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Quantized Rank R Matrices[1]

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First some old as well as new results about P.I. algebras, Ore extensions, and degrees are presented. Then quantized $n \times r$ matrices as well as certain quantized factor algebras $M_q^{r+1}(n)$ of $M_q(n)$ are analyzed. For $r = 1, \ldots, n-1$, $M_q^{r+1}(n)$ is the quantized function algebra of rank r matrices obtained by working modulo the ideal generated by all $(r+1) \times (r+1)$ quantum subdeterminants and a certain localization of this algebra is proved to be isomorphic to a more manageable one. In almost all cases, the quantum parameter is a primitive mth root of unity. The degrees and centers of the algebras are determined when m is a prime and the general structure is obtained for arbitrary m.

Key Words: quantized function algebra; quantum minor; factor algebra; P.I. algebra; block diagonal.

1. INTRODUCTION

Through the last several years, quantized function algebras have attracted a lot of attention ([1, 3–7, 10, 15, 16, 18, 20, 23], and many others). Among these, $M_q(n)$ has attracted the most attention. Since in fact a number of candidates for the quantized function algebra of $n \times n$ matrices have been proposed, we stress that the one we consider here is the "original" (or "standard" or "official") one introduced by Faddeev *et al.* in [6].

We wish to consider some natural subalgebras and quotients of this algebra, namely the subalgebra $M_q(n, r)$ of quantized $n \times r$ matrices, the subalgebra $A_{n,r}$ obtained by removing the $(n-r) \times (n-r)$ corner generated by those $Z_{i,j}$ for which $i, j \geq r+1$, and finally the quotients

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$M_q^{r+1}(n) = M_q(n)/I_q^{r+1}$ obtained by factoring out the ideal I_q^{r+1} generated by all $(r+1) \times (r+1)$ quantum subdeterminants. The emphasis will throughout be on the case where q is a primitive root of unity.

The major tool is the theory developed by De Concini and Procesi in [1] as well as the theory of P.I. algebras. We have found it convenient to collect these results, some corollaries to them, as well as some further developments in Section 2 following immediately after this introduction. In some sense, the results of De Concini and Procesi turn the problem into an elementary one, but which at the same time is of such a kind that one should not expect general results except possibly in special cases. Indeed, a major part of the procedure is to bring into block diagonal form an integer coefficient skewsymmetric form.

The new results we present relate to (iterated) skew polynomial extensions and are particularly useful for the algebras $A_{n,r}$. Even for the known case $M_q(n)$ they are sufficient (cf. [10]), and we have chosen to illustrate this in Section 3. Actually, the case of $M_q(n)$ was brought to a completion by the discovery of a very special phenomenon for the associated quasipolynomial algebra [10] and a substantial further development of this observation now makes it possible to attack $M_q(n,r)$. This is done in Section 4.

Section 5 is devoted to the proof of the isomorphism $A_{n,r}[d^{-1}] \simeq M_q^{r+1}(n)[d^{-1}]$, where d is the quantum $r \times r$ minor corresponding to the subalgebra generated by those $Z_{i,j}$ for which $1 \leq i, j \leq r$. A major tool is the representation theory of quantized enveloping algebras. Having established that, we turn our attention to $A_{n,r}$ in Section 6. Indeed, for questions relating to degree, center, etc., it is sufficient to consider this algebra, which is more manageable. The methods that worked well for $M_q(n,r)$ do not apply as easily but, fortunately, the results obtained in Section 2 are applicable, especially after some fortunate guesses relating to the center. As one consequence we obtain (combine Corollary 5.6 with Theorem 6.1): *If q is an odd primitive mth root of unity then* $\deg M_q^{r+1}(n) = m^{nr - r(r+1)/2}$.

2. P.I. ALGEBRAS AND THEIR DEGREES

In this section, unless explicitly stated, A denotes a prime P.I. algebra and k an algebraically closed field k of characteristic 0. We assume throughout that A is finitely generated (affine) as an algebra over k.

We start by recalling some basic results from the theory of P.I. algebras and then we show these results can be applied to calculations of the degree of an algebra. Let us first recall some basic definitions.

DEFINITION 2.1 (De Concini and Procesi [1, p. 50]). An algebra A is said to be of degree at most d, if A satisfies all identities of $(d \times d)$-matrices over a commutative ring. If no such d exists A is said to have infinite degree. In either case the smallest possible d is denoted by $\deg A$.

DEFINITION 2.2. The P.I. degree of an algebra A, $\text{p.i.}\deg A$, is $\frac{b}{2}$, where b is the smallest possible degree of a multilinear polynomial which vanishes on A.

If A is a prime P.I. algebra, such as ours, one gets

PROPOSITION 2.3 (McConnel and Robson [21, 13.6.7 (v)]). $\deg A = \text{p.i.}\deg A$.

For an affine prime P.I. algebra A one has several useful results concerning the P.I. degree. To state and prove these, let us first recall that the intersection of all primitive ideals of A is 0 [21, Theorem 13.10.3] and all primitive ideals are maximal by Kaplansky's Theorem [21, Theorem 13.3.8].

PROPOSITION 2.4. $\text{p.i.}\deg A = \sup_M \text{p.i.}\deg A/M$, *where* M *runs through the set of all maximal two-sided ideals.*

Proof. Since A/M is a factor algebra of A for all maximal ideals M, we get that the right hand side is bounded by the left hand side.

From [21, Corollary 13.6.7] we get that $\text{p.i.}\deg A/M \leq n$ if and only if S_{2n}, the standard identity, is an identity for A/M; thus $\sup_M \text{p.i.}\deg A/M \leq n$ implies that S_{2n} is an identity for A/M for all M. Therefore S_{2n} is an identity for $B = \Pi_M A/M$, where M ranges over all maximal ideals. But by the above remarks, A has a natural embedding into B and hence any identity of B is also an identity of A. ∎

Let M be a maximal two-sided ideal of A. Then $V_A = A/M$ is a simple P.I. algebra and hence of the form $M_n(D)$, where D is a division ring which is finite dimensional over its center C. Moreover, $D = \operatorname{End} V_A$ [21, 13.3].

If H is a maximal commutative subfield of D then $H = k$ since by [21, Theorem 13.10.3 (the proof)] it is finite dimensional over k and the latter is algebraically closed. Hence [21, Lemma 13.3.4], $A/M \cong M_n(k)$ for some n.

Thus we conclude

PROPOSITION 2.5. $\deg A = \sup_M \dim_k S$, *where* $S = A/M$ *runs through all irreducible* A*-modules.*

Remark 2.6. The Goldie quotient ring $Q(A)$ of A can be obtained by inverting the non-zero *central* elements of A [21, Corollary 13.6.7]. Thus,

A and $Q(A)$ have the same P.I. degree. Therefore, any ring B between A and $Q(A)$ has the same P.I. degree as A and in case B is affine over k, A and B have the same degree.

From [21, Corollary 13.3.5] we now get

PROPOSITION 2.7. p.i. deg $A = (\dim_{Q(Z)} Q(A))^{1/2}$, *where* $Q(Z)$ *denotes the quotient field of the center of* A.

As noted in [12] we have

PROPOSITION 2.8. *Let* $\{a_1, \dots, a_k\}$ *be a finite set of regular elements of* A. *There exists an irreducible representation* ρ *of* A *of maximal degree in which* $\rho(a_1), \dots, \rho(a_k)$ *are units.*

Proof. Let $B = A[a_1^{-1}, \dots, a_k^{-1}]$. Then $A \subseteq B \subseteq Q(A)$. By Remark 2.6 the P.I. degrees of A and B are equal.

Let ρ be an irreducible representation of B of maximal dimension on a finite-dimensional vector space V over k. Let ρ' denote the restriction of ρ to A. Then $\rho'(a_i) = \rho(a_i)$ for all $i = 1, \dots, k$. Moreover, each $\rho(a_i)$ is a regular linear map, hence by the Cayley–Hamilton Theorem its inverse is in $\rho'(A)$. Thus $\rho'(A) = \rho(B)$ and hence ρ' is irreducible. ∎

A different approach to finding the degree of certain algebras has been found by De Concini and Procesi [1, p. 60, Sect. 7]. We recall some of their results.

Let $J = (h_{i,j})$ be a skewsymmetric $n \times n$ matrix such that $\forall i, j : h_{i,j} \in \mathbb{Z}$. Given J, the quasipolynomial algebra $k_J[x_1, \dots, x]$ is the algebra over the field k generated by $x_1, \dots, x_n$ and with defining relations

$$x_i x_j = q^{h_{i,j}} x_j x_i, \qquad i < j. \tag{2.1}$$

We call J the **defining matrix** of the quasipolynomial algebra. In the following, $q \in k$ is always assumed to be a primitive mth root of unity.

De Concini and Procesi proved

THEOREM 2.9. $\deg k_J[x_1, \dots, x_n] = \sqrt{h}$, *where* h *is the cardinality of the image of the map induced by* J

$$\mathbb{Z}^n \mapsto (\mathbb{Z}/m\mathbb{Z})^n, \tag{2.2}$$

defined by $w = (w_1, \dots, w_n) \mapsto \overline{Jw}$, *where* $\bar{\ }$ *denotes taking residue class in each coordinate.*

Furthermore $k_J[x_1, \dots, x_n]$ *is a free module over its center of rank* $\sqrt{h}$.

