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The operational matrix of fractional integration for shifted Chebyshev polynomials

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ABSTRACT

A new shifted Chebyshev operational matrix (SCOM) of fractional integration of arbitrary order is introduced and applied together with spectral tau method for solving linear fractional differential equations (FDEs). The fractional integration is described in the Riemann–Liouville sense. The numerical approach is based on the shifted Chebyshev tau method. The main characteristic behind the approach using this technique is that only a small number of shifted Chebyshev polynomials is needed to obtain a satisfactory result. Illustrative examples reveal that the present method is very effective and convenient for linear multi-term FDEs.

1. Introduction

In the recent decades, the applied scientists and the engineers realize that FDEs provided a better approach to describe the complex phenomena in nature, such as non-Brownian motion, signal processing, systems identification, control, viscoelastic materials and polymers (see [1–3] and references therein).

Spectral methods are one of the principal methods of discretization for the numerical solutions of differential equations. The main advantage of these methods lies in their accuracy for a given number of unknowns (see, for instance, [4–6]). For smooth problems in simple geometries, they offer exponential rates of convergence/spectral accuracy.

From the numerical point of view, in [3], Podlubny introduced a numerical approach for the arbitrary order derivative by using the definition of Riemann–Liouville based on the relationship between the Grünwald–Letnikov derivative and the Riemann–Liouville derivative. The Legendre wavelet method is developed and used for solving FDEs in [7]. Moreover, the authors in [8–10] constructed an efficient spectral method for the numerical approximation of the multi-term FDEs based on tau and pseudo-spectral methods. Furthermore, Bhrawy et al. [11] introduced a quadrature shifted Legendre tau method based on Gauss–Lobatto interpolation for solving the multi-order FDEs with variable coefficients.

The operational matrix of fractional derivatives has been determined for some types of orthogonal polynomials, such as Chebyshev polynomials [12], Legendre polynomials [10]. The operational matrix of integration has been determined for several types of orthogonal polynomials, such as Laguerre series [13], Chebyshev polynomials [14], Legendre polynomials [15] and Bessel series [16]. Recently, Singh et al. [17] derived the Bernstein operational matrix of integration. The Bernstein operational matrix approach is developed for solving a system of high order linear Volterra–Fredholm integro-differential equations in [18].

Up until now, and to the best of our knowledge, many formulas corresponding to those mentioned previously are unknown and are traceless in the literature for fractional integration in the Riemann–Liouville sense. This partially motivates

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our interest in operational matrix of fractional integration for shifted Chebyshev polynomials. Another motivation is concerned with the direct solution techniques for solving the integrated forms of FDEs using shifted Chebyshev tau method based on operational matrix of fractional integration.

In this paper, an extension of the operational tau method is proposed to numerically solve the FDEs. The basic idea of this technique is as follows: (i) The FDE is converted to an fully integrated form via multiple fractional integration in the Riemann–Liouville sense. (ii) Subsequently, the various signals involved in the integrated form equation are approximated by representing them as linear combinations of shifted Chebyshev polynomials. (iii) Finally, the integrated form equation is converted to an algebraic equation by introducing the operational matrix of fractional integration of the shifted Chebyshev polynomials. The key idea of the technique depends on the following integral property of the basis vector $\phi(x)$:

$$I^{\nu}\phi(t) \simeq \mathbf{P}^{(\nu)}\phi(x),$$

where $\phi(x) = [T_{L,0}(x), T_{L,1}(x), \dots, T_{L,N}(x)]^T$, in which the elements $T_{L,i}(x)(i = 0, 1, \dots, N)$ are the shifted Chebyshev polynomials on a certain interval $[0, L]$ and $\mathbf{P}^{(\nu)}$ is the operational matrix of fractional integration of $\phi(x)$. Note that $\mathbf{P}^{(\nu)}$ is a constant matrix of order $(N+1)\times(N+1)$ and ν is arbitrary. Finally, the accuracy of the proposed algorithm is demonstrated by test problems.

The paper is organized as follows. In Section 2 we introduce some necessary definitions and give some relevant properties of Chebyshev polynomials. In Section 3 the SCOM of fractional integration is introduced. In Section 4 we apply SCOM of fractional integration for solving linear multi-order FDEs. In Section 5 the proposed method is applied to several examples. Also a conclusion is given in Section 6.

