ACADEMIC PRESS

Available at www.ComputerScienceWeb.com
POWERED BY SCIENCE @ DIRECT®

Journal of Computer and System Sciences 67 (2003) 174–182

JOURNAL OF COMPUTER AND SYSTEM SCIENCES

http://www.elsevier.com/locate/jcss

Remarks on permutive cellular automata

J.-P. Allouche[a,*] and G. Skordev[b]

[a] *CNRS, LRI, Bâtiment 490, F-91405 Orsay, Cedex France*
[b] *CEVIS, Universität Bremen, Universitätsallee 29, D-28359 Bremen, Germany*

Received 8 March 2002; revised 23 January 2003

Abstract

We prove that every two-dimensional permutive cellular automaton is conjugate to a one-sided shift with compact set of states.

MSC: 37B15; 68Q80

Keywords: Cellular automata; Permutivity; Permutativity; One-sided shift; Chaotic cellular automata

1. Preliminaries

1.1. Cellular automata on finitely generated abelian groups

In this section we recall the definition of a cellular automaton A with set of states $\mathcal{S}$ on an abelian group G of finite rank, and we give some notations (see [2,3] for example).

We denote by $\mathbb{Z}$ the set of integers and by $\mathbb{N}$ the set of nonnegative integers. Let G be a finitely generated abelian group of rank n (see for example [11, Chapter 3]). We fix a representation of G as a direct sum $G = \mathbb{Z}^n \oplus T$, where T is a finite abelian group. Then an element $g \in G$ is an $(n+1)$-tuple: $g = (m_1, \ldots, m_n, h)$, $m_1, \ldots, m_n \in \mathbb{Z}$, $h \in T$.

The *set of states* $\mathcal{S}$ of the cellular automaton A is a finite set with cardinality $|\mathcal{S}| \geqslant 1$.

A *configuration* of A is a map $\underline{x} : G \to \mathcal{S}$. The set of all configurations is denoted by $\mathcal{S}^G$ or by $M(G, \mathcal{S})$. The *shift maps* $\sigma_g : \mathcal{S} \to \mathcal{S}$ are defined on the set of configurations as follows. For a given element $g \in G$ the shift σ_g is defined by

$$\sigma_g(\underline{x})(a) := \underline{x}(a+g), \quad \forall a \in G.$$

*Corresponding author.
E-mail addresses: allouche@lri.fr (J.-P. Allouche), skordev@cevis.uni-bremen.de (G. Skordev).

0022-0000/03/$ - see front matter
doi:10.1016/S0022-0000(03)00049-7

The *cellular automaton* A is a map $A:\mathscr{S}^G\to\mathscr{S}^G$ which is *local* and *homogeneous*. This means that the map A is defined by a given *local* (generating) function (or *local rule*) $l:\mathscr{S}^{N_k}\to\mathscr{S}$, where $N_k=[-k,k]^n\times T$, $k\in\mathbb{N}$ ($\mathscr{S}^{N_k}$ is the set of all configurations/blocks on N_k with values in $\mathscr{S}$). For a given $\underline{x}\in\mathscr{S}^G$ we denote by $\underline{x}|_k$ the restriction of the configuration $\underline{x}$ to N_k. Then the map A is defined by

$$A(\underline{x}(g))=l(\sigma_{-g}(\underline{x}|_k)).$$

There is an elegant topological definition of cellular automata in [9, Theorem 3.4]. To recall it we introduce a product metric on the space of configurations $\mathscr{S}^G$ generated by the (discrete) metric on the sets S and G (see [13, p. 122]). The space $\mathscr{S}^G$ with this metric is compact and totally disconnected, and therefore homeomorphic to the triadic Cantor set (see for example [13, pp. 142, 165–166]). The shifts $\sigma_g:\mathscr{S}^G\to\mathscr{S}^G$ are homeomorphisms. The set $\mathscr{S}^G$ is a G-space, i.e., the group G acts on it as a group of transformations. The action is given by the embedding of G into the group of homeomorphims of the space $\mathscr{S}^G$ to itself: $g\to\sigma_g$ (see for example [10, pp. 112–113]). The natural maps $B:\mathscr{S}^G\to\mathscr{S}^G$ of the G-space $\mathscr{S}^G$ are the G-maps, i.e., the continuous maps which commute with all shifts σ_g:

$$B\sigma_g(\underline{x})=\sigma_g B(\underline{x}),\quad \forall g\in G,\ \forall \underline{x}\in\mathscr{S}^G.$$

In [9, Theorem 3.4], it is proved that *the cellular automata on $\mathscr{S}^G$ are exactly the G-maps*. The proof of Hedlund is given in the case where $G=\mathbb{Z}$, but the same proof works in general (see also [16]).

