}_{x_0}(f_2 df_1)$. Let

$$U(\hat{\Gamma}_{R,\hat{x}}) \to \hat{\Gamma}_{R,\hat{x},\kappa}$$

be the quotient by the ideal generated by the element $\mathbf{1} - c$. The expansion at the points x_i, $i \in S$, gives an R-algebra homomorphism $R[\mathbb{P}^1_{\hat{x}}] \to R((t_S))$ and an R-algebra homomorphism

$$\hat{\Gamma}_{R,\hat{x},2m-\kappa} \to \hat{\mathcal{G}}_{R,S,\kappa}. \tag{A.3}$$

The expansion at the point x_0 yields an R-algebra homomorphism $R[\mathbb{P}^1_{\hat{x}}] \to R((t))$ and an R-algebra homomorphism

$$\hat{\Gamma}_{R,\hat{x},2m-\kappa} \to \hat{\mathbf{g}}_{R,2m-\kappa}. \tag{A.4}$$

See [18, Secs. 4.6, 8.2] for details. Now, for each $M_i \in \mathcal{C}(\hat{\mathbf{g}}_{R,\kappa})$, $i \in S$, the tensor product

$$W = \bigotimes_{i\in S} M_i$$

(over R) has an obvious structure of $\hat{\mathcal{G}}_{R,S,\kappa}$-module. We equip W with the structure of a $\hat{\Gamma}_{R,\hat{x},2m-\kappa}$-module via the map (A.3). For each integer $r>0$ let

$$G_{R,r}\subset U(\hat{\Gamma}_R)$$

be the R-submodule generated by the products of r elements in

$$\mathfrak{g}\otimes\{f;\ f(x_0)=0\}.$$

Using (A.4) we equip the projective limit of R-modules

$$\hat{W}=\varprojlim_r W_r,\qquad W_r=W/G_{R,r}W$$

with the structure of a $\hat{\mathbf{g}}_{R,\kappa^\sharp}$-module. We define

$$M_1\dot{\otimes}_R M_2\dot{\otimes}_R\cdots\dot{\otimes}_R M_n={}^\sharp T(W),\qquad T(W)=\hat{W}(-\infty).$$

To summarize, if $n=2$ then for any almost smooth $\hat{\mathbf{g}}_{R,\kappa}$-modules M_1, M_2 the (smooth) $\hat{\mathbf{g}}_{R,\kappa}$-module $M_1\dot{\otimes}_R M_2$ is given by

$$M_1\dot{\otimes}_R M_2={}^\sharp T(W),\qquad T(W)=\hat{W}(-\infty),\qquad \hat{W}=\varprojlim_r W/G_rW,\qquad W=M_1\otimes_R M_2.$$

See [18, Secs. 4.9, 8.4] for details. As above, if $R=\mathbb{C}$ we simply forget the subscript R everywhere.

A.2.2. Remark. The functor $\dot{\otimes}$ depends on the choice of the tuple of distinct points $x=(x_i;\ i\in S)$ and on the choice of the coordinate z_i centered at x_i for each i, see [18, Sec. 9]. Unless mentioned otherwise, we'll choose once for all the coordinates as in Section 2.12, i.e., we set $z_i=z-x_i$ if $x_i\neq\infty$ and $z_i=-z^{-1}$ else. The systems of coordinates associated with the tuples in the set

$$\mathcal{C}=\{x\in\mathbb{R}^n;\ 0<x_1<\cdots<x_n\}$$

belong to the contractible real manifold introduced in [18, Sec. 13.1]. So the space of affine coinvariants, see Definition 2.13, is independent on the choice of $x\in\mathcal{C}$ by [18, Sec. 13.3]. We may abbreviate

$$\langle M_i;\ i\in S\rangle=\langle M_i;\ i\in S\rangle_x,\quad x\in\mathcal{C},$$

if this does not create any confusion.

A.2.3. Proof of Proposition 7.2(a). If $R=\mathbb{C}$ and $\kappa\notin\mathbb{Q}_{\geqslant 0}$ the bifunctor $\dot{\otimes}$ yields the Kazhdan–Lusztig's monoidal category [18, Sec. 31]

$$\big(\hat{\mathcal{O}}^{fg}_{\geqslant 0,\kappa},\dot{\otimes},a,M(c\omega_0)\big).$$

A.2.4. Proposition. *Assume that $R = \mathbb{C}$ and $\kappa \notin \mathbb{Q}_{\geqslant 0}$. The functor $\dot{\otimes}$ takes $\hat{\mathcal{O}}^{fg}_{\geqslant 0,\kappa} \times \hat{\mathcal{O}}^{fg}_{\nu,\kappa}$ and $\hat{\mathcal{O}}^{fg}_{\nu,\kappa} \times \hat{\mathcal{O}}^{fg}_{\geqslant 0,\kappa}$ into $\hat{\mathcal{O}}^{fg}_{\nu,\kappa}$. The tuple $(\hat{\mathcal{O}}^{fg}_{\nu,\kappa}, \dot{\otimes}, a, M(c\omega_0))$ is a bimodule over $(\hat{\mathcal{O}}^{fg}_{\geqslant 0,\kappa}, \dot{\otimes}, a, M(c\omega_0))$.*

Proof. The first part is proved in [30, Thm. 1.6], and the second one is claimed there. Since the construction of the associativity isomorphism is the same as in [18, Sec. 18.2] we'll not give more details there. Note that the axioms of a bimodule over a category imply that there is a canonical isomorphism from $M(c\omega_0) \dot{\otimes} -$ to the identity functor of $\hat{\mathcal{O}}^{fg}_{\nu,\kappa}$. It is given by the following chain of isomorphisms, for each modules M_1, M_2

$$\begin{aligned}
\operatorname{Hom}_{\hat{\mathbf{g}}}(M_1, M_2) &= \langle M_1, DM_2\rangle^* \\
&= \bigl\langle M(c\omega_0), M_1, DM_2\bigr\rangle^* \\
&= \bigl\langle M(c\omega_0) \dot{\otimes} M_1, DM_2\bigr\rangle^* \\
&= \operatorname{Hom}_{\hat{\mathbf{g}}}\bigl(M(c\omega_0) \dot{\otimes} M_1, M_2\bigr).
\end{aligned}$$

The second isomorphism is as in Proposition 2.14(c), the other ones are as in Proposition A.2.6 below. A similar construction yields an isomorphism from $- \dot{\otimes} M(c\omega_0)$ to the identity functor. □

A.2.5. Proof of Proposition 7.2(b). Let us prove the following proposition.

A.2.6. Proposition.

