Some Sufficient Conditions for the Division Property of Invariant Subspaces in Weighted Bergman Spaces

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A weighted Bergman space $B$ is a Banach space of the form $L^p(\mu) \cap \text{Hol}(\Omega)$, where $\mu$ is a Borel measure carried by the bounded region $\Omega$ in the complex plane. We consider closed subspaces $\mathcal{M}$ of $B$ that are invariant for multiplication by the independent variable $z$. We say $\mathcal{M}$ has the division property, if $\dim \mathcal{M}((z-\lambda) \cdot \mathcal{M}) = 1$ for each $\lambda \in \Omega$. In terms of the local boundary behavior of the functions in $\mathcal{M}$ we give several conditions which imply the division property. For example, this happens if $\mathcal{M}$ is generated by functions that extend analytically near a fixed boundary point and if $\Omega$ is nice near this point. "Analytic" may be replaced by "locally Nevanlinna." For the standard weights $(1-|z|)^\alpha \ \text{d}A$ on the unit disc we show that $\mathcal{M}$ has the division property if it contains one function that is locally Nevanlinna near a boundary point. Furthermore, in the unweighted case ($\alpha = 0$) the invariant subspace generated by two functions that are $L^r$ respectively $L^s$ near some boundary point, has the division property.

1. INTRODUCTION

Let $\Omega$ be a bounded region in the complex plane $\mathbb{C}$, and let $\mathcal{B}$ be a non-trivial Banach space consisting entirely of analytic functions on $\Omega$ such that the point evaluations $f \to f(\lambda)$ are continuous for each $\lambda \in \Omega$ and such that $\mathcal{B}$ is invariant under multiplication by the identity function $z$, i.e. $zf \in \mathcal{B}$, whenever $f \in \mathcal{B}$. It follows from a standard application of the closed graph theorem that the linear transformation $f \to zf$ defines a bounded linear

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operator on $\mathcal{B}$. We shall denote it by $M_\lambda|\mathcal{B}$ or simply by $M_\lambda$. An invariant subspace $\mathcal{M}$ of $M_\lambda|\mathcal{B}$ is a closed subspace of $\mathcal{B}$ such that $z.\mathcal{M} \subseteq \mathcal{M}$.

**Definition 1.1.** Let $\mathcal{M}$ be an invariant subspace of $M_\lambda|\mathcal{B}$, and let $Z(\mathcal{M}) \subseteq \Omega$ denote the set of common zeros of the functions in $\mathcal{M}$. We say that $\mathcal{M}$ has the division property, if $f(z)/(z-\lambda)$ defines a function in $\mathcal{M}$ whenever $\lambda \in \Omega \setminus Z(\mathcal{M})$ and $f \in \mathcal{M}$ with $f(\lambda) = 0$.

Note that under some additional hypothesis on $\mathcal{B}$ this property is equivalent to the property that in previous work has been called the codimension one property [11] or such subspaces have been said to have index one [8]. In fact, it follows from Lemma 2.1 of [11] that an invariant subspace $\mathcal{M}$ has the division property, if and only if for each $\lambda \in \Omega \setminus Z(\mathcal{M})$ the operator $M_\lambda|\mathcal{M}$ is bounded below and $\dim \mathcal{M}/(z-\lambda) \mathcal{M} = 1$. Thus, if one assumes that $M_{z-\lambda}$ is bounded below as an operator on $\mathcal{B}$, then the theory of semi-Fredholm operators implies that $\dim \mathcal{M}/(z-\lambda) \mathcal{M} = 1$ for all $\lambda \in \Omega \setminus Z(\mathcal{M})$, if and only if $\dim \mathcal{M}/(z-\lambda_0) \mathcal{M} = 1$ for a single $\lambda_0 \in \Omega \setminus Z(\mathcal{M})$ or if $\mathcal{M} = \{0\}$.

Now assume that $(z-\lambda) \mathcal{B}$ is closed in $\mathcal{B}$ for each $\lambda \in \Omega$. Since $M_{z-\lambda}$ is always 1–1, it follows that $M_{z-\lambda}$ is bounded below on $\mathcal{B}$. Hence the remarks of the previous paragraph imply that it suffices to check the division property at a single point $\lambda_0 \in \Omega$. We note that an argument for this fact not mentioning the Fredholm theory is given in the proof of Proposition 5.1 of [1].

It is always easy to construct invariant subspaces with the division property. For example, if $f \in \mathcal{B}$ and if $[f]$ denotes the smallest invariant subspace of $\mathcal{B}$ that contains $f$, then $[f]$ is called the cyclic invariant subspace generated by $f$ and it has the division property. In some cases, like in the Hardy spaces and certain weighted Dirichlet spaces of the unit disc, all invariant subspaces are of this form, so in these cases all invariant subspaces have the division property. For the unweighted Bergman space $L^2_a(D)$ it follows from the results of [2] that the cyclic invariant subspaces are exactly the invariant subspaces with the division property, but it is also known that $L^2_a$ contains noncyclic invariant subspaces (see [3]). For $p \neq 2$ it is an open problem to decide, whether every invariant subspace of the Bergman space $L^p_a(D)$ with the division property is cyclic.

It is easy to construct an “artificial” example of a space $\mathcal{B}$ that does not have the division property.