Later on we shall consider the cellular automaton $A:\mathscr{S}^G\to\mathscr{S}^G$ as a dynamical system and shall use the notation $(A,\mathscr{S}^G)$. Recall that a (discrete, topological) dynamical system (f,X) is a continuous map $f:X\to X$ on a (compact metric) space X (see [6, Chapter 2]).

1.2. Cellular automata on finitely generated abelian groups of rank n and on the n-dimensional lattice $\mathbb{Z}^n$

In this section we give a representation of any cellular automaton on an abelian group $G=\mathbb{Z}^n\oplus T$ with states $\mathscr{S}$ as a cellular automaton on $\mathbb{Z}^n$ with appropriate state space. For this purpose we will use the classical identification

$$i_1:M(\mathbb{Z}^n\oplus T,\mathscr{S})\to M(\mathbb{Z}^n,M(T,\mathscr{S})),$$

defined as follows. Let $\underline{x}:\mathbb{Z}^n\oplus T\to\mathscr{S}$ be a configuration. Then

$$i_1(\underline{x}):\mathbb{Z}^n\to M(T,\mathscr{S})$$

is given by

$$i_1(\underline{x})(m_1,\dots,m_n)(h)=\underline{x}(m_1,\dots,m_n,h)$$

for $(m_1,\dots,m_n)\in\mathbb{Z}^n$ and $h\in T$. The map i_1 is a homeomorphism (see for example [10, pp. 23–24]). Now with a given cellular automaton $A:\mathscr{S}^G\to\mathscr{S}^G$ we associate a cellular automaton

$$A_1:\mathscr{S}_1^{\mathbb{Z}^n}\to\mathscr{S}_1^{\mathbb{Z}^n}$$

with state space $\mathscr{S}_1 = \mathscr{S}^T$ defined by

$$A_1 = i_1 A i_1^{-1}.$$

This means that the dynamical systems $(A, \mathscr{S}^G)$ and $(A_1, \mathscr{S}_1^{\mathbb{Z}^n})$ are conjugate and therefore they have the same dynamics, see for example [6, p. 109].

1.3. *Representation of multi-dimensional cellular automata as one-dimensional cellular automata*

A cellular automaton $A : \mathscr{S}^{\mathbb{Z}^n} \to \mathscr{S}^{\mathbb{Z}^n}$ will be called an n-dimensional cellular automaton. Let $n \geqslant 2$. We will use the identification i_2

$$i_2 : M(\mathbb{Z}^n, \mathscr{S}) \to M(\mathbb{Z}, M(\mathbb{Z}^{n-1}, \mathscr{S})),$$

given by

$$i_2(\underline{x})(m_1)(m_2, \ldots, m_n) = \underline{x}(m_1, \ldots, m_n)$$

for $\underline{x} \in M(\mathbb{Z}^n, \mathscr{S})$. The map i_2 is bijective and bicontinuous. The elements $\underline{z}$ of the set $M(\mathbb{Z}, M(\mathbb{Z}^{n-1}, \mathscr{S}))$ are configurations on $\mathbb{Z}$ with values in the compact (totally disconnected) space $M(\mathbb{Z}^{n-1}, \mathscr{S})$.

With the cellular automaton $A : \mathscr{S}^{\mathbb{Z}^n} \to \mathscr{S}^{\mathbb{Z}^n}$ we associate the map $A_2 : \mathscr{S}_2^{\mathbb{Z}} \to \mathscr{S}_2^{\mathbb{Z}}$, where $\mathscr{S}_2 = M(\mathbb{Z}^{n-1}, \mathscr{S})$ such that

$$A_2 = i_2 A i_2^{-1}.$$

The (topological) dynamical systems $(A, \mathscr{S}^{\mathbb{Z}^n})$ and $(A_2, \mathscr{S}_2^{\mathbb{Z}})$ are conjugate and therefore have the same dynamics. The map A_2 is homogeneous and local, i.e., is a one-dimensional cellular automaton, but with states in a compact set. This leads to a qualitative difference between one-dimensional and higher dimensional cellular automata.

