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Schatten p-norm inequalities related to an extended operator parallelogram law

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ABSTRACT

Let $\mathcal{C}_p$ be the Schatten p-class for $p > 0$. Generalizations of the parallelogram law for the Schatten 2-norms have been given in the following form: if $\mathbf{A} = \{A_1, A_2, \ldots, A_n\}$ and $\mathbf{B} = \{B_1, B_2, \ldots, B_n\}$ are two sets of operators in $\mathcal{C}_2$, then

$$\sum_{i,j=1}^{n} \|A_i - A_j\|_2^2 + \sum_{i,j=1}^{n} \|B_i - B_j\|_2^2 = 2\sum_{i,j=1}^{n} \|A_i - B_j\|_2^2 - 2\left\|\sum_{i=1}^{n}(A_i - B_i)\right\|_2^2.$$

In this paper, we give generalizations of this as pairs of inequalities for Schatten p-norms, which hold for certain values of p and reduce to the equality above for $p = 2$. Moreover, we present some related inequalities for three sets of operators.

1. Introduction

Suppose that $\mathbb{B}(\mathcal{H})$ denotes the algebra of all bounded linear operators on a separable complex Hilbert space $\mathcal{H}$ endowed with an inner product $\langle \cdot, \cdot \rangle$. Let $A \in \mathbb{B}(\mathcal{H})$ be a compact operator and

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let $\{s_j(A)\}$ denote the sequence of decreasingly ordered singular values of A, i.e. the eigenvalues of $|A| = (A^*A)^{1/2}$. The Schatten p-norm (p-quasi-norm, respectively) for $1 \le p < \infty$ ($0 < p < 1$, respectively) is defined by

$$\|A\|_p = \left(\sum_{j=1}^{\infty} s_j^p(A)\right)^{1/p}.$$

For $p > 0$, the Schatten p-class, denoted by $\mathcal{C}_p$, is defined to be the two-sided ideal in $\mathbb{B}(\mathcal{H})$ of those compact operators A for which $\|A\|_p$ is finite. Clearly

$$\left\| \, |A|^2 \, \right\|_{p/2} = \|A\|_p^2 \tag{1.1}$$

for $p > 0$. In particular, $\mathcal{C}_1$ and $\mathcal{C}_2$ are the trace class and the Hilbert–Schmidt class, respectively. For $1 \le p < \infty$, $\mathcal{C}_p$ is a Banach space; in particular the triangle inequality holds. For $0 < p < 1$, the quasi-norm $\|\cdot\|_p$ does not satisfy the triangle inequality, but however satisfies the inequality $\|A+B\|_p^p \leqslant \|A\|_p^p + \|B\|_p^p$. For more information on the theory of Schatten p-norms the reader is referred to [8, Chapter 2].

It follows from [7, Corollary 2.7] that for $A_1, \ldots, A_n, B_1, \ldots, B_n \in \mathbb{B}(\mathcal{H})$

$$\sum_{i,j=1}^{n} \|A_i - A_j\|^2 + \sum_{i,j=1}^{n} \|B_i - B_j\|^2 = 2\sum_{i,j=1}^{n} \|A_i - B_j\|^2 - 2\left\|\sum_{i=1}^{n}(A_i - B_i)\right\|^2, \tag{1.2}$$

which is indeed a generalization of the classical *parallelogram law*:

$$|z+w|^2 + |z-w|^2 = 2|z|^2 + 2|w|^2 \quad (z, w \in \mathbb{C}).$$

There are several extensions of parallelogram law among them we could refer the interested reader to [2,3,6,9,10]. Generalizations of the parallelogram law for the Schatten p-norms have been given in the form of the celebrated Clarkson inequalities (see [4] and references therein). Since $\mathcal{C}_2$ is a Hilbert space under the inner product $\langle A, B\rangle = \mathrm{tr}(B^*A)$, it follows from an equality similar to (1.2) stated for vectors of a Hilbert space (see Corollary 2.7 [7]) that if $A_1, \ldots, A_n, B_1, \ldots, B_n \in \mathcal{C}_2$, then

$$\sum_{i,j=1}^{n} \|A_i - A_j\|_2^2 + \sum_{i,j=1}^{n} \|B_i - B_j\|_2^2 = 2\sum_{i,j=1}^{n} \|A_i - B_j\|_2^2 - 2\left\|\sum_{i=1}^{n}(A_i - B_i)\right\|_2^2. \tag{1.3}$$