**Example 1.2.** Let $\varphi$ be an analytic function in the open unit disc $\mathbb{D}$ that is not in the Nevanlinna class of $\mathbb{D}$, let $H^2$ denote the usual Hardy space of the disc, and set $\mathcal{B} = \{f + g\varphi : f, g \in H^2\}$, $\|f + g\varphi\|_\mathcal{B}^2 = \|f\|_{H^2}^2 + \|g\|_{H^2}^2$. One checks that $\mathcal{B}$ is a Banach space satisfying all of our
hypotheses, but that $\dim \mathcal{H}/z = 2$, so $\mathcal{H}$ does not have the division property.

Furthermore, if $\mathcal{H}$ is a closed subspace of some $L^p_\mu(D)$, $1 \leq p < \infty$, then $\mathcal{H}$ may have the division property, but it will have an invariant subspace without the division property. For $p = 2$, this follows from the results of [3] and for general $p$ from [6]. Such results are also available for other regions [4]. These results are abstract existence proofs, for more concrete constructions see [7] and [8].

The results in this paper will shed some more light on the reasons why examples of invariant subspaces without the division property are difficult to construct. In order to be able to state two of our theorems we need some notation. For $1 \leq p < \infty$ and $\alpha > -1$ the weighted Bergman spaces $A^\alpha_p$ are the Banach spaces of analytic functions $f$ in the open unit disc $D$ such that

$$\|f\|_{A^\alpha_p}^p = \int_D |f|^p (1-|z|)^\alpha \, dA < \infty,$$

where $dA$ denotes 2-dimensional Lebesgue measure. Thus, when $\alpha = 0$ we obtain the unweighted Bergman spaces $L^p_\mu(D) = A^0_p$. For a region $G \subseteq \mathbb{C}$ we denote by $N(G)$ the Nevanlinna class of $G$, i.e. the set of meromorphic functions $h$ on $G$ with the property that $\log |h|$ has a harmonic majorant in $G$. Furthermore, an analytic function $f$ on $D$ will be called locally Nevanlinna at $z_0 \in D$, if there is an $\varepsilon > 0$ such that $f$ is in the Nevanlinna class of the region $D \cap \{z : |z - z_0| < \varepsilon\}$.

Among other things we shall prove the following two theorems:

**THEOREM 3.2.** Let $1 \leq p < \infty$ and $\alpha > -1$. If an invariant subspace $\mathcal{M}$ of $A^\alpha_p$ contains a nonzero function that is locally Nevanlinna at some boundary point of $D$, then $\mathcal{M}$ has the division property.

**THEOREM 4.1.** Let $1 \leq p < \infty$ and let $f$ and $g$ be two nonzero functions in $L^p_\mu(D)$. If there exist $\lambda_0 \in \partial D$, a neighborhood $V$ of $\lambda_0$ such that $f \in L^p(V \cap D, d\mu)$ and $g \in L^p(V \cap D, d\mu)$, where $1/s + 1/s' = 1/p$, then $[f] \vee [g]$ has the division property. Here $[f] \vee [g]$ denotes the closed linear span of $[f]$ and $[g]$ in $L^p_\mu(D)$.

An easy inspection of the proof of Hedenmalm's example [7] together with Seip's characterization of sampling and interpolating sequences for the Bergman $L^p_\mu$-spaces [12] shows that for any $\varepsilon > 0$ there exist functions $f, g \in L^p_\mu$ such that the closed linear span of $[f]$ and $[g]$ in $L^p_\mu$ does not have the division property. Thus, Theorem 4.1 is sharp in a certain sense.

Note that the idea for considering local conditions on the boundary came from [16], where a related result is proved, also see [15].
We shall now try to give the reader some more intuition and motivation for our approach to the problem. First, note that every invariant subspace \( \mathcal{M} \) is the closed linear span of the cyclic invariant subspaces \([f], f \in \mathcal{M}\), which have the division property. It follows from [11, Thm 3.13 (b)] that if \( \mathcal{M} \) does not have the division property, then there must be functions \( f, g \in \mathcal{M} \) such that \([f] \cup [g]\) does not have the division property. Thus, it becomes important to be able to decide when \([f] \cup [g]\) has the division property for arbitrary \( f, g \in \mathcal{M}\).

Now fix \( f, g \in \mathcal{M} \). By use of an approximation argument one can easily see that it is enough to check the division property on the dense set \( \{ pf + qg : p, q \text{ polynomials} \} \) (see [11], Theorem 3.2). Thus, let \( \lambda \in \mathbb{Q} \setminus (\mathbb{Z}(f) \cup \mathbb{Z}(g)) \) and let \( p \) and \( q \) be polynomials such that \((pf + qg)(\lambda) = 0\). We would like to know whether or not \((pf + qg)/(z - \lambda)\) is in \([f] \cup [g]\). Note that \( q(\lambda) = -p(\lambda)f(g)(\lambda) \), hence

\[
\frac{pf + qg}{z - \lambda} = \frac{p - p(\lambda)}{z - \lambda} f + \frac{q - q(\lambda)}{z - \lambda} g + p(\lambda) \frac{f - f(g)(\lambda) g}{z - \lambda}.
\]

This implies that \([f] \cup [g]\) has the division property, if and only if \((f - f(g)(\lambda) g)/(z - \lambda) \in [f] \cup [g]\) for some (and hence for all) \( \lambda \in \mathbb{Q} \setminus (\mathbb{Z}(f) \cup \mathbb{Z}(g)) \). In what follows we shall use this observation without further reference.