2. Preliminaries and notation

2.1. The fractional integration in the Riemann–Liouville sense

There are several definitions of a fractional integration of order $\nu > 0$, and not necessarily equivalent to each other, see [19]. The most used definition is due to Riemann–Liouville, which is defined as

$$\begin{aligned} I^{\nu}f(x) &= \frac{1}{\Gamma(\nu)}\int_0^x (x-t)^{\nu-1}f(t)dt, \quad \nu > 0, x > 0, \\ I^0 f(x) &= f(x). \end{aligned} \tag{2.1}$$

One of the basic property of the operator I^{ν} is

$$I^{\nu}x^{\beta} = \frac{\Gamma(\beta+1)}{\Gamma(\beta+1+\nu)}\, x^{\beta+\nu}. \tag{2.2}$$

The Riemann–Liouville fractional derivative of order ν will be denoted by D^{ν}. The next equation define Riemann–Liouville fractional derivative of order ν

$$D^{\nu}f(x) = \frac{d^m}{dx^m}(I^{m-\nu}f(x)), \tag{2.3}$$

where $m-1 < \nu \le m$, $m \in N$ and m is the smallest integer greater than ν.

Lemma 2.1. *If $m-1 < \nu \le m$, $m \in N$, then*

$$D^{\nu}I^{\nu}f(x) = f(x), \qquad I^{\nu}D^{\nu}f(x) = f(x) - \sum_{i=0}^{m-1} f^{(i)}(0^+)\frac{x^i}{i!}, \quad x > 0. \tag{2.4}$$

2.2. Properties of shifted Chebyshev polynomials

Let $T_{L,i}(x)$; $x \in (0, L)$ be the shifted Chebyshev polynomials. Then $T_{L,i}(x)$ can be obtained with the aid of the following recurrence formula:

$$T_{L,i+1}(x) = 2\left(\frac{2x}{L}-1\right)T_{L,i}(x) - T_{L,i-1}(x), \quad i = 1, 2, \dots, \tag{2.5}$$

where $T_{L,0}(x) = 1$ and $T_{L,1}(x) = \frac{2x}{L} - 1$. The analytic form of the shifted Chebyshev polynomials $T_{L,i}(x)$ of degree i is given by

$$T_{L,i}(x) = i\sum_{k=0}^{i}(-1)^{i-k}\frac{(i+k-1)!\,2^{2k}}{(i-k)!\,(2k)!\,L^k}\,x^k, \tag{2.6}$$

where $T_{L,i}(0) = (-1)^i$ and $T_{L,i}(L) = 1$. The orthogonality condition is

$$\int_0^L T_{L,j}(x)T_{L,k}(x)w_L(x)dx = h_k\,\delta_{jk}, \tag{2.7}$$

where $w_L(x) = \frac{1}{\sqrt{Lx-x^2}}$ and $h_j = \frac{b_j}{2}\pi$, $b_0 = 2$, $b_j = 1, j \geq 1$.

The special values

$$T_{L,i}^{(q)}(0) = (-1)^{(i-q)}\frac{i\,(i+q-1)!}{\Gamma\left(q+\frac{1}{2}\right)\,(i-q)!\,L^q}\sqrt{\pi}, \quad q \leq i, \tag{2.8}$$

will be of important use later.

A function $u(x)$, square integrable in $(0, L)$, may be expressed in terms of shifted Chebyshev polynomials as

$$u(x) = \sum_{j=0}^{\infty} c_j T_{L,j}(x),$$

where the coefficients c_j are given by

$$c_j = \frac{1}{h_j}\int_0^L u(x)T_{L,j}(x)w_L(x)dx, \quad j = 0, 1, 2, \ldots. \tag{2.9}$$

In practice, only the first $(N + 1)$-terms shifted Chebyshev polynomials are considered. Hence $u(x)$ can be expanded in the form

$$u_N(x) \simeq \sum_{j=0}^{N} c_j T_{L,j}(x) = C^T\phi(x), \tag{2.10}$$

where the shifted Chebyshev coefficient vector C and the shifted Chebyshev vector $\phi(x)$ are given by

$$\begin{aligned} C^T &= [c_0, c_1, \ldots, c_N], \\ \phi(x) &= [T_{L,0}(x), T_{L,1}(x), \ldots, T_{L,N}(x)]^T, \end{aligned} \tag{2.11}$$

If we define the q times repeated integration of Chebyshev vector $\phi(x)$ by $I^q\phi(x)$, then (cf. Paraskevopoulos [14])

$$I^q\phi(x) \simeq \mathbf{P}^{(q)}\phi(x), \tag{2.12}$$

where q is an integer value and $\mathbf{P}^{(q)}$ is the operational matrix of integration of $\phi(x)$. For more details see [14].