(a) *Let $M_1, \ldots, M_n \in \hat{\mathcal{O}}^{fg}_{\geqslant 0,\kappa}$ and $M_0, M_{n+1} \in \hat{\mathcal{O}}^{fg}_{\nu,\kappa}$. We have natural isomorphisms of vector spaces*

$$\begin{aligned}
\bigl\langle M_0 \dot{\otimes} M_1 \dot{\otimes} \cdots \dot{\otimes} M_n, {}^{\dagger}M_{n+1}\bigr\rangle &= \operatorname{Hom}_{\hat{\mathbf{g}}}\bigl(M_0 \dot{\otimes} M_1 \dot{\otimes} \cdots \dot{\otimes} M_n, {}^{\dagger} DM_{n+1}\bigr)^* \\
&= \bigl\langle M_0, M_1, \ldots, {}^{\dagger}M_{n+1}\bigr\rangle.
\end{aligned}$$

(b) *If $M_i = (N_i)_\kappa$ is a generalized Weyl module for each $i \in S$ then we have a natural isomorphism of vector spaces $\langle M_1, M_2, \ldots, M_n\rangle = H_0(\mathfrak{g}, N_1 \otimes N_2 \otimes \cdots \otimes N_n)$.*

Proof. First, observe that the module $M_0 \dot{\otimes} M_1 \dot{\otimes} \cdots \dot{\otimes} M_n$ lies in the category $\hat{\mathcal{O}}^{fg}_{\nu,\kappa}$ by Proposition A.2.4(a). Thus to prove the first equality in (a) it is enough to check that for each modules M, N in $\hat{\mathcal{O}}^{fg}_{\nu,\kappa}$ we have a natural isomorphism of (finite dimensional) vector spaces

$$\bigl\langle M, {}^{\dagger} N\bigr\rangle^* = \operatorname{Hom}_{\hat{\mathbf{g}}}\bigl(M, {}^{\dagger}DN\bigr). \tag{A.5}$$

The right-hand side of (A.5) consists of the families of linear forms

$$(f_{\lambda,\mu}) \in \prod_{\lambda} \bigoplus_{\mu} (M_\lambda \otimes N_\mu)^*$$

which vanish on the set

$$\big\{(\xi m)\otimes n + m\otimes(\ddagger\xi n);\ \forall\xi\in\hat{\mathbf{g}},\ m\in M,\ n\in N\big\}.$$

We claim that the obvious inclusion

$$\prod_\lambda\bigoplus_\mu(M_\lambda\otimes N_\mu)^*\subset\prod_{\lambda,\mu}(M_\lambda\otimes N_\mu)^*$$

factors to a natural isomorphism

$$\mathrm{Hom}_{\hat{\mathbf{g}}}\big(M,\,{}^\dagger DN\big)\to\mathrm{Hom}_{\hat{\mathbf{g}}}\big(M\otimes{}^\ddagger N,\mathbb{C}\big).$$

The definition of the set of affine coinvariants yields

$$\big\langle M,\,{}^\dagger N\big\rangle = H_0\big(\hat{\mathbf{g}},M\otimes{}^\ddagger N\big).$$

Therefore there is a natural isomorphism

$$\mathrm{Hom}_{\hat{\mathbf{g}}}\big(M\otimes{}^\ddagger N,\mathbb{C}\big)=\big\langle M,\,{}^\dagger N\big\rangle^*.$$

Now we check our claim. The injectivity is clear. To prove surjectivity, fix

$$(f_{\lambda,\mu})\in\mathrm{Hom}_{\hat{\mathbf{g}}}\big(M\otimes{}^\ddagger N,\mathbb{C}\big),\qquad f_{\lambda,\mu}\in(M_\lambda\otimes N_\mu)^*,$$

and fix a finite subset $\mathcal{S}\subset\mathbf{t}^*$ such that N is generated by the subspace $\bigoplus_{\mu\in\mathcal{S}}N_\mu$. Fix also λ and $x\in M_\lambda$. For each μ the weight space N_μ is spanned by elements of the form $y=\xi z$ with $z\in N_\gamma$, $\gamma\in\mathcal{S}$ and $\xi\in U(\hat{\mathbf{g}}_\kappa)$ of weight $\mu-\gamma$. For all μ except a finite number the weight space $M_{\lambda-\mu+\gamma}$ vanishes, hence

$$f_{\lambda,\mu}(x\otimes y)=-f_{\lambda,\mu}\big((\ddagger\xi)x\otimes z\big)=0.$$

So $f_{\lambda,\mu}=0$ for all μ except a finite number. This proves the claim.

The vector space $\langle M_0,M_1,\dots,{}^\dagger M_{n+1}\rangle$ is finite dimensional, because the modules $M_0,M_1,\dots,{}^\dagger M_{n+1}$ are quotient of generalized Weyl modules. Therefore, the same argument as in [18, Sec. 13.4] yields the second isomorphism in (a).

Now we concentrate on (b). The inclusions $N_i\subset M_i$, $i\in S$, yield an inclusion $\bigotimes_{i\in S}N_i\subset\bigotimes_{i\in S}M_i$. Taking the coinvariants we get a natural map

$$H_0(\mathfrak{g},N_1\otimes N_2\otimes\cdots\otimes N_n)\to\langle M_1,M_2,\dots,M_n\rangle$$

which is invertible by [18, Prop. 9.15]. □

A.2.7. Proof of Proposition 7.2(c).

A.2.8. Proposition. *If $\kappa\notin\mathbb{Q}_{\geqslant0}$ the functor $\dot\otimes$ takes $\hat{\mathcal{O}}^\Delta_{\geqslant0,\kappa}\times\hat{\mathcal{O}}^\Delta_{\nu,\kappa}$ and $\hat{\mathcal{O}}^\Delta_{\nu,\kappa}\times\hat{\mathcal{O}}^\Delta_{\geqslant0,\kappa}$ into $\hat{\mathcal{O}}^\Delta_{\nu,\kappa}$.*

Proof. Fix a module M_1 in $\hat{\mathcal{O}}^{\Delta}_{\geqslant 0,\kappa}$ and a module M_2 in $\hat{\mathcal{O}}^{\Delta}_{\nu,\kappa}$. The module $M_1 \dot{\otimes} M_2$ belongs to the category $\hat{\mathcal{O}}^{fg}_{\nu,\kappa}$ by Proposition A.2.4. Fix a finite set B such that M_2 and $M_1 \dot{\otimes} M_2$ belong to the subcategory ${}^B\hat{\mathcal{O}}^{\Delta}_{\nu,\kappa}$. To unburden notation we'll write $\mathcal{B} = {}^B\hat{\mathcal{O}}^{fg}_{\nu,\kappa}$. Recall that $\mathcal{B}$ is a highest weight category with a weak duality functor ${}^\dagger D$. To prove the claim it is enough to check that we have

$$\mathrm{Ext}^1_{\mathcal{B}}\big(M_1 \dot{\otimes} M_2,\ {}^\dagger DM\big) = 0, \quad \forall M \in \Delta_{\mathcal{B}}.$$

See [4, Prop. A.2.2(iii)]. Fix a module $P \in \mathcal{B}^{\mathrm{proj}}$ which maps onto M. Since P, M both lie in $\mathcal{B}^{\Delta}$, the kernel K of the surjective map $P \to M$ lies also in $\mathcal{B}^{\Delta}$ [4, Prop. A.2.2(v)]. Since ${}^\dagger DP$ is injective we have also

$$\mathrm{Ext}^1_{\mathcal{B}}\big(M_1 \dot{\otimes} M_2,\ {}^\dagger DP\big) = 0.$$

