Exponentially many nonisomorphic orientable triangular embeddings of $K_{12s}$

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Abstract

It was proved earlier that there are constants $M, c > 0$ such that for every $n \geq M$ (resp., every $n \geq M, n \not\equiv 0, 3 \mod 12$) there are at least $c 2^{n/6}$ nonisomorphic nonorientable (resp., orientable) genus embeddings of $K_n$. In the present paper we show that for $s \geq 6$ there are at least $2^{s-6}$ nonisomorphic OT-embeddings of $K_{12s}$. As a byproduct, we give a relatively simple method of constructing index four current graphs with current group $Z_{12s}$ generating orientable triangular embeddings of $K_{12s}$.

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1. Introduction

The orientable (resp., nonorientable) genus of a graph is the smallest genus of an orientable (resp., nonorientable) surface in which the graph can be embedded. Any such embedding is called an orientable (resp., nonorientable) genus embedding of the graph. In the course of the proof of the Map Color Theorem [8] one orientable and one nonorientable genus embeddings were constructed for every complete graph. In this paper we consider the natural question on the rate of growth of the number of nonisomorphic genus embeddings of complete graphs.

Until now, there were surprisingly few results about the number of nonisomorphic genus embeddings of complete graphs and all the results are related to triangular embeddings in surfaces of small genus. Only in the last five years was it shown [1,2,4–6] that there is at least exponentially many (in $n$) nonisomorphic genus embeddings of some complete graphs $K_n$. There are two approaches to construct the embeddings.

The first approach [1,2] uses recursive constructions and a cut-and-paste technique. This approach establishes the existence of at least $2^{2n^2 - O(n)}$ nonisomorphic orientable (resp., nonorientable) triangular embeddings of $K_n$ for some families of $n$ such that $n \equiv 3$ or $7 \mod 12$ (resp., $n \equiv 1$ or $3 \mod 6$). This approach has some restrictions: using recursive constructions does not make it possible to obtain $2^{2n^2 - O(n)}$ lower bound for all values of $n \equiv 3$ or $7 \mod 12$ in the orientable case and $n \equiv 1$ or $3 \mod 6$ in the nonorientable case, and does not make it possible to obtain lower bounds on the number of nonisomorphic genus embeddings for other cases $n \mod 12$.

The second approach [4–6] uses the index one current graph technique. This approach yields only $c 2^{bn}$ lower bound on the number of nonisomorphic genus embeddings of $K_n$, but the approach applies to the large range of cases $n$.
modulo 12, yields families of nonisomorphic nontriangular genus embeddings as well as nonisomorphic triangular ones, gives us a possibility for any given \( k \geq 3 \) to construct exponentially many (in \( n \)) nonisomorphic embeddings of \( K_n \) such that all faces are \( k \)-gonal, etc. In this approach an equivalence relation of isomorphism is defined on the set of all index one current graphs with abelian current group and it is shown that two such current graphs generate isomorphic embeddings of a complete graph if and only if the current graphs are isomorphic. Changing rotations of vertices of such an index one current graph we can obtain exponentially many nonisomorphic current graphs, thereby obtaining exponentially many nonisomorphic embeddings of a complete graph. Using the approach it was shown that there are constants \( M, c > 0 \) such that for every \( n \geq M \) (resp., every \( n \geq M, n \not\equiv 0, 3 \mod 12 \)) there are at least \( c2^n/n^6 \) nonisomorphic nonorientable (resp., orientable) genus embeddings of \( K_n \). This approach cannot be directly applied to the orientable cases \( n \equiv 0, 3 \mod 12 \) since orientable triangular embeddings (OT-embeddings, for short) of \( K_{12s} \) and \( K_{12s+3} \) cannot be constructed by using index one current graphs with abelian current group. So exponentially many nonisomorphic OT-embeddings have not been obtained in the two cases. In the present paper we show that for \( s \geq 6 \) there are at least \( 2^{s-6} \) nonisomorphic OT-embeddings of \( K_{12s} \).

Note that the orientable case \( n \equiv 0 \mod 12 \) was particularly difficult to solve when proving the Map Color Theorem. An OT-embedding of \( K_{12s} \) was first constructed [9] by using an index one current graph with nonabelian current group. But the current group is complicated and it requires some knowledge in Galois field theory for readers to understand it. It was shown [7] that index one, two or three current graphs with current group \( \mathbb{Z}_{12} \) generating an OT-embedding of \( K_{12s} \) do not exist. An index four current graph with current group \( \mathbb{Z}_{12} \), generating an OT-embedding of \( K_{12s} \), was given in [7]. This current graph is complicated, the construction breaks down into a number of cases depending on \( n \mod 8 \), and the details are given only for \( s \equiv 0 \mod 8 \).

In the present paper we give a new index four current graph with current group \( \mathbb{Z}_{12s} \), generating an OT-embedding of \( K_{12s} \) for \( s = 4 \) and every \( s \geq 6 \). The construction breaks down into four cases depending on \( s \mod 4 \). All four cases are given in detail. The current graph is constructed from some more simple constructions, a basic ladder and four side blocks, and is simpler than the current graph from [7] in several ways. The reader using the same basic ladder but choosing other side blocks can construct many other (with different geometry) index four current graphs generating OT-embeddings of \( K_{12s} \). Using this approach it is not difficult to construct OT-embeddings of \( K_{12s} \) for \( s = 1, 2, 3, 5 \) (we do not do it in the present paper for reasons of space).

The current graphs constructed in the paper have the property that changing rotations of some vertices of the current graphs we can obtain exponentially many (in \( n \)) distinct index four current graphs generating OT-embeddings of \( K_{12s} \). To show that there are exponentially many (in \( s \)) nonisomorphic embeddings among the embeddings we proceed as follows. The vertices, whose rotations are changed, belong to subgraphs of a constructed main current graph such that, roughly speaking, the subgraphs can be considered as index one current graphs incorporated in the main index four current graph, and the embeddings generated by the index one current graphs are incorporated in the embedding generated by the main current graph. The embeddings generated by the main current graph with different chosen rotations of the subgraphs have the property that if two such embeddings are isomorphic then the embeddings generated by the index one current graphs incorporated in the main current graph are isomorphic also. As a result, the problem of deciding whether the embeddings generated by the main current graph with different chosen rotations are isomorphic is reduced to the problem of deciding whether the embeddings generated by the incorporated index one current graphs with corresponding chosen rotations are isomorphic. To solve the last problem we use the approach in [4].

The paper is organized as follows. In Section 2 we briefly review some material about current graphs in the form used in the paper. In Section 3 we show how the problem of deciding whether two triangular embeddings are isomorphic is connected with examining some properties of the set of all pairs of adjacent triangular faces of the embeddings. An index four current graph generating an OT-embedding of \( K_{12s} \) is constructed in Section 4, and it is shown in Sections 5 and 6 how to use the current graph to obtain exponentially many nonisomorphic OT-embeddings of \( K_{12s} \).

2. Current graphs

In this section we briefly review some material about index one and four current graphs in the form used in the paper. The reader is referred to [3,8] for a more detailed development of the material sketched herein. We assume the reader is familiar with current graphs, the log of a circuit, derived graphs and their derived embeddings generated by current graphs. At the end of the section we consider transformations T1–T3 of some fragments of index four current
graphs. The transformations will be used in Section 4 to construct index four current graphs from some simple building blocks.

Let $G$ be a connected graph (multiple edges and loops are allowed) with the vertex set $V(G)$ whose edges have been given plus and minus direction. Hence each edge $e$ gives rise to two reverse arcs $e^+$ and $e^-$ of $G$. The involutary permutation $\theta$ of the arc set $A(G)$ of the graph $G$ that permutes reverse arcs is called the involution of $G$. By a current assignment on $G$ we mean a function $\lambda$ from $A(G)$ into a group $\Phi$ such that $\lambda(e^-) = (\lambda(e^+))^{-1}$ for every edge $e$. The values of $\lambda$ are called currents and $\Phi$ is called the current group. If an edge $e$ is a loop and $\lambda(e^-) = \lambda(e^+)$ (that is, $\lambda(e^+)$ is of order 2 in $\Phi$), then the arcs $e^+$ and $e^-$ are identified and this arc is called an end arc.

A rotation $D$ of $A(G)$ is a permutation of $A(G)$ whose orbits cyclically permute the arcs directed from each vertex. The rotation $D$ can be represented as $D = \{D_v : v \in V(G)\}$, where $D_v$, called a rotation of the vertex $v$, is a cyclic permutation of the arcs directed from $v$. Consider the permutation $D\theta$ of $A(G)$. It is easy to see that the terminal vertex of an arc $a$ is the initial vertex of the arc $D\theta a$, hence a cycle $(a_1, a_2, \ldots, a_m)$ of $D\theta$ can be considered as an oriented path in $G$ called a circuit induced by the rotation $D$ of $G$, and we say that the circuit traverses the arcs $a_1, a_2, \ldots, a_m$ in this order. By a one-rotation of $G$ we mean a rotation of $G$ inducing exactly one circuit.

A triple $(G, \lambda, D)$ is called a current graph. The index of the current graph is the number of circuits induced by $D$. By the log of a circuit $(a_1, a_2, \ldots, a_m)$ we mean the cyclic sequence $(\lambda(a_1), \lambda(a_2), \ldots, \lambda(a_m))$.

An index one current graph $(G, \lambda, D)$ with current group $\mathbb{Z}_3$ generates an orientable embedding of $K_{3s}$ with vertex set $\{0, 1, \ldots, 3s - 1\}$ (the set of all elements of $\mathbb{Z}_3$) if the log of the circuit of the current graph contains every nonzero element of $\mathbb{Z}_3$ exactly once. Theorem 1 (Section 3) determines when the orientable embeddings of $K_{3s}$ generated by such index one current graphs are isomorphic.

Given an index four current graph $(G, \lambda, D)$ with current group $\mathbb{Z}_{12s}$, the circuits induced by $D$ are denoted by $[i]$, $i = 0, 1, 2, 3$. The circuits having been denoted, we will distinguish between an arc of $G$ and an arc of the current graph. By an arc $([i], [j])$ of the current graph we mean an arc $a$ of $G$ such that the arc carries current $\delta$ is traversed by the circuit $[i]$, and the reverse arc is traversed by the circuit $[j]$. If we consider the triple $([i], [\delta], [j])$ as a notation of the arc $a$, then in what follows, in every current graph under consideration no two arcs have the same such a notation. Hence, when we speak about an arc $([i], [\delta], [j])$ of a current graph $(G, \lambda, D)$, the triple $([i], [\delta], [j])$ uniquely determines the arc $a$ of $G$. Note, that arcs $([i], [\delta], [j])$ and $([j], [-\delta], [i])$ of a current graph are reverse arcs. For our purposes it is more convenient to formulate some properties of current graphs in terms of arc $([i], [\delta], [j])$.