In [1, 7.2 Proposition, p. 61] it was shown that $k_J[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$ is an Azumaya algebra under the assumption that k is an algebraically closed field of characteristic 0 and arbitrary J.

Since the algebras $k_J[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$ all are affine prime P.I. algebras we can use some of the results from above to prove that the assumptions on k made by De Concini and Procesi are superfluous.

PROPOSITION 2.10. *Let k be a field and J a skewsymmetric matrix with integer coefficients. The algebra $k_J[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$ is an Azumaya algebra.*

Proof. We wish to use the Artin-Procesi Theorem [21, Theorem 13.7.14]. Thus we have to show that S_m is an identity for A if and only if S_m is an identity for A/P for all primes of A [21, Corollary 13.6.7].

S_m is an identity for A or A/P if S_m vanishes on all m-tuples of monomials in the $x_j^{\pm 1}$'s, $1 \leq j \leq n$.

For any such an m-tuple $(a_1, \ldots, a_m)$ we get

$$S_m(a_1, \ldots, a_m) = f_g \cdot x_1^{l_1} \cdots x_n^{l_n} \tag{2.3}$$

with $l_j \in \mathbb{Z}$, $f_g \in k$ depending on $a_1, \ldots, a_m$. Since no $x_1^{l_1} \cdots x_n^{l_n}$ can belong to a prime P, the claim follows. ∎

Many of the algebras considered in the following are iterated Ore extensions. We therefore recall some results on the degree of iterated Ore extensions or skew polynomial algebras.

In [1, p. 59, Theorem; 12, Theorem 1] it was proved that if R is an affine prime algebra over a field of characteristic 0, then $\deg R[\theta; \alpha, \delta] = \deg R[\theta; \alpha]$ provided $\deg R[\theta; \alpha, \delta]$ is finite. Here, α is a k-automorphism of R and δ an α-derivation on R.

Combining this result with Proposition 2.7, the degrees of the so-called Dipper−Donkin algebras $D_q(n)$ and the quantized ("official") matrix algebras $M_q(n)$ were found by Jakobsen and Zhang [10, 11] (for the quantum parameter q a primitive mth root of unity).

For later purposes we need a few more results concerning skew polynomial algebras (implicitly in [12]).

We consider an affine prime P.I. algebra R and a skew polynomial algebra

$$A = R[\theta; \alpha, \delta], \tag{2.4}$$

where α is a k-automorphism of R and δ an α-derivation.

We assume A is a P.I. algebra and α has finite order.

Notice that for any regular element $r \in R$ (resp. $r \in A$) $R[r^{-1}]$ (resp. $A[r^{-1}]$) is a k-affine prime P.I. subalgebra of the Goldie quotient ring of R (resp. A) hence by the previous results has the same degree as R (resp. A).

The next lemma follows easily and is very well known.

LEMMA 2.11.

$$\text{p.i.}\deg A = \text{p.i.}\deg A[t]. \tag{2.5}$$

Combining these results for $A = R[\theta; \alpha, \delta]$ we obtain

LEMMA 2.12. *In case there exist regular r, t in R and s in R such that $r\theta + st^{-1}$ commutes with all elements of A then*

$$\deg R[\theta; \alpha, \delta] = \deg R. \tag{2.6}$$

In [12, Sect. 4] it was proved that such r, s, and t exist when α induces the identity on $Z(R)$, the center of R.

Remark 2.13. In the present article this result will only be used in situations in which there exist a regular element $r \in R$ and an element $s \in R$ such that $r\Theta + s = z$ is central in A. Observe that r is regular in A. In an irreducible representation ρ of A of maximal degree, we may by Proposition 2.8 assume that r is invertible. Then $\rho(\Theta) = \rho(r)^{-1}\rho(z - s)$ and hence ρ remains irreducible when restricted to R. Thus, $\deg A = \deg R$.

It was also shown in [12, Proof of Theorem 3.1] that in case α is not the identity on $Z(R)$, then there exists a finitely generated multiplicatively closed α-invariant set T of central elements of R such that

$$R[T^{-1}][\theta; \alpha, \delta] \cong R[T^{-1}][\theta'; \alpha], \tag{2.7}$$

where $\theta' = \theta - a$ for some $a \in Z(R[T^{-1}])$, and such that $R[T^{-1}]$ is k-affine.

In case there exists a subalgebra Z_0 of the center Z of R such that (i) Z is a finite Z_0 module, (ii) $\delta(Z_0) = 0$, and (iii) $\alpha_{|Z_0} = 1_{Z_0}$, De Concini and Procesi proved [1, Theorem p. 58] that

$$\deg R[\Theta, \alpha, \delta] = (\deg R) \cdot k, \tag{2.8}$$

where k is the order of α's restriction to $Z(R)$.

(By the methods of [12, 13] one can in fact show that the special assumptions on R, α, δ are superfluous. One just needs that $R[\Theta, \alpha, \delta]$ is a prime P.I. algebra.)

In particular, we get, provided $R[\Theta, \alpha, \delta]$ is a P.I. algebra,

PROPOSITION 2.14. *Let R be a prime P.I. algebra and α an automorphism of R of order k. If there exists an element $c \in Z(R)$ such that the α orbit of c has order k, then*

$$\deg(R[\Theta, \alpha, \delta]) = (\deg R) \cdot k.$$

3. THE QUANTIZED FUNCTION ALGEBRA $M_q(n)$

The "standard" quantized function algebra $M_q(n)$ of $n \times n$ matrices is the quadratic algebra generated by n^2 elements $Z_{i,j}$, $i, j = 1, \dots, n$, and with defining relations

$$\begin{aligned} Z_{i,j}Z_{i,k} &= qZ_{i,k}Z_{i,j} && \text{if } j < k, \\ Z_{i,j}Z_{k,j} &= qZ_{k,j}Z_{i,j} && \text{if } i < k, \\ Z_{i,j}Z_{s,t} &= Z_{s,t}Z_{i,j} && \text{if } i < s, t < j, \\ Z_{i,j}Z_{s,t} &= Z_{s,t}Z_{i,j} + \left(q - q^{-1}\right)Z_{i,t}Z_{s,j} && \text{if } i < s, j < t, \end{aligned} \tag{3.1}$$

for $i, j, k, s, t = 1, 2, \dots, n$.

It is well known that the monomials $Z^A = Z_{1,1}^{a_{1,1}} \cdots Z_{1,n}^{a_{1,n}} \cdots Z_{2,1}^{a_{2,1}} \cdots Z_{2,n}^{a_{2,n}} \cdots Z_{n,1}^{a_{n,1}} \cdots Z_{n,n}^{a_{n,n}}$ for $A = \{a_{i,j}\}_{i,j=1,1}^{n,n} \in Mat(n^2, \mathbb{N}_0)$ form a PBW-type basis of $M_q(n)$ for any $q \neq 0$. The quantum determinant

$$\det{}_q = \sum_{\sigma \in S_n} (-q)^{l(\sigma)} Z_{1,\sigma(1)}Z_{2,\sigma(2)} \cdots Z_{n,\sigma(n)} \tag{3.2}$$

is central for any $0 \neq q \in k$.

Viewing $M_q(n)$ as an iterated skew polynomial algebra (cf. below), it follows that *the associated quasipolynomial algebra* $\overline{M_q(n)}$ is given in terms of the same generators, but with defining relations

$$\begin{aligned} Z_{i,j}Z_{i,k} &= qZ_{i,k}Z_{i,j} && \text{if } j < k, \\ Z_{i,j}Z_{k,j} &= qZ_{k,j}Z_{i,j} && \text{if } i < k, \\ Z_{i,j}Z_{s,t} &= Z_{s,t}Z_{i,j} && \text{if } i < s, t < j, \\ Z_{i,j}Z_{s,t} &= Z_{s,t}Z_{i,j} && \text{if } i < s, j < t, \end{aligned} \tag{3.3}$$

for $i, j, k, s, t = 1, 2, \dots, n$.

Later on, we shall encounter a number of subalgebras B of $M_q(n)$. For each of these, analogously to the above, the associated quasipolynomial algebra $\overline{B}$ is the algebra with the same generators but where in the defining relations, all terms of the form $(q - q^{-1})Z_{i,t}Z_{s,j}$ have been dropped.

The degree of $M_q(n)$ was found in [10] to be $m^{n(n-1)/2}$ for q an mth root of unity, m odd. The approach there utilized the result of Procesi and De Concini [1, Theorem p. 59] according to which, as a special case, $\deg M_q(n) = \deg \overline{M_q(n)}$. We will reprove this result by utilizing the results of Section 2.

Before doing so let us introduce some notation, which will be used also later in this paper.

We view $M_q(n-1)$ as the k-algebra generated by the elements $Z_{i,j}$, $1 \le i, j \le n-1$ and we will then view $M_q(n)$ as an iterated Ore extension of $M_q(n-1)$ obtained by adjoining the indeterminates as

$$M_q(n) = M_q(n-l)[Z_{n,1}; \alpha_{n,1}] \cdots [Z_{n,n-1}; \alpha_{n,n-1}\delta_{n,n-1}]$$
$$\cdot [Z_{1,n}; \alpha_{1n}] \cdots [Z_{n,n}; \alpha_{n,n}, \delta_n], \tag{3.4}$$

where $Z_{n,k}Z_{i,j} = \alpha(Z_{i,j})Z_{n,k} + \delta(Z_{i,j})$ for $1 \le i, j < n$ or $i = n, j < k$, and where $Z_{k,n}Z_{i,j} = \alpha(Z_{i,j})Z_{k,n} + \delta(Z_{i,j})$ for $1 \le i, j < n$ or $j = n$, $i < k$.