So the long exact sequence of the Ext-group and Proposition A.2.6(a) yield

$$\dim \mathrm{Ext}^1_{\mathcal{B}}\big(M_1 \dot{\otimes} M_2,\ {}^\dagger DM\big) = \dim\langle M_1, M_2,\ {}^\dagger P\rangle - \dim\langle M_1, M_2,\ {}^\dagger K\rangle - \dim\langle M_1, M_2,\ {}^\dagger M\rangle.$$

The right-hand side is zero by Lemma A.2.10 below. □

A.2.9. Definition. If $M \in \hat{\mathcal{O}}^{fg}_{\nu,\kappa}$ we write $(M : \lambda)$ for the coefficient of $[M]$ along the element $[M(\hat{\lambda})_\nu]$ of the basis of the free Abelian group $[\hat{\mathcal{O}}^{fg}_{\nu,\kappa}]$ consisting of the standard modules. If $M \in \mathcal{O}^{fg}_{\nu}$ we write $(M : \lambda)$ for the coefficient of the element $[M]$ along the element $[M(\lambda)_\nu]$ of the basis of the free Abelian group $[\mathcal{O}^{fg}_{\nu}]$ consisting of the standard modules.

A.2.10. Lemma. Let $\kappa \notin \mathbb{Q}_{\geqslant 0}$. For $M_1, M_2 \in \hat{\mathcal{O}}^{\Delta}_{\nu,\kappa}$ and $M \in \hat{\mathcal{O}}^{\Delta}_{\geqslant 0,\kappa}$ we have

$$\dim\langle M_1, M,\ {}^\dagger M_2\rangle = \sum_{\lambda_1,\lambda_2,\mu} (M_1 : \lambda_1)(M_2 : \lambda_2)(M : \mu)\big(M(\lambda_1)_\nu \otimes L(\mu) : \lambda_2\big),$$

where μ runs over $\mathbb{Z}^m_{\geqslant 0}$ and λ_1, λ_2 run over $\mathbb{Z}^\nu_{\geqslant 0}$.

Proof. First assume that $M = M(\hat{\mu})$, $M_1 = M(\hat{\lambda}_1)_\nu$ and $M_2 = M(\hat{\lambda}_2)_\nu$. Note that ${}^\dagger M_2$ is again a generalized Weyl module. Thus Proposition A.2.6(b) yields

$$\begin{aligned}\langle M_1, M,\ {}^\dagger M_2\rangle &= H_0\big(\mathfrak{g}, M(\lambda_1)_\nu \otimes L(\mu) \otimes^\dagger M(\lambda_2)_\nu\big)\\ &= \mathrm{Hom}_{\mathfrak{g}}\big(M(\lambda_1)_\nu \otimes L(\mu),\ {}^\dagger DM(\lambda_2)_\nu\big)^*.\end{aligned}$$

Since the module $M(\lambda_1)_\nu \otimes L(\mu)$ lies in $\mathcal{O}^{\Delta}_{\nu}$, the Hom space above has dimension

$$\big(M(\lambda_1)_\nu \otimes L(\mu) : \lambda_2\big)$$

by [4, Prop. A.2.2(ii)]. The proof is the same if M, M_1, M_2 are generalized Weyl modules. The general case follows as in [18, Lem. 28.1]. □

A.2.11. Proof of Corollary 7.3. The duality functor D equip $(\mathcal{O}^{fg}_{\geqslant 0,\kappa}, \dot{\otimes}, a)$ with the structure of a rigid monoidal category, see [18]. This means that there are natural morphisms in $\mathcal{O}^{fg}_{\geqslant 0,\kappa}$

$$i_M : M(c\omega_0) \to M \dot{\otimes} DM, \qquad e_M : DM \dot{\otimes} M \to M(c\omega_0)$$

such that the following hold

(a) for any $M_1, M_2 \in \mathcal{O}^{fg}_{\geqslant 0,\kappa}$ the map $\mathrm{Hom}_{\hat{\mathfrak{g}}}(M_1, DM_2) \to \mathrm{Hom}_{\hat{\mathfrak{g}}}(M_1 \dot{\otimes} M_2, M(c\omega_0))$ such that $f \mapsto e_{M_2} \circ (f \otimes 1)$ is an isomorphism,
(b) the compositions below are equal to the identity

$$M = M(c\omega_0) \dot{\otimes} M \xrightarrow{i_M \dot{\otimes} 1} M \dot{\otimes} DM \dot{\otimes} M \xrightarrow{1 \dot{\otimes} e_M} M \dot{\otimes} M(c\omega_0) = M,$$

$$DM = DM \dot{\otimes} M(c\omega_0) \xrightarrow{1 \dot{\otimes} i_M} DM \dot{\otimes} M \dot{\otimes} DM \xrightarrow{e_M \dot{\otimes} 1} M(c\omega_0) \dot{\otimes} DM = DM.$$

Therefore, for any $M_1, M_2 \in \hat{\mathcal{O}}^{fg}_{\nu,\kappa}$ and any $M \in \hat{\mathcal{O}}^{fg}_{\geqslant 0,\kappa}$ the composed map

$$\begin{aligned}\mathrm{Hom}_{\hat{\mathfrak{g}}}(M_1, M_2 \dot{\otimes} M) &\longrightarrow \mathrm{Hom}_{\hat{\mathfrak{g}}}(M_1 \dot{\otimes} DM, M_2 \dot{\otimes} M \dot{\otimes} DM) \\ &\longrightarrow \mathrm{Hom}_{\hat{\mathfrak{g}}}\big(M_1 \dot{\otimes} DM, M_2 \dot{\otimes} M(c\omega_0)\big) = \mathrm{Hom}_{\hat{\mathfrak{g}}}(M_1 \dot{\otimes} DM, M_2)\end{aligned}$$

is an isomorphism whose inverse map is the composition of the chain of maps

$$\begin{aligned}\mathrm{Hom}_{\hat{\mathfrak{g}}}(M_1 \dot{\otimes} DM, M_2) &\longrightarrow \mathrm{Hom}_{\hat{\mathfrak{g}}}(M_1 \dot{\otimes} DM \dot{\otimes} M, M_2 \dot{\otimes} M) \\ &\longrightarrow \mathrm{Hom}_{\hat{\mathfrak{g}}}\big(M_1 \dot{\otimes} M(c\omega_0), M_2 \dot{\otimes} M\big) = \mathrm{Hom}_{\hat{\mathfrak{g}}}(M_1, M_2 \dot{\otimes} M).\end{aligned}$$

Thus the functors $- \dot{\otimes} M$, $- \dot{\otimes} DM$ are adjoint (left and right) to each other. Since every right (resp. left) adjoint is right (resp. left) exact this implies that the functor $- \dot{\otimes} M$ is exact for each module M in $\hat{\mathcal{O}}^{fg}_{\geqslant 0,\kappa}$. The same holds for the functor $M \dot{\otimes} -$.