Consider an index four current graph $(G, \lambda, D)$ with current group $\mathbb{Z}_{12s}$ satisfying the following construction principles (A1)–(A5):

(A1) Each vertex is onevalent or trivalent.
(A2) There are exactly two onevalent vertices, each has excess 4s or 8s. For even $s$, there are exactly four end arcs, each has current 6s. For odd $s$, there are no end arcs.
(A3) Every trivalent vertex satisfies KCL.
(A4) There are exactly four circuits: $[0], [1], [2],$ and $[3]$.
(A5) The arc set of the current graph consists of all arcs $([i], [\delta], [j])$, where $i, j \in \{0, 1, 2, 3\}, \delta \in \{1, 2, \ldots, 12s - 1\}$, and $\delta \equiv j - i \mod 4$. 
By [8], the current graph generates an OT-embedding of $K_{12s}$ whose vertex set is the set $\{0, 1, \ldots, 12s - 1\}$ of all elements of $\mathbb{Z}_{12s}$. The four sets $V_i(12s) = \{i, i + 4, \ldots, i + 12s - 4\}$, $i = 0, 1, 2, 3$, are called the vertex parts of the vertex set.

It is easy to see that the condition (A5) is equivalent to the following condition (more usual for the reader): a nonzero current from $\mathbb{Z}_{12s}$ is assigned to each arc; the log of every circuit contains every nonzero current from $\mathbb{Z}_{12s}$ exactly once; for every arc $a$, if a circuit $[i]$ traverses the arc, then the reverse arc is traversed by the circuit $[j]$ such that $j - i \equiv \lambda(a) \mod 4$.

Let $K$ be a graph without loops and multiple edges. A face of a cellular embedding of $K$ will be designated as a cyclic sequence $[x_1, x_2, \ldots, x_m]$ of vertices (for convenience, we enclose the sequence in brackets) obtained by listing the incident vertices when traversing the boundary cycle of the face in some chosen direction. The sequences $[x_1, x_2, \ldots, x_m]$ and $[x_m, \ldots, x_2, x_1]$ designate the same face. The edge joining vertices $x$ and $y$ is denoted by $(x, y)$.

For later use, taking into account Theorem 4.4.1 [3], we describe in the following claim the faces induced by onevalent and trivalent vertices of the current graphs under consideration. In the claim, the circuit of an index one current graph is denoted by $[0]$.

**Claim 1.** Consider an index $k$ current graph with current group $\mathbb{Z}_n$ (here either $k = 1$ or $4$).

A trivalent vertex with current rotation $(\beta, \gamma, \delta)$, satisfying KCL and shown in Fig. 1(a), induces $n/k$ trivalent faces $[x, x + \beta, x + \beta + \gamma]$, $x = i, i + k, i + 2k, \ldots, i + n - k$, shown in Fig. 1(b). Note that for $k = 4$, $x \in V_i(n)$ and $x + \beta + \gamma \in V_h(n)$.

If $n$ is divisible by $3$, then a onevalent vertex with excess $\delta = \pm n/3$ of order $3$ passed by a circuit $[i]$ induces $n/3k$ triangular faces $[x, x + \delta, x + 2\delta]$, $x = i, i + k, \ldots, i + n/3 - k$.

In what follows, considering index one and four current graphs, the reader should also bear in mind that onevalent vertices with excess of order not 3, trivalent vertices not satisfying KCL, and twovalent vertices, induce nontriangular faces.

The figures of the current graphs under consideration will have fragments of the form shown in Fig. 2(a) (the orientations of all vertical or all horizontal arcs can be reversed). The fragment shown in Fig. 2(a) designates the ladder-like fragment shown in Fig. 2(b) such that if we consider the vertical arcs from left to right, then the arcs are directed in alternating fashion up and down, and carry currents $\mu, \mu + \delta, \mu + 2\delta, \ldots, \eta - \delta, \eta$, where $\delta \in \{4, -4\}$. The horizontal arcs form two horizontal lines: the top and bottom horizontals of the fragment. All vertices on the same horizontal of such a ladder-like fragment have the same rotation (clockwise or counterclockwise).
Now we consider some special ladder-like fragments of depicted index four current graphs. A proper ladder-like fragment (pl-fragment, for short) is a fragment of the form shown in Fig. 2(b) such that:

(i) If the vertices on the top horizontal have clockwise (resp., counterclockwise) rotation, then the vertices on the bottom horizontal have counterclockwise (resp., clockwise) rotation. The top (resp., bottom) horizontal arcs are directed from left to right (resp., from right to left) and every pair of horizontal arcs forming top and bottom of a quadrangle carry the same current.

(ii) All four circuits traverse arcs of the fragment.

An example of a pl-fragment is given in Fig. 3(a), the pl-fragment can be schematically depicted as shown in Fig. 4 (note that, taking (i) into account, we do not indicate currents on the bottom horizontal arcs). Similar schematic designations will be used for other orientations of vertical arcs and for the case when the currents on the vertical arcs decrease from left to right. Sometimes to fit a long graph on a page we will leave in the schematic designation only two (the leftmost and the rightmost) vertical arcs connected by dotted line, it will be clear from the context whether the currents on vertical arcs decrease from left to right or not.
By a maximal pl-fragment of an index four current graph we mean a pl-fragment which is not a subgraph of other pl-fragments with larger number of vertical arcs. In the figures of current graphs we usually indicate maximal pl-fragments.

In the description of transformations of some pl-fragments of index four current graphs, we will single out pl-fragments by using the following definition. In Fig. 3(a), for $0 \leq h \leq h' \leq 2t + 1$, consider the pl-fragment whose set of depicted arcs consists of vertical arcs $((i], \beta + 4(h + 1), [j]), ((i], \beta + 4(h + 2), [j]), \ldots, ((i], \beta + 4(h' − 1), [j])$ and of all horizontal arcs adjacent to the vertical arcs. This fragment is called the pl-fragment lying between the vertical arcs $((i], \beta + 4h, [j])$ and $((i], \beta + 4h', [j])$, or is called the pl-fragment with extreme vertical arcs $((i], \beta + 4(h + 1), [j])$ and $((i], \beta + 4(h' − 1), [j])$. The horizontal arcs of the fragment adjacent to two vertical arcs of the fragment are called the inner horizontal arcs of the fragment.

Now we describe transformations T1–T3 of pl-fragments.

Consider the pl-fragment in Fig. 3(a). The transformations T1 and T2 are applied to a pl-fragment with an odd number of depicted vertical arcs. Fig. 3(b) (resp., (c)) shows the result of the transformation T1 (resp., T2) applied to the pl-fragment lying between the vertical arcs $((i], \beta + 4, [j])$ and $((i], \beta + 8t + 4, [j])$. As is easily seen, the arc set of the fragment in Fig. 3(b) (resp., (c)) consists of all arcs of the fragment in Fig. 3(a) except the two vertical arcs $((i], \beta + 4, [j])$ and $((i], \beta + 8t + 4, [j])$, and their reverse arcs (we say that the four arcs are deleted during the transformation T1 or T2).

In Fig. 3, as in what follows in Figs. 7, 11, 13, 15, 16, 19, and 20, the vertical dashed arcs incident with twovalent vertices indicate a possibility of attaching new edges to obtain trivalent vertices satisfying KCL.

The transformation T3 is applied to a pl-fragment with an even number of depicted vertical arcs. Fig. 3(d) shows the result of the transformation T3 applied to the pl-fragment lying between the vertical arcs $((i], \beta + 4, [j])$ and $((i], \beta + 8t + 4, [j])$. As is easily seen, the arc set of the fragment in Fig. 3(d) consists of all arcs of the fragment in Fig. 3(a) except the two vertical arcs $((i], \beta + 4, [j])$ and $((i], \beta + 8t + 8, [j])$, the two horizontal arcs $((k], \delta − 4t − 4, [i])$ and $((j], \delta − 4t − 4, [\ell])$, and their reverse arcs (we say that the eight arcs are deleted during the transformation T3).

The transformations T1–T3 are defined in a similar way when we consider a pl-fragment which differs from the pl-fragment in Fig. 3(a) by some of the following properties (i)–(iii):

(i) the rotations of all vertices are reversed;
(ii) the orientations of all vertical arcs are reversed;
(iii) the currents on the vertical arcs, when we consider them from left to right, are $\beta', \beta' − 4, \beta' − 8, \ldots$.

If in Fig. 3 we reverse the rotations of all vertices and the orientations of all four circuits, then the obtained figure describes T1–T3 in case (i). If in Figs. 3(a)–(d) we rotate each of the figures around a horizontal axis through 180°, then the obtained figures describe T1–T3 in case (ii). If in Fig. 3 we put $\beta = −\beta'$ and then replace every vertical arc by the reverse arc, then the obtained figure describes T1–T3 in case (iii).

3. Isomorphisms and links

In this section we consider links of the embeddings generated by current graphs. A link is a pair of adjacent triangular faces of an embedding. In Section 5 we will use links in the following way. Let $f$ and $f'$ be the embeddings generated by main current graphs $I$ and $I'$, respectively, such that the current graphs differ from one another by rotations of some vertices only. The knowledge of the link sets of the embeddings will give us a possibility (Lemma 1) to show that if
there is an isomorphism from \( f \) onto \( f' \), then the isomorphism takes the vertices of the embedding \( f \) generated by an index one current graph incorporated in \( f \) to the vertices of the embedding \( f'' \) generated by an index one current graph incorporated in \( f' \), and, as a consequence, we will obtain that the embeddings \( f \) and \( f'' \) are to be isomorphic. Hence, to prove that \( f \) and \( f' \) are nonisomorphic it will suffice to show that \( f \) and \( f'' \) are nonisomorphic (here we will need only to show the nonisomorphism of embeddings generated by index one current graphs such that the current graphs differ from one another by rotations of some vertices only).

At the end of this section we briefly review some results from [4] that are helpful to decide whether the embeddings of a complete graph generated by index one current graphs are isomorphic.

Let \( K \) be a graph without loops and multiple edges. One can distinguish between cellular embeddings of \( K \) as labeled objects (in this case we speak about different embeddings, they have different face sets) and as unlabeled objects (in this case we speak about nonisomorphic embeddings).

Two cellular embeddings \( f \) and \( f' \) of the graph \( K \) in a surface are isomorphic if there is an automorphism \( \varphi \) of \( K \) such that if \( [x_1, x_2, \ldots, x_m] \) is a face of \( f \), then \( [\varphi(x_1), \varphi(x_2), \ldots, \varphi(x_m)] \) is a face of \( f' \). The automorphism \( \varphi \) is called an isomorphism from the embedding \( f \) onto the embedding \( f' \). For a complete graph every bijection between the vertices is an automorphism of the graph.

Two faces of an embedding are adjacent if they share a common edge. A link for vertices \( x \) and \( y \) of an embedding is every pair \( [x, z_1, z_2], [y, z_1, z_2] \) of adjacent triangular faces of the embedding.

Let \( f \) and \( f' \) be two isomorphic embeddings of a graph. Let \( \varphi \) be an isomorphism from \( f \) onto \( f' \). The following claim is obvious.