In [7] a joint operator extension of the Bohr and parallelogram inequalities is presented. In particular, it follows from [7, Corollary 2.3] that if $A_1, \ldots, A_n, B_1, \ldots, B_n \in \mathbb{B}(\mathcal{H})$, then

$$\sum_{1\le i<j\le n} |A_i - A_j|^2 + \sum_{1\le i<j\le n} |B_i - B_j|^2 = \sum_{i,j=1}^{n} |A_i - B_j|^2 - \left|\sum_{i=1}^{n}(A_i - B_i)\right|^2. \tag{1.4}$$

In this paper, we give a generalization of the equality (1.3) for the Schatten p-norms ($p > 0$). First we present similar consideration for three sets of operators. In addition, we provide pairs of complementary inequalities that reduce to (1.3) for the certain value $p = 2$.

2. Schatten *p*-norm inequalities

To accomplish our results, we need the following lemma which can be deduced from [1, Lemma 4] and [8, p. 20]:

Lemma A. *Let $A_1, \ldots, A_n \in \mathcal{C}_p$ for some $p > 0$. If $A_1, \ldots, A_n$ are positive, then*

$$n^{p-1}\sum_{i=1}^{n}\|A_i\|_p^p \le \left\|\sum_{i=1}^{n}A_i\right\|_p^p \le \sum_{i=1}^{n}\|A_i\|_p^p \tag{2.1}$$

for $0 < p \le 1$ and the reverse inequalities hold for $1 \leqslant p < \infty$.

Let us define a constant $D_{\mathbf{A}}$ for a set of operators $\mathbf{A} = \{A_1, A_2, \ldots, A_n\}$ as follows:

$$D_{\mathbf{A}} := \sum_{i=1}^{n}\delta(A_i) \quad \text{where} \quad \delta(A_i) = \begin{cases} 1 & (A_i \neq 0) \\ 0 & (A_i = 0) \end{cases}.$$

If there exists $1 \le i \le n$ with $A_i = 0$, then Lemma A is refined as follows:

$$D_{\mathbf{A}}^{p-1}\sum_{i=1}^{n}\|A_i\|_p^p \le \left\|\sum_{i=1}^{n}A_i\right\|_p^p \le \sum_{i=1}^{n}\|A_i\|_p^p \tag{2.2}$$

for $0 < p \le 1$ and the reverse inequalities holds for $1 \leqslant p < \infty$.

We also put $\mathbf{A} - \mathbf{B} := \{A_i - B_j : 1 \le i, j \le n\}$ for sets of operators $\mathbf{A} = \{A_1, A_2, \ldots, A_n\}$ and $\mathbf{B} = \{B_1, B_2, \ldots, B_n\}$. Then we remark that $0 \le D_{\mathbf{A}-\mathbf{B}} \le n^2$.

Now we give our main results that involve three sets of operators.

Theorem 2.1. *Let $\mathbf{A} = \{A_1, A_2, \ldots, A_n\}$, $\mathbf{B} = \{B_1, B_2, \ldots, B_n\}$, $\mathbf{C} = \{C_1, C_2, \ldots, C_n\} \subset \mathcal{C}_p$ for some $p > 0$. Then*