A.3. Reminder on induction. Let R be a commutative ring with 1. An R-split induction datum is a quadruple $(\mathfrak{a}, \mathfrak{b}, \mathfrak{c}, F)$ where $\mathfrak{a}$ is an R-Lie algebra which is free as an R-module, $\mathfrak{b}$, $\mathfrak{c}$ are supplementary R-Lie subalgebras of $\mathfrak{a}$ and F is a $\mathfrak{b}$-module. The corresponding induced module is

$$\Gamma^{\mathfrak{a}}_{\mathfrak{b}}(F) = U(\mathfrak{a}) \otimes_{U(\mathfrak{b})} F.$$

For each $\mathfrak{a}$-module E the assignment $a \otimes e \otimes f \mapsto \sum a_1 e \otimes a_2 \otimes f$, where $\sum a_1 \otimes a_2$ is the coproduct of a, yields an $\mathfrak{a}$-module isomorphism

$$\Gamma^{\mathfrak{a}}_{\mathfrak{b}}(E \otimes_R F) \to E \otimes_R \Gamma^{\mathfrak{a}}_{\mathfrak{b}}(F).$$

This isomorphism is called the *tensor identity*. There is also a $\mathfrak{c}$-module isomorphism [18, Sec. II.A]

$$\Gamma^{\mathfrak{c}}(F) = U(\mathfrak{c}) \otimes_R F \to \Gamma_{\mathfrak{b}}^{\mathfrak{a}}(F).$$

A.4. Reminder on the Fock space. This section is a reminder from [27]. Fix an integer $e > 1$ and a tuple $s \in \mathbb{Z}^\ell$. First, we recall the definition of the Fock space $Fock_{s^\circ,e}$, a $\widehat{\mathfrak{sl}}_e$-module of level ℓ equipped with three remarkable bases. As a vector space, it is the limit of a filtering inductive system of vector spaces Λ^k with $k > 0$. Each Λ^k is equipped with a level zero representation of $\widehat{\mathfrak{sl}}_e$. We give the definition of Λ^k in the second section. Next, we introduce a particular submodule $\Lambda^\nu \subset \Lambda^m$ for each integer $m > 0$ and each composition $\nu \in \mathcal{C}_{m,\ell}$. We do not give the construction of the inductive system above, we'll not use it. However, in the particular case where $\nu = s$ we define explicitly the linear map

$$\Lambda^s \to Fock_{s^\circ,e}.$$

This section does not contain new results. Most proofs can be found in [27] and will be omitted.

A.4.1. The Fock space and its canonical bases. Let $e_a,\ f_a,\ a \in \mathbb{Z}/e\mathbb{Z}$, be the Chevalley generators of the affine Lie algebra $\widehat{\mathfrak{sl}}_e$. For each $s \in \mathbb{Z}^\ell$ the vector space

$$Fock_{s^\circ,e} = \bigoplus_{\lambda \in \mathcal{P}^\ell} \mathbb{C}\big|\lambda, s^\circ, e\big\rangle$$

is equipped with a level ℓ representation of $\widehat{\mathfrak{sl}}_e$ and with two canonical bases

$$\big(\mathcal{G}\big(\lambda, s^\circ, e\big)^\pm;\ \lambda \in \mathcal{P}^\ell\big).$$

We define matrices $\nabla^\pm = (\nabla^\pm_{\lambda,\mu,s^\circ,e})$ and $\Delta^\pm = (\Delta^\pm_{\lambda,\mu,s^\circ,e})$ such that

$$\mathcal{G}\big(\lambda, s^\circ, e\big)^\pm = \sum_\mu \Delta^\pm_{\lambda,\mu,s^\circ,e}\big|\mu, s^\circ, e\big\rangle, \qquad \Delta^\pm = \big(\nabla^\pm\big)^{-1}.$$

We have the following formula [27, Thm. 5.15]

$$\Delta^+_{{}^t\mu,{}^t\lambda,-s,e} = \nabla^-_{\lambda,\mu,s^\circ,e}. \tag{A.6}$$

A.4.2. The $\widehat{\mathfrak{sl}}_e$-modules Λ^m, Λ^ν. We define a representation of $\widehat{\mathfrak{sl}}_e$ on the vector space $U = \bigoplus_{a\in\mathbb{Z}} \mathbb{C}u_a$ by the following formulas: for $b \in \mathbb{Z}/e\mathbb{Z}$ and $a \in \mathbb{Z}$ we set

$$e_b(u_{a+1}) = u_a, \qquad f_b(u_a) = u_{a+1} \quad \text{if } b \neq 0,\ a \in b,$$
$$e_0(u_{a+1}) = u_{a+e-e\ell}, \qquad f_0(u_a) = u_{a+1-e+e\ell} \quad \text{if } a \in e\mathbb{Z},$$
$$e_b(u_a) = f_b(u_a) = 0 \quad \text{else.}$$

It yields a representation of $\widehat{\mathfrak{sl}}_e$ on the m-th exterior power $\Lambda^m = \bigwedge^m U$ for each $m > 0$. Write

$$|\underline{\alpha}\rangle = u_{a_1} \wedge u_{a_2} \wedge \cdots \wedge u_{a_m}, \quad \forall \underline{\alpha} = (a_1, a_2, \ldots, a_m) \in \mathbb{Z}^m_{>0}.$$

The vectors $|\underline{\alpha}\rangle$ with $\alpha \in \mathbb{Z}^m_{>0}$ form a basis of Λ^m. Let $(\mathcal{G}(\underline{\alpha})^+)$, $(\mathcal{G}(\underline{\alpha})^-)$ denote the *canonical bases* of Λ^m introduced in [27].

Now, fix a composition $\nu \in \mathcal{C}_{m,\ell}$. We'll define a $\widehat{\mathfrak{sl}}_e$-submodule $\Lambda^\nu \subset \Lambda^m$. First, for each integer a let

$$p_a \in \Lambda, \qquad c_a \in \mathbb{Z}/e\mathbb{Z}, \qquad r_a, \phi_a \in \mathbb{Z},$$

be defined by

$$a = c_a + e(p_a - 1) + e\ell r_a, \qquad \phi_a = c_a + e r_a.$$

This yields the bijection

$$\mathbb{Z}^m_{>0} \to \bigsqcup_{\nu \in \mathcal{C}_{m,\ell}} \mathbb{Z}^{\nu^\circ}_{>0} \times \{\nu\}, \qquad \underline{\alpha} \mapsto (\alpha, \nu), \tag{A.7}$$

where $\underline{\alpha} = (a_1, a_2, \ldots, a_m)$, $\alpha = (\phi_{a_{w(1)}}, \ldots, \phi_{a_{w(m)}})$ and $w \in \mathfrak{S}$ is minimal with

$$(p_{a_{w(1)}}, \ldots, p_{a_{w(m)}}) = \left(\ell^{\nu_\ell}, \ldots, 2^{\nu_2}, 1^{\nu_1}\right).$$

For example take $e = 2$, $\ell = 3$, $m = 7$, $\underline{\alpha} = (3, 1, 0, -2, -4, -6, -7)$. Then we get $\nu = (2, 2, 3)$ and $\alpha = (0, -2, -3, 1, 0, 1, 0)$.