Changing slightly the notation from Parshall and Wang [23, Sect. 4] we define $I = \{n-i+1, \dots, n\}$ and $J = \{1, \dots, i\}$ and let $\tilde{\theta}_{n+1-i} = D(I, J)$ (the quantum determinant based on the rows in I and columns in J) and $\theta_{i+1} = A(I, J)$. Later on, we shall introduce some more general elements with these names but letting $r = n$ in Fig. 2 (see Section 6), the elements $\theta_k, \tilde{\theta}_t$ in that figure are precisely what have been defined here (with $k = i+1$ and $t = n+1-i$).

In [10] the following notion was introduced

DEFINITION 3.1. *An element $x \in M_q(n)$ is called covariant if for any $Z_{i,j}$ there exists an integer $n_{i,j}$ such that*

$$xZ_{i,j} = q^{n_{i,j}} Z_{i,j} x. \tag{3.5}$$

Clearly, $Z_{1,n}$ *and* $Z_{n,1}$ *are covariant.*

It was then shown that the elements $\tilde{\theta}_t$ and θ_k are covariant. Utilizing this, the following elements were also found to belong to the center,

$$c_{i+1} = \tilde{\theta}_{n+1-i}\theta_{i+1}^{m-1} \qquad \text{and} \qquad d_{i+1} = \tilde{\theta}_{n+1-i}^{m-1}\theta_{i+1} \tag{3.6}$$

for $1 \le i \le n-1$.

For the convenience of the reader we list the covariance properties of the elements $\theta_{i+1}, \tilde{\theta}_{n+1-i}$ (from which the centrality of c_{i+1} and d_{i+1} also is obvious). Let $I = \{n-i+1, \dots, n\}$ and $J = \{1, \dots, i\}$ as previously:

$$Z_{a,b}\theta_{i+1} = q\theta_{i+1}Z_{a,b} \qquad \text{and} \qquad Z_{a,b}\tilde{\theta}_{n+1-i} = q\tilde{\theta}_{n+1-i}Z_{a,b} \qquad \text{for } a \notin I, b \in J, \tag{3.7}$$

$$Z_{a,b}\theta_{i+1} = q^{-1}\theta_{i+1}Z_{a,b} \qquad \text{and} \qquad Z_{a,b}\tilde{\theta}_{n+1-i} = q^{-1}\tilde{\theta}_{n+1-i}Z_{a,b} \qquad \text{for } a \in I, b \notin J, \tag{3.8}$$

$$Z_{a,b}\theta_{i+1} = \theta_{i+1}Z_{a,b} \qquad \text{and} \qquad Z_{a,b}\tilde{\theta}_{n+1-i} = \tilde{\theta}_{n+1-i}Z_{a,b} \qquad \text{in all other cases.} \tag{3.9}$$

It is well known that $M_q(n)$ (being an iterated Ore extension) is a domain; thus the r, t in Lemma 2.12 are automatically regular if non-zero.

THEOREM 3.2. $\deg M_q(n) = m^{n(n-1)/2}$, *where* m *is an odd integer and* q *is a primitive* m*th root of unity.*

Proof. We use induction on n. Since the formula clearly holds for $n = 1$, it suffices to prove

$$\deg M_q(n) = m^{n-1} \deg M_q(n-1). \tag{3.10}$$

First notice that

$$\det{}_q = rZ_{n,n} + s, \tag{3.11}$$

where in fact r up to a sign is $\det_q$ for $M_q(n-1)$, and where s, when expanded in the PBW basis, does not contain $Z_{n,n}$ either.

By Lemma 2.12 we see that $\deg M_q(n)$ is the same as the degree of the algebra where $Z_{n,n}$ is excluded.

The same argument works for $Z_{n-i,n}$. One just has to replace $\det_q$ by $d_i = \tilde{\theta}_{n+1-i}^{m-1}\theta_{i+1}$ in the procedure. Thus,

$$\deg M_q(n) = \deg M_q(n-1)[Z_{n,1}; \alpha_{n,1}] \cdots [Z_{n,n-1}; \alpha_{n,n-1}, \delta_{n,n-1}]. \tag{3.12}$$

Let R_j be the algebra obtained by adjoining $Z_{n,1}, \dots, Z_{n,j}$ to $M_q(n-1)$. Let $\underline{\tilde{\theta}}_i$ and $\underline{\theta}_i$ be the quantities in $M_q(n-1)$ analogous to $\tilde{\theta}_i$ and θ_i. Then notice that

$$\underline{c}_{j+1} = \underline{\tilde{\theta}}_{n+1-j}\underline{\theta}_{j+1}^{m-1}\left(\underline{\det}_q\right)^{m-1} \tag{3.13}$$

is a central element in R_j and

$$\alpha_{n,j+1}(\underline{c}_{j+1}) = q^2\underline{c}_{j+1}. \tag{3.14}$$

Therefore by Proposition 2.14 and because m is odd, $\deg R_{j+1} = m \deg R_j$.

In the case of R_1 one may just use $\underline{\det}_q$ as c_0. Since $\alpha_{n,1}(c_0) = q^{-1}c_0$, there is a similar conclusion. The proof is thus completed. ∎

4. THE CASE OF $n \times r$

We consider here the quadratic algebra $M_q(n, r)$ consisting of $n \times r$ matrices ($r \leq n$). We determine the degree and the center in case m is "good" and also obtain insight, in some cases even full, into the cases where m is not so "good."

4.1. *Central Elements.* We assume that $r = x \cdot s$ and $n = y \cdot s$ with $x \cdot y$ odd and s as big as possible. We display s central elements of $M_q(n, r)$. (For a hint of how these were discovered, see the proof of Proposition 4.9 below. Also recall from [10] that the centres of $M_q(n, r)$ and $\overline{M_q(n, r)}$ are in bijective correspondence via the leading term.)

We begin by defining elements $\Psi_t \in M_q(n, r)$ for $t = 1 - r, \dots, n$. First set $\Psi_{1-r} = 1$. Ψ_{2-j} is the quantum $(r - j + 1) \times (r - j + 1)$ minor involving the rows $1, 2, \dots, r - j + 1$ and columns $j, j + 1, \dots, r$ (for $j = 2, \dots, r$). Then consider $i = 1, \dots, n - r + 1$ and let Ψ_i be the quantum $r \times r$ determinant involving the rows $i, i + 1, \dots, i + r - 1$. Finally, Ψ_{n-k+1} is the quantum $k \times k$ determinant involving columns $1, \dots, k$ and rows $n - k + 1, n - k + 2, \dots, n$ for $k = 1, \dots, r - 1$.

LEMMA 4.1. *For $a = 1, \dots, s$ the elements Z_a defined by*

$$Z_a := \prod_{\ell = -x}^{y-1} (\Psi_{a + \ell \cdot s})^{(-1)^{\ell}} \tag{4.1}$$

are central.

Proof. Let us consider the case $a = 1$. We may then view our $n \times r$ matrix as being built up of $y \cdot x$ blocks $B_{i,j}$ of size s, block $B_{1,1}$ consisting of rows and columns $1, \dots, s$, block $B_{2,1}$ consisting of rows $s + 1, \dots 2s$ and columns $1, \dots, s$, etc. Let us now look at some $X_{a,b} \in B_{i,j}$. Due to the covariance of the various determinants it is possible to see that the commutativity of Z_1 with $X_{a,b}$ is equivalent to picking up a factor of $q^{\pm 1}$ for each block $B_{\alpha, j}$ and each $B_{i, \beta}$ and that indeed the whole computation may be viewed as the computation for commutativity of the analogous expression computed in $\overline{M_q(y, x)}$. Here it is a matter of investigating the matrix $B = \{b_{i,j}\}_{i,j=1}^{y,x}$ given by $b_{i,j} = (-1)^{i+j}$ and checking that $\overline{Z}_1^B$ is central. But since x and y are odd, this is straightforward. Indeed, the computation is reduced to ascertaining that if w is odd and $1 \le i \le w$ then $(w - i) - (i - 1) = 0$ in $\mathbb{Z}_2$.