**Claim 2.** Vertices \( x \) and \( y \) of \( f \) and the vertices \( \varphi(x) \) and \( \varphi(y) \) of \( f' \) have the same number of links.

Two vertices \( x \) and \( y \) of an embedded graph are chain \( h \)-linked \((h \geq 0)\) if \( h \) is the maximal integer \( t \) such that there is a sequence \( x = z_0, z_1, \ldots, z_r, z_r = y \) \((r \geq 1)\) of vertices of the embedding such that \( z_i \) and \( z_{i+1} \) have \( t \) links for \( i = 0, 1, \ldots, r - 1 \). Now Claim 2 implies the following claim.

**Claim 3.** Vertices \( x \) and \( y \) of \( f \) are chain \( h \)-linked if and only if the vertices \( \varphi(x) \) and \( \varphi(y) \) of \( f' \) are chain \( h \)-linked.

For every bijection \( \varphi : V(K_n) \rightarrow V(K_n) \) and every subset \( \{x_1, x_2, \ldots, x_t\} \subseteq V(K_n) \) denote \( \varphi(\{x_1, x_2, \ldots, x_t\}) = \{\varphi(x_1), \varphi(x_2), \ldots, \varphi(x_t)\} \).

**Lemma 1.** Let \( f \) and \( f' \) be isomorphic OT-embeddings of the graph \( K_{12s} \) with vertex set \( \{0, 1, \ldots, 12s - 1\} \). Suppose that there are integers \( m < M \) such that in each of the two embeddings every two vertices from different vertex parts have at most \( m \) links, and every two vertices \( x \) and \( x + 4 \) from the same vertex part have at least \( M \) links. Let \( \varphi \) be an isomorphism from \( f \) onto \( f' \). Then there is a permutation \( \Omega \) on elements \( 0, 1, 2, 3 \) such that \( \varphi(V_i(12s)) = V_{\Omega(i)}(12s) \) for every \( i \in \{0, 1, 2, 3\} \).

**Proof.** For each of the embeddings \( f \) and \( f' \), the vertices of every vertex part \( V_i(12s) \) can be placed in the cyclic order \((i, i + 4, \ldots, i + 12s - 4)\) such that every two consecutive vertices have at least \( M \) links, hence every two vertices from the same vertex part are chain \( h \)-linked for some \( h \geq M \). Every two vertices from different vertex parts are chain \( h' \)-linked for some \( h' \leq m < M \leq h \). Now, by Claim 3, the lemma follows.

In what follows, vertex parts of the embeddings of \( K_{12s} \) generated by index four current graphs will be the vertex sets of the embeddings generated by index one current graphs incorporated in the main current graphs.

To apply Lemma 1 we need to know how to determine the number of links between pairs of vertices in the embedding generated by an index four current graph. Every link of a derived embedding is uniquely determined by the common edge of the adjacent triangular faces, we will say that the edge determines the link. Given an edge \( e \) of a current graph, the dual edge \( e^* \) of the (dual) voltage graph lifts to the edges of the derived embedding which are in the fiber over the edge \( e^* \) (see [3]), and either each of the edges of the derived embedding determines a link or none of the edges determines a link. If each of the edges determines a link, the links are called the links of the derived embedding induced by the edge \( e \) of the current graph. Clearly, the link set of the derived embedding consists of the links induced by edges of the current graph.
For the index four current graphs under consideration, every link of the derived embedding is induced either by an edge joining two trivalent vertices or by an edge joining a trivalent vertex and a onevalent vertex with excess of order 3. In a figure of a current graph, every edge is represented by one of its arcs. For a given arc of an edge of an index four current graph under consideration, we show (Claims 4 and 6) how to determine the links induced by the edge (in what follows we will say that the links are induced by the arc).

For an arc joining two trivalent vertices, define the type of the arc in the following way. Given an index four current graph with current group \( \mathbb{Z}_{12s} \), consider an arc \( a \) directed from a trivalent vertex \( v \) to a trivalent vertex \( w \) (see Fig. 5(a)). If the current rotations of \( v \) and \( w \) are \( (\beta, \gamma, \delta) \) and \( (\delta, \varepsilon, \mu) \), respectively, then the arc is said to be of type \( T(i|\varepsilon|j) \), where \( [i] \) and \( [j] \) are circuits passing the vertices \( v \) and \( w \), respectively, as shown in Fig. 5(a). By Claim 1, the edge whose arc is \( a \) induces exactly 3\( s \) links shown in Fig. 5(b). We see that the type of an arc joining two trivalent vertices satisfies the following claim.

**Claim 4.** Given an index four current graph with current group \( \mathbb{Z}_{12s} \), an arc of type \( T(i|\varepsilon|j) \) induces exactly 3\( s \) links: one link between the vertices \( x \) and \( x + \Delta \) for every \( x \in V_i(12s) \) (here \( x + \Delta \in V_j(12s) \)).

The following obvious claim will be often used.

**Claim 5.** The arcs of type \( T(i|\Delta|j) \) and \( T(j|\Delta|i) \) induce links between the same pairs of vertices.

Note that if in the figure of a current graph we replace an arc \( a \) of type \( T(i|\Delta|j) \) by the reverse arc, then, since KCL holds at every trivalent vertex, we obtain that the reverse arc is of type \( T(j|\Delta|i) \) and in accordance with Claim 4 induces, as expected, the same links that the arc \( a \) induces.

By Claim 1, for an edge joining a trivalent vertex and a onevalent vertex with excess of order 3, we have the following.

**Claim 6.** The arc with current \( \delta \) of order 3 shown in Fig. 5(c) induces exactly 3\( s \) links shown in Fig. 5(d): one link between the vertices \( x \) and \( x + \delta + \beta \) for every \( x \in V_i(12s) \) (here \( x + \delta + \beta \in V_j(12s) \)).

In Section 5 we will need some results from [4] about isomorphism of the embeddings of a complete graph generated by index one current graphs with abelian current group. Two graphs \( G \) and \( G' \) with involutions \( \theta \) and \( \theta' \), respectively, are isomorphic if there are bijections \( \omega : V(G) \to V(G') \) and \( \tilde{\omega} : A(G) \to A(G') \) such that if an arbitrary arc \( a \in A(G) \) is directed from a vertex \( v \) to a vertex \( u \), then the arc \( \tilde{\omega}(a) \) is directed from the vertex \( \omega(v) \) to the vertex \( \omega(u) \) and

\[ \tilde{\omega}(\theta a) = \theta' \tilde{\omega}(a). \]
The pair \((\omega, \tilde{\omega})\) is called an isomorphism from \(G\) onto \(G'\). An isomorphism from \(G\) onto itself is an automorphism of \(G\).

Two pairs \((G, \lambda)\) and \((G', \lambda')\) (where \(\lambda\) and \(\lambda'\) are current assignments on \(G\) and \(G'\), respectively, with the same current group \(\Phi\)) are strong isomorphic if there is an isomorphism \((\omega, \tilde{\omega})\) from \(G\) onto \(G'\) and an automorphism \(\psi\) of \(\Phi\) such that

\[
\lambda' \tilde{\omega}(a) = \psi \lambda(a)
\]

for every arc \(a\) of \(G\). This pair \((\omega, \tilde{\omega})\) is called a strong isomorphism from \((G, \lambda)\) onto \((G', \lambda')\), and the automorphism \(\psi\) is said to be associated with the isomorphism. An isomorphism from \((G, \lambda)\) onto itself is an automorphism of \((G, \lambda)\).

Two index one current graphs \((G, \lambda, D)\) and \((G', \lambda', D')\) with the same current group are said to be isomorphic, written as \((G, \lambda, D) \sim (G', \lambda', D')\), if there is a strong isomorphism \((\omega, \tilde{\omega})\) from \((G, \lambda)\) onto \((G', \lambda')\) such that either (i) or (ii) holds:

(i) For every vertex \(v\) of \(G\), if \(D_v = (a_1, a_2, \ldots, a_m)\), then \(D'_{\omega(v)} = (\tilde{\omega}(a_1), \tilde{\omega}(a_2), \ldots, \tilde{\omega}(a_m))\).

(ii) For every vertex \(v\) of \(G\), if \(D_v = (a_1, a_2, \ldots, a_m)\), then \(D'_{\tilde{\omega}(v)} = (\tilde{\omega}(a_m), \tilde{\omega}(a_2), \tilde{\omega}(a_1))\).

The pair \((\omega, \tilde{\omega})\) is called a strong isomorphism from \((G, \lambda, D)\) onto \((G', \lambda', D')\). The reader can easily check that all above-mentioned relations of isomorphism or strong isomorphism are equivalence relations.

Now Theorems 1 and 2 from [4] immediately imply the following theorem which will be used in Section 5.

**Theorem 1.** Let \((G, \lambda, D)\) and \((G', \lambda', D')\) be index one current graphs with current group \(\mathbb{Z}_{3s}\) generating orientable embeddings of \(K_{3s}\). Then the two derived embeddings are isomorphic if and only if the current graphs are isomorphic.

### 4. Current graphs generating OT-embeddings of \(K_{12s}\)

In this section we construct a current graph \(\Gamma(12s), s \geq 6\), with current group \(\mathbb{Z}_{12s}\) satisfying the construction principles (A1)–(A5). The current graph generates an OT-embedding of \(K_{12s}\).

The current graph \(\Gamma(12s)\) is constructed from five more simple constructions: a basic ladder \(L(s)\) and four side blocks \(B_i(s), i = 0, 1, 2, 3\). Roughly speaking, the basic ladder \(L(s)\) contains all or almost all arcs of the form \([(i), \delta, [j]]\), \(i \neq j\), of \(\Gamma(12s)\), and for \(i = 0, 1, 2, 3\), the side block \(B_i(s)\) contains all arcs of the form \([(i), \delta, [i]]\) of \(\Gamma(12s)\).

The basic ladder \(L(s)\) is shown in Fig. 6 for \(s = 2n, n \geq 1\), and in Fig. 7 for \(s = 2n + 1, n \geq 1\) (the ends labeled by the same letter, A, B, C, or D, are to be identified). The basic ladder \(L(s)\) is an index four current graph with current group \(\mathbb{Z}_{12s}\) satisfying the following conditions (B1)–(B3):

(B1) For even \(s\), all vertices are trivalent. For odd \(s\), there are four twovalent vertices, all other vertices are trivalent.

(B2) The properties (A3) and (A4) are satisfied.

(B3) For even \(s\), the arc set of the current graph consists of all arcs \([(i), \delta, [j]]\), where \(i, j \in \{0, 1, 2, 3\}\), \(i \neq j\), \(\delta \in \{1, 2, \ldots, 12s - 1\}\), and \(\delta \equiv j - i \mod 4\). For odd \(s = 2n + 1\), the arc set is the same as for even \(s\) except the arcs \([(11), 24n + 5, [2]], [11), 24n + 9, [2]]\), and the reverse arcs, the four arcs are absent (the four absent arcs and the corresponding two edges with these arcs are called the missing arcs and edges, respectively, of \(L(s)\) for \(s = 2n + 1\)).