Suppose now that we intend to show that under some hypothesis on \( f \) and \( g \), the subspace \([f] \cup [g]\) has the division property, and suppose for a moment that \( \mathcal{M} \) itself has the division property. Let \( \phi \) be a linear functional on \( \mathcal{M} \) that annihilates \([f] \cup [g]\), then we must show that the analytic function

\[
H(\lambda) = \phi \left( \frac{f - f(g)(\lambda) g}{z - \lambda} \right)
\]

vanishes identically on \( \mathbb{Q} \setminus (\mathbb{Z}(f) \cup \mathbb{Z}(g)) \). For sufficiently large values of \(|\lambda|\) we have \( f(z - \lambda) \in [f] \), so \( \phi(f(z - \lambda)) = 0 \). Similarly, \( \phi(g(z - \lambda)) = 0 \). So, if there were a way to extend the definition of \( H \) into the exterior of \( \mathbb{Q} \), then \( H \) should be zero there. Thus, the conditions on \( f \) and \( g \) that we will investigate in this paper are such that they imply an analytic continuation of \( H \) or some related function across \( \partial \Omega \).

2. INVARIANT SUBSPACES OF \( L^p(\mu) \)

In this Section we will give some results that are valid for a wide range of Banach spaces of analytic functions.
Let $\Omega$ be a bounded region in the complex plane $\mathbb{C}$, and let $\mathcal{B}$ be a non-trivial Banach space consisting entirely of analytic functions on $\Omega$ such that:

There exists a finite Borel measure $\mu$ that is concentrated on $\Omega$, and a number $p$, $1 \leq p < \infty$ such that $\mathcal{B}$ is a closed subspace of $L^p(\mu)$.

For every $\lambda \in \Omega$ the rule $f \mapsto f(\lambda)$ defines a bounded linear functional on $\mathcal{B}$.

$zf \in \mathcal{B}$ whenever $f \in \mathcal{B}$, where $z$ is the identity on $\Omega$, and for all $\lambda \in \Omega$ $(z-\lambda)\mathcal{B}$ is closed in $\mathcal{B}$.

It follows from these axioms and the closed graph theorem that the linear transformation $f \mapsto zf$ defines a bounded linear operator on $\mathcal{B}$. We shall denote it by $M_z|\mathcal{B}$ or simply by $M_z$.

We begin with a general result that has a number of interesting consequences.

**Theorem 2.1.** Let $\rho(M_z|\mathcal{B})$ denote the unbounded component of the resolvent of $M_z|\mathcal{B}$, and let $f$, $g \in \mathcal{B}$. If there exists a $\lambda_0 \in \Omega \cap \rho(M_z|\mathcal{B})$ such that $fg$ extends analytically in a neighborhood of $\lambda_0$, then $[f] \cap [g]$ has the division property.

Notice that we did not assume that $\mathcal{B}$ itself has the division property. However, also note that the functions $\varphi$ and $1$ in Example 1.2 may have an analytic quotient near some boundary points, yet $[1] \cap [\varphi] = \mathcal{B}$ does not have the division property. Thus, the space $\mathcal{B}$ in Example 1.2 is not a subspace of any $L^p(\mu)$.

We also would like to point out that one obtains a theorem like Theorem 2.1 for boundary points of other components of the resolvent of $M_z|\mathcal{B}$, if the invariant subspace under consideration is invariant under multiplication by rational functions with poles in the corresponding components of $\rho(M_z|\mathcal{B})$. We leave the details to the reader.

**Proof.** Let $h$ be a function in $L^q(\mu)$ that annihilates $[f] \cap [g]$. Let $\lambda_0$ be as in the statement and $V$ be an open neighborhood of $\lambda_0$ such that $fg$ extends analytically to $V$. Consider the function $H$ defined on $V \cup \{\lambda \in \Omega, g(\lambda) \neq 0\}$ by

$$H(\lambda) = \frac{f - (fg)(\lambda)}{z - \lambda} g \, d\mu.$$  

Note that for every $\lambda \in \rho(M_z|\mathcal{B})$ we have $(M_z - \lambda)^{-1}([f] \cap [g]) \subseteq [f] \cap [g]$, which shows that $H$ vanishes on $V \cap \rho(M_z|\mathcal{B})$. By what was
said in the Introduction and the Hahn–Banach theorem, it will be sufficient to prove that \( H = 0 \). This will follow once we show that \( H \) is analytic in \( V \).

Let \( u \) be any \( C^\infty \)-function supported on a compact subset of \( V \cup \{ \lambda \in \Omega, g(\lambda) \neq 0 \} \) and \( \partial u = \partial u/\partial \bar{z} \). Then

\[
    u(z) = \frac{1}{\pi} \int \frac{\partial u(\lambda)}{z - \lambda} dA(\lambda)
\]

hence from (2.4) and Fubini’s theorem we obtain

\[
    \int \partial u(\lambda) H(\lambda) dA(\lambda)
    = \int f \partial h \frac{\partial u(\lambda)}{z - \lambda} dA(\lambda) d\mu - \int g \partial h \frac{\partial u(\lambda)}{z - \lambda} dA(\lambda) d\mu
    = \pi \int u f \partial h d\mu - \pi \int g \partial h d\mu = 0.
\]

Thus, by Weyl’s lemma \( H \) is analytic in the open connected set \( V \cup \{ \lambda \in \Omega, g(\lambda) \neq 0 \} \) and the proof is complete. \qed

The condition on \( f/g \) in Theorem 2.1 can be weakened if \( \partial \Omega \) is “nice” near \( \lambda_0 \). A result of this type is given below.