The case $a \ge 2$ is similar though slightly more complicated. One may view the relevant diagram over the minors as being obtained by a translation along the line from the upper right corner to the lower left. Only near the boundaries does one need to check carefully that the signs match up. The details are left to the reader; cf. Fig. 1. ∎

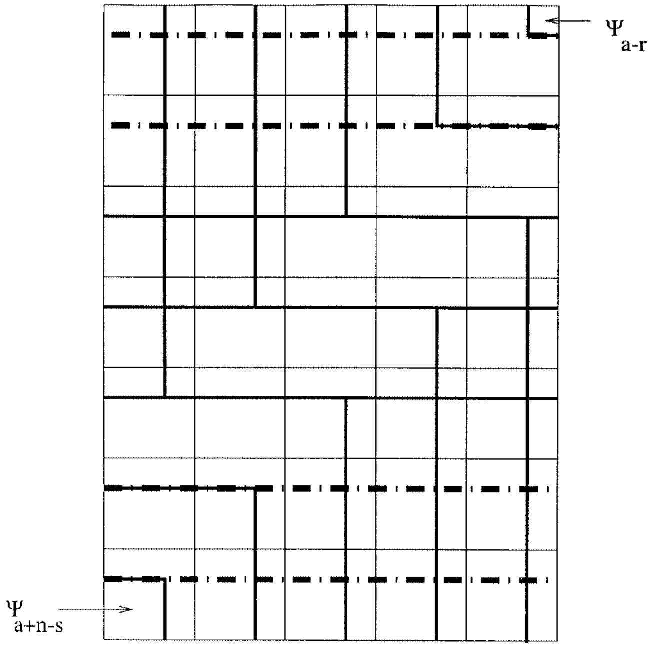

FIG. 1. The factors Ψ_i for $a \geq 2$. Beginning at the upper right hand corner, the first is an $(a-1) \times (a-1)$ matrix, then they grow larger in steps of size s, reach full size as indicated by the dashed lines, and then decrease again until reaching the bottom left corner which is a minor of size $(s-a+1) \times (s-a+1)$. The thin lines represent blocks of size $s \times s$. At $a = 1$ the factor Ψ_{1-r} deteriorates to a 1 and the lines defining the factors run into those defining the blocks. The reader is now invited to take a stroll in the figure: Place yourself in the position of some $Z_{i,j}$, say $Z_{n,1}$, and count covariance-q's from each factor. Convince yourself that the exponents are exactly right for giving a total factor q^0. Then move to a neighboring element and so forth. When crossing a "boundary" to a new factor, there will precisely be one new q and one new q^{-1}.

Finally observe that one has the following result, the first part of which is just as in [10]:

LEMMA 4.2. (i) *Let m be even and q a primitive mth root of unity. Any element of the form*

$$Z_{k,i}^{m/2} Z_{k,j}^{m/2} Z_{\ell,i}^{m/2} Z_{\ell,j}^{m/2}, \tag{4.2}$$

where $1 \leq k < \ell \leq n$ *and* $1 \leq i < j \leq r$, *is in the center of* $\overline{M_q(n,r)}$.

(ii) *If* $n + r$ *is even then*

$$\begin{aligned} &(Z_{n,1} \cdots Z_{2,1} Z_{1,1} Z_{1,2} \cdots Z_{1,r})^{m/2} \quad \text{is central if } n, r \text{ are odd, and} \\ &(Z_{n,1} \cdots Z_{2,1} Z_{1,2} \cdots Z_{1,r})^{m/2} \quad \text{is central if } n, r \text{ are even.} \end{aligned} \tag{4.3}$$

Remark 4.3. The analogous elements of $M_q(n, r)$ (i.e., the same expressions but where the generators are in that algebra) are also central in this case.

Remark 4.4. It follows by an argument similar to the one in [10, Theorem 6.2] that in case m is "good" (cf. (4.6) and Proposition 4.5 below) then the elements Z_a; $a = 1, \dots, s$ together with the elements $Z_{i,j}^m$; $1 \le i \le n$; $1 \le j \le r$ generate the center.

4.2. *Degree and Block Diagonal Form*. Following [10, pp. 469–470], the defining matrix J of the associated quasipolynomial algebra $\overline{M_q(n, r)}$, with respect to a natural basis $\{E_{i,j}\}$, is in fact the matrix of the map

$$A \overset{J}{\to} H_n A - A H_r, \tag{4.4}$$

where $H_k = \sum_{1 \le j < i \le k} (E_{i,j} - E_{j,i})$ for $k = n, r$. We start by computing the rank of this map or, equivalently, the dimension of the kernel. Let

$$c_{n,r} := \operatorname{corank}(J). \tag{4.5}$$

Thus, if m is "good," e.g., a large prime,

$$\deg\big(M_q(n, r)\big) = m^{(1/2)(nr - \operatorname{corank})} = m^{(1/2)(nr - c_{n,r})}. \tag{4.6}$$

Indeed, $\deg(M_q(n, r)) = \deg\overline{(M_n(n, r))}$ by [1, Theorem p. 59], and the latter is given by Theorem 2.9. The cardinality of J is clearly the same as for any block diagonal form of J. Now consider a 2×2 matrix $\left(\begin{smallmatrix} 0 & r \\ -r & 0 \end{smallmatrix}\right)$ for some $r \in \mathbb{N}$. Considered as a map from $\mathbb{Z}^2$ to $(\mathbb{Z}/m\mathbb{Z})^2$, the image clearly has cardinality $(m/\text{g.c.d.}(m, r))^2$, where g.c.d.$(m, r)$ denotes the greatest common divisor of m and r. This means that if, e.g., m is a prime bigger than all non-zero elements in a block form of J, then each non-trivial block contributes with a factor m to the total degree. But clearly, there are $\frac{1}{2} \cdot \operatorname{rank} J$ such blocks.

Recall from [10, p. 470] that $H_k = S_k + \cdots + S_k^{k-1} = (1 + S_k)/(1 - S_k)$ where $S_k = -E_{1,k} + \sum_{i=2}^{k} E_{i,i-1}$. In particular, $S_k^k = -1$. Now observe that

$$H_n A - A H_r = M' \Leftrightarrow 2(S_n A - A S_r) = (1 - S_n) M' (1 - S_r), \tag{4.7}$$

where $1 - S_k$ is invertible, indeed, $(1 - S_k)^{-1} = \frac{1}{2}(1 + S_k + \cdots + S_k^{k-1})$.

This observation will also be used later, but it follows immediately that the kernel is given by those $n \times r$ matrices A for which

$$S_n A = A S_r. \tag{4.8}$$

If $A = \sum_{i,j} a_{i,j} E_{i,j}$ a straightforward computation gives that (4.8) is equivalent to

$$\begin{aligned} &\forall i = 2,\dots,n, \forall j = 1,\dots,r-1, && a_{i-1,j} = a_{i,j+1}, \\ &\forall i = 2,\dots,n, && a_{i-1,r} = -a_{i,1}, \\ &\forall j = 1,\dots,r-1, && a_{1,j+1} = -a_{n,j}, \text{ and } a_{n,r} = a_{1,1}. \end{aligned} \tag{4.9}$$

If we define, for all $\alpha, \gamma \in \mathbb{Z}$, for $\beta = 0,\dots,n-1$, and for $\delta = 0,\dots,r-1$

$$a_{\alpha n+\beta, \gamma r+\delta} = (-1)^{\alpha+\gamma} \alpha_{\beta,\delta}, \tag{4.10}$$

then (4.8) is equivalent to

$$\forall \beta, \delta, t, \qquad a_{\beta+t, \delta+t} = a_{\beta,\delta}. \tag{4.11}$$

PROPOSITION 4.5. *Let $s = g.c.d.(n, r)$. Specifically, let $n = x \cdot s$ and $r = y \cdot s$. Then J is non-invertible if and only if both x and y are odd. In this case,*

$$c_{n,r} = \operatorname{corank} J = s.$$

Proof. Observe that by definition, x and y cannot both be even. Now, according to (4.10) and (4.11)

$$a_{y\cdot n, x\cdot r} = (-1)^{x+y} a_{0,0} = (-1)^{x+y} a_{1,1} = a_{x\cdot y\cdot s, x\cdot y\cdot s} = a_{1,1}. \tag{4.12}$$

Thus, if a solution to (4.8) is to exist with $a_{1,1} \neq 0$ then $(-1)^{x+y} = 1$. In this case, a solution is given by $\forall t$, $a_{t,t} = a_{1,1} = 1$ and all other $a_{i,j} = 0$. More generally, a non-zero solution exists if and only if $(-1)^{x+y} = 1$. In this case there are s independent solutions given by

$$\forall t, \qquad a_{t+i,t} = a_{1+i,i} \text{ for } i = 0, 1, \dots, s-1 \text{ and all other } a_{i,j} = 0. \tag{4.13}$$

The claim now follows. ∎

We wish to obtain more precise information about the blocks of a diagonal form of the associated matrix of the quasipolynomial algebra $\overline{M_q(n,r)}$.

PROPOSITION 4.6. *In the case where $q = -1$ there is an irreducible module of dimension 2^{d_0} where*

$$d_0 = \left[\frac{n+r-1}{2}\right].$$

Proof. In this case the term $(q - q^{-1})$ disappears and so $\overline{M_{-1}(n,r)} = M_{-1}(n,r)$. Hence, by covariance, in an irreducible module, $Z_{i,j}$ is either zero or invertible. Consider the subalgebra

$$\mathscr{S}_{n,r} = \langle Z_{1,j}, Z_{i,1} \mid 1 \leq i \leq n \text{ and } 1 \leq j \leq r \rangle. \tag{4.14}$$

By Proposition 2.8 there is an irreducible representation of maximal dimension of this algebra in which all the generators are invertible. Given such an irreducible module, the recipe

$$Z_{i,j} = c_{i,j} \cdot Z_{1,1}^{-1} Z_{i,1} Z_{1,j}, \tag{4.15}$$

where $2 \leq i \leq n$, $2 \leq j \leq r$, and the $c_{i,j}$'s are arbitrary constants, defines an irreducible representation of $\overline{M_q(n,r)}$. The proof is thus completed if we can establish that there are precisely $[\frac{n+r-1}{2}]$ non-trivial blocks in the block diagonal form of the associated matrix of $\mathscr{S}_{n,r}$ (and hence that the degree is $m^{[(n+r-1)/2]}$). For this purpose, let $x_j = Z_{1,j}$ for $j = 1, \ldots, r$ and let $y_j = Z_{i,1}$ for $i = 2, \ldots, n$. Upon the replacements $x_j \mapsto x_1 x_2 x_j$ $(j = 3, \ldots, r)$, $y_2 \mapsto y_2 x_2$, and $y_i \mapsto y_2 y_i$ $(i = 3, \ldots, n)$, the pair x_1, x_2 decouples completely leaving us with an algebra which is isomorphic to $\mathscr{S}_{n-1,r-1}$. It is well known (and elementary) to see that there are $[\frac{x}{2}]$ non-trivial blocks in the block diagonal form corresponding to $\mathscr{S}_{x,1}$. The result follows directly from these observations. ∎