Note that for even \(s\), the current graph \(L(s)\) generates an OT-embedding of the complete 4-partite graph \(K(3s, 3s, 3s, 3s)\). For odd \(s = 2n + 1\), the current graph \(L(s)\) generates an orientable embedding of the graph \(K(3s, 3s, 3s, 3s)\) without 6s edges \((x, x + 24n + 5)\) and \((x, x + 24n + 9), x = 1, 5, \ldots, 12s - 3\).

The side blocks \(B_i(s), i = 0, 1, 2, 3\), are shown in Figs. 10, 14, 12, 18, respectively (now ignore transverse strokes on arcs). Every side block \(B_i(s)\) is a triple \((G, \lambda, D)\) where \(\lambda\) is a current assignment with the current group \(\mathbb{Z}_{12s}\). The graph \(G\) either is connected or consists of two connected components one of which is a single edge with incident vertices (the component is called the isolated edge of \(B_i(s)\)) and the other component is called the main component of \(B_i(s)\). The triple \((G, \lambda, D)\) satisfies the following conditions (C1)–(C3):

(C1) All arcs of \(G\) have distinct currents and the set of currents on the arcs of \(G\) is \(\{4, 8, \ldots, 12s - 4\}\).
(C2) The valence of the vertices is at most 3. Every connected component of $G$ has exactly one starred vertex. Every unstarred trivalent vertex satisfies KCL.

(C3) The rotation of the main component of $G$ induces exactly one circuit, the circuit is denoted by $[i]$. If $G$ is connected, then $B_i(s)$ is an index one current graph with current group $Z_{12s}$.
Remark 1. If we denote the circuit of the isolated edge of $B_i(s)$ by $[i]$ as well, then the condition (C1) implies that the arc set of $B_i(s)$ consists of all arcs $([i], \delta, [i])$, $\delta = 4, 8, \ldots, 12s - 4$.

The current graph $L(s)$ has at most four twovalent vertices. To construct $\Gamma(12s)$ we first transform $L(s)$ to obtain a new index four current graph $\overline{L}(s)$ with larger number of twovalent vertices so that it becomes possible, using the twovalent vertices, to attach all side blocks to $\overline{L}(s)$. When transforming $L(s)$, we use transformations T1–T3, so some edges are deleted.

By attaching a side block to $\overline{L}(s)$ we mean the following. Every onevalent starred vertex of the side block is identified with a twovalent vertex of $\overline{L}(s)$ to produce a new trivalent vertex satisfying KCL. If the side block has a twovalent or trivalent starred vertex, then the vertex is deleted and the obtained “free” ends of edges incident with the starred vertex are glued to twovalent vertices of $\overline{L}(s)$ as shown in Fig. 8 to produce new trivalent vertices satisfying KCL.

The current graph $\overline{L}(s)$ has four circuits $[0]$, $[1]$, $[2]$, and $[3]$. For every $i \in \{0, 1, 2, 3\}$, the twovalent vertices of $\overline{L}(s)$ are passed by the circuit $[i]$ in some order such that when we delete the twovalent or trivalent starred vertex of $B_i(s)$ and glue the “free” ends of edges to the corresponding twovalent vertices of $\overline{L}(s)$, we can choose the rotations of the obtained trivalent vertices so that the circuit $[i]$ of $B_i(s)$ and the circuit $[i]$ of $\overline{L}(s)$ are combined together to form the circuit $[i]$ of $\Gamma(12s)$ as shown in Fig. 8.

The deleted and missing edges are paired so that in $\Gamma(12s)$ the edges of every pair form a cycle of length 2 joining onevalent unstarred vertices of two distinct attached side blocks.

Knowing how the side blocks are attached to $\overline{L}(s)$, by inspection of the figures of $\overline{L}(s)$ and the side blocks it is easy to check the properties (A1)–(A4) of the obtained $\Gamma(12s)$. Note that transforming $L(s)$ into $\overline{L}(s)$ does not violate (B2).

Constructing $\Gamma(12s)$ in such a manner (first $L(s)$, then $\overline{L}(s)$, and, finally, $\Gamma(12s)$) facilitates checking the property (A5) of $\Gamma(12s)$ in the following way. The basic ladder $L(s)$ is a simple construction consisting of two pl-fragments, and it is easy to check the property (B3). When transforming $L(s)$ into $\overline{L}(s)$, some edges (at most 8) are deleted from $L(s)$, but every arc $([i], \delta, [j])$ of $L(s)$ which is not deleted becomes the arc $([i], \delta, [j])$ of $\overline{L}(s)$ (where $[i]$ and $[j]$ are now circuits of $\overline{L}(s)$) and then becomes the arc $([i], \delta, [j])$ of $\Gamma(12s)$ (where $[i]$ and $[j]$ are now circuits of $\Gamma(12s)$). For every $i = 0, 1, 2, 3$, every arc $([i], \delta, [i])$ of $B_i(s)$ (here take Remark 1 into account) becomes the arc $([i], \delta, [i])$ of $\Gamma(12s)$ where $[i]$ is now a circuit of $\Gamma(12s)$. Every deleted or missing arc $([i], \delta, [j])$ becomes the arc $([i], \delta, [j])$ of $\Gamma(12s)$ joining onevalent unstarred vertices of attached $B_i(s)$ and $B_j(s)$. Hence the property (B3) of $\overline{L}(s)$ and the property (C1) of the side blocks imply the property (A5) of the obtained $\Gamma(12s)$.

In what follows, describing how to construct $\Gamma(12s)$, we will need some definitions.
Fig. 9. Symmetric horizontal arcs of an index four current graph.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig9.png}
\caption{Symmetric horizontal arcs of an index four current graph.}
\end{figure}

The figure of $L(s)$ or $\overline{L}(s)$ consists of two parts $\mathcal{P}_1$ and $\mathcal{P}_2$ such that the part $\mathcal{P}_1$ is depicted above the part $\mathcal{P}_2$ and the parts are connected with each other by identifying the ends labeled by the same letter, A, B, C, or D. The two parts $\mathcal{P}_1$ and $\mathcal{P}_2$ are called the upper and lower parts, respectively, of the depicted current graph.

The depicted current graphs $L(s)$ and $\overline{L}(s)$ have the property that in every part of $L(s)$ or $\overline{L}(s)$, for every top horizontal arc $a$, there is a bottom horizontal arc $a'$ such that $a'$ lies just below the arc $a$ and the two arcs carry the same current (see Fig. 9). The two horizontal arcs form a pair of (mutually) symmetrical horizontal arcs. To avoid clattering figures of $L(s)$ and $\overline{L}(s)$, for every pair of symmetrical arcs we indicate the current only of the top horizontal arc.

Now we begin to construct the current graphs $\Gamma(12s)$.

Case $s = 4t$, \( t \geq 1 \): The side block $B_3(4t)$ and its schematic designation are given in Figs. 10(a) and (b), respectively. The side block $B_1(4t)$ and its schematic designation are given in Figs. 10(c) and (d), respectively. The side blocks $B_0(4t)$ and $B_2(4t)$ are obtained from $B_3(4t)$ and $B_1(4t)$, respectively, by reversing the orientations of all arcs and the rotations of all vertices.

Consider a fragment $\mathcal{H}$ of the upper part of $L(4t)$ shown in Fig. 11(a) and apply the transformation $T_1$ to the pl-fragment lying between the vertical arcs $([1], 5, [2])$ and $([1], 32t + 5, [2])$; the two arcs and their reverse arcs are deleted during the transformation. After the transformation, the fragment $\mathcal{H}$ is replaced by the fragment $\mathcal{H}'$ shown in Fig. 11(b). Now apply $T_1$ to the pl-fragment of $\mathcal{H}'$ lying between the vertical arcs $([1], 8t + 1, [2])$ and $([1], 24t + 1, [2])$; the two arcs and their reverse arcs are deleted during the transformation. We obtain $\overline{L}(4t)$, its upper part is shown in Fig. 11(c) (for $t = 1$, the vertical arcs with currents $8t - 3, 8t - 7, \ldots, 9$ are absent in the figure). Attaching the side blocks as shown in Fig. 11(d), where $\overline{L}(4t)$ is represented by its two horizontals (as in what follows, when we give a figure of attaching side blocks to $\overline{L}(s)$ we obtain $\Gamma(12s)$, \( t \geq 1 \).

Case $s = 4t + 2$, \( t \geq 1 \): The side blocks $B_2(4t + 2)$ and $B_3(4t + 2)$ are given in Fig. 12(a) and their schematic designation is given in Fig. 12(b). The side blocks $B_0(4t + 2)$ and $B_1(4t + 2)$ are obtained from the side block in Fig. 12 by reversing the orientations of all arcs and the rotations of all vertices.

Consider a fragment $\mathcal{H}$ of the upper part of $L(4t + 2)$ shown in Fig. 13(a) and apply $T_1$ to the pl-fragment lying between the vertical arcs $([1], 1, [2])$ and $([1], 16t + 9, [2])$; the two arcs and their reverse arcs are deleted during the transformation. We obtain $\overline{L}(4t + 2)$, its upper part is shown in Fig. 13(b). Attaching the side blocks as shown in Fig. 13(c), we obtain $\Gamma(12(4t + 2), t \geq 1)$. 

Fig. 10. The side blocks $B_i(s)$, \( s = 4t \).
Case $s = 4t + 1$, $t \geq 2$: The side block $B_1(4t + 1)$ and its schematic designation are given in Figs. 14(a) and (b), respectively. The side block $B_3(4t + 1)$ and its schematic designation are given in Figs. 14(c) and (d), respectively. Note that for $t = 2$ the side blocks do not contain the ladder-like fragment with vertical arcs carrying currents $12, 16, \ldots, 8t - 8$. The side blocks $B_2(4t + 1)$ and $B_0(4t + 1)$ are obtained from $B_1(4t + 1)$ and $B_3(4t + 1)$, respectively, by reversing the orientations of all arcs and the rotations of all vertices.
Fig. 12. The side blocks $B_i(s)$, $s = 4t + 2$.

Fig. 13. Constructing the current graph $\Gamma(48t + 24)$, $t \geq 1$. 
Fig. 14. The side blocks $B_i(s), s = 4t + 1$.

Consider the upper part of $L(4t + 1)$ shown in Fig. 15(a). Apply the transformation T1 to two pl-fragments: one with extreme vertical arcs $([1], 1, [2])$ and $([1], 24t - 3, [2])$; the other with extreme vertical arcs $([1], 24t + 5, [2])$ and $([1], 48t + 1, [2])$. We obtain the upper part of $L(4t + 1)$ shown in Fig. 15(b). No arcs are deleted during the transformations.

Consider the lower part of $L(4t + 1)$ shown in Fig. 16(a). Apply T1 to the pl-fragment lying between the vertical arcs $([3], 24t + 1, [0])$ and $([3], 32t + 5, [0])$; the two arcs and their reverse arcs are deleted during the transformation.
Fig. 16. Transforming the lower part of $L(4t+1)$ into the lower part of $Z(4t+1)$.