$$\begin{aligned} &\sum_{i,j=1}^{n}\|A_i - A_j\|_p^p + \sum_{i,j=1}^{n}\|B_i - B_j\|_p^p + \sum_{i,j=1}^{n}\|C_i - C_j\|_p^p \\ &\ge \left(D_{\mathbf{A}-\mathbf{B}}^{\frac{p-2}{2}}\sum_{i,j=1}^{n}\|A_i - B_j\|_p^p + D_{\mathbf{B}-\mathbf{C}}^{\frac{p-2}{2}}\sum_{i,j=1}^{n}\|B_i - C_j\|_p^p + D_{\mathbf{C}-\mathbf{A}}^{\frac{p-2}{2}}\sum_{i,j=1}^{n}\|C_i - A_j\|_p^p\right) \\ &\quad - \left(\left\|\sum_{i=1}^{n}(A_i - B_i)\right\|_p^p + \left\|\sum_{i=1}^{n}(B_i - C_i)\right\|_p^p + \left\|\sum_{i=1}^{n}(C_i - A_i)\right\|_p^p\right) \end{aligned} \tag{2.3}$$

for $0 < p \le 2$ and the reverse inequality holds for $2 \le p < \infty$.

Proof. We only prove the case when $0 < p \le 2$. The other case can be proved by a similar argument. We have

$$\begin{aligned} &\sum_{i,j=1}^{n}\|A_i - A_j\|_p^p + \sum_{i,j=1}^{n}\|B_i - B_j\|_p^p + \sum_{i,j=1}^{n}\|C_i - C_j\|_p^p \\ &\quad + \left(\left\|\sum_{i=1}^{n}(A_i - B_i)\right\|_p^p + \left\|\sum_{i=1}^{n}(B_i - C_i)\right\|_p^p + \left\|\sum_{i=1}^{n}(C_i - A_i)\right\|_p^p\right) \\ &= 2\left(\sum_{1\le i<j\le n}\|A_i - A_j\|_p^p + \sum_{1\le i<j\le n}\|B_i - B_j\|_p^p + \sum_{1\le i<j\le n}\|C_i - C_j\|_p^p\right) \end{aligned}$$

$$+\left(\left\|\sum_{i=1}^{n}(A_i-B_i)\right\|_p^p+\left\|\sum_{i=1}^{n}(B_i-C_i)\right\|_p^p+\left\|\sum_{i=1}^{n}(C_i-A_i)\right\|_p^p\right)$$

$$=2\left(\sum_{1\le i<j\le n}\left\||A_i-A_j|^2\right\|_{p/2}^{p/2}+\sum_{1\le i<j\le n}\left\||B_i-B_j|^2\right\|_{p/2}^{p/2}+\sum_{1\le i<j\le n}\left\||C_i-C_j|^2\right\|_{p/2}^{p/2}\right)$$

$$+\left(\left\|\left|\sum_{i=1}^{n}(A_i-B_i)\right|^2\right\|_{p/2}^{p/2}+\left\|\left|\sum_{i=1}^{n}(B_i-C_i)\right|^2\right\|_{p/2}^{p/2}+\left\|\left|\sum_{i=1}^{n}(C_i-A_i)\right|^2\right\|_{p/2}^{p/2}\right)$$

(by relation (1.1))

$$\ge\left\|\sum_{1\le i<j\le n}|A_i-A_j|^2+\sum_{1\le i<j\le n}|B_i-B_j|^2+\left|\sum_{i=1}^{n}(A_i-B_i)\right|^2\right\|_{p/2}^{p/2}$$

$$+\left\|\sum_{1\le i<j\le n}|B_i-B_j|^2+\sum_{1\le i<j\le n}|C_i-C_j|^2+\left|\sum_{i=1}^{n}(B_i-C_i)\right|^2\right\|_{p/2}^{p/2}$$

$$+\left\|\sum_{1\le i<j\le n}|C_i-C_j|^2+\sum_{1\le i<j\le n}|A_i-A_j|^2+\left|\sum_{i=1}^{n}(C_i-A_i)\right|^2\right\|_{p/2}^{p/2}$$