Actually, we did not use anything about the algebra except that it was contained in a box of size $n \times r$, hence we get the following corollary to the proof:

COROLLARY 4.7. *Let S be a subalgebra of $M_q(n,r)$ such that $\forall i = 1, \ldots, n$, $Z_{i,1} \in S$ and such that $\forall i = j \ldots, r$, $Z_{1,j} \in S$. Then in case $q = -1$ there is an irreducible module of dimension 2^{d_0} where*

$$d_0 = \left[\frac{n+r-1}{2}\right].$$

Remark 4.8. For general q we get a result similar to Proposition 4.6. Specifically, given a representation of $\mathscr{S}_{n,r}$ in which the generators are denoted $\overline{Z}_{i,j}$ and in which $\overline{Z}_{1,1}$ is invertible, the recipe

$$Z_{i,j} = q \overline{Z}_{1,1}^{-1} \overline{Z}_{i,1} \overline{Z}_{1,j} \qquad \text{if } i,j > 1$$

$$Z_{i,j} = \overline{Z}_{i,j} \qquad \text{else}$$

defines a representation of $M_q(n)$ as can be seen by a straightforward but tedious computation.

4.3. *The General Form of the Center and The Blocks.* We wish to take a closer look at the degree and center in the case where m is not necessarily a prime and where n, r are arbitrary. For this purpose we need more information about the pertinent block diagonal form.

PROPOSITION 4.9. *The non-trivial blocks in a block diagonal form of the defining matrix J of $\overline{M_q(n,r)}$ are either of the form* $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, $\begin{pmatrix} 0 & 2 \\ -2 & 0 \end{pmatrix}$ *or of the form* $\begin{pmatrix} 0 & 4 \\ -4 & 0 \end{pmatrix}$.

Proof. We will begin by studying the center of $\overline{M_q(n,r)}$ at a primitive mth root of unity. If A is an $n \times r$ integer matrix, the condition for a monomial $u^A = Z_{1,1}^{a_{1,1}} Z_{1,2}^{a_{1,2}} \cdots Z_{n,r}^{a_{n,r}}$ to be in the center is precisely (in the notation of (4.4)) that

$$H_n A - A H_r = 0 \qquad \bmod m. \tag{4.16}$$

Returning to (4.7), it follows that

$$S_n A - A S_r = \tfrac{1}{2} \cdot M, \tag{4.17}$$

where $M = (1 - S_n) M' (1 - S_r)$ is an integer matrix whose entries all are multiples of m. However, basically due to the $\frac{1}{2}$ in $(1 - S_k)^{-1}$, not all such matrices M need define a solution A to (4.16). Returning now to Eqs. (4.9)–(4.11), these remain valid when reinterpreted as equations *modulo* $\frac{m}{2}$. In case $(-1)^{x+y} = 1$, with x, y as in Proposition 4.5, we just get the old solutions possibly with some elements of the form (4.2) or (4.3) superimposed. But the case $(-1)^{x+y} = -1$ now is possible since we just need $2a_{1,1} = 0 \bmod \frac{m}{2}$. Thus, it should be proportional to $\frac{m}{4}$. In all cases it follows that the entries of A are integer multiples of $\frac{m}{4}$ and hence that the element u^A satisfies that its fourth power is in the central subalgebra generated by the elements $Z_{i,j}^m$. But suppose that there is a block of the form $\begin{pmatrix} 0 & s \\ -s & 0 \end{pmatrix}$ with $s \neq 1, 2, 4$. Then there are monomials u^A, u^B such that $u^A u^B = q^s u^B u^A$ and such that u^A commutes with all other generators. But then the element $(u^A)^{m/s}$ is central for any m which is a multiple of s and this is a contradiction since $(u^A)^{4m/s}$ will not be in the above mentioned central subalgebra. ∎

Remark 4.10. It follows from (4.17) that if u^A is central, then so is u^B for any $B = S_n^i A S_r^j$ with $i, j \in \mathbb{Z}$. This symmetry can be used to construct new solutions from given ones; cf. below.

PROPOSITION 4.11. *The non-trivial blocks in a block diagonal form of the defining matrix J of $\overline{M_q(n,r)}$ are d_0 matrices of the form* $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ *and* $\max\{0, ((nr - c_{n,r})/2) - d_0\}$ *matrices of the form* $\begin{pmatrix} 0 & 2 \\ -2 & 0 \end{pmatrix}$ *or* $\begin{pmatrix} 0 & 4 \\ -4 & 0 \end{pmatrix}$.

Proof. This follows immediately from Proposition 2.10, Proposition 4.6, and Proposition 4.9. ∎

We finish with some remarks about the occurrence of "4"s: Suppose that r, n are relatively prime. Following the reasoning just before Remark 4.10, if we are to have a genuine solution to (4.17) involving $\frac{m}{4}$ then by (4.11) we must have $a_{\beta,\delta} = \frac{m}{4} \bmod \frac{m}{2}$ for all β, δ. We may assume that $a_{\beta,\delta} = \frac{m}{4}$ for all $\beta = 2, \dots, r$ and all $\delta = 2, \dots, n$. This follows since if some $a_{\beta,\delta} = \frac{3m}{4}$ then, by Lemma 4.2 the $\frac{m}{2}$ is part of a central element involving $a_{1,1}$, $a_{1,\delta}$, $a_{\beta,1}$, and $a_{\beta,\delta}$ and may thus be discarded. We furthermore assume that $a_{1,j} = \frac{m}{4} + \alpha_j \frac{m}{4}$ for $j = 1, \dots, n$ and $a_{i,1} = \frac{m}{4} + \beta_i \frac{m}{4}$ for $i = 2, \dots, r$, where each β_i and α_j is 0 or 2 modulo 4.

Consider first a pair of indices (i, j) with $i, j > 1$. Then the condition for u^A to commute with $Z_{i,j}$ in the quasipolynomial algebra is

$$(r - i) + (n - j) - (i - 1) - (j - 1) - \alpha_j - \beta_i = 0 \qquad \bmod 4.$$

By subtracting consecutive terms it follows that

$$\alpha_j = 2j + c \qquad \text{for } j = 2, \dots, n,$$

$$\beta_i = 2i + d \qquad \text{for } i = 2, \dots, r, \text{ and}$$

$$\alpha_1 = f.$$

Also observe that

$$n + r \text{ must be even}$$

and

$$n + r + 2 + c + d = 0 \qquad \bmod 4.$$

At a point $(1, j)$ with $j > 1$ we get, utilizing the parity properties,

$$n + r + nc + n(n+1) + f = 0 \qquad \bmod 4.$$

Likewise, at $(i, 1)$ with $i > 1$ we get

$$n + r + rd + r(r+1) + f = 0 \qquad \bmod 4,$$

and, finally, at $(1, 1)$ we get

$$n + r - 2 + (n-1)c + (r-1)d + n(n+1) + r(r+1) = 0 \qquad \bmod 4.$$

For these equations to have solutions, r and n must have the the same parity. If both are odd, there are no further restrictions for these equations to have a solution which also solves (4.17).

Returning to the general situation, if both n, r are even, it turns out to be a further necessary (and also sufficient) condition for (4.17) to have a genuine $\frac{m}{4}$ solution that they are equal modulo 4. It should now be observed that if there is a general center, that is, if J is singular, among these solutions there will be some which are actually of no interest, namely those solutions that correspond to a zero on the right hand side of (4.17).

Let us now assume that r is prime.

If r = 2, n odd does not contribute by the above analysis. If it is even, it is forced to be of the form $n = 4t + 2$ which means that the central elements already have been picked up by the general central elements. Suppose then that r is an odd prime. If r does not divide n, is it also forced to be odd and hence J is singular and hence the solutions we pick up are general central elements raised to the power $\frac{m}{4}$. This takes care of all cases except $n = zr$ for some positive integer z. If z is odd we have an r-dimensional center: again nothing new. Finally, if z is even, the previous elements do not give anything. However, there are in fact some non-trivial $\frac{m}{4}$-central elements. Specifically, let A_1 be the matrix whose non-zero coefficients $a_{i,j}^{(1)}$ satisfy

$$a_{1,r}^{(1)} = a_{i,i+jr+e}^{(1)} = \frac{m}{4}$$

$$\text{for } i = 1, \ldots r;\ j = 0, \ldots, z-1,\ e = 0, 1, \text{ and } i + jr + e \leq z \cdot r,$$

and add to that the matrix A_2 whose whose non-zero coefficients $a_{i,j}^{(2)}$ satisfy

$$a_{i,1}^{(2)} = \frac{m}{2} = a_{r,1+jr}^{(2)} \qquad \text{for } i = 1, \ldots r \text{ and } j = 1, \ldots z-1.$$

Then $A_1 + A_2$ defines a central element. Moreover, using the symmetries of the original equation, we get in fact $(r-1)$ solutions (which clearly is as much as could be hoped for).