3. Operational matrix of fractional integration

The main objective of this section is to generalize the SCOM of integration (2.12) for fractional calculus.

Theorem 3.1. *Let $\phi(x)$ be the shifted Chebyshev vector and $\nu > 0$ then*

$$I^\nu\phi(x) \simeq \mathbf{P}^{(\nu)}\phi(x), \tag{3.1}$$

where $\mathbf{P}^{(\nu)}$ is the $(N + 1) \times (N + 1)$ operational matrix of fractional integration of order ν in the Riemann–Liouville sense and is defined as follows:

$$\mathbf{P}^{(\nu)} = \begin{pmatrix} \Omega_\nu(0,0) & \Omega_\nu(0,1) & \cdots & \Omega_\nu(0,N) \\ \Omega_\nu(1,0) & \Omega_\nu(1,1) & \cdots & \Omega_\nu(1,N) \\ \vdots & \vdots & \cdots & \vdots \\ \Omega_\nu(i,0) & \Omega_\nu(i,1) & \cdots & \Omega_\nu(i,N) \\ \vdots & \vdots & \cdots & \vdots \\ \Omega_\nu(N,0) & \Omega_\nu(N,1) & \cdots & \Omega_\nu(N,N) \end{pmatrix} \tag{3.2}$$

where

$$\Omega_\nu(i,j) = \sum_{k=0}^{i} \frac{(-1)^{i-k}\,2i\,L^\nu\,(i+k-1)!\,\Gamma\left(k+\nu+\frac{1}{2}\right)}{b_j\,\Gamma\left(k+\frac{1}{2}\right)\,(i-k)!\,\Gamma(k+\nu-j+1)\,\Gamma(k+j+\nu+1)}.$$

Proof. The analytic form of the shifted Chebyshev polynomials $T_{L,i}(x)$ of degree i is given by (2.6), Using Eqs. (2.1) and (2.2), and since the Riemann–Liouville's fractional integration is a linear operation, then

$$I^{\nu}T_{L,i}(x) = i\sum_{k=0}^{i} (-1)^{i-k} \frac{(i+k-1)!\,2^{2k}}{(i-k)!\,(2k)!\,L^k} I^{\nu}x^k$$
$$= i\sum_{k=0}^{i}(-1)^{i-k} \frac{(i+k-1)!\,2^{2k}\,k!}{(i-k)!\,(2k)!\,L^k\,\Gamma(k+\nu+1)} x^{k+\nu}, \quad i=0,1,\ldots,N. \tag{3.3}$$

Now, approximate $x^{k+\nu}$ by $N+1$ terms of shifted Chebyshev series, we have

$$x^{k+\nu} = \sum_{j=0}^{N} c_{kj}T_{L,j}(x), \tag{3.4}$$

where c_{kj} is given from (2.9) with $u(x) = x^{k+\nu}$, that is

$$c_{kj} = \begin{cases} \dfrac{1}{\sqrt{\pi}} \dfrac{L^{k+\nu}\,\Gamma\left(k+\nu+\frac{1}{2}\right)}{\Gamma(k+\nu+1)}, & j=0, \\ \dfrac{j\,L^{k+\nu}}{\sqrt{\pi}} \displaystyle\sum_{r=0}^{j}(-1)^{j-r}\, \frac{(j+r-1)!\,2^{2r+1}\,\Gamma\left(k+r+\nu+\frac{1}{2}\right)}{(j-r)!\,(2r)!\,\Gamma(k+r+\nu+1)}, & j=1,2,\ldots,N. \end{cases} \tag{3.5}$$