**Corollary 2.2.** Let \( \Gamma \) be a nonvoid open arc contained in the unit circle \( \mathbb{T} \). Suppose there exists \( \varepsilon > 0 \) with

\[
    G = \{ \zeta_0, \zeta \in \Gamma, 1 - \varepsilon < \eta < 1 \} \subseteq \Omega,
\]

and

\[
    G^* = \{ \zeta_0, \zeta \in \Gamma, 1 < \eta < 1 + \varepsilon \} \subseteq \rho_d(M_z) \setminus \overline{B}.
\]

If \( f, g \in \mathbb{B} \) are such that \( f/g \in N(G) \), then \( [f] \cup [g] \) has the division property.

**Proof.** We are going to construct bounded analytic functions \( h_1, h_2 \neq 0 \) in \( \mathbb{C} \setminus \overline{G^*} \) such that \( h_1 f/h_2 g \) extends analytically across some open arc contained in \( \Gamma \). Since \( h_1, h_2 \) can be approximated boundedly pointwise in \( \mathbb{C} \setminus G^* \) by rational functions with poles in \( G^* \), it follows immediately by the dominated convergence theorem that \( h_1 f \in [f], h_2 g \in [g] \) and by Theorem 2.1 \( [h_1 f] \cup [h_2 g] \) has the division property. By [11], Theorem 3.13 this implies that \( [f] \cup [g] \) has the division property. A way to construct the functions \( h_1, h_2 \) is described below. Let \( f/g = b_1 s_1 F/b_2 s_2 \) be the canonical factorization of \( f/g \) in \( N(G) \), with \( b_1, b_2 \) Blaschke products, \( s_1, s_2 \) singular inner functions and \( F \) outer in the simply connected domain \( G \). Let
Let \( \Gamma \) be an open nonvoid arc on \( \mathbb{T} \) with \( \Gamma' \subset \overline{\mathbb{T}} \subset \Gamma \). Since the conformal mapping \( \phi \) from \( G \) onto \( \mathbb{D} \) has an analytic (and injective) extension across \( \Gamma \) there is a neighborhood \( V \) of \( \Gamma' \) such that the zeros of \( h_1 \) and \( h_2 \) that lie in \( V \) satisfy the Blaschke condition in \( \mathbb{D} \). If \( B_1, B_2 \) are the corresponding Blaschke products in \( \mathbb{D} \) then both are analytic and bounded in \( \mathbb{C} \setminus \overline{\mathbb{G}} \) and \( b_1/B_1, B_2/b_2 \) extend analytically across \( \Gamma' \). Further, the functions \( s_k, k = 1, 2 \) can be written as

\[
s_k(\zeta) = \exp \int_{\phi(\zeta)}^{\zeta} \phi^{-1}(e^{it}) d\theta_k(\theta),
\]

with \( \nu_k, k = 1, 2 \) singular measures on \( \mathbb{T} \). Using again the properties of \( \phi \) it follows by simple computations that the singular inner functions \( S_1, S_2 \) defined in \( \mathbb{D} \) have the property that \( S_1, S_2 \) are analytic and bounded in \( \mathbb{C} \setminus \overline{\mathbb{G}} \) and that \( s_k/S_1, s_k/S_2 \) extend analytically across \( \Gamma' \). Finally, if we define outer functions \( F_1, F_2 \) in \( \mathbb{D} \) by

\[
|F_1| = \min\{ |gf|, 1 \}, \quad |F_2|^{-1} = \max\{ |gf|, 1 \}, \quad \text{ a.e. on } \Gamma',
\]

then \( F_1, F_2 \) are analytic and bounded in \( \mathbb{C} \setminus \overline{\mathbb{G}} \) and \( FF_1/F_2 \) extends analytically across \( \Gamma' \). Thus, the functions \( h_1 = B_2S_2F_1, h_2 = B_1S_1F_2 \) have the desired properties.

Remarks. (1) Corollary 2.2 continues to hold for analytic arcs \( \Gamma \) with appropriate modifications in the definition of \( G \) and \( G^* \).

(2) As pointed out in the Introduction, these results yield sufficient-conditions for \( \mathcal{B} \) to have the division property. If \( \mathcal{B} \) is generated by a set \( \mathcal{S} \) (i.e. \( \mathcal{B} \) is the smallest hyperinvariant subspace containing \( \mathcal{S} \)) such that for any two functions \( f, g \in \mathcal{S} \setminus \{0\} \) we can find a point \( \lambda_0 \in \partial \Omega \) where either the condition in Theorem 2.1, or the condition in Corollary 2.2 is satisfied then \( \mathcal{B} \) has the division property.

The next result is a special case of the last remark. Recall that an analytic function \( f \) in \( \mathbb{D} \) will be called locally Nevanlinna at \( \lambda_0 \in \partial \mathbb{D} \) if there exists \( \epsilon > 0 \) such that \( f \in N(\mathbb{D} \cap \{ |z - \lambda_0| < \epsilon \}) \).

**Corollary 2.3.** Assume that \( \Omega = \mathbb{D} \) and that \( \mathcal{B} \) is generated by a set \( \mathcal{S} \) of functions that are locally Nevanlinna at some fixed point \( \lambda_0 \in \partial \mathbb{D} \). Then \( \mathcal{B} \) has the division property.
3. SOME WEIGHTED BERGMAN SPACES ON THE DISC

For \( p \geq 1 \) and \( x > 1 \) let \( A^p_x \) be the space of analytic functions \( f \) in \( \mathbb{D} \) with the property that

\[
\|f\|_p^p = \int_{\mathbb{D}} |f|^p (1 - |z|)^x \, dA < \infty.
\]