We thus have the following partial result

PROPOSITION 4.12. *Let r be a prime. Then $\begin{pmatrix} 0 & 4 \\ -4 & 0 \end{pmatrix}$ occurs in the block diagonal form of the defining matrix J of $\overline{M_q(n, r)}$ if and only if r is odd and $n = z \cdot r$ for some even integer z. In this case, there are $\frac{r-1}{2}$ such blocks.*

5. QUANTIZED MINORS

For each $\ell = 1, \ldots, n$, I_q^{ℓ} denotes the ideal generated by all $\ell \times \ell$ quantum minors. We consider here the function algebra of rank r matrices. Specifically, let

$$M_q^{r+1}(n) = M_q(n)/I_q^{r+1}. \tag{5.1}$$

For each $t = 1, \dots, n$, let d_t denote the $t \times t$ quantum determinant of the subalgebra generated by the elements $Z_{i,j}$ with $1 \le i, j \le t$. Set $d = d_r$. The natural candidate for quantized rank r matrices is then

$$M_q^{r+1}(n)[d^{-1}], \tag{5.2}$$

where we shall return to the issue of inverting d shortly.

We wish to compare this algebra to a somewhat more manageable one, namely $A_{n,r}$, where

DEFINITION 5.1. The algebra $A_{n,r}$ is the subalgebra of $M_q(n)$ generated by those $Z_{i,j}$ for which $(i, j) \notin \{r+1, \dots, n\} \times \{r+1, \dots, n\}$.

In [9], Goodearl and Lenagan proved that I_q^{r+1} is completely prime. Moreover, Rigal proved that $A_{n,1}[d^{-1}] \simeq M_q^2(n)[d^{-1}]$ [24]. We shall prove below that

$$A_{n,r}[d^{-1}] \simeq M_q^{r+1}(n)[d^{-1}] \tag{5.3}$$

for a general r and on the way give a new proof of the former result.

For (5.2) to make sense we first of all need the following:

PROPOSITION 5.2. *$d = d_r$ is regular in $M_q(n)/I_q^{r+1}$ both in the case of q generic and the case where $q = \varepsilon$ is a primitive mth root of unity.*

Proof. It is proved in [9, Theorem 2.5] that $M_q(n)/I_q^{r+1}$ is a domain. Since d clearly is non-zero in $M_q(n)/I_q^{r+1}$ we get the result. ∎

We now offer an alternative argument in the generic case. We need this result to prove (5.3) in case q is generic. Later on, we also obtain (5.3) in the case of a primitive mth root of unity by ring theoretic methods.

Proof of Proposition 5.2 for q Generic. Our proof relies on representation theory. First of all, for this case it was proved in [22] that $M_q(n)$ is a bi-module of a version $U_q(gl(n, \mathbb{C}))$ of the quantized enveloping algebra of $gl(n, \mathbb{C})$. Essentially, this version is what results if one starts from the quantized Serre relations and view the q entering there as a complex number. Furthermore, it is assumed that $q \neq 0$ and that q is not a root of unity.

The results obtained by [22] reveal that the same general picture holds as in the well-known case for $q = 1$ [2]. Specifically, each I_q^s is a $U_q(gl(n, \mathbb{C}))$ sub-bi-module. Moreover, there is a decomposition

$$M_q(n) = \bigoplus_{\lambda} W(\lambda) \tag{5.4}$$

as a bi-module. Here, each $W(\lambda)$ is an irreducible $U_q(gl(n)) \times U_q(gl(n))$ module. The highest weight vector in W_λ is given by

$$w_\lambda = d_1^{a_1} \cdot d_2^{a_2} \cdots d_s^{a_s} \tag{5.5}$$

for $1 \le s \le n$ and $a_1, a_2, \ldots, a_s \in \mathbb{N} \cup \{0\}$. For each $i \in \{1, \ldots, n\}$ let

$$\lambda_i = (\underbrace{1, \ldots, 1}_{i}, 0, \ldots, 0).$$

Then the weight λ of the w_λ in (5.5) is given by $\lambda = a_1\lambda_1 + \cdots + a_s\lambda_s$. There are no multiplicities.

Furthermore, each I_q^s is invariant. If $W_{q,s}$ denotes the direct sum of the highest weight modules whose highest weight vectors are of the form $d_1^{a_1} \cdot d_2^{a_2} \cdots d_s^{a_{\hat{s}}}$ with $\hat{s} \le s$, then this is precisely equal to $I_q^s \setminus I_q^{s+1}$.

We are now ready to prove that d is regular: *Suppose that* $d \cdot u \in I_q^{r+1}$. *Then* $u \in I_q^{r+1}$. Assume that $u_q \notin I_q^{r+1}$. With no loss of generality we may assume that $u_q \in W_{q,r}$ and $u_q \neq 0$. Observe that $d = d_q$ is a primitive vector for all $q \neq 0$. The 2-sided action of the Borel subalgebra $\mathscr{U}_q^+(gl(n))$ at q generic then preserves the general form of (5.4) and hence we may assume that u_q is a sum of highest weight vectors of different highest weights, and using weight considerations, we may assume that u_q is a single highest weight vector. This is then of the form (5.5) with $s = r$. But we must still have that $d \cdot u_q \in I_q^{r+1}$ and this is a contradiction. ∎

Below we show that the powers of d can be inverted in a manageable manner.

Let π be the natural homomorphism from $A_{n,r}$ to $M_q^{r+1}(n)$.

PROPOSITION 5.3. *Let* $S = \{q^{-i}d^j \mid i, j = 0, 1, 2, \ldots\}$. *Then* $\pi(S)$ *is an Ore set of regular elements* $M_q^{r+1}(n)$.

Proof. Since $\pi(d)$ is regular, it suffices to prove that S is an Ore set in $M_q(n)$. Clearly, S is an Ore set in $\mathbb{C}\{Z_{i,j} \mid 1 \le i, j \le r\}$ since d is central in that algebra. The remaining indeterminates are now added in a suitable order, i.e., in such a way that $M_q(n)$ is an iterated Ore extension of $\mathbb{C}\{Z_{i,j} \mid 1 \le i, j \le r\}$. If $\alpha_{i,j}$ denotes the automorphism corresponding to $Z_{i,j}$, then either $\alpha_{i,j}(d) = d$ or $\alpha_{i,j}(d) = q^{-1}d$. The result then follows by [8, Lemma 1.4]. ∎

In case q is a primitive mth root of unity d^m is central by [23, Lemma 7.23] and the localization is then a central localization.

It follows now that π induces a homomorphism $\pi[S^{-1}]$: $A_{n,r}[S^{-1}] \mapsto M_q^{r+1}[S^{-1}]$, which is onto since for each $r < i \le n$ or $r < j \le n$ there exists a $t_{i,j} \in A_{n,r}$ such that $d \cdot Z_{i,j} + t_{i,j}$ is a quantum $(r+1) \times (r+1)$ minor.

We can now prove

PROPOSITION 5.4. *The natural homomorphism* $\pi\colon A_{n,r} \mapsto M_q^{r+1}(n)$, *is injective*.

Proof. We first give the details for the "q generic" case.

Under the bi-module action of $U_q(gl(n,\mathbb{C}))$ on $M_q(n)$, the algebra $A_{n,r}$ is invariant under a Borel subalgebra from one side and under the opposite Borel subalgebra from the other side. Since I_q^{r+1} is invariant we may argue exactly as in the proof of Proposition 5.2. Thus, if there is a non-zero element $p \in A_{n,r}$ such that $\pi(p) \in I_q^{r+1}$, there is also a non-zero highest weight vector $p_h \in A_{n,r}$ such that $\pi(p_h) \in I_q^{r+1}$. But then p_h is of the form (5.5) with $s \geq r+1$ and by looking at leading terms in these quantized determinants, this is easily seen to be impossible.

Next suppose q is a primitive mth root of unity.

Here all the algebras

$$A_{n,r},\ A_{n,r}[d^{-1}],\ M_q^{r+1}(n), \text{ and } M_q^{r+1}(n)[d^{-1}]$$

are prime affine P.I. algebras.

We recall that for such algebras Krull-dimension, transcendence degree and G-K-dimension coincide [21, Proposition 10.6]. In the sequel, dim denotes one of these.

By the definition of transcendence degree any affine algebra between the original algebra, B, and the simple quotient algebra will have the same dimension as B.

By [21, Proposition 6.5.4]

$$\dim A_{n,r}[d^{-1}] \leq n^2 - (n-r)^2,$$

because it is a localization of a $n^2 - (n-r)^2$ iterated Ore extension.

By a recent result [14]

$$\dim M_q^{r+1}(n) \geq n^2 - (n-r)^2,$$

and therefore

$$\dim M_q^{r+1}(n)[d^{-1}] \geq n^2 - (n-r)^2.$$

If Ker $\pi[S^{-1}] \neq 0$ then the Krull-dimension of $\pi[S^{-1}](A_{n,r}[S^{-1}])$ must be strictly less than that of $A_{n,r}[S^{-1}]$; therefore $\pi[S^{-1}]$ is injective and so is π. ∎

COROLLARY 5.5. *For q either generic or a primitive root of unity*,

$$A_{n,r}[d^{-1}] \simeq M_q^{r+1}(n)[d^{-1}].$$

COROLLARY 5.6. *If q is a primitive root of unity,*

$$\deg M_q^{r+1} = \deg A_{n,r}.$$

6. QUANTIZED FACTOR ALGEBRAS OF $M_q(n)$

In this section we determine the degree of $A_{n,r}$.