(by the second inequality of (2.1))

$$=\left\|\sum_{i,j=1}^{n}|A_i-B_j|^2\right\|_{p/2}^{p/2}+\left\|\sum_{i,j=1}^{n}|B_i-C_j|^2\right\|_{p/2}^{p/2}+\left\|\sum_{i,j=1}^{n}|C_i-A_j|^2\right\|_{p/2}^{p/2}$$

(by (1.4))

$$\ge D_{\mathbf{A}-\mathbf{B}}^{\frac{p}{2}-1}\sum_{i,j=1}^{n}\left\||A_i-B_j|^2\right\|_{p/2}^{p/2}+D_{\mathbf{B}-\mathbf{C}}^{\frac{p}{2}-1}\sum_{i,j=1}^{n}\left\||B_i-C_j|^2\right\|_{p/2}^{p/2}+D_{\mathbf{C}-\mathbf{A}}^{\frac{p}{2}-1}\sum_{i,j=1}^{n}\left\||C_i-A_j|^2\right\|_{p/2}^{p/2}$$

(by the first inequality of (2.2))

$$=D_{\mathbf{A}-\mathbf{B}}^{\frac{p-2}{2}}\sum_{i,j=1}^{n}\|A_i-B_j\|_p^p+D_{\mathbf{B}-\mathbf{C}}^{\frac{p-2}{2}}\sum_{i,j=1}^{n}\|B_i-C_j\|_p^p+D_{\mathbf{C}-\mathbf{A}}^{\frac{p-2}{2}}\sum_{i,j=1}^{n}\|C_i-A_j\|_p^p.$$

So we have the desired inequality (2.3). □

The next result can be regarded as a generalization of (1.3).

Proposition 2.2. *Let $A_1,\ldots,A_n,B_1,\ldots,B_n\in\mathcal{C}_p$ for some $p>0$. Then*

$$\sum_{i,j=1}^{n}\|A_i-A_j\|_p^p+\sum_{i,j=1}^{n}\|B_i-B_j\|_p^p\ge 2n^{p-2}\sum_{i,j=1}^{n}\|A_i-B_j\|_p^p-2\left\|\sum_{i=1}^{n}(A_i-B_i)\right\|_p^p$$

for $0<p\le 2$ and the reverse inequality holds for $2\le p<\infty$.

Proof. We only prove the case where $0 < p \le 2$. Putting $C_i = 0$ for $i = 1, 2, \ldots, n$, we have

$$D_{\mathbf{B}-\mathbf{C}}^{\frac{p-2}{2}} \sum_{i,j=1}^{n} \|B_i - C_j\|_p^p = nD_{\mathbf{B}}^{\frac{p-2}{2}} \sum_{i=1}^{n} \|B_i\|_p^p, \quad D_{\mathbf{C}-\mathbf{A}}^{\frac{p-2}{2}} \sum_{i,j=1}^{n} \|C_i - A_j\|_p^p = nD_{\mathbf{A}}^{\frac{p-2}{2}} \sum_{i=1}^{n} \|A_i\|_p^p.$$