1.4. *One-sided shifts with compact state space*

Let K be a compact set with a metric $\rho_K(.,.)$. On the set $K^{\mathbb{N}} = M(\mathbb{N}, K)$ we define the metric

$$d_1(\underline{u}, \underline{v}) = \sum_{n \geqslant 0} \frac{\rho_K(u(n), v(n))}{2^n},$$

where $\underline{u} = (u(n))_{n \geqslant 0}$ and $v = (v(n))_{n \geqslant 0}$ (see [6, p. 102] for the case $K = \{0, 1\}$).

The map $\sigma_K : K^{\mathbb{N}} \to K^{\mathbb{N}}$ defined by

$$\sigma_K(\underline{u})(m) = \underline{u}(m+1), \quad \forall m \in \mathbb{N}$$

is called the *shift map* on $K^{\mathbb{N}}$. The dynamical system $(\sigma_K, K^{\mathbb{N}})$ is called the *one-sided shift* (shift on $\mathbb{N}$) with state space K.

1.5. Chaotic dynamical systems

Let X be a compact metric space and let $f : X \to X$ be a continuous map. The dynamical system (f, X) is called *chaotic* (*in the sense of Devaney*) (see [6, p. 119]; [4]) if and only if

- the set of periodic points of the map f is dense in X;
- (f, X) is mixing, i.e., for any two (nonempty) open sets $U, V \subset X$ there exists $k \in \mathbb{N}$ with $f^k(U) \cap V \neq \emptyset$.

Note that the one-sided shift is a chaotic dynamical system (see [6, p. 119] in the case of finite K, but the same proof works in general).

Remark 1. (1) The dynamical system (f, X) is called *expanding* if there exists a number $\delta > 0$ such that for any two different points $x, y \in X$ there exists $n \in \mathbb{N}$ with $\rho_X(f^n(x), f^n(y)) > \delta$ (here $\rho_X(.,.)$ is the metric on the space X). For an expanding map f on a compact metric space X the dynamical system (f, X) is conjugate to a one-sided subshift with finite state space [9, Theorem 2.1]. A one-sided subshift is a closed subset of the one-sided shift dynamical system invariant under the shift.

(2) The one-sided shift with state space K is expanding if and only if the set K is finite. We give a simple argument for this assertion. Since the state space K is compact, but not finite, there exist a point $k_0 \in K$ and a sequence $(k_m)_{m \geqslant 1}$ with $\lim_{m \to \infty} \rho_K(k_0, k_m) = 0$. Assume that the dynamical system $(\sigma_K, K^{\mathbb{N}})$ is expanding. Then there exists a number $\delta > 0$ such that for any two different points $\underline{x}, \underline{y} \in K^{\mathbb{N}}$ there is $n \in \mathbb{N}$ with $d_1(\sigma_K^n(\underline{x}), \sigma_K^n(\underline{y})) > \delta$. Choose m so large that $\rho_K(k_0, k_m) < \delta$. Consider the points $\underline{x} = (x(k))_{k \geqslant 0} \in K^{\mathbb{N}}$, $x(k) = k_0$, $k \geqslant 0$ and $\underline{y} = (y(k))_{k \geqslant 0} \in K^{\mathbb{N}}$, $x(0) = k_m$, $x(k) = k_0$, $k \geqslant 1$. Then $d_1(\sigma_K^l(\underline{x}), \sigma_K^l(\underline{y})) < \delta$ for all $l \geqslant 0$.

2. Multidimensional cellular automata are conjugated to one-sided shifts with compact state space

Using the notion of permutive cellular automata, that was introduced in [9], Gilman proved in [8] that (bi)permutive one-dimensional cellular automata are topologically conjugate to one-sided shifts with appropriate finite state space. (Note that Gilman uses the expression "linear automata" for "one-dimensional automata". Usually, "linear automata" are automata whose local rule is linear.) This implies that (bi)permutive cellular automata are chaotic as dynamical systems. This theorem was rediscovered several times, e.g., see [1,5,7,14,18]. We will generalize this result to two-dimensional cellular automata. Then, following the discussion of the two-dimensional case, the generalization to n-dimensional automata is also possible.