Fig. 17. The current graph $\Gamma(48t+12)$, $t \geq 2$. 
Fig. 18. The side blocks $B_i(s), s = 4t + 3$.

Fig. 19. Transforming the upper part of $L(4t + 3)$ into the upper part of $\mathcal{Z}(4t + 3)$. 
Fig. 20. Transforming the lower part of $L(4t+3)$ into the lower part of $\overline{L}(4t+3)$.

Fig. 21. The current graph $\Gamma(48t+36), t \geq 1$.

After the transformation, the lower part takes form $\mathcal{H}$ shown in Fig. 16(b). Now apply $T2$ to the pl-fragment of $\mathcal{H}$ lying between vertical arcs $([3], 8t-3, [0])$ and $([3], 16t+5, [0])$; the two arcs and their reverse arcs are deleted during the transformation. We obtain the lower part of $\overline{L}(4t+1)$ shown in Fig. 16(c). Attaching the side blocks as shown in Fig. 17, we obtain $\Gamma(48t+12), t \geq 2$.

Case $s = 4t+3, t \geq 1$: The side block $B_3(4t+3)$ and its schematic designation are given in Figs. 18(a) and (b), respectively. The side block $B_1(4t+3)$ and its schematic designation are given in Figs. 18(c) and (d), respectively. Note that for $t = 1$ the side blocks do not contain the ladder-like fragment with vertical arcs carrying currents $12, 16, \ldots, 8t-4$. The side blocks $B_0(4t+3)$ and $B_2(4t+3)$ are obtained from $B_3(4t+3)$ and $B_1(4t+3)$, respectively, by reversing the orientations of all arcs and the rotations of all vertices.

Consider the upper part of $L(4t+3)$ shown in Fig. 19(a). Interchange, as shown in Fig. 19(b), two pl-fragments: the one with extreme vertical arcs $([1], 1, [2])$ and $([1], 24t+9, [2])$; the other with extreme vertical arcs $([1], 24t+17, [2])$.
and ([1], 48t + 25, [2]). No arcs are deleted during the transformations. The upper part takes form $M$ shown in Fig. 19(b). Now apply $T_3$ to the pl-fragment of $M$ lying between vertical arcs ([1], 40t + 29, [2]) and ([1], 48t + 25, [2]). The following four arcs and their reverse arcs are deleted during the transformation: the vertical arcs ([1], 40t + 29, [2]) and ([1], 48t + 25, [2]); the horizontal arcs ([0], 24t + 25, [1]) and ([2], 24t + 5, [3]). We obtain the upper part of $L(4t + 3)$ shown in Fig. 19(c). Consider the lower part of $L(4t + 3)$ shown in Fig. 20(a). Removing four arcs ([3], 16t − 3, [0]), ([3], 16t − 7, [0]), ([3], 44t + 23, [2]), ([1], 44t + 23, [0]), and their reverse arcs, and forming two new twovalent vertices as shown in Fig. 20(b), we obtain the lower part of $L(4t + 3)$. Attaching the side blocks as shown in Fig. 21, we obtain $\Gamma(48t + 36)$, $t \geq 1$.

5. Nonisomorphic OT-embeddings of $K_{12s}$

In this section we first show that by changing the rotations of some trivalent vertices of side blocks in $\Gamma(12s)$, $s \geq 6$, we can obtain at least $2^{s-3}$ different index four current graphs generating different OT-embeddings of $K_{12s}$. Then we prove (Theorem 2) that among these embeddings there are at least $2^{s-6}$ nonisomorphic embeddings.

When constructing $\Gamma(12s)$, the unstarred vertices and some arcs of the side blocks and $L(s)$ become vertices and arcs of $\Gamma(12s)$ (we will speak about vertices and arcs of the side blocks and $L(s)$ in $\Gamma(12s)$). Given a vertex of $\Gamma(12s)$, we can say that the vertex has some valence as a vertex of $\Gamma(12s)$ and has another (different) valence as a vertex of a side block $B_i(s)$ or $\bar{L}(s)$.

Let $B_i(s) = (G_i, \lambda_i, P_i)$, $i = 0, 1, 2, 3$. If $B_i(s)$ is an index one current graph (when $G_i$ is connected), then an admissible one-rotation for $B_i(s)$ is a one-rotation $Q_i$ of $G_i$ such that $(Q_i)_w = (P_i)_w$ for the starred vertex $w$ of $G_i$. Denote $B_i(s|Q_i) = (G_i, \lambda_i, Q_i)$. It should be observed here that, by construction of $\Gamma(12s)$, every unstarred trivalent vertex $w$ of $B_i(s)$ becomes a trivalent vertex of $\Gamma(12s)$ and has rotation $(P_i)_w$ in $\Gamma(12s)$.

Let $B_h(s)$ and $B_r(s)$, $h < r$, be index one current graphs ($h$ and $r$ depend on $s$; in the case $s \equiv 2 \mod 4$, when all side blocks are current graphs, we put $h = 1$ and $r = 2$). Let $Q_h$ and $Q_r$ be admissible one-rotations of $B_h(s)$ and $B_r(s)$, respectively. By the construction of $\Gamma(12s)$, if for $i = h, r$, for every trivalent vertex $w$ of $B_i(s)$ in $\Gamma(12s)$, we change the rotation $(P_i)_w$ of the vertex to the rotation $(Q_i)_w$, then we obtain an index four current graph, denoted by $\Gamma(12s|Q_h, Q_r)$, generating an OT-embedding of $K_{12s}$.

Given a current graph $B_i(s|Q_i)$, $i \in \{h, r\}$, with current group $\bar{K}_{12s}$, the derived embedding consists of four isomorphic orientable embeddings of $K_{3s}$, the embeddings have vertex sets $V_j(12s)$, $j = 0, 1, 2, 3$, respectively. The embedding with vertex set $V_j(12s)$ will be referred to as the embedding generated by $B_i(s|Q_i)$, and the embedding is considered to be incorporated in the embedding of $K_{12s}$ generated by $\Gamma(12s|Q_h, Q_r)$.

Now we show that there are at least $2^{s-3}$ different current graphs $\Gamma(12s|Q_h, Q_r)$, where $Q_h$ and $Q_r$ are admissible rotations of $B_h(s)$ and $B_r(s)$, respectively.

Lemma 2. Let a rotation $D$ of a graph $G$ induce exactly one circuit. Let an edge $e$ of $G$ be incident to distinct trivalent vertices $v$ and $w$. Then there are two ways to choose rotations of $v$ and $w$, not changing the rotations of other vertices, such that the obtained rotation of $G$ induces exactly one circuit.

Proof. Remove the edge $e$ from $G$. We obtain a graph with the rotation that induces exactly two circuits, say $X$ and $Y$, such that each of these circuits passes through one of the obtained twovalent vertices $v$ and $w$. Here we have the two possible cases shown in Fig. 22 (now ignore the dashed lines). It is seen that in both cases we can insert the edge $e$ (depicted as a dashed line) in two ways to obtain a rotation of $G$ inducing exactly one circuit.

![Fig. 22. The both cases of inserting the edge.](image-url)
Given an index one current graph $B_i(s) = (G_i, \lambda_i, P_i)$, two pairs of adjacent trivalent unstarred vertices of $G_i$ are said to be disjoint if the pairs have no common vertices. In a figure of $B_i(s)$ we can indicate some set of mutually disjoint pairs of adjacent trivalent unstarred vertices of $G_i$ in the following way: for every pair, the arc joining the vertices of the pair has transverse stroke. Since $G_i$ is obtained from $G_0$ by reversing the orientations of all arcs, the two graphs have the same sets of mutually disjoint pairs of adjacent trivalent unstarred vertices. By inspection of Figs. 10, 12, 14, and 18, the reader can find that $B_h(s)$ has exactly $p(s)$ arcs with transverse stroke, where $p(4t) = 2t - 2$ and $p(4t + i) = 2t$ for $i = 1, 2, 3$.

By applying Lemma 2, we observe that there are at least $2\|s\|$ different admissible one-rotations $Q_h$ (resp., $Q_r$) for $B_h(s)$ (resp., $B_r(s)$), hence there are at least $2\|s\|$ different current graphs $\Gamma(12s|Q_h, Q_r)$ generating different OT-embeddings of $K_{12s}$. It is easy to see that $2p(s)$ equals $s - 2, s - 1, s - 2,$ and $s - 3$ for $s = 4t, 4t + 1, 4t + 2,$ and $4t + 3$, respectively, hence $2p(s) \geq s - 3$.

The following theorem shows that among these (at least $2^{s-3}$) different OT-embeddings of $K_{12s}$ generated by the $2\|s\|$ different current graphs $\Gamma(12s|Q_h, Q_r)$ there are at least $(\frac{1}{8})2^{s-3}$ mutually nonisomorphic embeddings.

**Theorem 2.** There are at least $(\frac{1}{8})2^{s-3} = 2^{s-6}$ nonisomorphic OT-embeddings of $K_{12s}, s \geq 6$.

**Proof.** In Section 6 we prove the following lemma.

**Lemma 3.** Let $f$ and $f'$ be two isomorphic OT-embeddings of $K_{12s}, s \geq 6$, generated by $\Gamma(12s|Q_h, Q_r)$ and $\Gamma(12s|Q_h', Q_r')$, respectively. Let $\varphi$ be an isomorphism from $f$ onto $f'$.

(a) For $i = 0, 1, 2, 3$, $\varphi(V_i(12s)) = V_{\Omega(i)}(12s)$, where $\Omega$ is a permutation on elements 0, 1, 2, 3.

(b) $\{\Omega(h), \Omega(r)\} = \{h, r\}$.

(c) For every $i \in \{h, r\}$, the restriction $\tilde{\varphi}$ of $\varphi$ on $V_i(12s)$ is an isomorphism from the embedding of $K_{3s}$ generated by $B_i(s|Q_i)$ onto the embedding of $K_{3s}$ generated by $B_{\Omega(i)}(s|Q_{\Omega(i)})$.

An index one current graph $B_i(s|Q_i) = (G_i, \lambda_i, Q_i), i \in \{h, r\}$, with current group $\mathbb{Z}_{12s}$ generates a cellular embedding of $K_{3s}$ with vertex set $\{i, i + 4, \ldots, i + 12s - 4\}$. Now in $B_i(s|Q_i)$ replace every current $4k$ by $k$, $k = 1, 2, \ldots, 3s - 1$. We obtain an index one current graph $\tilde{B}_i(s|Q_i) = (\tilde{G}_i, \lambda_i, \tilde{Q}_i), i \in \{h, r\}$, with current group $\mathbb{Z}_{3s}$ generating a cellular embedding of $K_{3s}$ with vertex set $\{0, 1, \ldots, 3s - 1\}$. Clearly, if $(4k_1, 4k_2, \ldots, 4k_{3s-1})$ is the log of the circuit of $B_i(s|Q_i)$, then $h, k_2, \ldots, k_{3s-1}$ is the log of the circuit of $\tilde{B}_i(s|Q_i)$ (by (A1), the log $(k_1, k_2, \ldots, k_{3s-1})$ contains every nonzero element of $\mathbb{Z}_3$ exactly once). Taking into account how the log determines the rotations of the vertices of the derived embedding, we obtain that $[i + 4t_1, i + 4t_2, \ldots, i + 4t_m]$ is a face of the embedding generated by $B_i(s|Q_i)$ if and only if $[\ell_1, \ell_2, \ldots, \ell_m]$ is a face of the embedding generated by $\tilde{B}_i(s|Q_i)$, hence the two embeddings are isomorphic (an isomorphism from the first embedding onto the second embedding takes the vertex $i + 4\ell$ to the vertex $\ell$ for $\ell = 0, 1, \ldots, 3s - 1$).