The norm \( \| \cdot \|_p \) makes \( A^p_x \) into a Banach space and every invariant subspace of \( A^p_x \) satisfies the conditions (2.1)–(2.3). It turns out that Corollary 2.3 can be considerably improved for these spaces. Before we get to the main result we recall some facts about duality on \( A^p_x \) (see [5], [10]). With the pairing

\[
\langle f, h \rangle = \lim_{r \to 1^-} \sum_{k=0}^{\infty} a_k \overline{b_k} r^{2k} = \lim_{r \to 1^-} \int_{|z| < 1} f(r\zeta) \overline{h(r\zeta)} \frac{|d\zeta|}{2\pi},
\]

where \( f = \sum a_k z^k \in A^p_x \) and \( h = \sum b_k z^k \), the dual of \( A^p_x \) can be identified with the space \( X^* \), of analytic functions \( h \) in \( \mathbb{D} \) with the property that \( h^{(n+1)}(1 - |z|)^{n-x} \in L^n((1 - |z|)^x \, dA) \), where \( n > x \) is an integer and \( 1/p + 1/q = 1 \). The definition does not depend on the choice of the integer \( n \), in fact, these are well known spaces of functions, i.e. Besov classes for \( p > 1 \) and Lipschitz or Zygmund classes for \( p = 1 \). We will use the following identity that can be derived from (3.1) by a computation with power series

\[
\langle f, h \rangle = c_n \int_{\mathbb{D}} f(z^{n+1}h)^{n+1} (1 - |z|^2)^n \, dA,
\]

where \( n > x \) and \( c_n \) is a positive constant depending only on \( n \). Note that by the definition of \( X^* \), we have \( f(z^{n+1}h)^{n+1} (1 - |z|^2)^n \in L^1(dA) \). The basic additional information that we obtain from the duality described above is contained in the next lemma.

**Lemma 3.1.** Suppose that \( f \in A^p_x \), \( f \neq 0 \), extends analytically in a neighborhood of a point \( \lambda_0 \in \partial \mathbb{D} \). If \( h \in X^* \) annihilates \( [f] \) then \( h \) extends analytically in a neighborhood of \( \lambda_0 \).

**Proof.** We can assume without loss of generality that \( f(\lambda_0) \neq 0 \). Indeed, if \( f(\lambda_0) = 0 \) one shows easily that \( [f(1/\lambda)] = [f] \). For \( \lambda \) near \( \lambda_0 \) we have by (3.1)

\[
0 = \left\langle \frac{1}{1-\lambda z}, f, h \right\rangle = \left\langle \frac{f - f(1/\lambda)}{1 - \lambda z}, h \right\rangle + \left\langle \frac{1}{\lambda z} \overline{h(\lambda)} \right\rangle.
\]
and by (3.2) it follows that
\[
 f\left(\frac{1}{\lambda}\right)h(\lambda) = -\int_{\mathbb{D}} \frac{f - f(1/\lambda)}{1 - \lambda z} \left(\frac{z^n + 1}{z^{n+1}}\right) (1 - |z|^2)^v dA. \tag{3.3}
\]

If \( f \) extends analytically in a neighborhood \( V \) of \( \lambda_0 \) and has no zeros there then the function
\[
 F(\lambda) = \frac{1}{f(1/\lambda)} \int_{\mathbb{D}} \frac{f - f(1/\lambda)}{1 - \lambda z} \left(\frac{z^n + 1}{z^{n+1}}\right) (1 - |z|^2)^v dA
\]
is antianalytic in \( V \) and by (3.3) we have \( F = \tilde{h} \) on \( V \cap \mathbb{D} \).

**Theorem 3.2.** If the invariant subspace \( \mathcal{M} \) of \( A^v_\mathbb{D} \) contains a nonzero function that is locally Nevanlinna at a boundary point, then \( \mathcal{M} \) has the division property.

**Proof.** We shall first prove the theorem in the case where \( \mathcal{M} \) contains a nonzero function \( f \) with an analytic continuation across an arc containing the point \( \lambda_0 \in \partial \mathbb{D} \). Let \( h \in X_{\lambda, \rho} \) be a function that annihilates \( \mathcal{M} \) and let \( g \in \mathcal{M} \). For \( n > \nu \) denote by \( h_n = (z^{n+1}h)^{(n+1)} \) and note that by Lemma 3.1 \( h_n \) extends analytically near \( \lambda_0 \). Define the function \( H \) in \( \mathbb{D} \) by
\[
 H(\lambda) = \frac{1}{f(1/\lambda)} \int_{\mathbb{D}} \frac{f - f(1/\lambda)}{1 - \lambda z} \left(\frac{z^n + 1}{z^{n+1}}\right) (1 - |z|^2)^v dA. \tag{3.4}
\]

The result will follow by (3.2) once we show that \( H = 0 \) for arbitrary \( g \in \mathcal{M} \) and \( h \in \mathcal{M} \). To this end, note first that \( H \) is analytic in \( \mathbb{D} \). Further, if \( g \in \mathcal{M} \subseteq A^v_\mathbb{D} \) a direct application of Hölder’s inequality implies that \( |g(\zeta)| = O(1 - |\zeta|)^{-\nu - 1} \) as \( |\zeta| \to 1^- \). Then choose \( n > \nu + 2 \) in (3.4) and use the fact that \( f \) and \( h_n \) are analytic in neighborhood \( V \) of \( \lambda_0 \) to conclude that the function
\[
 H_1(\lambda) = f(\lambda) \int_{\mathbb{D}} \frac{g h_n}{z - \lambda} (1 - |z|^2)^v dA
\]
is continuous in \( \bar{V} \). Also, by assumption \( H_1 \) vanishes in \( V \setminus \mathbb{D} \). Moreover, the fact that \( f \) and \( h_n \) are analytic in \( V \) implies also that the function
\[
 H_2(\lambda) = \int_{\mathbb{D}} \frac{g h_n}{z - \lambda} (1 - |z|^2)^v dA
\]
is \( C^\infty \) in \( V \) and since \( H_2 \) vanishes identically outside \( \mathbb{D} \) we obtain that \( |H_2(\lambda)| = o((1 - |\lambda|)^{\beta}) \) when \( |\lambda| \to 1^- \) for every \( \beta > 0 \). Then the function \( gH_2 \) is continuous in \( V \cap \mathbb{D} \) and vanishes on \( V \cap \partial \mathbb{D} \). Thus, the function
\( H = H_1 - gH_2 \) is analytic in \( V \cap D \), continuous in \( V \cap \partial D \) and vanishes on \( V \cap \partial D \), hence \( H = 0 \).