Let us first recall the definition of *permutivity* in the one-dimensional case [9, Definition 6.3]: a one-dimensional cellular automaton is called *permutive* if and only if the local function has the property that, when all its variables but the leftmost (resp. the rightmost) take any fixed values, then the resulting one-variable function is a bijection. Note that permutivity is sometimes called *permutativity* in the literature; it is also called the *class M property* in [1].

We now introduce an appropriate notion of permutive two-dimensional cellular automaton.

Let $A : \mathscr{S}^{\mathbb{Z}^2} \to \mathscr{S}^{\mathbb{Z}^2}$ be a two-dimensional cellular automaton with states in a finite set $\mathscr{S}$, induced by a local generating function $\lambda : \mathscr{S}^{U_{\mathbf{k},\mathbf{l}}} \to \mathscr{S}$, where $\mathbf{k} = (-k_1, k_2)$, $\mathbf{l} = (-l_1, l_2)$, with $k_1, k_2, l_1, l_2 \in \mathbb{N}$, and $U_{\mathbf{k},\mathbf{l}} = [-k_1, k_2] \times [-l_1, l_2]$.

For $(a, b) \in U_{\mathbf{k},\mathbf{l}}$ denote by $M_{(a,b)}$ the set $U_{\mathbf{k},\mathbf{l}} \backslash \{(a, b)\}$. For any configuration $\underline{c} \in \mathscr{S}^{M_{(a,b)}}$ we define the map

$$i_{\underline{c}} : \mathscr{S} \to \mathscr{S}^{U_{\mathbf{k},\mathbf{l}}}$$

by

$$i_{\underline{c}}(s)(x, y) = \begin{cases} \underline{c}(x, y) & \text{if } (x, y) \neq (a, b), \\ s & \text{if } (x, y) = (a, b). \end{cases}$$

For every $\underline{c} \in \mathscr{S}^{M_{(a,b)}}$ we define the map

$$\mu_{\underline{c}} : \mathscr{S} \to \mathscr{S}$$

by

$$\mu_{\underline{c}}(s) = \lambda(i_{\underline{c}}(s)).$$

Definition 1. The local generating function λ is *permutive* at (a, b) if the map $\mu_{\underline{c}}$ is bijective for all $\underline{c} \in \mathscr{S}^{M_{(a,b)}}$.

Definition 2. The point $(a, b) \in U_{\mathbf{k},\mathbf{l}}$ is called *not essential* for the local generating function λ if $\mu_{\underline{c}} \equiv \mathit{Constant}$ for all $\underline{c} \in \mathscr{S}^{M_{(a,b)}}$.

Definition 3. The cellular automaton A with local generating function $\lambda : \mathscr{S}^{U_{\mathbf{k},\mathbf{l}}} \to \mathscr{S}$ is called *permutive* if and only if

- there exist $u, v \geqslant 1$ and $a_1, \ldots, a_u, b_1, \ldots, b_v$ with

  $$-k_1 \leqslant a_1 < \cdots < a_u \leqslant k_2 \quad \text{and} \quad -k_1 \leqslant b_1 < \cdots < b_v \leqslant k_2$$

  such that the set of essential points of λ of the form (α, l_2) or $(\beta, -l_1)$ is equal to the set

  $$\{(a_1, l_2), \ldots, (a_u, l_2), (b_1, -l_1), \ldots, (b_v, -l_1)\};$$

- the inequalities $a_1 < 0 < a_u$, $b_1 < 0 < b_v$ hold;
- the map λ is permutive at the four points $(a_1, l_2), (a_u, l_2), (b_1, -l_1), (b_v, -l_1)$.

Example 1. Let $\mathscr{S} = GF(q)$ be the finite field with g elements. Then $\mathscr{S}^{U_{\mathbf{k},\mathbf{l}}}$ is a vector space over $GF(q)$. The cellular automaton A with local generating function λ is called a *linear cellular automaton* if the map $\lambda : \mathscr{S}^{U_{\mathbf{k},\mathbf{l}}} \to \mathscr{S}$ is linear (over $GF(q)$). The linear cellular automaton A is permutive if and only if λ has essential points $(a_1, l_2), (a_u, l_2), (b_1, -l_1), (b_v, -l_1)$ with $a_1 < 0 < a_u$, $b_1 < 0 < b_v$.