Let $\Lambda^\nu \subset \Lambda^m$ be the subspace spanned by the basis elements $|\underline{\alpha}\rangle$ such that $\underline{\alpha}$ maps to $\mathbb{Z}^\nu_{>0} \times \{\nu^\circ\}$ under (A.7). One can prove that it is a $\widehat{\mathfrak{sl}}_e$-submodule. Let us described explicitly this module. Fix a tuple $s = (s_p) \in \mathbb{Z}^\ell$. We'll define three bases of Λ^ν whose elements are labeled by $\mathbb{Z}^\nu_{\geqslant 0}$. The $\widehat{\mathfrak{sl}}_e$-module Λ^ν does not depend on s. The tuple s enters only in the labelling of the bases elements. Let $J_p = J_{\nu,p}$ be as in Section 1.4. Given a m-tuple $\lambda \in \mathbb{Z}^\nu_{\geqslant 0}$ let

$$\alpha(\lambda, \nu, s) \in \mathbb{Z}^\nu_{>0}$$

be the m-tuple whose j-th entry is $\lambda_j + i_p - j + s_p$ for $j \in J_p$, and let

$$\underline{\alpha}(\lambda, \nu, s) \in \mathbb{Z}^m_{>0}$$

be the unique m-tuple such that (A.7) maps $\underline{\alpha}(\lambda, \nu, s)$ to $(\alpha(\lambda, \nu, s), \nu^\circ)$.

For example take $e = 2$, $\ell = 3$, $m = 7$, $\nu = (2, 2, 3)$, $\lambda = (2, 0, 1, -3, 1, -2, -4)$ and $s = (1, 1, 4)$. Then we get $\alpha(\lambda, \nu, s) = (3, 0, 2, -3, 5, 1, -2)$ and $\underline{\alpha}(\lambda, \nu, s) = (13, 11, 4, 1, 0, -9, -10)$.

We define the following elements of Λ^m

$$\left|\lambda, \nu, s^\circ, e\right\rangle = \left|\underline{\alpha}(\lambda, \nu, s)\right\rangle, \qquad \mathcal{G}\left(\lambda, \nu, s^\circ, e\right)^\pm = \mathcal{G}\left(\underline{\alpha}(\lambda, \nu, s)\right)^\pm.$$

For each $\lambda, \mu \in \mathbb{Z}^m$ and each $b \in \mathbb{Z}$ we write $\lambda \xrightarrow{b} \mu$ if there exist $j \in J$ such that $\lambda_j = b$, $\mu_j = b + 1$, and $\lambda_i = \mu_i$ if $i \neq j$.

A.4.3. Proposition.

(a) *The elements* $|\lambda, \nu, s^\circ, e\rangle$, $\mathcal{G}(\lambda, \nu, s^\circ, e)^+$ *and* $\mathcal{G}(\lambda, \nu, s^\circ, e)^-$ *are basis vectors of* Λ^ν *when* λ *runs over* $\mathbb{Z}^\nu_{\geqslant 0}$. *The representation of* $\widehat{\mathfrak{sl}}_e$ *in* Λ^ν *is given by*

$$e_a\big(|\mu, \nu, s^\circ, e\rangle\big) = \sum_{b,\lambda} |\lambda, \nu, s^\circ, e\rangle, \qquad f_a\big(|\lambda, \nu, s^\circ, e\rangle\big) = \sum_{b,\mu} |\mu, \nu, s^\circ, e\rangle, \quad a \in \mathbb{Z}/e\mathbb{Z},$$

summing over all integer b *and all* λ, μ *such that* $b \in a$ *and* $\alpha(\lambda, \nu, s) \xrightarrow{b} \alpha(\mu, \nu, s)$.

(b) *For each tuple* $\mu \in \mathbb{Z}^\nu_{\geqslant 0}$ *we have*

$$\mathcal{G}\big(\mu, \nu, s^\circ, e\big)^- = \sum_\lambda (-1)^{l(v_\lambda) - l(v_\mu)} P^{\gamma, -1}_{v_\lambda, v_\mu}(1) \big|\lambda, \nu, s^\circ, e\big\rangle.$$

The sum is over all tuples λ *such that* $v_\lambda \bullet \gamma = \alpha(\widehat{\lambda, \nu, s}) - \hat{\rho}$ *and* $v_\mu \bullet \gamma = \alpha(\widehat{\mu, \nu, s}) - \hat{\rho}$ *with* $\gamma \in \mathfrak{A}$, $v_\lambda, v_\mu \in \hat{\mathfrak{S}}^\gamma$ *and* $v_\mu \geqslant v_\lambda$.

(c) *If* $\nu = s$ *there is a unique linear map* $\Lambda^s \to \mathit{Fock}_{s^\circ, e}$ *such that* $|\lambda, s, s^\circ, e\rangle \mapsto |\lambda^\circ, s^\circ, e\rangle$, $|\mu, s, s^\circ, e\rangle \mapsto 0$ *and* $\mathcal{G}(\lambda, s, s^\circ, e)^\pm \mapsto \mathcal{G}(\lambda^\circ, s^\circ, e)^\pm$ *for* $\lambda \in \mathbb{N}^s_{\geqslant 0}$ *and* $\mu \in \mathbb{Z}^s_{\geqslant 0} \setminus \mathbb{N}^s_{\geqslant 0}$.

A.4.4. Remarks. (a) If $\nu = s$ then we have

$$\alpha(\lambda, s, s) - \rho = \lambda_\pi,$$

where π is as in Section 8.1.

(b) Proposition A.4.3(b) is [27, Thm. 3.25–3.26]. The tuple $\underline{\alpha}(\lambda, s, s)$ which is used in Proposition A.4.3(c) differs from the combinatorial definition in [27]. Let us explain this and let us explain how to deduce the lemma from [27]. For any ℓ-partition λ and any tuple $s \in \mathbb{Z}^\ell$ we write

$$\alpha(\lambda, s) = \big\{\mathrm{c}_s(i, \lambda_{p,i} + 1, p);\ i > 0,\ p \in \Lambda\big\}, \quad \mathrm{c}_s(i, j, p) = s_p + j - i.$$

Set $\mathcal{A} = \{\alpha(\lambda, s);\ \lambda \in \mathcal{P}^\ell,\ s \in \mathbb{Z}^\ell\}$. In the particular case where $\ell = 1$ we write $\underline{\lambda}$, $\underline{\alpha}$, $\underline{\mathcal{A}}$ for λ, α, $\mathcal{A}$. Consider the bijection

$$\underline{\mathcal{A}} \to \mathcal{A}, \qquad \underline{\alpha} \mapsto \alpha = \big\{(\phi_a, p_a);\ a \in \underline{\alpha}\big\}.$$

Let $\alpha \mapsto \underline{\alpha}$ denote the inverse map.