Now, considering Lemma 3(c), define a bijection $\tilde{\varphi}'$ between the elements 0, 1, 3s as follows: if $\varphi(4x + i) = 4y + \Omega(i)$, then $\tilde{\varphi}'(x) = y$. We obtain that $\tilde{\varphi}'$ is an isomorphism from the embedding of $K_{3s}$ generated by $\tilde{B}_i(s|Q_i)$ onto the embedding of $K_{3s}$ generated by $\tilde{B}_{\Omega(i)}(s|Q_{\Omega(i)})$ (by Theorem 1, the two embeddings are isomorphic if and only if $\tilde{B}_i(s|Q_i) \sim \tilde{B}_{\Omega(i)}(s|Q_{\Omega(i)})$). Now Lemma 3 implies the following:

(D) If the OT-embeddings of $K_{12s}$ generated by $\Gamma(12s|Q_h, Q_r)$ and $\Gamma(12s|Q'_h, Q'_r)$ are isomorphic, then there is a permutation $\tilde{\Omega}$ on elements $\{h, r\}$ such that $\tilde{B}_{\Omega(i)}(s|Q_{\Omega(i)})$ for every $i \in \{h, r\}$.

Consider the following equivalence relation on the set of all ordered pairs $\langle \tilde{B}_h(s|Q_h), \tilde{B}_r(s|Q_r) \rangle$; two pairs $\langle \tilde{B}_h(s|Q_h), \tilde{B}_r(s|Q_r) \rangle$ and $\langle \tilde{B}_h(s|Q'_h), \tilde{B}_r(s|Q'_r) \rangle$ are equivalent if and only if $\tilde{B}_h(s|Q_h) \sim \tilde{B}_h(s|Q'_h)$ and $\tilde{B}_r(s|Q_r) \sim \tilde{B}_r(s|Q'_r)$.

It follows from (D) that if the embeddings generated by $\Gamma(12s|Q_h, Q_r)$ and $\Gamma(12s|Q'_h, Q'_r)$ are isomorphic, then the pairs $\langle \tilde{B}_h(s|Q_h), \tilde{B}_r(s|Q_r) \rangle$ and $\langle \tilde{B}_h(s|Q'_h), \tilde{B}_r(s|Q'_r) \rangle$ are equivalent. It was shown that there are at least $2^{s-3}$ different ordered pairs $\langle Q_h, Q_r \rangle$, thus there are at least $2^{s-3}$ different pairs $\langle \tilde{B}_h(s|Q_h), \tilde{B}_r(s|Q_r) \rangle$. Hence to prove the theorem it suffices to show that among the $2^{s-3}$ pairs $\langle \tilde{B}_h(s|Q_h), \tilde{B}_r(s|Q_r) \rangle$ there are at least $(\frac{1}{8})2^{s-3}$ nonequivalent
pairs, that is, it suffices to show that:

(a) At most eight different pairs \( \langle \tilde{B}_h(s|Q_h), \tilde{B}_r(s|Q_r) \rangle \) can be mutually equivalent.

To prove (a) it suffices to prove the following:

(b) For \( \{i, j\} = \{h, r\} \) and for every \( \tilde{B}_i(s|Q_i) \), there is at most one another \( \tilde{B}_i(s|Q'_i) \) isomorphic to \( \tilde{B}_i(s|Q_i) \), and there are at most two different \( \tilde{B}_j(s|Q_j) \) isomorphic to \( \tilde{B}_i(s|Q_i) \).

Having proved (b), for every pair \( \langle \tilde{B}_h(s|Q_h), \tilde{B}_r(s|Q_r) \rangle \) there are at most four pairs \( \langle \tilde{B}_h(s|Q'_h), \tilde{B}_r(s|Q'_r) \rangle \) such that \( \tilde{B}_h(s|Q_h) \sim \tilde{B}_h(s|Q'_h) \) and \( \tilde{B}_r(s|Q_r) \sim \tilde{B}_r(s|Q'_r) \) (resp., \( \tilde{B}_h(s|Q_h) \sim \tilde{B}_r(s|Q'_r) \) and \( \tilde{B}_r(s|Q_r) \sim \tilde{B}_h(s|Q'_h) \)), whence (a) follows.

Now, we aim to prove (b). Examining \( B_h(s) = \langle G_h, \lambda_h, P_h \rangle \) and \( B_r(s) = \langle G_r, \lambda_r, P_r \rangle \) in Figs. 10, 12, 14, 18, the reader can see that for \( s \neq 2 \mod 4 \) the group of automorphisms of \( G_i (i = h, r) \) consists of the identical automorphism only. For \( s \equiv 2 \mod 4 \), when we consider \( G_i, \lambda_i \) with current group \( Z_{12t+6} \), we have \( \varphi(6t+3) = 6t+3 \) for every automorphism \( \varphi \) of \( Z_{12t+6} \), hence every strong automorphism of \( G_i, \lambda_i \) takes the end arc to itself. Now it is easy to check the following:

(E) For every \( s \), the group of strong automorphisms of \( G_i, \lambda_i \), \( i = h, r \), consists of the identical strong automorphism only (the automorphism takes each vertex and arc to itself).

The current graph \( B_i(s) \) is obtained from \( B_h(s) \) by reversing the orientations of all arcs and the rotations of all vertices, hence, for every \( i, j \in \{h, r\} \), there is a strong isomorphism from \( G_i, \lambda_i \) onto \( G_j, \lambda_j \) (for \( i \neq j \), the automorphism of the current group associated with the strong isomorphism takes every element of the group to the inverse element). Suppose that there exist two different strong isomorphisms \( (\omega_1, \tilde{\omega}_1) \) and \( (\omega_2, \tilde{\omega}_2) \) from \( G_i, \lambda_i \) onto \( G_j, \lambda_j \), where \( i, j \in \{h, r\} \). Then there is a nonidentical strong automorphism \( (\omega, \tilde{\omega}) \) of \( G_j, \lambda_j \) defined by \( \omega_1(v) = \omega_2(v) \) and \( \tilde{\omega}_1(a) = \tilde{\omega}_2(a) \) for every vertex \( v \) and arc \( a \) of \( G_j \). It contradicts (E). Hence we obtain the following:

(F) For every \( i, j \in \{h, r\} \), there is exactly one strong isomorphism from \( G_i, \lambda_i \) onto \( G_j, \lambda_j \).

If \( (\omega, \tilde{\omega}) \) is an isomorphism from \( G_i, \lambda_i, Q_i \) onto \( G_j, \lambda_j, Q_j \) (i.e., \( i, j \in \{h, r\} \)), then \( (\omega, \tilde{\omega}) \) is a strong isomorphism from \( G_i, \lambda_i \) onto \( G_j, \lambda_j \). For a one-rotation \( Q_1 \) of \( G_i \) and for every strong isomorphism from \( G_i, \lambda_i \) onto \( G_j, \lambda_j \), there are exactly two one-rotations \( Q'_j \) (they are reverse) of \( G_j \) such that this \( (\omega, \tilde{\omega}) \) is an isomorphism from \( G_i, \lambda_i, Q_i \) onto \( G_j, \lambda_j, Q'_j \). Now, taking (F) into account, we obtain (b). Note that if \( G_j \) has a trivalent starred vertex, then reversing an admissible one-rotation for \( B_j(s) \) does not yield an admissible one-rotation for \( B_j(s) \).

It is possible, for every \( s \mod 4 \), to construct \( I'(12s) \) such that all side blocks are current graphs (as in the case \( s \equiv 2 \mod 4 \)). It would give us a possibility to obtain the lower bound of \( 2^{2s-1} \) on the number of nonisomorphic OT-embeddings of \( K_{12s} \). But this solution seems to be rather cumbersome. So in the present paper we restrict ourselves to a more simple solution that gives the lower bound of \( 2^{s-1} \).

6. Proof of Lemma 3

In the proof of Lemma 3(a) and (b), by an arc of a current graph we mean an arc depicted in the figure of the current graph or the fragment. When in the proof of Lemma 3(a) we say “...there are \( k \) arcs (\( k \) links)…” or “...contains \( k \) arcs…” we always mean “...there are exactly \( k \) arcs (\( k \) links)…” or “...contains exactly \( k \) arcs...”, respectively.

The following important property of pl-fragments is easily verified:

(G) For a pl-fragment with \( m \) vertical arcs (see, for example, Fig. 3(a), where \( m = 2t+4 \)), all upward-directed (resp., downward-directed) vertical arcs are of the same \( T(l|\ell|\ell') \) (resp., \( T(k|\ell'|\ell) \)). All inner horizontal arcs are of type \( T(i|\pm 4|i) \) or \( T(j|\pm 4|j) \). An inner horizontal arc is of type \( T(i|\pm 4|i) \) if and only if the symmetrical inner horizontal arc is of type \( T(j|\pm 4|j) \), hence there are \( m-1 \) inner horizontal arcs of type \( T(i|\pm 4|i) \) (resp., \( T(j|\pm 4|j) \)).
In the figure of \(L(s)\) and \(\bar{L}(s)\) there are four horizontal arcs, each of which is represented by two halves belonging to distinct, upper and lower, parts. The arcs are called the broken arcs, they are not inner horizontal arcs of any minimal pl-fragment of the current graph.

In what follows, for simplicity, we will write \(V_i\) instead of \(V_i(12x)\) (it will be clear from the context what \(s\) is meant). Denote by \(\tilde{T}(12s)\) an arbitrary \(\Gamma(12s|Q_h, Q_r)\). By a 3-arc of \(\bar{L}(s)\) or \(\tilde{T}(12s)\) we mean an arc joining two trivalent vertices. Among the arcs of \(\bar{L}(s)\) only 3-arcs induce links.

**Proof of Lemma 3(a).** By Lemma 1, it suffices to prove the following lemma.

**Lemma 4.** There are integers \(m(s) < M(s)\) such that for every \(s \geq 6\), in the OT-embedding of \(K_{12s}\) generated by an arbitrary \(\Gamma(12s|Q_h, Q_r)\) every two vertices from distinct vertex parts have at most \(m(s)\) links, and every two vertices \(x, y\) and \(x + 4\) from the same vertex part have at least \(M(s)\) links.