Remark 2. Permutivity was defined only for cellular automata on the grid $\mathbb{Z}^n$. It can be defined in a similar way for a cellular automaton on a finitely generated abelian group of rank n, either directly, or by using the conjugation of such cellular automata with appropriate cellular automata on the grid $\mathbb{Z}^n$.

Proposition 1. *Let A be a two-dimensional permutive cellular automaton with state space $\mathcal{S}$. Then, the dynamical system $(A, \mathcal{S}^{\mathbb{Z}^2})$ is conjugate to a one-sided shift $(\sigma_K, K^{\mathbb{N}})$ with appropriate compact state space K.*

Proof. We use an idea of Gilman [8] and consider the map

$$h : \mathcal{S}^{\mathbb{Z}^2} \to K^{\mathbb{N}},$$

where $K = \mathcal{S}^M$ and $M = (\mathbb{Z} \times [-l_1 + 1, l_2]) \cup ([a_1, a_u - 1] \times [l_2 + 1, \infty)) \cup ([b_1, b_v - 1] \times [-l_1, -\infty))$ (in the above notations), defined by

$$h(\underline{x})(n) = A^n(\underline{c})|_M, \quad \forall n \in \mathbb{N}$$

for the configuration $\underline{x} : \mathbb{Z}^2 \to \mathcal{S}$ (here A^n is the nth iteration of the cellular automaton A).

The map h is continuous and $hA = \sigma_K h$. Note that if the map h is defined from a continuous map that is *not* a cellular automaton (i.e., that does not commute with the shifts), then the map h is not necessarily surjective or bijective. But here we prove that h is bijective.

Step 1: *The map h is injective.*

Assume that $h(\underline{x}_1) = h(\underline{x}_2)$ for $\underline{x}_1, \underline{x}_2 \in \mathcal{S}^{\mathbb{Z}^2}$. We will prove that $\underline{x}_1 = \underline{x}_2$. The proof is by induction and we will only give the first two steps. Since $h(\underline{x}_1) = h(\underline{x}_2)$, we have $h(\underline{x}_1)(0) = h(\underline{x}_2)(0)$. This is equivalent to $\underline{x}_1(m,n) = \underline{x}_2(m,n)$ for $(m,n) \in M$. We will prove by induction on m that $\underline{x}_1(m, -l_1) = \underline{x}_2(m, -l_1)$ for all $m \geqslant b_v$. For $m = b_v$: the assumption implies $h(\underline{x}_1)(1) = h(\underline{x}_2)(1)$.

Therefore $h(\underline{x}_1)(1)(0,0) = h(\underline{x}_2)(1)(0,0)$ or $A(\underline{x}_1)(0,0) = A(\underline{x}_2)(0,0)$. From the definition of the cellular automaton A we have $A(\underline{x}_j)(0,0) = \lambda(\underline{x}_j | U_{\mathbf{k},\mathbf{l}})$. Since $\underline{x}_1(a,b) = \underline{x}_2(a,b)$ for $(a,b) \in U_{\mathbf{k},\mathbf{l}} \setminus \{(b_v, -l_1)\}$, the permutivity of λ at $(b_v, -l_1)$ implies $\underline{x}_1(b_v, -l_1) = \underline{x}_2(b_v, -l_1)$. The next steps of the induction are similar. Having $\underline{x}_1 | M = \underline{x}_2 | M$ and $\underline{x}_1(m,n) = \underline{x}_2(m,n)$ for $m \in \mathbb{Z}$, $-l_1 \leqslant n \leqslant l_2$ we prove as above that $\underline{x}_1(m,n) = \underline{x}_2(m,n)$ for $m \in \mathbb{Z}$, $-\infty < n \leqslant -l_1$.

In the same way we prove that $\underline{x}_1(m, -l_1) = \underline{x}_2(m, -l_1)$ for $m \leqslant b_1$.

In the same manner, using the permutivity of λ at (a_1, l_2) and (a_u, l_2), we prove that $\underline{x}_1(m,n) = \underline{x}_2(m,n)$ for $m \in \mathbb{Z}$, $l_2 \leqslant n < \infty$.