THEOREM 6.1. *If q is an odd primitive mth root of unity then* $\deg A_{n,r} = m^{nr-r(r+1)/2}$.

Proof. We fix $r \geq 1$ and use induction on $n \geq r$.

If $n = r$ the formula holds by Theorem 3.2 in Section 3. A closer look at the beginning of the proof of that result yields the validity of the formula for $n = r + 1$ also.

We view $A_{n,r}$ as an iterated Ore extension:

$$\begin{aligned} A_{n,r} = A_{n-1,r}[Z_{1,n}; \alpha_{1,n}] \cdots [Z_{r,n}; \alpha_{r,n}, \delta_{r,n}] \\ \cdot [Z_{n,1}; \alpha_{n,1}] \cdots [Z_{n,r}; \alpha_{n,r}, \delta_{n,r}], \end{aligned} \tag{6.1}$$

The general strategy of the proof is similar to the proof of Theorem 3.2. We begin by adjoining the variables $Z_{1,n} \cdots Z_{r,n}$ to $A_{n-1,r}$. Let $A_{n-1,r}^{(i)}$ denote the algebra obtained by adjoining $Z_{1,n}, \dots Z_{i,n}$ so that $A_{n-1,r}^{(0)} = A_{n-1,r}$. We show that there exist suitable central elements $c_1, \dots, c_r$ in $A_{n-1,r}$ which behave nicely under the automorphisms induced by each of the variables $Z_{1,n} \cdots Z_{r,n}$. This makes it possible to construct a central element for each $A_{n-1,r}^{(i)}$ which has an mth order orbit under $\alpha_{i+1,n}$. Thus, an application of Proposition 2.14 is possible with the conclusion that the degree of $A_{n-1,r}^{(r)} = A_{n-1,r}[Z_{1,n}; \alpha_{1,n}] \cdots [Z_{r,n}; \alpha_{r,n}, \delta_{r,n}]$ is m^r times the degree of $A_{n-1,r}$.

After that we construct r central elements of $A_{n,r}$ In the general situation they will have the same shape as those for $A_{n-1,r}$. For each $Z_{n,i}$ there will in fact be a central element of $A_{n,r}$ which is an inhomogeneous polynomial of degree 1 in that variable. Thus, Lemma 2.12 implies that the degree does not go up by adjoining $Z_{n,1}, \dots, Z_{n,r}$ to $A_{n-1,r}^{(r)}$. Having established this, the theorem is proved.

To make the argument clearer we consider the algebra in a diagrammatic fashion; see Fig. 2.

We will first consider the cases where $r + 1 \leq n < 3r$. Extend the previous definitions of $\theta_k, \tilde{\theta}_t$ as follows: For $k > n + 1 - r$, let θ_k denote the quantum determinant of the sub-algebra generated by

$$Z_{1,k}, \dots, Z_{1,n}, \dots, Z_{n-k+1,k}, \dots Z_{n-k+1,n}. \tag{6.2}$$

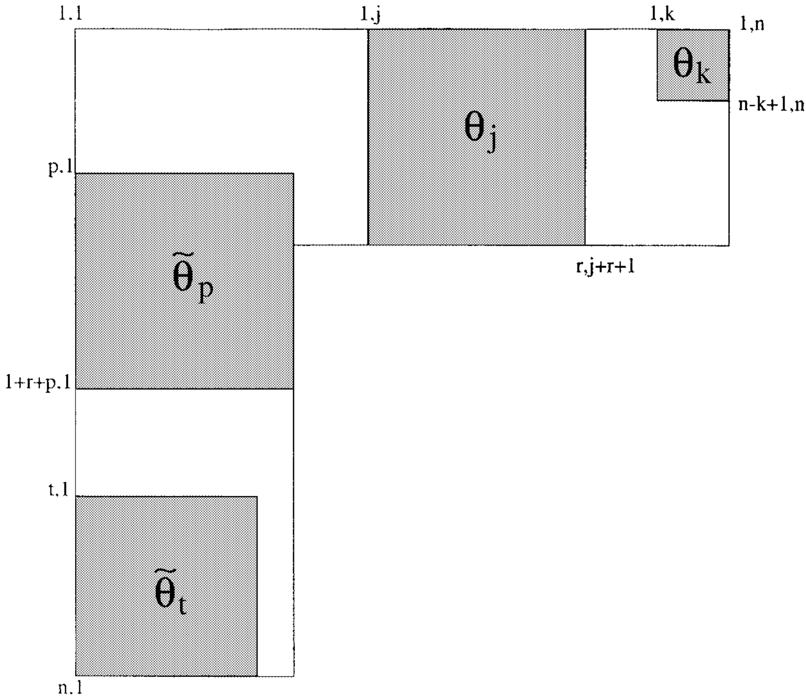

FIG. 2. The various building blocks.

For $j \leq n+1-r$, θ_j denotes the quantum determinant (quantum r-minor) of the algebra generated by $Z_{1,j}, \dots, Z_{1,j+r-1}, \dots, Z_{r,j}, \dots, Z_{r,j+r-1}$.

Elements $\tilde{\theta}_j$ are defined analogously for $1 \leq j \leq n$ by interchanging all $Z_{i,j}$ with $Z_{j,i}$ in θ_j. It follows easily that the θ_i and $\tilde{\theta}_i$ are covariant for all $i = 1, \dots, n$. Notice that $Z_{i,j}\theta_k = \theta_k Z_{i,j}$ in case $Z_{i,j}$ belongs to the subalgebra involved in defining θ_k. Also, if $Z_{i,j}$ is not in the subalgebra used to define θ_k then $Z_{i,j}\theta_k = q\theta_k Z_{i,j}$ if $j < k$ and $i \leq \min\{r, n-k+1\}$. Similarly, $Z_{i,j}\theta_k = q^{-1}\theta_k Z_{i,j}$ in case there exists a $Z_{a,b}$ used in the definition of θ_k such that $(i,j) = (a, b+x)$ or $(i,j) = (a+x, b)$ for some positive integer x.

In order to make the arguments and ideas as clear as possible, we will first consider the case where we, starting from $\mathscr{A}_{r+1,r}$, determine the degree of $\mathscr{A}_{r+2,r}$. We begin by adjoining the elements $Z_{1,r+2}, \dots, Z_{r,r+2}$ to $\mathscr{A}_{r+1,1}$ in order of increasing first index. Thus, we get a sequence of skewpolynomial algebras $\mathscr{A}^{(i)}_{r+1,r}$ for $i = 1, \ldots, r$. For convenience, let $\mathscr{A}^{(0)}_{r+1,r} = \mathscr{A}_{r+1,r}$. Now, since $\mathscr{A}_{r+1,r} \subset M_q(r+1)$ we can use those central elements of the latter that do not involve $Z_{r+1,r+1}$. In this way we get r central elements,

$$c_1 = \theta_{r+1}\left(\tilde{\theta}_2^{-1}\right), \ldots, c_r = \theta_2\left(\tilde{\theta}_{r+1}\right)^{-1}, \tag{6.3}$$

where we write $(\tilde{\theta}_j)^{-1}$ for $(\tilde{\theta}_j)^{m-1}$.

We let α_j denote the automorphism connected with adjoining $Z_{j,r+2}$ to $\mathscr{A}^{(j-1)}_{r+1,r}$ for $j = 1, \ldots, r$. Clearly, these automorphisms have order m when acting on the relevant full algebras.

We get

$$\begin{array}{llcl} \alpha_1(c_1) = q^{-1}c_1 & \alpha_1(c_2) = q^{-1}c_2 & \cdots & \alpha_1(c_r) = q^{-1}c_r \\ & \alpha_2(c_2) = q^{-1}c_2 & \cdots & \alpha_2(c_r) = q^{-1}c_r \\ & & \vdots & \vdots \\ & & & \alpha_r(c_r) = q^{-1}c_r. \end{array} \tag{6.4}$$

Since c_1 is central in $\mathscr{A}_{r+1,r}$, and since the length of the orbit of α_1's action on c_1 is m, we get that adjoining $Z_{1,r+2}$ raises the degree by a factor m. Next, $(c_1c_2^{-1})$ is clearly central in $\mathscr{A}^{(1)}_{r+1,r}$ and $\alpha_2(c_1c_2^{-1}) = q^2c_1c_2^{-1}$. Thus, since m is odd, we get that when we adjoin $Z_{2,r+2}$ to the previously constructed algebra the degree again goes up by a factor m.

Replacing $c_1c_2^{-1}$ by $c_jc_{j+1}^{-1}$ and repeating the argument, we get

$$\deg \mathscr{A}^{(r)}_{r+1,r} = m^r \deg \mathscr{A}_{r+1,r}. \tag{6.5}$$

We will now construct r central elements of $\mathscr{A}_{r+2,r}$. The actual form of these will, in connection with Lemma 2.12, imply that adjoining $Z_{r+2,1}, \ldots, Z_{r+2,r}$ to $\mathscr{A}^{(r)}_{r+1,r}$ does not increase the degree.