Now we prove Lemma 4. For the embedding generated by \(\bar{L}(s)\) denote by \(r(s)\) the maximal number of links between two vertices from distinct vertex parts, and denote by \(R(s)\) the minimal number of links between vertices \(x\) and \(x + 4\) from the same vertex part. For every case \(s \mod 4\), we will examine the figure of \(\bar{L}(s)\) and, using Claims 4, 6 and (G), determine the links induced by the arcs of the current graph, thereby obtaining \(r(s)\) and \(R(s)\). Then, knowing how \(\bar{L}(s)\) is incorporated into \(\bar{T}(12s)\), it is easy to evaluate \(m(s)\) and \(M(s)\).

Before considering the four cases we explain briefly the main reason why Lemma 4 is true. The lower part of \(L(s)\) contains 3s vertical arcs, the upper part of \(L(s)\) contains 3s (resp., \(3s - 2\)) vertical arcs for even (resp., odd) \(s\). As a result, every part of \(\bar{L}(s)\) contains about 3s horizontal arcs of type \(T(i| \pm 4|i)\) for \(i = 0, 1, 2, 3\). In every part of \(\bar{L}(s)\) almost all vertical arcs are vertical arcs of pl-fragments. By (G), in every pl-fragment all upward-directed (resp., downward-directed) vertical arcs are of the same type \(T(\ell | A(k)|)\) (resp., \(T(k| A'(\ell))\), \(\ell \neq k\), and in our constructions we have \(A \neq A'\). As a result, in the embedding generated by \(\bar{L}(s)\) every two vertices from distinct vertex parts have at most \(3s/2\) links, and every two vertices \(x\) and \(x + 4\) from the same vertex part have about 3s links. When transforming \(\bar{L}(s)\) into \(\bar{T}(12s)\) we form no more than 42 new arcs which induce links between vertices from distinct vertex parts, and examining the types of the arcs we obtain Lemma 4.

**Case \(s = 2n, n \geq 2\):** For \(n = 2t\) and \(n = 2t + 1\), the current graph \(L(2n)\) has the same lower part given in Fig. 6. The lower part is a pl-fragment with 6n vertical arcs. For \(i = 0, 3\), there are \(6n - 1\) inner horizontal arcs of type \(T(i| \pm 4|i)\), hence there are \(6n - 1\) links between \(x\) and \(x + 4\) for every \(x \in V_i\). All \(3n\) upward-directed (resp., downward-directed) vertical arcs of the lower part are of type \(T(2|12n - 3|1)\) (resp., \(T(1|12n + 1|2)\)), hence there are \(3n\) links between \(y\) and \(y + 12n + 3\), and between \(y\) and \(y + 12n + 1\) for every \(y \in V_i\).

The upper part of \(\bar{L}(4t)\) given in Fig. 11(c) contains \(12t - 4\) vertical arcs: \(6t - 2\) vertical arcs of type \(T(3|1|0)\) or \(T(0|1|3)\), and \(6t - 2\) vertical arcs of type \(T(0|3|3)\) or \(T(3|3|0)\). The vertical arcs induce \(6t - 2\) links between \(y\) and \(y + 1\), and between \(y + 2\) and \(y + 3\) for every \(y \in V_0\).

The upper part of \(\bar{L}(4t + 2)\) given in Fig. 13(b) contains \(12t + 4\) vertical arcs: \(6t + 3\) vertical arcs of type \(T(3|1|0)\) or \(T(0|1|3)\), and \(6t + 1\) vertical arcs of type \(T(0|3|3)\) or \(T(3|3|0)\). The vertical arcs induce \(6t + 3\) links between \(y\) and \(y + 1\), and \(6t + 1\) links between \(y + 2\) and \(y + 3\) for every \(y \in V_0\).

In the upper part of \(\bar{L}(2n)\) every unbroken horizontal arc inducing links is an inner horizontal arc of a pl-fragment. For \(i = 1, 2\) and for \(n = 2t\) (resp., \(n = 2t + 1\)), the upper part of \(\bar{L}(2n)\) has \(6n - 9\) (resp., \(6n - 4\)) inner horizontal arcs of type \(T(i| \pm 4|i)\), hence there are \(6n - 9\) (resp., \(6n - 4\)) links between \(x\) and \(x + 4\) for every \(x \in V_i\).

It remains to consider the four broken horizontal arcs. Two of them are 3-arcs for every \(n\) and have types \(T(0|1|1)\) and \(T(2|1|3)\), respectively. The other two arcs are 3-arcs for \(n = 2t\) (in this case they have types \(T(3|12n - 1|2)\) and \(T(1|12n - 1|0)\), respectively) and are not 3-arcs for \(n = 2t + 1\). Hence, the broken arcs induce one link between \(y\) and \(y + 1\) for every \(y \in V_0 \cup V_2\) and, for \(n = 2t\), they, in addition, induce one link between \(y\) and \(y + 12n + 1\) for every \(y \in V_0 \cup V_2\).

We obtain \(r(4t) = 6t, R(4t) = 12t - 9, r(4t + 2) = 6t + 3, R(4t + 2) = 12t + 2\).

**Case \(s = 4t + 1, t \geq 2\):** In the upper part of \(\bar{L}(4t + 1)\) given in Fig. 15(b) all 6t upward-directed (resp., downward-directed) arcs of the two maximal pl-fragments are of type \(T(3|1|0)\) (resp., \(T(0|3|3)\)), hence there are \(6t\) links between \(x\) and \(x - 1\), and between \(x\) and \(x + 3\) for every \(x \in V_0\). The vertical arc \((1), 24t + 1, [2]\) is of type \(T(3|24t + 9|0)\) and induces one link between \(x\) and \(x + 24t + 3\) for every \(x \in V_0\). For \(i = 1, 2\), the upper part has \(12t - 2\) inner horizontal arcs of type \(T(i| \pm 4|i)\), hence there are \(12t - 2\) links between \(x\) and \(x + 4\) for every \(x \in V_i\). There are four
horizontal 3-arcs of the upper part which are not inner horizontal arcs of pl-fragments. These four arcs are adjacent to the vertical arc ([1], 24t + 1, [2]): two of the arcs are of type $T(1|24t+1)$ or $T(1|24t+12|1)$; the other two arcs are of type $T(2|24t+2)$ or $T(2|24t+12|2)$. Hence there are two links between vertices $x$ and $x + 24t$ for every $x \in V_1 \cup V_2$.

In the lower part of $\overline{L}(4r + 1)$ given in Fig. 16(c) there are $4t - 1$ and $2r + 1$ upward-directed vertical arcs of type $T(1|24r + 5|2)$ and $T(1|24r + 1|2)$, respectively, and there are $4t - 1$ downward-directed vertical arcs of type $T(2|24r + 11|1)$ and $t$ downward-directed vertical arcs of type $T(2|24t + 15|1)$ (resp., $T(2|24t + 7|1)$). Hence, for every $x \in V_1$, there are $6t$ links between $x$ and $x + 24t + 1$, $5t - 1$ links between $x$ and $x + 24t + 5$, and $t$ links between $x$ and $x + 24t - 3$. In the lower part all four maximal pl-fragments contain together exactly $2(12t - 5)$ inner horizontal arcs: $12t - 5$ inner horizontal arcs of type $T(j \pm 4|j)$ inducing $12t - 5$ links between $x$ and $x + 4$ for every $x \in V_j$, $j = 0, 3$.

We obtain $r(4t + 1) = 6t$ and $R(4t + 1) = 12t - 5$.

Case $s = 4t + 3, t \geq 1$: In the upper part of $\overline{L}(4r + 3)$ given in Fig. 19(c) there are $6t + 3$ vertical arcs of type $T(0|3|3)$ or $T(3|3|0)$, $5t + 2$ vertical arcs of type $T(0|0|3)$, and $t - 1$ vertical arcs of type $T(3|3|7|0)$. Hence there are $6t + 3$ links between $x$ and $x + 3$, $5t + 2$ links between $x$ and $x - 1$, and $t - 1$ links between $x$ and $x + 7$ for every $x \in V_0$. The vertical arc ([1], 24t + 13, [2]) is of type $T(0|24t + 15|3)$ and induces one link between $x$ and $x + 24t + 15$ for every $x \in V_0$. The upper part contains $2(12t + 1)$ inner horizontal arcs of pl-fragments: $12t + 1$ inner horizontal arcs of type $T(i \pm 4|i)$ inducing $12t + 1$ links between $x$ and $x + 4$ for every $x \in V_i, i = 1, 2$.

There are four horizontal 3-arcs of the upper part which are not inner horizontal arcs of pl-fragments. The four arcs are adjacent to the vertical arc ([1], 24t + 13, [2]) and have types $T(1|24|8|1)$, $T(1|24|24|1)$, $T(2|24|8|2)$, and $T(2|24|24|2)$, respectively. Hence, there is one link between $x$ and $x + 24t + 8$, and between $x$ and $x + 24t + 24$ for every $x \in V_1 \cup V_2$.

In the lower part of $\overline{L}(4t + 3)$ given in Fig. 20(b) all $6t + 4$ upward-directed (resp., $6t + 3$ downward-directed) vertical arcs are of type $T(1|24t + 13|2)$ (resp., $T(2|24t + 19|1)$), hence there are $6t + 4$ links between $x$ and $x + 24t + 13$, and $6t + 3$ links between $x$ and $x + 24t + 17$ for every $x \in V_1$. The lower part contains $2(12t + 5)$ inner horizontal arcs of pl-fragments: for $i = 0, 3$ there are $12t + 5$ inner horizontal arcs of type $T(i|4|i)$ inducing $12t + 5$ links between $x$ and $x + 4$ for every $x \in V_i$.

We obtain $r(4t + 3) = 6t + 4$ and $R(4t + 3) = 12t + 1$.

When constructing $\tilde{T}(12s)$, all 3-arcs of $\overline{L}(s)$ become 3-arcs of $\tilde{T}(12s)$ and their types are unchanged. It follows that all links of the embedding generated by $\overline{L}(s)$ are links of the embedding generated by $\tilde{T}(12s)$, hence we can put $M(s) = R(s)$.

The arcs of $\tilde{T}(12s)$ which are not 3-arcs of $\overline{L}(s)$ and which induce links between vertices from different vertex parts (the arcs are of type $T(|i|4|j), i \neq j$), are called the essential arcs of $\tilde{T}(12s)$.

By the construction of $\tilde{T}(12s)$, the set of essential arcs of $\tilde{T}(12s)$ consists of the following arcs:

(i) every arc incident to exactly one vertex such that the vertex is a trivalent vertex in $\overline{L}(s)$;
(ii) every arc incident to a onevalent unstarred vertex of a side block such that the vertex is a trivalent vertex in $\tilde{T}(12s)$ (this arc is contained in a 2-cycle of $\tilde{T}(12s)$ such that the vertices of the cycle are from different side blocks).

The reader can easily see that the number $e(s)$ of essential arcs of $\tilde{T}(12s)$ is 32, 42, 16, and 42 for $s$ equal to $4t$, $4t + 1$, $4t + 2$, and $4t + 3$, respectively. It is significant that, given $s$, the set of all essential arcs (and thus the number $e(s)$) is the same for every $\tilde{T}(12s)$, that is, does not depend on the choice of $Q_b$ and $Q_r$.