It follows from Theorem 2.1 that

$$\begin{aligned}
0 &\le \sum_{i,j=1}^{n} \|A_i - A_j\|_p^p + \sum_{i,j=1}^{n} \|B_i - B_j\|_p^p - D_{\mathbf{A}-\mathbf{B}}^{\frac{p-2}{2}} \sum_{i,j=1}^{n} \|A_i - B_j\|_p^p \\
&\quad -n\left(D_{\mathbf{B}}^{\frac{p-2}{2}} \sum_{i=1}^{n} \|B_i\|_p^p + D_{\mathbf{A}}^{\frac{p-2}{2}} \sum_{i=1}^{n} \|A_j\|_p^p\right) + \left(\left\|\sum_{i=1}^{n}(A_i - B_i)\right\|_p^p + \left\|\sum_{i=1}^{n} B_i\right\|_p^p + \left\|\sum_{i=1}^{n} A_i\right\|_p^p\right) \\
&= \sum_{i,j=1}^{n} \|A_i - A_j\|_p^p + \sum_{i,j=1}^{n} \|B_i - B_j\|_p^p - 2D_{\mathbf{A}-\mathbf{B}}^{\frac{p-2}{2}} \sum_{i,j=1}^{n} \|A_i - B_j\|_p^p + 2\left\|\sum_{i=1}^{n}(A_i - B_i)\right\|_p^p \\
&\quad + \left(D_{\mathbf{A}-\mathbf{B}}^{\frac{p-2}{2}} \sum_{i,j=1}^{n} \|A_i - B_j\|_p^p - \left\|\sum_{i=1}^{n}(A_i - B_i)\right\|_p^p\right) \\
&\quad + \left(\left\|\sum_{i=1}^{n} A_i\right\|_p^p - nD_{\mathbf{A}}^{\frac{p-2}{2}} \sum_{i=1}^{n} \|A_i\|_p^p\right) + \left(\left\|\sum_{i=1}^{n} B_i\right\|_p^p - nD_{\mathbf{B}}^{\frac{p-2}{2}} \sum_{i=1}^{n} \|B_i\|_p^p\right).
\end{aligned}$$

Here the inequality (2.2) implies

$$D_{\mathbf{A}-\mathbf{B}}^{\frac{p-2}{2}} \sum_{i,j=1}^{n} \|A_i - B_j\|_p^p - \left\|\sum_{i=1}^{n}(A_i - B_i)\right\|_p^p = D_{\mathbf{A}-\mathbf{B}}^{\frac{p}{2}-1} \sum_{i,j=1}^{n} \left\||A_i - B_j|^2\right\|_{p/2}^{p/2} - \left\|\sum_{i=1}^{n} |A_i - B_i|^2\right\|_{p/2}^{p/2} \le 0$$

by relation (1.1). Due to $nD_{\mathbf{A}}^{\frac{p-2}{2}}$ is no less than 1, we deduce from Lemma A that

$$\left\|\sum_{i=1}^{n} A_i\right\|_p^p - nD_{\mathbf{A}}^{\frac{p-2}{2}} \sum_{i=1}^{n} \|A_i\|_p^p \le \left\|\sum_{i=1}^{n} A_i\right\|_p^p - \sum_{i=1}^{n} \|A_i\|_p^p \le 0.$$

Similarly, we have $\|\sum_{i=1}^{n} B_i\|_p^p \le nD_{\mathbf{B}}^{\frac{p-2}{2}} \sum_{i=1}^{n} \|B_i\|_p^p$. It therefore implies that

$$\begin{aligned}
\sum_{i,j=1}^{n} \|A_i - A_j\|_p^p + \sum_{i,j=1}^{n} \|B_i - B_j\|_p^p &\ge 2D_{\mathbf{A}-\mathbf{B}}^{\frac{p-2}{2}} \sum_{i,j=1}^{n} \|A_i - B_j\|_p^p - 2\left\|\sum_{i=1}^{n}(A_i - B_i)\right\|_p^p \\
&\ge 2n^{p-2} \sum_{i,j=1}^{n} \|A_i - B_j\|_p^p - 2\left\|\sum_{i=1}^{n}(A_i - B_i)\right\|_p^p \\
&\ge 0 \qquad (\text{by } n^2 \ge D_{\mathbf{A}-\mathbf{B}}(\ge 0)).
\end{aligned}$$

Thus we obtain the desired inequality. □

Corollary 2.3. *Let $A_1, \ldots, A_n, B_1, \ldots, B_n \in \mathcal{C}_p$ for some $p > 0$ and $\sum_{i=1}^n A_i = \sum_{i=1}^n B_i$. Then*

$$\sum_{i,j=1}^{n} \|A_i - A_j\|_p^p + \sum_{i,j=1}^{n} \|B_i - B_j\|_p^p \geq 2n^{p-2} \sum_{i,j=1}^{n} \|A_i - B_j\|_p^p$$

for $0 < p \leq 2$ and the reverse inequality holds for $2 \leq p < \infty$.