Step 2: *The map h is surjective.*

Let $\underline{c} : \mathbb{N} \to \mathcal{S}^M$. We have to find an extension $\tilde{c} : \mathbb{Z}^2 \to \mathcal{S}$ of $\underline{c}(0)$ with $h(\tilde{c}) = \underline{c}$. This is done in several steps by induction, using the permutivity of the cellular automaton A. First, we extend $\underline{c}(0)$ to $M \cup (\mathbb{Z} \times \{l_2 + 1\})$. For all (m, l_2), with $m \geqslant a_u$, the procedure is the same as for (a_u, l_2). We show only this step. The permutivity of λ at (a_u, l) implies that there exists only one $s \in \mathcal{S}$ such that for $c_1 : U_{\mathbf{k},\mathbf{1}} \to \mathcal{S}$ defined by

$$c_1(a,b) = \begin{cases} \underline{c}(0)(a,b) & \text{if } (a,b) \neq (a_u, l_2), \\ s & \text{if } (a,b) = (a_u, l_2), \end{cases}$$

we have $\underline{c}(1)(0,0) = \lambda(c_1)$. Then we define $\tilde{c}(a_u, l_2) = s$.

Using the permutivity of λ at (a_1, l_2) we define $\tilde{c}$ for (m, l_2), $m \leqslant a_1$. In the same way the extension $\tilde{c}$ is defined for (m,n), $m \in \mathbb{Z}$, $n \geqslant l_2$. Using the permutivity of λ at $(b_1, -l_1), (b_v, -l_1)$ we define the extension $\tilde{c}$ for (m,n), $m \in \mathbb{Z}$, $n \leqslant -l_1$.

Remark 3. (1) A consequence of our Proposition 1 above is that, for any permutive cellular automaton $A : \mathcal{S}^{\mathbb{Z}^2} \to \mathcal{S}^{\mathbb{Z}^2}$, the dynamical system $(A, \mathcal{S}^{\mathbb{Z}^2})$ is chaotic (in the sense of Devaney), since the dynamical systems $(A, \mathcal{S}^{\mathbb{Z}^2})$ and $(\sigma_K, K^{\mathbb{N}})$ are conjugate. This assertion can be compared to [20, Theorem A, p. 137]; [19, Theorem 3.4, p. 604].

(2) As we mentioned before, the one-sided shift with infinite compact state space is not expanding. The n-dimensional cellular automata, $n \geqslant 2$, are conjugate to one-dimensional cellular automata with a Cantor set as state space. This is at least an intuitive reason for the result in [17] that n-dimensional cellular automata are not expanding for $n \geqslant 2$.

3. Continuous maps commuting with some powers of the shift

Here we consider a continuous map $B : \mathcal{S}^{\mathbb{Z}} \to \mathcal{S}^{\mathbb{Z}}$ for which there exists an integer $l \in \mathbb{N}$, $l \geqslant 2$ such that $B\sigma^l = \sigma^l B$. We call such maps *l-cellular automata*; they are also called *place-dependent cellular automata* [15]. In the case where the cellular automaton is linear, the reader can look at [12].

Example 2. Let $f_0 : \mathcal{S}^{2u+1} \to \mathcal{S}$ and $f_1 : \mathcal{S}^{2v+1} \to \mathcal{S}$ be two local generating functions. They generate a map $B : \mathcal{S}^{\mathbb{Z}} \to \mathcal{S}^{\mathbb{Z}}$ as follows: for $\underline{x} \in \mathcal{S}^{\mathbb{Z}}$

$$B(\underline{x})(2n) := f_0(\underline{x}(2n-u), \ldots, \underline{x}(2n+u)),$$

$$B(\underline{x})(2n+1) := f_1(\underline{x}(2n+1-v), \ldots, \underline{x}(2n+1+v)).$$

The map B is continuous and satisfies $\sigma^2 B = B\sigma^2$, i.e., it is a 2-cellular automaton. We say that the map B is generated by two local generating functions. In a similar way, we define maps induced by l generating functions.