Given two vertices from distinct vertex parts, every essential arc can induce at most one link between the vertices, hence it would be possible to put $m(s) = r(s) + e(s)$. We have shown that the pair $(r(s), R(s))$ is $(6t, 12t - 9)$, $(6t, 12t - 5)$, $(6t + 3, 12t + 2)$, and $(6t + 4, 12t + 1)$ for $s$ equal to $4t, 4t + 1, 4t + 2$, and $4t + 3$, respectively. Simple calculations show that $M(s) = R(s) > r(s) + e(s) = m(s)$ holds for $s \geq 32$ only, that is, this choice of $m(s)$ is not good enough.

By tedious but routine inspection of the links induced by the essential arcs, the reader can check that every $\tilde{T}(12s)$, $s \geq 6$, has the following two properties:

- the essential arcs do not induce a link between every two vertices $x$ and $y$ such that in the embedding generated by $\overline{L}(s)$ there is a link between $x$ and $y$;
- at most six different essential arcs induce links between the same pair of vertices.
Hence, we can put \( m(s) = \max(r(s), 6) \). Simple calculations show that \( M(s) = R(s) > \max(r(s), 6) = m(s) \) for \( s \geq 6 \). This completes the proof of Lemma 3(a). □

**Proof of Lemma 3(b).** Case \( s \neq 4t + 2 \): In this case exactly two side blocks, \( B_h(s) \) and \( B_r(s) \), are current graphs.

Note that every trivalent vertex of \( B_i(s), i \in \{h, r\} \), in \( \tilde{T}(12s) \) is passed by the circuit \([i]\) only.

By the construction of \( \tilde{T}(12s) \), if \( B_i(s) \) is a current graph, then every trivalent vertex of \( B_i(s) \) in \( \tilde{T}(12s) \) is adjacent to some another trivalent vertex of \( B_j(s) \) in \( \tilde{T}(12s) \), and every edge of \( \tilde{T}(12s) \) traversed twice by the circuit \([i]\) is incident with a trivalent vertex of \( B_i(s) \) in \( \tilde{T}(12s) \). As a result, taking into account Fig. 5, the derived embedding has the following properties:

(H1) Every triangular face incident with vertices from \( V_i \) only is adjacent to some another triangular face incident with vertices from \( V_j \) only.

(H2) Every edge joining two vertices from \( V_i \) is incident with a face incident with vertices from \( V_j \) only.

If \( B_j(s) \) is not a current graph, then \( B_j(s) \) has an isolated edge which in \( \tilde{T}(12s) \) is either incident with onevalent vertex passed by \([j]\) and with trivalent vertex passed by \([k]\), \( j \neq k \) (in this case, by Fig. 5, (H1) does not hold if we replace \( i \) by \( j \)) or incident with two trivalent vertices passed by circuits \([k]\) and \([\ell]\), respectively, such that \( k, \ell \neq j \) (in this case (H2) does not hold if we replace \( i \) by \( j \)).

By Lemma 3(a), if \( \phi \) is an isomorphism of \( f \) onto \( f' \), and if (H1) and (H2) hold for some \( i \in \{0, 1, 2, 3\} \), then (H1) and (H2) hold for \( \Omega(i) \) also. Hence, if \( i \in \{h, r\} \), then \( \Omega(i) \in \{h, r\} \).

Case \( s = 4t + 2 \): In this case all side blocks are current graphs. Recall that we have chosen \( h \) = 1 and \( r = 2 \). For \( i, j \in \{0, 1, 2, 3\} \) denote by \( N(i, j) \) (resp., \( n(i, j) \)) the number of arcs (resp., of essential arcs) of type \( T(i|A|j) \) or \( T(j|A'|i) \) in the figure of \( I'(48t + 24) \). The current graph \( I'(48t + 24) \) has only 16 essential arcs and it is easy to check that

\[
\begin{align*}
n(0, 1) &= n(2, 3) = 1, \\
n(0, 2) &= n(1, 3) = n(0, 3) = 2, \\
n(1, 3) &= 8.
\end{align*}
\]

Taking into account the description of the link set of the embedding generated by \( \tilde{L}(4t + 2) \) given in the proof of Lemma 3(a), we can obtain

\[
\begin{align*}
2 &= N(0, 2) = N(1, 3) < 3 = N(0, 1) = N(2, 3) < 12t + 5 \\
&= N(0, 3) < 12t + 4 = N(1, 2).
\end{align*}
\]

The number \( N(i, j) \) is the total number of links that a vertex from \( V_i \) has with all vertices from \( V_j \) in the embedding generated by \( I'(48t + 24) \). Hence, by Lemma 3(a), \( N(\Omega(1), \Omega(j)) = N(i, j) \) for all \( i \) and \( j \). Now it follows from (1) that \( N(\Omega(1), \Omega(2)) = N(1, 2) \), hence \( \{\Omega(1), \Omega(2)\} = \{1, 2\} \). □

**Proof of Lemma 3(c).** First we consider how the embedding \( \mathcal{F}_i \) of \( K_{3s} \) (with vertex set \( V_i \)) generated by a current graph \( B_i(s|\bar{Q}_i) \), \( i \in \{h, r\} \), is incorporated in the embedding \( \mathcal{F} \) of \( K_{12s} \) generated by \( I'(12s|\bar{Q}_h, \bar{Q}_r) \) for arbitrary \( \bar{Q}_h \) and \( \bar{Q}_r \).

Now we show that all triangular faces of \( \mathcal{F}_i \) become all triangular faces of \( \mathcal{F} \) incident with vertices from \( V_i \) only. In \( B_i(s|\bar{Q}_i) \) the starred trivalent vertex does not satisfy KCL, and every onevalent vertex that becomes a trivalent vertex in \( I'(12s|\bar{Q}_h, \bar{Q}_r) \) does not have excess \( \pm 4s \) in \( B_i(s|\bar{Q}_i) \). Hence, the onevalent vertex with excess \( \pm 4s \) and every unstarred trivalent vertex are the only vertices of \( B_i(s|\bar{Q}_i) \) inducing triangular faces of \( \mathcal{F}_i \) and, by the construction of \( I'(12s|\bar{Q}_h, \bar{Q}_r) \), the vertices become the only vertices of \( I'(12s|\bar{Q}_h, \bar{Q}_r) \) inducing triangular faces of \( \mathcal{F} \) incident with vertices from \( V_i \) only. It is easy to see the following:

(J) [\( x, y, z \)] is a face of \( \mathcal{F}_i \), if and only if \( [x, y, z] \), is a face of \( \mathcal{F} \) such that \( x, y, z \in V_i \).

Consider nontriangular faces of \( \mathcal{F}_i \). The edges \( (x, y) \) and \( (y, z) \) are neighboring i-boundary edges of \( \mathcal{F} \) if the following holds: \( x, y, z \in V_i \); in \( \mathcal{F} \) there are faces \( [x, y, x'] \) and \( [y, z, z'] \), where \( x', z' \in V_i \); there is a sequence \( x = v_0, v_1, \ldots, v_m, v_{m+1} = z \) (\( m \geq 1 \)) of vertices such that \( v_1, v_2, \ldots, v_m \notin V_i \) and \( [y, v_j, v_{j+1}] \) is a face of \( \mathcal{F} \) for \( j = 0, 1, \ldots, m \) (see Fig. 23).
Take the faces of $V_i$ of $F$ and consider an arbitrary subsequence $(a_j, a_{j+1}, \ldots, a_k)$ of the circuit $[i]$ such that the arc $a_j$ (resp., $a_k$) is directed from (resp., to) a vertex of $B_i(s|\overline{Q}_i)$ inducing nontriangular faces, and the arcs $a_{j+1}, a_{j+2}, \ldots, a_{k-1}$ are not incident with vertices of $B_i(s|\overline{Q}_i)$ inducing nontriangular faces. We have $k > j$ since in $B_i(s|\overline{Q}_i)$ no two vertices inducing nontriangular faces are adjacent. By the construction of $\Gamma(12s|\overline{Q}_h, \overline{Q}_r)$, it is easy to see that the circuit $[i]$ of $\Gamma(12s|\overline{Q}_h, \overline{Q}_r)$ is of the form $(a_j, a_{j+1}, \ldots, a_k, b_1, b_2, \ldots, b_t)$, where the arcs $b_1, b_2, \ldots, b_t$ are arranged on the surface as shown in Fig. 23(b) where the edges $x + \lambda_i(a_k)$, $x + \lambda_i(a_{k+1})$, and $x + \lambda_i(a_{k+2})$ are neighboring edges on the boundary of a nontriangular face of $F$ and the edges are neighboring $i$-boundary edges of $F$. We see that two edges joining vertices from $V_i$ are neighboring edges on the boundary of a nontriangular face of $F$ if and only if the edges are neighboring $i$-boundary edges of $F$. Hence we have the following:

(K) $[x_1, x_2, \ldots, x_n], n \geq 4$ is a nontriangular face of $F$ if and only if $(x_1, x_2, \ldots, x_n)$ is an $i$-boundary cycle of $F$.

The statements (J) and (K) show that the embedding $F_i$ is incorporated in the embedding $F$ in the following way. Take $F_i$ and remove the interior of every nontriangular face. We obtain a surface with boundaries. All faces on the surface are all the faces of $F_i$ that become faces of $F$, and the edges of the boundaries are exactly the edges which the faces of $F_i$ share with other faces of $F$.

Now we are ready to prove Lemma 3(c).

By Lemma 3(a), if $[x_1, x_2, x_3]$ is a face of $f$ incident with vertices from $V_i, i \in \{h, r\}$ only, then $[\varphi(x_1), \varphi(x_2), \varphi(x_3)]$ is a face of $f'$ incident with vertices from $V_{\Omega(i)}$ only. By Lemma 3(a), if $(x, y)$ and $(y, z)$ are neighboring $i$-boundary edges of $f$, then $[\varphi(x), \varphi(y)]$ and $[\varphi(y), \varphi(z)]$ are neighboring $\Omega(i)$-boundary edges of $f'$ (resp., $[\varphi(x), \varphi(y)]$ and $[\varphi(y), \varphi(z)]$ are neighboring $\Omega(i)$-boundary edges of $f'$). Hence, if $(x_1, x_2, \ldots, x_n), n \geq 4$, is an $i$-boundary cycle of $f$, then $[\varphi(x_1), \varphi(x_2), \ldots, \varphi(x_n)]$ is an $\Omega(i)$-boundary cycle of $f'$. Taking (J) and (K) into account, we obtain that if $[x_1, x_2, \ldots, x_n], n \geq 3$, is a face of the embedding generated by $B_i(s|\overline{Q}_i)$, then $[\varphi(x_1), \varphi(x_2), \ldots, \varphi(x_n)]$ is a face of the embedding generated by $B_{\Omega(i)}(s|\overline{Q}_i)$. This completes the proof. □
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References