In virtue of (3.3) and (3.4), we get

$$I^{\nu}T_{L,i}(x) = \sum_{j=0}^{N} \Omega_{\nu}(i,j)T_{L,j}(x), \quad i=0,1,\ldots,N, \tag{3.6}$$

where $\Omega_{\nu}(i,j) = \sum_{k=0}^{i} \zeta_{ijk}$, and

$$\zeta_{ijk} = \begin{cases} \dfrac{i\,(-1)^{i-k}\,L^{\nu}\,(i+k-1)!\,2^{2k}\,k!\,\Gamma\left(k+\nu+\frac{1}{2}\right)}{(i-k)!\,(2k)!\,\sqrt{\pi}\,(\Gamma(k+\nu+1))^2}, & j=0, \\ \dfrac{(-1)^{i-k}\,i\,j\,L^{\nu}\,(i+k-1)!\,2^{2k+1}\,k!}{(i-k)!\,(2k)!\,\Gamma(k+\nu+1)\,\sqrt{\pi}} & \\ \quad\times \displaystyle\sum_{r=0}^{j} \frac{(-1)^{j-r}\,(j+r-1)!\,2^{2r}\,\Gamma\left(k+r+\nu+\frac{1}{2}\right)}{(j-r)!\,(2r)!\,\Gamma(k+r+\nu+1)}, & j=1,2,\ldots N. \end{cases}$$

After some lengthy manipulation, $\zeta_{i,j,k}$ may be put in the following explicit form

$$\zeta_{ijk} = \frac{(-1)^{i-k}\,2i\,L^{\nu}\,(i+k-1)!\,\Gamma\left(k+\nu+\frac{1}{2}\right)}{b_j\,\Gamma\left(k+\frac{1}{2}\right)\,(i-k)!\,\Gamma(k+\nu-j+1)\,\Gamma(k+j+\nu+1)}, \quad j=0,1,\ldots,N, \tag{3.7}$$

where $b_0 = 2$, $b_j = 1, j \geq 1$.

Accordingly, Eq. (3.6) can be written in a vector form as follows:

$$I^{\nu}T_{L,i}(x) \simeq [\Omega_{\nu}(i,0), \Omega_{\nu}(i,1), \Omega_{\nu}(i,2), \ldots, \Omega_{\nu}(i,N)]\,\phi(x), \quad i=0,1,\ldots,N. \tag{3.8}$$

Eq. (3.8) leads to the desired result. □

4. Fractional SCOM for solving linear multi-order FDEs

In this section, the proposed multi-order FDE is integrated ν times, in the Riemann–Liouville sense, where ν is the highest fractional-order and making use of the formula relating the expansion coefficients of fractional integration appearing in this integrated form of the proposed multi-order FDE to shifted Chebyshev polynomials themselves, and then we apply tau approximations based on operational matrix. In order to show the fundamental importance of SCOM of fractional integration, we apply it to solve the following multi-order FDE:

$$D^{\nu}u(x) = \sum_{i=1}^{k} \gamma_j D^{\beta_i}u(x) + \gamma_{k+1}u(x) + f(x), \quad \text{in } I=(0,L), \tag{4.1}$$

with initial conditions

$$u^{(i)}(0) = d_i, \quad i = 0, \ldots, m-1, \tag{4.2}$$

where γ_i $(i = 1, \ldots, k+1)$ are real constant coefficients and also $m-1 < \nu \le m$, $0 < \beta_1 < \beta_2 < \cdots < \beta_k < \nu$. Moreover $D^\nu u(x) \equiv u^{(\nu)}(x)$ denotes the Riemann–Liouville fractional derivative of order ν for $u(x)$ and the values of d_i $(i = 0, \ldots, m-1)$ describe the initial state of $u(x)$ and $g(x)$ is a given source function. For the existence and uniqueness and continuous dependence of the solution to the problem, see [20].

If we apply the Riemann–Liouville integral of order ν on (4.1) and after making use of (2.4), we get the integrated form of (4.1), namely

$$\begin{aligned} & u(x) - \sum_{j=0}^{m-1} u^{(j)}(0^+)\frac{x^j}{j!} = \sum_{i=1}^{k} \gamma_i I^{\nu-\beta_i}\left[u(x) - \sum_{j=0}^{m_i-1} u^{(j)}(0^+)\frac{x^j}{j!}\right] + \gamma_{k+1} I^\nu u(x) + I^\nu f(x), \\ & u^{(i)}(0) = d_i, \quad i = 0, \ldots, m-1, \end{aligned} \tag{4.3}$$

where $m_i - 1 < \beta_i \le m_i$, $m_i \in N$, this implies that

$$\begin{aligned} & u(x) = \sum_{i=1}^{k} \gamma_i I^{\nu-\beta_i} u(x) + \gamma_{k+1} I^\nu u(x) + g(x), \\ & u^{(i)}(0) = d_i, \quad i = 0, \ldots, m-1, \end{aligned} \tag{4.4}$$

where

$$g(x) = I^\nu f(x) + \sum_{j=0}^{m-1} d_j \frac{x^j}{j!} + \sum_{i=1}^{k} \gamma_i I^{\nu-\beta_i}\left(\sum_{j=0}^{m_i-1} d_j \frac{x^j}{j!}\right).$$