Next, fix an integer $m > 0$ and a composition $s \in \mathcal{C}_{m,\ell}$. For each $\lambda \in \mathcal{P}^\ell$ there is a unique $\underline{\lambda} \in \mathcal{P}$ such that $\underline{\alpha}(\underline{\lambda}, m) = \underline{\alpha(\lambda, s)}$. This yields a map $\mathcal{P}^\ell \to \mathcal{P}$, $\lambda \mapsto \underline{\lambda}$.

Now let $\lambda \in \mathcal{P}^\ell$. The set $\underline{\alpha(\lambda^\circ, s^\circ)}$ is well defined, it belongs to $\underline{\mathcal{A}}$. For each integer $k > 0$ let $\underline{\alpha}(\lambda, k, s^\circ)$ be the tuple consisting of the k largest entries of $\underline{\alpha(\lambda^\circ, s^\circ)}$ arranged in decreasing order. This tuple belongs to $\mathbb{Z}^k_{>0}$.

Assume further that λ is indeed a tuple in $\mathbb{N}^s_{\geqslant 0}$ which is viewed as an ℓ-partition as in Section 1.5. Then a direct computation yields $m \geqslant l(\underline{\lambda})$. Using this inequality, another computation yields

$$\underline{\alpha}\big(\lambda, m, s^\circ\big) = \underline{\alpha}(\lambda, s, s).$$

So Proposition A.4.3(c) follows from [27, Sec. 4.4].

For example take

$$e = 2, \qquad \ell = 3, \qquad m = 6, \qquad \lambda = \big((1,1),(2,1),(1)\big), \qquad s = (2,3,1).$$

We can identify λ with the tuple $(1,1,2,1,0,1) \in \mathbb{N}^s_{\geqslant 0}$. Since $\underline{\lambda} = (9,4,3^2,1^2)$ we have $m = 6 = l(\underline{\lambda})$. We have also $\alpha(\lambda,s,s) = (3,2,5,3,1,2)$. Thus $\underline{\alpha}(\lambda,s,s) = (15,11,9,6,3,2)$ by definition of (A.7). The tuple $\underline{\alpha}(\lambda,s,s)$ coincides with the first six entries of $\underline{\alpha(\lambda^\circ,s^\circ)}$ arranged in decreasing order.

A.5. The functor $\mathfrak{E}$ for $\ell = 1$. In this section we'll set $\ell = 1$, $\nu = (m)$ and $n \leqslant m$. Recall that we have identified the set of partitions of n with a set of integral dominant weights in (1.1). Write

$$\Delta_{\lambda,\geqslant 0,\kappa} = \Delta_{\lambda,\nu,\kappa}, \qquad S_{\lambda,\geqslant 0,\kappa} = S_{\lambda,\nu,\kappa}$$

for each dominant weight λ. See Section 6.1 for details. Let

$$\mathcal{A}_{n,\geqslant 0,\kappa} \subset \hat{\mathcal{O}}_{\geqslant 0,\kappa}$$

be the Serre subcategory generated by the modules $S_{\lambda,\geqslant 0,\kappa}$ with $\lambda \in \mathcal{P}_n$. It is a quasi-hereditary category with respect to the order $\trianglelefteq$, see e.g., Proposition 8.7. Since $\ell = 1$ the algebra $\mathbf{H}_{h,H}$ is the rational DAHA of $\mathrm{GL}_n(\mathbb{C})$ with the parameter h and $\mathbf{A}_{q,Q}$ is the Hecke algebra of $\mathrm{GL}_n(\mathbb{C})$ with the parameter q. Set $q = \exp(2i\pi h)$. Note that

$$\mathfrak{E} : \hat{\mathcal{O}}_{\geqslant 0,\kappa} \to \mathcal{H}_{h,H}$$

is Suzuki's functor. The aim of this section is to prove the following theorem.

A.5.1. Theorem. *Assume that $\kappa \notin \mathbb{Q}_{\geqslant 0}$, $h = 1/\kappa$ and $n \leqslant m$. Assume also that $(q+1)(q^2+q+1) \neq 0$. The functor $\mathfrak{E}$ restricts to an equivalence of quasi-hereditary categories $\mathcal{A}_{n,\geqslant 0,\kappa} \to \mathcal{H}_{h,H}$ which takes $\Delta_{\lambda,\geqslant 0,\kappa}$ to $\Delta_{\lambda,h,H}$ for each $\lambda \in \mathcal{P}_n$.*

To prove Theorem A.5.1 we need some material. Let $\mathbf{A}$ be a finite dimensional $\mathbb{C}$-algebra with 1. Let $(\mathcal{A}, \Delta_{\mathcal{A}}, F)$, $(\mathcal{B}, \Delta_{\mathcal{B}}, G)$ be $\mathbb{C}$-linear 1-faithful covers of $\mathbf{A}$. See [22, Def. 4.37] for the terminology. Hence

$$F : \mathcal{A} \to \mathbf{A}\text{-}\mathbf{mod}^{fg}, \qquad G : \mathcal{B} \to \mathbf{A}\text{-}\mathbf{mod}^{fg}$$

are functors which restrict to equivalences of exact categories

$$\mathcal{A}^\Delta \to (\mathbf{A}\text{-}\mathbf{mod})^{F(\Delta_{\mathcal{A}})}, \qquad \mathcal{B}^\Delta \to (\mathbf{A}\text{-}\mathbf{mod})^{G(\Delta_{\mathcal{B}})}$$

by [22, Prop. 4.41(2)]. The following lemma follows easily from [22].

A.5.2. Lemma. *Assume there is a functor $\phi : \mathcal{A} \to \mathcal{B}$ such that $F = G \circ \phi$ and $\phi(\Delta_{\mathcal{A}}) = \Delta_{\mathcal{B}}$. Then ϕ yields an equivalence of exact categories $\mathcal{A}^\Delta \to \mathcal{B}^\Delta$. In particular ϕ is fully faithful on $\mathcal{A}^{\mathrm{proj}}$ and it takes a projective generator of $\mathcal{A}$ to a projective generator of $\mathcal{B}$.*

We'll also use the following lemma.

A.5.3. Lemma. *Let $\phi : \mathcal{A} \to \mathcal{B}$ be a functor of Artinian Abelian categories and P be a projective generator of $\mathcal{A}$. Set $Q = \phi(P)$. Assume that*

(a) *the module Q is a projective generator of $\mathcal{B}$,*
(b) *the functor ϕ is fully faithful on $\mathcal{A}^{\mathrm{proj}}$,*
(c) *the category $\mathcal{A}$ has finite projective dimension,*
(d) *the functor ϕ is right exact.*

Then ϕ is an equivalence of categories.