First of all, there are $r-1$ central elements coming from $M_q(r+2)$:

$$c_j = \left(\tilde{\theta}_{r+3-j}\right)\theta_{j+1}^{-1} \qquad \text{for } j = 2, \ldots, r. \tag{6.6}$$

The remaining central element c_1 has got to involve $Z_{r+2,1} = \tilde{\theta}_{r+2}$. We claim that the element $c_1 = \theta_{r+2}\theta_2^{-1}\tilde{\theta}_{r+2}(\tilde{\theta}_2)^{-1}\theta_1$ fulfills the requirements. First of all, clearly $Z_{i,j}c_1 = q^{\alpha_{i,j}}c_1Z_{i,j}$ for all i, j. To prove commutativity in all details would involve checking that the five factors of c_1 are in such a balance with each other that the $q^{\alpha_{i,j}}$, while being the product of five terms of the form q^{*k} and while each q^{*k} depends on the actual form of (i, j), in the end always equals q^0. We leave the somewhat tedious (and somewhat amusing) details of this, as well as similar claims later on, to the reader.

We can now start adjoining the elements $Z_{r+2,i}$. Since for each i, $Z_{r+2,i}$ occurs in the summands of c_i to either the power 1 or 0, Lemma 2.12 applies and the degree remains unchanged.

Now suppose $r + 1 \leq n \leq 2r$.

We have $2r - n + 1$ central elements from $M_q(n)$,

$$c_{j+1} = \theta_{n-j}\left(\tilde{\theta}_{j+2}\right)^{-1}; \qquad j = n - r - 1, \dots, r - 1. \tag{6.7}$$

The remaining $n - r - 1$ central elements can be chosen as

$$c_{n+1-j} = \theta_1 \theta_j \left(\tilde{\theta}_{n-j+2}\right)^{-1} \left(\theta_{j-r}\right)^{-1} \left(\tilde{\theta}_{n-j+r+2}\right), \quad \text{where } r + 1 < j \leq n. \tag{6.8}$$

As far as the proof goes, these central elements have the same properties as the previously constructed. Hence, our strategy applies and we get that $\deg \mathscr{A}_{n+1,r} = m^r \deg \mathscr{A}_{n,r}$.

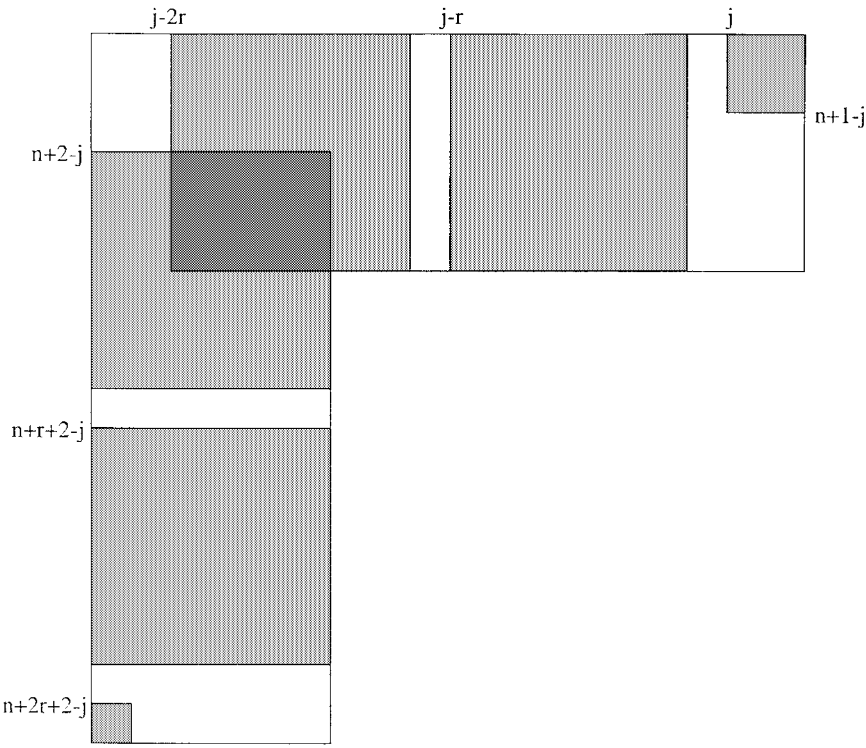

FIG. 3. Region covered by $\theta_j(\theta_{j-r})^{-1}(\theta_{j-2r})(\tilde{\theta}_{n-j+2})^{-1}(\tilde{\theta}_{n-j+2+r})(\tilde{\theta}_{n-j+2+2r})^{-1}$.

In the cases $2r < n \leq 3r$ there are no central elements coming from $M_q(n)$, but there are 2 types of central elements still yielding a total of r central elements. Specifically for each j with $n - r + 1 \leq j \leq 2r + 1$ we have the central elements

$$\theta_j(\theta_{j-r})^{-1}\theta_1(\tilde{\theta}_{n-j+2})^{-1}(\tilde{\theta}_{n-j+2+r}). \tag{6.9}$$

The remaining central elements where $j > 2r + 1$ can be obtained by (see Fig. 3)

$$\theta_j(\theta_{j-r})^{-1}(\theta_{j-2r})(\tilde{\theta}_{n-j+2})^{-1}(\tilde{\theta}_{n-j+2+r})(\tilde{\theta}_{n-j+2+2r})^{-1}. \tag{6.10}$$

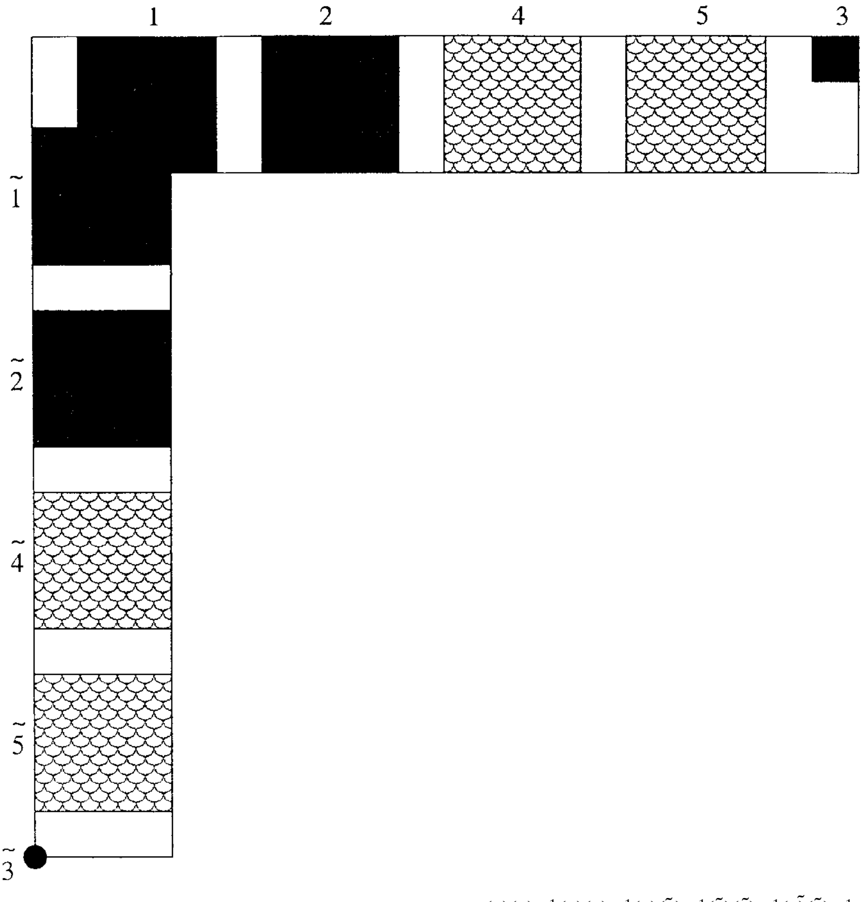

FIG. 4. A central element in the general region: $(3)(5)^{-1}(4)(2)^{-1}(1)(\tilde{1})^{-1}(\tilde{2})(\tilde{4})^{-1}(\tilde{5})(\tilde{3})^{-1}$. Compare with Fig. 2.

Now let us finally comment on the case of a general $n > 3r$: Here one can easily construct r central elements from the previous recipes. Indeed, observe that each previously constructed central element contains exactly one pair of factors $\theta, \tilde{\theta}$ which are not full $r \times r$ quantum determinants. If, say, θ is $i \times i$ then $\tilde{\theta}$ is $(r-i) \times (r-i)$. Even more precisely, $\theta = \theta_{n-i+1}$ and $\tilde{\theta} = \tilde{\theta}_{n-r+i+1}$. Then a number of factors of the form $(\theta_{n+1-i-k\cdot r})^{\pm 1}$ and $(\hat{\theta}_{n-r+i+1-k\cdot r})^{\pm 1}$ are inserted for $k \in \mathbb{N}$—as long as the resulting indices are positive. Clearly this procedure continues to work for a general n. We omit the finer details and refer to Fig. 4. That concludes the proof. ∎

Analogously to Proposition 4.11 one gets

PROPOSITION 6.2. *The non-trivial blocks in a block diagonal form of the defining matrix* J_A *of* $\overline{A_{n,r}}$ *are* $n-1$ *matrices of the form* $\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$ *and* $nr - (r(r+1)/2) - (n-1)$ *matrices of the form* $\left(\begin{smallmatrix} 0 & 2 \\ -2 & 0 \end{smallmatrix}\right)$. *Let* $m' = m$ *if* m *is odd and* $m' = \frac{m}{2}$ *if* m *is even. Then, in particular, the degree is* $m^{n-1}(m')^{nr-(r(r+1)/2)-(n-1)}$.

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