With a small modification of the proof of [9, Theorem 3.4] we obtain

Proposition 2. *The l-cellular automata $B : \mathcal{S}^{\mathbb{Z}} \to \mathcal{S}^{\mathbb{Z}}$ are exactly the maps induced by l generating functions.*

The theorem of Gilman [8] that we generalized above also holds for l-cellular automata. Consider an l-cellular automaton B induced by the generating functions $f_0, \dots, f_{l-1} : \mathscr{S}^{2k+1} \to \mathscr{S}$. We say that the l-cellular automaton B is permutive if $f_0, \dots, f_{l-1}$ are permutive at the leftmost and the rightmost arguments.

Proposition 3. *Every permutive l-cellular automaton is conjugate to an appropriate one-dimensional shift with finite state space.*

The proof is similar to the proof of Gilman [8].

Acknowledgments

This paper was written during a stay of JPA in Bremen and a stay of GS in Orsay. Both authors want to thank heartily H.-O. Peitgen and D. Gouyou-Beauchamps for their warm support and for their interest. We also want to thank the referee for having suggested an extension of our initial result.

References

[1] V.S. Afraĭmovich, M.A. Shereshevsky, Cellular automata as dynamical systems, in: A.V. Gaponov-Grekhov, M.I. Rabinovich, J. Engelbrecht (Eds.), Nonlinear Waves 3, Physics and Astrophysics, Proceedings of Ninth All-Union School on Nonlinear Physics, Gorky/USSR, 1989, Research Reports in Physics, Springer, Berlin, 1990, pp. 296–300.

[2] J.-P. Allouche, M. Courbage, G. Skordev, Notes on cellular automata, Cubo Mat. Educacional 3 (2001) 213–244.

[3] H. Aso, N. Honda, Dynamical characteristics of linear cellular automata, J. Comput. System Sci. 30 (1985) 291–317.

[4] J. Banks, J. Brooks, J.G. Cairns, G. Davis, P. Stacey, On Devaney's definition of chaos, Amer. Math. Monthly 99 (1992) 332–334.

[5] G. Cattaneo, M. Finelli, L. Margara, Investigating topological chaos by elementary cellular automata dynamics, Theoret. Comput. Sci. 244 (2000) 219–241.

[6] R.L. Devaney, A First Course in Chaotic Dynamical Systems, Theory and Experiment, Studies in Nonlinearity, Addison-Wesley, Reading, 1992.

[7] F. Fagnani, L. Margara, Expansivity, permutivity and chaos for cellular automata, Theory Comput. System 31 (1998) 663–677.

[8] R. Gilman, Periodic behavior of linear automata, in: J.C. Alexander (Ed.), Dynamical Systems, Lecture Notes in Mathematics, Vol. 1342, Springer, New York, 1988, pp. 216–219.

[9] G. Hedlund, Endomorphisms and automorphisms of the shift dynamical systems, Math. Systems Theory 3 (1969) 320–375.

[10] I.M. James, General Topology and Homotopy Theory, Springer, New York, 1984.

[11] M.I. Kargapolov, Ju.I. Merzljakov, Fundamentals of the Theory of Groups, Springer, New York, 1979.

[12] J. Kari, Linear cellular automata with multiple state variables, in: H. Reichel, S. Tison (Eds.), Stacs 2000, Lecture Notes in Computer Science, Vol. 1770, Springer, New York, 2000, pp. 110–121.

[13] J.L. Kelley, General Topology, Springer, New York, 1975 (reprint of the 1955 edition).

[14] R. Kleveland, Mixing properties of one-dimensional cellular automata, Proc. Amer. Math. Soc. 125 (1997) 1755–1766.

[15] E. Lange, H.-O. Peitgen, G. Skordev, Fractal patterns in Gaussian and Stirling number tables, Ars Combin. 48 (1998) 3–26.
[16] D. Richardson, Tessellation with local transformations, J. Comput. System Sci. 6 (1972) 373–388.
[17] M. Shereshevsky, Expansiveness, entropy and polynomial growth for groups acting on subshifts by automorphisms, Indag. Math. 4 (1993) 203–210.
[18] M.A. Shereshevsky, V.S. Afraĭmovich, Bipermutative cellular automata are topologically conjugate to the one-sided Bernoulli shift, Random Comput. Dyn. 1 (1992) 91–98.
[19] M. Shirvani, T.D. Rogers, On ergodic one-dimensional cellular automata, Comm. Math. Phys. 136 (1991) 599–605.
[20] S.J. Willson, On the ergodic theory of cellular automata, Math. Systems Theory 9 (1975) 132–141.