Proof. Note that (a), (b) imply that the categories $\mathcal{A}$, $\mathcal{B}$ are equivalent. We must prove that the functor ϕ is an equivalence. Set

$$\mathbf{A} = (\mathrm{End}_{\mathcal{A}}\, P)^{\mathrm{op}}, \qquad \mathbf{B} = (\mathrm{End}_{\mathcal{B}}\, Q)^{\mathrm{op}}.$$

The functor

$$\mathrm{Hom}_{\mathcal{B}}(Q, -) : \mathcal{B} \to \mathbf{B}\text{-}\mathbf{mod}^{fg}$$

is an equivalence by (a). Thus it is enough to check that the functor

$$G = \mathrm{Hom}_{\mathcal{B}}\big(Q, \phi(-)\big) : \mathcal{A} \to \mathbf{B}\text{-}\mathbf{mod}^{fg}$$

is an equivalence. Note that ϕ yields a ring homomorphism $\mathbf{A} \to \mathbf{B}$. So we have the functor

$$F = \mathbf{B} \otimes_{\mathbf{A}} \mathrm{Hom}_{\mathcal{A}}(P, -) : \mathcal{A} \to \mathbf{B}\text{-}\mathbf{mod}^{fg},$$

and the morphism of functors

$$\Phi : F \to G, \qquad r \otimes f \mapsto r\phi(f).$$

The ring homomorphism $\phi : \mathbf{A} \to \mathbf{B}$ is invertible by part (b). The functor

$$\mathrm{Hom}_{\mathcal{A}}(P, -) : \mathcal{A} \to \mathbf{A}\text{-}\mathbf{mod}^{fg}$$

is an equivalence. Thus F is also an equivalence. Therefore it is enough to prove that Φ is an isomorphism of functors.

First, assume that $M = P$. Then $F(M) = G(M) = \mathbf{B}$ and $\Phi(M)$ is the identity of $\mathbf{B}$. Next, let M be a projective object of $\mathcal{A}$. Then the morphism in $\mathbf{B}\text{-}\mathbf{mod}^{fg}$

$$\Phi(M) : F(M) \to G(M)$$

is invertible. Indeed, we may assume that M is indecomposable. Then M is a direct summand of P. So $\Phi(M)$ is the identity of the $\mathbf{B}$-module $\mathbf{B}\phi(a)$ for some idempotent $a \in \mathbf{A}$. Finally, let M be any object of $\mathcal{A}$. By (c) the projective dimension of M is $e < \infty$. Fix an exact sequence

$$0 \to M_2 \to M_1 \to M \to 0$$

with $M_1 \in \mathcal{A}^{\mathrm{proj}}$ and $M_2 \in \mathcal{A}$ of projective dimension $< e$. Consider the diagram

$$\begin{array}{ccccccc} F(M_2) & \to & F(M_1) & \to & F(M) & \to & 0 \\ \downarrow & & \downarrow & & \downarrow & & \\ G(M_2) & \to & G(M_1) & \to & G(M) & \to & 0. \end{array}$$

Here the vertical maps are $\Phi(M_2)$, $\Phi(M_1)$ and $\Phi(M)$. The functor G is right exact by (d). Thus both rows are exact. We may assume that $\Phi(M_2)$, $\Phi(M_1)$ are both invertible by induction on e. Thus $\Phi(M)$ is also invertible by the five lemma. We are done. □

A.5.4. Proof of Theorem A.5.1. First, observe that $\mathfrak{E}(\Delta_{\lambda,\geqslant 0,\kappa}) = \Delta_{\lambda,h,H}$ for each $\lambda \in \mathcal{P}_n$ by Proposition 6.2(a). Thus it is enough to check that $\mathfrak{E}$ is an equivalence of categories $\mathcal{A}^{fg}_{n,\geqslant 0,\kappa} \to \mathcal{H}^{fg}_{h,H}$. Set

$$\mathcal{A} = \mathcal{A}^{fg}_{n,\geqslant 0,\kappa}, \qquad \mathcal{B} = \mathcal{H}^{fg}_{h,H}, \qquad \mathbf{A} = \mathbf{A}_{q,Q}, \qquad \phi = \mathfrak{E}.$$

The hypotheses (c), (d) in Lemma A.5.3 are obviously true. By [22, Thm. 5.3] the functor

$$G = \mathrm{KZ} \circ \heartsuit : \mathcal{B} \to \mathbf{A}\text{-}\mathbf{mod}^{fg}$$

is a 1-faithful cover. We claim that the functor

$$F = G \circ \phi : \mathcal{A} \to \mathbf{A}\text{-}\mathbf{mod}^{fg}$$

is also a 1-faithful cover. Therefore ϕ satisfies also the hypotheses (a), (b) in Lemma A.5.3, by Lemma A.5.2. Hence ϕ is an equivalence of categories. Now we prove the claim. Write

$$\mathbf{V}_{n,\geqslant 0,\kappa} = (\mathbf{V}_\kappa)^{\dot{\otimes} n}, \qquad \mathbf{A}_{n,\geqslant 0,\kappa} = \mathrm{End}_{\hat{\mathbf{g}}}(\mathbf{V}_{n,\geqslant 0,\kappa}).$$

Proposition 7.6 gives an algebra homomorphism

$$\mathbf{A} \to \mathbf{A}_{n,\kappa,\geqslant 0} \tag{A.8}$$

such that

$$F = \mathrm{Hom}_{\hat{\mathbf{g}}}(\mathbf{V}_{n,\kappa,\geqslant 0}, -),$$

up to a twist by some duality functor that we omit to simplify. Let $U_q(\mathfrak{g})$ be the quantized enveloping algebra of $\mathfrak{g}$ with the parameter q and let $\mathbf{V}_q$ be its vectorial representation. Under the Kazhdan–Lusztig tensor equivalence [18, Thm. IV.38.1] the category $\hat{\mathcal{O}}^{fg}_{\geqslant 0,\kappa}$ is equivalent to the category of finite dimensional $U_q(\mathfrak{g})$-modules. Therefore the ring homomorphism (A.8) is invertible, because it is taken to the isomorphism

$$\mathbf{A} \to \mathrm{End}_{U_q(\mathfrak{g})}\big(\mathbf{V}_q^{\otimes n}\big)$$

given by the Schur–Weyl duality. Thus the functor F is taken to the Schur functor

$$M \mapsto \mathrm{Hom}_{U_q(\mathfrak{g})}\big(\mathbf{V}_q^{\otimes n}, M\big).$$

It is well known that the Schur functor is a 1-faithful cover, see [22, Rem. 6.7] and the reference there for instance. Hence F is also a 1-faithful cover.

A.5.5. Remark. The idea to use the Kazhdan–Lusztig equivalence to prove Theorem A.5.1 is not new. However we have not found any proof of Theorem A.5.1 in the literature.

A.6. Proof of Proposition 8.7. By [3, Thm 3.5(a)] it is enough to prove the following.