Now suppose \( \mathcal{H} \) contains an arbitrary nonzero function that is locally Nevanlinna near \( \lambda_0 \in \partial D \). Then we can use the same argument as in the proof of Corollary 2.2 to conclude that \( \mathcal{H} \) contains a nonzero function of the form \( SF \), where \( S \) is an inner function and \( f \) extends analytically across some arc containing \( \lambda_0 \). By Theorem 3.13 of [11] it suffices to show that \( [SF] \vee [g] \) has the division property for every \( g \in \mathcal{H} \).

Let \( g \in \mathcal{H} \) and fix \( \lambda \in D \), such that \( g(\lambda) \neq 0 \) and \( (SF)(\lambda) \neq 0 \). We have to show that \( \{ (SF - (SF/g)(\lambda) g)/(z - \lambda) \} \in \mathcal{H} \). By the first part of the proof \( \{ (f - (f/g)(\lambda) g)/(z - \lambda) \} \in [f] \vee [g] \) hence \( \{ (SF - (f/g)(\lambda) Sg)/(z - \lambda) \} \in [SF] \vee [Sg] \subseteq \mathcal{H} \). Furthermore,

\[
\frac{SF - (SF/g)(\lambda) g}{z - \lambda} = \frac{SF - (f/g)(\lambda) Sg}{z - \lambda} + \frac{f(\lambda)}{g(\lambda)} \frac{S - S(\lambda)}{z - \lambda} g,
\]

and the second summand is in \( \mathcal{H} \), because \( \{(S - S(\lambda))/(z - \lambda)\} \in H^\infty \). 

This result has a nice application to invariant subspaces generated by zero sets in the unweighted Bergman space \( L_p^w = A_p^0 \). For a sequence \( \lambda = (\lambda_n) \) of (not necessarily distinct) points in \( D \) let \( \mathcal{H}(\lambda) = \{ f \in A_p^0, \ f(\lambda_n) = 0, \ n \geq 1 \} \), where as usual, multiplicities are counted, that is, the multiplicity of the zero of \( f \in \mathcal{H}(\lambda) \) at \( \lambda_n \) is equal to the number of occurrences of the value \( \lambda_n \) in \( \lambda \). We call \( \lambda \) a zero sequence for \( A_p^0 \) if \( \mathcal{H}(\lambda) \neq \{0\} \). Recently, Hedenmalm, Richter and Seip [8] have constructed for arbitrary positive integers \( n \), or \( n = \infty \), zero sequences \( \lambda_j \), \( j = 1, \ldots, n \), such that the invariant subspace \( \mathcal{H} = \bigvee^n_{j = 1} \mathcal{H}(\lambda_j) \) satisfies \( \dim \mathcal{H} = n \). Using Theorem 3.2 we can show that such sequences \( \lambda_j \) must have a "high density" near every boundary point.

**Corollary 3.3.** Let \( \lambda = (\lambda_n) \) be a zero sequence for \( A_p^0 \) and suppose that for some \( \lambda_0 \in \partial D \) and \( \varepsilon > 0 \) we have

\[
\sum_{|\lambda_n - \lambda_0| < \varepsilon} (1 - |\lambda_n|) < \infty.
\]

Then every invariant subspace \( \mathcal{H} \) of \( A_p^0 \) that contains \( \mathcal{H}(\lambda) \) has the division property.

**Proof.** The condition in the statement is equivalent to the fact that there exists a Blaschke product \( B \) that vanishes at all points \( \lambda_n \) with \( |\lambda_n - \lambda_0| < \varepsilon \) counting multiplicities. Let \( \mathcal{A}_1 = \{ \lambda_n, \ |\lambda_n - \lambda_0| \geq \varepsilon \} \). It is known (see [9], Theorem 7.9) that \( \mathcal{H}(\lambda_1) \neq \{0\} \). Let \( \lambda \in D \setminus A_1 \) and \( \varphi \in \mathcal{H}(\lambda_1) \) be the reproducing kernel for \( \lambda \) in \( \mathcal{H}(\lambda_1) \), that is, the function
determined uniquely by the relation $\langle f, \varphi \rangle_{A^p_0} = f(\lambda)$, $f \in \mathcal{H}(A_1)$. From the results obtained by Sundberg ([14]) it follows that $\varphi$ extends analytically in a neighborhood of $\lambda_0$ and if $B$ is the Blaschke product defined above, then $M(A)$ contains the locally Nevanlinna function $B\varphi$.

4. THE UNWEIGHTED CASE

In the present section we will concentrate on the spaces $A^p_0$. It turns out that for invariant subspaces of $A^p_0$ one can obtain sufficient conditions that imply the division property by means of growth restrictions. Roughly speaking, the main result shows that the invariant subspace generated by two functions in $A^p_0$ that do not grow too fast near some boundary point has the division property.