In order to use the tau method with SCOM for solving the fully integrated problem (4.4) with initial conditions (4.2). We approximate $u(x)$ and $g(x)$ by the shifted Chebyshev polynomials as

$$u_N(x) \simeq \sum_{i=0}^{N} c_i T_{L,i}(x) = C^T \phi(x), \tag{4.5}$$

$$g(x) \simeq \sum_{i=0}^{N} g_i T_{L,i}(x) = G^T \phi(x), \tag{4.6}$$

where the vector $G = [g_0, \ldots, g_N]^T$ is given but $C = [c_0, \ldots, c_N]^T$ is an unknown vector.

Now, the Riemann–Liouville integral of orders ν- and $(\nu - \beta_j)$ of the approximate solution (4.5), after making use of Theorem 3.1 (relation (3.1)), can be written as

$$I^\nu u_N(x) \simeq C^T I^\nu \phi(x) \simeq C^T \mathbf{P}^{(\nu)} \phi(x), \tag{4.7}$$

and

$$I^{\nu-\beta_j} u_N(x) \simeq C^T I^{\nu-\beta_j} \phi(x) \simeq C^T \mathbf{P}^{(\nu-\beta_j)} \phi(x), \quad j = 1, \ldots, k, \tag{4.8}$$

respectively, where $\mathbf{P}^{(\nu)}$ is the $(N+1) \times (N+1)$ operational matrix of fractional integration of order ν.

Employing Eqs. (4.5)–(4.8) the residual $R_N(x)$ for Eq. (4.4) can be written as

$$R_N(x) = \left(C^T - C^T \sum_{j=1}^{k} \gamma_j \mathbf{P}^{(\nu-\beta_j)} - \gamma_{k+1} C^T \mathbf{P}^{(\nu)} - G^T\right) \phi(x). \tag{4.9}$$

As in a typical tau method, see [4,12], we generate $N - m + 1$ linear algebraic equations by applying

$$\langle R_N(x), T_{L,j}(x) \rangle = \int_0^L R_N(x) T_{L,j}(x) dx = 0, \quad j = 0, 1, \ldots, N-m. \tag{4.10}$$

Also by substituting Eqs. (2.8) and (4.5) in Eq. (4.2), we get

$$u^{(i)}(0) = \sum_{i=0}^{N} c_i T_{L,i}^{(i)}(0) = d_i, \quad i = 0, 1, \ldots, m-1. \tag{4.11}$$

Eqs. (4.10) and (4.11) generate $N - m + 1$ and m set of linear equations, respectively. These linear equations can be solved for unknown coefficients of the vector C. Consequently, $u_N(x)$ given in Eq. (4.5) can be calculated, which give a solution of Eq. (4.1) with the initial conditions (4.2).

5. Illustrative examples

To illustrate the effectiveness of the proposed method in the present paper, some test examples are carried out in this section. The results obtained by the present methods reveal that the present method is very effective and convenient for linear FDEs.

Example 1. As the first example, we consider the following initial value problem,

$$D^{\frac{3}{2}}u(x) + 3u(x) = 3x^3 + \frac{8}{\Gamma(0.5)}x^{1.5}, \qquad u(0) = 0, \qquad u'(0) = 0, \quad x \in [0, L], \tag{5.1}$$

whose exact solution is given by $u(x) = x^3$.

By applying the technique described in Section 4.1 with $N = 3$, we may write the approximate solution and the right hand side in the forms

$$u(x) = \sum_{i=0}^{3} c_i T_{L,i}(x) = C^T \phi(x), \quad \text{and} \quad g(x) \simeq \sum_{i=0}^{3} g_i T_{L,i}(x) = G^T \phi(x).$$

From Eq. (3.2) one can write

$$\mathbf{P}^{(\frac{3}{2})} = \frac{4L^{\frac{3}{2}}}{\pi^{\frac{3}{2}}} \begin{pmatrix} \frac{4}{9} & \frac{8}{15} & \frac{8}{105} & \frac{-8}{945} \\ \frac{-4}{25} & \frac{-8}{63} & \frac{8}{135} & \frac{8}{385} \\ \frac{-20}{147} & \frac{-152}{675} & \frac{-248}{3465} & \frac{136}{4095} \\ \frac{188}{2025} & \frac{136}{1617} & \frac{-56}{975} & \frac{-664}{17325} \end{pmatrix}, \qquad G = \begin{pmatrix} g_0 \\ g_1 \\ g_2 \\ g_3 \end{pmatrix}.$$