A.6.1. Proposition. *If $\mu+\pi \trianglelefteq \lambda+\pi$, $\lambda\in\mathcal{P}_{n,s}$ and $\mu\in\mathbb{Z}^s_{\geqslant 0}$, then $\mu\in\mathcal{P}_{n,s}$.*

Proof. Recall that $\lambda \triangleleft \mu$ iff there are $\mu_1,\mu_2,\ldots,\mu_r\in\mathbb{Z}^s_{\geqslant 0}$ such that

$$\tilde\lambda=\tilde\mu_1<\tilde\mu_2<\tilde\mu_3<\cdots<\tilde\mu_r=\tilde\mu$$

and such that $\tilde\mu_{i+1}=w_i s_{\alpha_i}\bullet\tilde\mu_i$ for some $\alpha_i\in\hat\Pi_{\mathrm{re}}\setminus\Pi_s$ and some $w_i\in\mathfrak{S}_s$. Now, assume that $\mu+\pi\trianglelefteq\lambda+\pi$, $\lambda\in\mathcal{P}_{n,s}$, and $\mu\in\mathbb{Z}^s_{\geqslant 0}$. By an easy induction we may assume that

$$\tilde\mu+\pi=ws_\alpha\bullet(\tilde\lambda+\pi),\qquad \langle\tilde\lambda+\hat\rho+\pi:\alpha\rangle\in\mathbb{Z}_{>0},\quad \alpha\in\hat\Pi^+_{\mathrm{re}}\setminus\Pi_s,\ w\in\mathfrak{S}_s.$$

So we have $|\mu|=n$, and we must prove that $\mu_1,\mu_2,\ldots,\mu_m$ are $\geqslant 0$. There is a unique map

$$\mathbb{C}^m\to\mathbb{C}^{\mathbb{Z}},\qquad \lambda\mapsto\bar\lambda$$

such that $\bar\lambda_1,\bar\lambda_2,\ldots,\bar\lambda_m$ are the entries of $\lambda+\pi+\rho$ and $\bar\lambda_{j+m}=\bar\lambda_j-\kappa$ for all $j\in\mathbb{Z}$. Under this map the dot action of the affine reflection s_α is taken to the linear operator which switches the $(a+km)$-th and the $(b+km)$-th entries of any sequence for each $k\in\mathbb{Z}$ and some fixed integers $a\neq b$. We have $\bar\lambda_j\geqslant\bar 0_j$ for all $j\in\mathbb{Z}$. We must check that the same holds for the entries of $\bar\mu$.

Recall the partition $J=\bigsqcup_{p\in\Lambda}J_{s,p}$ with $J_{s,p}=[i_p,j_p]$. Since $\mu\in\mathbb{Z}^s_{\geqslant 0}$ it is enough to prove that we have

$$\bar\mu_{j_1}\geqslant\bar 0_{j_1},\quad \bar\mu_{j_2}\geqslant\bar 0_{j_2},\quad \ldots,\quad \bar\mu_{j_\ell}\geqslant\bar 0_{j_\ell}.$$

The ℓ-tuples (i_p), (j_p) can be regarded as sequences of integers such that

$$i_{p+\ell}=i_p+m,\qquad j_{p+\ell}=j_p+m,\quad \forall p\in\mathbb{Z}.$$

For all $p\in\mathbb{Z}$ we set also $J_{s,p+\ell}=J_{s,p}+m$. Now fix p,q such that

$$a\in J_{s,p},\qquad b\in J_{s,q}.$$

It is enough to prove that $\bar\mu_{j_p}\geqslant\bar 0_{j_p}$ and that $\bar\mu_{j_q}\geqslant\bar 0_{j_q}$. Assume that $b>a$. Then $q>p$ because $\alpha\notin\Pi_s$. Since $\bar\lambda_a>\bar\lambda_b$ we have $\bar 0_{j_p}-\bar 0_{j_q}\in\mathbb{Z}$. Since $\kappa\notin\mathbb{R}_{\geqslant 0}$ we have $\bar 0_{j_q}\geqslant\bar 0_{j_p}$. Note that

$$\{\bar\mu_i;\ i\in J_{s,p}\}=\{\bar\lambda_i;\ i\in J_{s,p}\}\setminus\{\bar\lambda_a\}\cup\{\bar\lambda_b\},$$
$$\{\bar\mu_i;\ i\in J_{s,q}\}=\{\bar\lambda_i;\ i\in J_{s,q}\}\setminus\{\bar\lambda_b\}\cup\{\bar\lambda_a\}.$$

Therefore we have

$$\bar{\mu}_{j_p} = \inf\{\bar{\mu}_i;\ i \in J_{s,p}\} \geqslant \inf\bigl(\{\bar{\lambda}_i;\ i \in J_{s,p}\} \cup \{\bar{\lambda}_b\}\bigr) \geqslant \inf\{\bar{0}_{j_p}, \bar{0}_{j_q}\} \geqslant \bar{0}_{j_p},$$

$$\bar{\mu}_{j_q} = \inf\{\bar{\mu}_i;\ i \in J_{s,q}\} \geqslant \inf\bigl(\{\bar{\lambda}_i;\ i \in J_{s,q}\} \cup \{\bar{\lambda}_a\}\bigr) \geqslant \inf\{\bar{\lambda}_i;\ i \in J_{s,q}\} \geqslant \bar{0}_{j_q}. \qquad \square$$

Index of notation

5.7: $\mathfrak{A}$,
5.8: $P^{\lambda,-1}_{v,w}$,
6.1: $\mathfrak{E}$, $\tilde{\lambda}_\pi$, $\Delta_{\lambda,\nu,\kappa}$, $S_{\lambda,\nu,\kappa}$, ${}^B P_{\lambda,\nu,\kappa}$,
7.1: $(\hat{\mathcal{O}}^{fg}_{\geqslant 0,\kappa}, \dot{\otimes}, a, M(c\omega_0))$,
7.4: $\mathbf{A}_{q,Q}$, KZ, $\heartsuit$, $\mathfrak{E}_{\mathrm{KZ}}$,
7.6: $\mathbf{V}_{n,\nu,\kappa}$, $\mathbf{A}_{n,\nu,\kappa}$, $\mathfrak{F}$,
8.1: $\mathcal{A}_{s,-e}$,
8.3: $\mathbf{S}_{q,Q}$, $\Delta_{\lambda,q,Q}$, $S_{\lambda,q,Q}$,
8.5: $\mathcal{A}_{n,s,-e}$,
8.9: $\mathcal{A}_{n,\nu,\kappa}$,
A.2: $\dot{\otimes}$, $\mathcal{C}$,
A.3: $\Gamma^{\mathfrak{a}}_{\mathfrak{b}}$,
A.4: $Fock_{s^\circ,e}$, $\mathcal{G}(\lambda, s^\circ, e)^\pm$, $|\lambda, s^\circ, e\rangle$, $\nabla^\pm_{\lambda,\mu,s^\circ,e}$, $\Delta^\pm_{\lambda,\mu,s^\circ,e}$, Λ^m, Λ^ν, $\lambda \xrightarrow{b} \mu$,
A.5: $\Delta_{\lambda,\geqslant 0,\kappa}$, $S_{\lambda,\geqslant 0,\kappa}$, $\mathcal{A}_{n,\geqslant 0,\kappa}$.

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