**Theorem 4.1.** Let $1 \leq p < \infty$ and let $f, g$ be two nonzero functions in $A^p_0 = L^p_a$. Suppose there exists $\lambda_0 \in \partial \mathbb{D}$ and a neighborhood $V$ of $\lambda_0$ such that $f \in L^s(V \cap \mathbb{D}, dA)$ and $g \in L^{s'}(V \cap \mathbb{D}, dA)$, where $1/s + 1/s' = 1/p$. Then $[f] \cap [g]$ has the division property.

For the proof we need two lemmas concerning Cauchy transforms.

**Lemma 4.2.** Let $f$ and $u$ be compactly supported functions with $f \in L^p(dA)$, $p > 1$, and $u \in C^\infty$. Let $F$ be defined a.e. on $\mathbb{C}$ by

$$F(\lambda) = -\frac{1}{\pi} \int \frac{f(z)}{z - \lambda} dA.$$  \hspace{1cm} (4.1)

Then

(i) For almost every $\lambda \in \mathbb{C}$ we have

$$u(\lambda) F(\lambda) = -\frac{1}{\pi} \int \frac{F \partial u + fu}{z - \lambda} dA.$$

(ii) If $G$ is a simply connected domain bounded by a simple closed $C^2$-curve such that $uF$ vanishes a.e. outside $G$ then there exists a sequence $(v_n)$ of $C^\infty$ functions compactly supported on $G$ such that $\partial v_n \rightarrow F \partial u + fu$ and $v_n \rightarrow uF$ in $L^p(G, dA)$.

**Proof.** (i) follows immediately by Fubini’s theorem and Green’s formula. We have
\[
\frac{1}{\pi} \int \frac{F(w)}{w-\lambda} \tilde{\partial}u(w) \, dA(w)
\]
\[
= -\frac{1}{\pi^2} \int \int f(z) \tilde{\partial}u(w) \left( \frac{1}{w-\lambda} + \frac{1}{z-\omega} \right) \, dA(z) \, dA(w)
\]
\[
= -u(\lambda) F(\lambda) - \frac{1}{\pi} \int \frac{fu(z)}{z-\lambda} \, dA(z).
\]

(ii) is a consequence of the Hahn–Banach theorem. Since the convolution with \(1/z\) is a continuous operation on \(L^p(G, dA)\) it suffices to find a sequence \((v_n)\) that satisfies the first condition. If \(h \in L^q(G, dA)\) annihilates all functions of the form \(\tilde{\partial}v\) with \(v \in C^\infty\) and compactly supported on \(G\) then by Weyl’s lemma \(h\) is analytic in \(G\). Then \(h\) can be approximated in \(L^q(G, dA)\) by rational functions with poles outside \(G\). By (i) the \(L^p\)-function \(F \tilde{\partial}u + fu\) annihilates all these rational functions hence it annihilates also \(h\). Thus, \(F \tilde{\partial}u + fu\) is in the closed span of the functions \(\tilde{\partial}v\) considered above and the result follows.

**Lemma 4.3.** Let \(\Gamma\) be an open arc on \(\mathbb{T}\) and \(G\) be a simply connected domain bounded by a \(C^2\)-curve with \(\Gamma \subset \partial G\) and such that \(\{t\zeta, \zeta \in \Gamma, 1-\varepsilon < t < 1\} \subset G\) for some \(\varepsilon > 0\). Let \(f \in C(G) \cap L^p(G, dA), p > 1\), and \(F\) the function defined by (4.1). If \(F = 0\) a.e. outside \(G\), then
\[
\int_\Gamma |F(t\zeta)|^p |d\zeta| = O((1-t)^{p-1}),
\]
when \(t \to 1^-\).

**Proof.** Let \(U = \{t\zeta, \zeta \in \Gamma, 1-\varepsilon < t < 1\}\). For a \(C^\infty\)-function \(v\) compactly supported on \(G\) and \(t > \frac{1}{2}, 1/p + 1/q = 1\), we have
\[
\int_\Gamma |v(t\zeta)|^p |d\zeta| \leq \int_\Gamma \left( \int_\Gamma \frac{\partial}{\partial r} v(r\zeta) \, dr \right)^p |d\zeta|
\]
\[
\leq (1-t)^{pq} \int_\Gamma \left( \int_\Gamma \frac{\partial}{\partial r} v(r\zeta) \, dr \right)^p |d\zeta|
\]
\[
\leq 2(1-t)^{pq} \int_{U_t} |v| \, dA,
\]
where \(v\) is the gradient of \(v\). By the Calderon–Zygmund estimates ([13], p. 60) the last integral above is dominated by the \(L^q\)-norm of \(\tilde{\partial}v\), hence we obtain the inequality
\[
\int_\Gamma |v(t\zeta)|^p |d\zeta| \leq c_p(1-t)^{pq} \int |\tilde{\partial}v|^p \, dA,
\]
where $c_p$ is a positive constant depending only on $p$. If $f$ is a function with the properties in the statement we choose a compactly supported function $u \in C^\infty$ with $u = 1$ in a neighborhood of $G$ and use the approximation given by Lemma 4.2 (ii) in order to find a sequence $(v_n)$ of $C^\infty$-functions compactly supported on $G$ such that $\tilde{\partial} v_n \to G \tilde{\partial} u + F u = f$ and $v_n \to u F = F$ in $L^p(G, dA)$. Then by (4.2) we obtain for arbitrary $\frac{1}{2} < t < 1$ and $0 < \varepsilon < 1 - t$

$$\int_G |F(r_\varepsilon)|^p \, dA \leq \frac{1}{t^+e} \int_G |f|^p \, dA.$$

Finally, since $f \in C(G)$ we have $F \in C(G)$, and the result follows if we let $\varepsilon \to 0$ in the last inequality.