Therefore using Eqs. (4.9) and (4.10) we obtain

$$\left(\frac{96L^{\frac{3}{2}}}{105\pi^{\frac{3}{2}}}\right) c_0 + \frac{96L^{\frac{3}{2}}}{135\pi^{\frac{3}{2}}} c_1 + \left(\frac{2976L^{\frac{3}{2}}}{3465\pi^{\frac{3}{2}}}\right) c_2 - \left(\frac{672L^{\frac{3}{2}}}{975\pi^{\frac{3}{2}}}\right) c_3 + c_2 - g_2 = 0, \tag{5.2}$$

$$\left(-\frac{96L^{\frac{3}{2}}}{945\pi^{\frac{3}{2}}}\right) c_0 + \frac{96L^{\frac{3}{2}}}{385\pi^{\frac{3}{2}}} c_1 + \left(\frac{1632L^{\frac{3}{2}}}{4095\pi^{\frac{3}{2}}}\right) c_2 - \left(\frac{7968L^{\frac{3}{2}}}{17325\pi^{\frac{3}{2}}}\right) c_3 + c_3 - g_3 = 0. \tag{5.3}$$

Now, by applying Eq. (4.11) for the initial conditions we have

$$\begin{aligned} &C^T \phi(0) = c_0 - c_1 + c_2 - c_3 = 0, \\ &C^T D^{(1)} \phi(0) = \frac{2}{L} c_1 - \frac{8}{L} c_2 + \frac{18}{L} c_3 = 0. \end{aligned} \tag{5.4}$$

Finally by solving Eqs. (5.2)–(5.4) we get

$$c_0 = \frac{5L^3}{16}, \qquad c_1 = \frac{15L^3}{32}, \qquad c_2 = \frac{3L^3}{16}, \qquad c_1 = \frac{L^3}{32}.$$

Thus we can write

$$u(x) = \sum_{i=0}^{3} c_i T_{L,i}(x) = x^3.$$

Numerical results will not be presented since the exact solution is obtained.

Example 2. Consider the following initial value problem,

$$D^{\frac{3}{2}}u(x) + 3u(x) = 3x^3 + \frac{8}{\Gamma(0.5)}x^{1.5}, \qquad u(0) = 0, \qquad u'(0) = 0, \quad x \in [0, L], \tag{5.5}$$

whose exact solution is given by $u(x) = x^3$.

If we use the technique described in the previous example with $N = 3$, we get

$$c_0 = \frac{5L^3}{16}, \qquad c_1 = \frac{15L^3}{32}, \qquad c_2 = \frac{3L^3}{16}, \qquad c_1 = \frac{L^3}{32}.$$

Thus, we can write

$$u(x) = \sum_{i=0}^{3} c_i T_{L,i}(x) = x^3,$$

which is the exact solution.

Example 3. Consider the equation

$$D^2u(x) - 2Du(x) + D^{\frac{1}{2}}u(x) + u(x) = x^3 - 6x^2 + 6x + \frac{16}{5\sqrt{\pi}}x^{2.5}, \qquad u(0) = 0, \qquad u'(0) = 0, \quad x \in [0, L], \tag{5.6}$$

whose exact solution is given by $u(x) = x^3$.

Now, we can apply the technique described in Example 1 with $N = 3$, and the 4 unknown coefficients will be in the form

$$c_0 = \frac{5L^3}{16}, \qquad c_1 = \frac{15L^3}{32}, \qquad c_2 = \frac{3L^3}{16}, \qquad c_3 = \frac{L^3}{32}.$$

Thus we can write, $u(x) = \sum_{i=0}^{3} c_i T_{L,i}(x) = x^3$, which is the exact solution.

6. Conclusion

A general formulation for the Chebyshev operational matrix of fractional integration has been derived. The fractional integration is described in the Riemann–Liouville sense. This matrix is used to approximate numerical solution of linear multi-term FDEs. Our approach was based on the shifted Chebyshev tau method. The solution obtained using the suggested method shows that this approach can solve the problem effectively. We note that similar technique can be applied to tau method using Legendre polynomials or other Jacobi polynomials.

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