Utilizing Corollary 2.3 with $B_i = 0\,(1 \leq i \leq n)$, we obtain the following result which is a refinement of [5, Corollary 2.3].

Corollary 2.4. *Let $A_1, \ldots, A_n \in \mathcal{C}_p$ for some $p > 0$ such that $\sum_{i=1}^n A_i = 0$. Then*

$$\sum_{i,j=1}^{n} \|A_i - A_j\|_p^p \geq 2n^{p-1} \sum_{i=1}^{n} \|A_i\|_p^p$$

for $0 < p \leq 2$ and the reverse inequality holds for $2 \leq p < \infty$.

Next, we have the following reverse inequalities of Proposition 2.2:

Proposition 2.5. *Let $A_1, \ldots, A_n, B_1, \ldots, B_n \in \mathcal{C}_p$ for some $p > 0$. Then*

$$2\left(n^2 - n + 1\right)^{\frac{2-p}{2}} \sum_{i,j=1}^{n} \|A_i - B_j\|_p^p - 2\left\|\sum_{i=1}^{n}(A_i - B_i)\right\|_p^p \geq \sum_{i,j=1}^{n} \|A_i - A_j\|_p^p + \sum_{i,j=1}^{n} \|B_i - B_j\|_p^p$$

for $0 < p \leq 2$ and the reverse inequality holds for $2 \leq p < \infty$.

Proof. We only consider the case when $0 < p \leq 2$. We have

$$\begin{aligned}
&\sum_{1 \leq i<j \leq n} \|A_i - A_j\|_p^p + \sum_{1 \leq i<j \leq n} \|B_i - B_j\|_p^p + \left\|\sum_{i=1}^{n}(A_i - B_i)\right\|_p^p \\
&= \sum_{1 \leq i<j \leq n} \left\||A_i - A_j|^2\right\|_{p/2}^{p/2} + \sum_{1 \leq i<j \leq n} \left\||B_i - B_j|^2\right\|_{p/2}^{p/2} + \left\|\left|\sum_{i=1}^{n}(A_i - B_i)\right|^2\right\|_{p/2}^{p/2} \\
&\qquad\qquad \text{(by relation (1.1))} \\
&\leq \left(2 \cdot \frac{n^2 - n}{2} + 1\right)^{1-\frac{p}{2}} \left\|\sum_{1 \leq i<j \leq n} |A_i - A_j|^2 + \sum_{1 \leq i<j \leq n} |B_i - B_j|^2 + \left|\sum_{i=1}^{n}(A_i - B_i)\right|^2\right\|_{p/2}^{p/2} \\
&\qquad\qquad \text{(by the first inequality of (2.1))} \\
&= \left(n^2 - n + 1\right)^{\frac{2-p}{2}} \left\|\sum_{i,j=1}^{n} |A_i - B_j|^2\right\|_{p/2}^{p/2} \qquad \text{(by (1.4))}
\end{aligned}$$

$$\leq \left(n^2 - n + 1\right)^{\frac{2-p}{2}} \sum_{i,j=1}^{n} \left\| |A_i - B_j|^2 \right\|_{p/2}^{p/2} \qquad \text{(by the second inequality of (2.1))}$$

$$= \left(n^2 - n + 1\right)^{\frac{2-p}{2}} \sum_{i,j=1}^{n} \|A_i - B_j\|_p^p.$$

So we have the desired inequality. □

Remark 2.6. (i) By an easily calculation, we have the inequality $n^{p-2} < \left(2n^2 - n + 1\right)^{\frac{2-p}{2}} < \left(n^{p-2}\right)^{-1}$ for $0 < p \leq 2$.

(ii) The values of $2n^{p-2}$, $2\left(n^2 - n + 1\right)^{\frac{2-p}{2}}$ of Propositions 2.2 and 2.5 are 2, if $p = 2$. So these results ensured the equality (1.3).

Finally, we would like to give a problem for further research.

Problem 2.7. What the form of the identity is for the general case of k sets of operators?

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