**Proof of Theorem 4.1.** Recall that the dual of $A_p^0$ can be identified with the Blochspace $B_p$ if $p = 1$, or with $A_q^0$, $1/p + 1/q = 1$ if $p > 1$. So let $h$ a function in the appropriate space that annihilates $[f] \vee [g]$. Consider the analytic function $H$ defined on $D$ by

$$H(\lambda) = \int_D \frac{g(z) - g(\lambda)}{z - \lambda} f(z) \, dA = f(\lambda) H_1(\lambda) - g(\lambda) H_2(\lambda),$$

where $H_1$, $H_2$ are the Cauchy transforms of the measures $g h \, dA$ and $f h \, dA$ respectively. As we did in the proofs of the previous results we will show that $H = 0$. Let $\Gamma$ be a closed arc on $\mathbb{T}$ with endpoints $\zeta_1$, $\zeta_2$ that is contained in $V$. For $\frac{1}{2} < t < 1$, let $C_t$ be the closed curve determined by the arcs $\frac{1}{2} \Gamma$, $t \Gamma$, and the radii $\{r_j \gamma, \frac{1}{2} < r < t\}$, $j = 1, 2$. Then for each point $\lambda$ that is interior to $C_t$ we have by Cauchy’s formula

$$H(\lambda) = \frac{1}{2\pi i} \int_{C_t} \frac{H(\zeta)}{\zeta - \lambda} \, d\zeta. \quad (4.3)$$

We claim that

$$\lim_{t \to 0} \int_{C_t} \frac{H(\zeta)}{\zeta - \lambda} \, d\zeta = 0. \quad (4.4)$$

We shall now show that the claim implies that $H \equiv 0$ in $D$. Since $H$ is analytic in $D$ (4.4) will imply that $H \in L^1(G, dA)$, where $G = \{z \in D : \exists \gamma \in \Gamma \}$. Thus, there are $w_1, w_2 \in I$ such that $H(w_i) \in L^1(\,d\gamma)$, $i = 1, 2$. Without loss of generality we may assume $w_i = \zeta_i$, $i = 1, 2$. Hence it follows from (4.3) and (4.4) that

$$H(\lambda) = \frac{1}{2\pi i} \int_{\zeta_i} \frac{H(\zeta)}{\zeta - \lambda} \, d\zeta,$$
where \( \gamma \) is the curve determined by the arc \( \frac{1}{2} \Gamma \) and the two radii \( \{ r_i \zeta; \frac{1}{2} < r \leq 1 \} \), \( i = 1, 2 \). This means that \( H \) has an analytic continuation across \( \Gamma \) and by Fatou’s Lemma

\[
\int_{\Gamma} |H(\zeta)| \, |d\zeta| \leq \lim_{r \to 1^+} \int_{\Gamma} |H(t \zeta)| \, |d\zeta| = 0,
\]

so \( H = 0 \) on \( \Gamma \) and hence \( H \equiv 0 \) in \( D \).

To see the claim (4.4), let \( \Gamma' \) be an open arc containing \( \Gamma \) and \( G \) be a simply connected domain bounded by a \( C^2 \)-curve with \( \partial G \subset V \odot \partial \) and such that \( \partial G \cap \partial D \) contains \( \Gamma' \). Let \( u \) be a compactly supported \( C^\infty \)-function with \( u = 1 \) in a neighborhood of \( \Gamma \) and \( u = 0 \) on \( D \setminus G \). Using Lemma 4.2 (i) and the fact that \( gh \in L^{(r-1)}(V, dA) \) we can write

\[
u H_1(\zeta) = \frac{1}{\pi} \int_{\partial G} \frac{gh}{z-w} \, dA = \frac{1}{\pi} \int_{\partial G} \frac{h_1}{z-w} \, dA,
\]

where \( h_1 = \partial u H_1 + gh u \) is a function in \( L^{(r-1)}(G, dA) \cap C(G) \) whose Cauchy transform \( \nu H_1 \) vanishes outside \( G \). Then by Lemma 4.3 and the fact that \( u = 1 \) near \( \Gamma \) we obtain

\[
\left( \int_{\partial G} |H_1(\zeta)|^{(r-1)} \, |d\zeta| \right)^{(r-1)/r} = O((1-t)^{1/r}), \quad (4.5)
\]

when \( t \to 1^- \). Using the fact that \( f \in L^t(G, dA) \) and that the conformal map from \( G \) onto \( \mathbb{D} \) extends analytically across \( \Gamma' \) it is a standard matter to show that

\[
\int_{\partial G} |f(\zeta)|^t \, |d\zeta| = o(1-t)^{-1},
\]

when \( t \to 1^- \). Together with (4.5) and Hölder’s inequality this implies

\[
\lim_{t \to 1^-} \int_{\partial G} |f(\zeta)|^{1-t} H_1(\zeta) \, |d\zeta| = 0.
\]

Moreover, exactly the same reasoning shows that

\[
\lim_{t \to 1^-} \int_{\partial G} |g(\zeta)|^{1-t} H_2(\zeta) \, |d\zeta| = 0
\]

and the claim (4.4) is proved.
REFERENCES


