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A stochastic approach to a multivalued Dirichlet–Neumann problem

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Abstract

We prove the existence and uniqueness of a viscosity solution of the parabolic variational inequality (PVI) with a mixed nonlinear multivalued Neumann–Dirichlet boundary condition:

$$\begin{cases} \dfrac{\partial u(t,x)}{\partial t} - \mathcal{L}_t u\,(t,x) + \partial\varphi\Big(u(t,x)\Big) \ni f\Big(t,x,u(t,x),(\nabla u\sigma)(t,x)\Big), & t>0, x\in\mathcal{D}, \\ \dfrac{\partial u(t,x)}{\partial n} + \partial\psi\Big(u(t,x)\Big) \ni g\Big(t,x,u(t,x)\Big), & t>0, x\in Bd\,(\mathcal{D}), \\ u(0,x) = h(x), \quad x\in\overline{\mathcal{D}}, \end{cases}$$

where $\partial\varphi$ and $\partial\psi$ are subdifferential operators and $\mathcal{L}_t$ is a second-differential operator given by

$$\mathcal{L}_t v\,(x) = \frac{1}{2}\sum_{i,j=1}^{d}(\sigma\sigma^*)_{ij}(t,x)\frac{\partial^2 v\,(x)}{\partial x_i\partial x_j} + \sum_{i=1}^{d} b_i(t,x)\frac{\partial v\,(x)}{\partial x_i}.$$

The result is obtained by a stochastic approach. First we study the following backward stochastic generalized variational inequality:

$$\begin{cases} \mathrm{d}Y_t + F\,(t,Y_t,Z_t)\,\mathrm{d}t + G\,(t,Y_t)\,\mathrm{d}A_t \in \partial\varphi\,(Y_t)\,\mathrm{d}t + \partial\psi\,(Y_t)\,\mathrm{d}A_t + Z_t\mathrm{d}W_t, & 0\le t\le T, \\ Y_T = \xi, \end{cases}$$

where $(A_t)_{t\ge 0}$ is a continuous one-dimensional increasing measurable process, and then we obtain a Feynman–Kaç representation formula for the viscosity solution of the PVI problem.

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1. Introduction

Viscosity solutions were introduced by Crandall and Lions in [1], and then developed in the classical work of Crandall, Ishii, and Lions [2], where several equivalent formulations are presented. The framework of this theory allows for merely continuous functions to be the solutions of fully nonlinear equations of second order which provides very general existence and uniqueness theorems.

In 1992 Pardoux and Peng [10] introduced backward stochastic differential equations (BSDE) and supplied probabilistic formulas for the viscosity solutions of semilinear partial differential equations, both of parabolic and elliptic type in the whole space. Elliptic equations with Dirichlet boundary conditions have been treated by Darling and Pardoux in [3] and with a homogeneous Neumann boundary condition by Hu in [4].

The parabolic (and elliptic) systems of partial differential equations (PVI without the subdifferential operator) with nonlinear Neumann boundary conditions were the subject of the paper Pardoux and Zhang [11]. The case of systems of variational inequalities for partial differential equations in the whole space was studied by Maticiuc, Pardoux, Răşcanu and Zălinescu in [7].

The main idea for proving the existence of the viscosity solutions for PDE and PVI is the stochastic approach. Using a suitable BSDE, or backward stochastic variational inequality (BSVI) for the PVI case, one can obtain a generalization of the Feynman–Kaç formula (i.e. a stochastic representation formula of the viscosity solution for deterministic problems).

The origin of our study comes from the PDE

$$\begin{cases} \dfrac{\partial u}{\partial t} - \mathcal{L}_t u = f, & t > 0, x \in \mathcal{D}, \\ \dfrac{\partial u}{\partial n} = g, & t > 0, x \in Bd\left(\mathcal{D}\right), \\ u(0,x) = h(x), & x \in \overline{\mathcal{D}}, \end{cases}$$

which is a mathematical model for the evolution of a state $u(t,x) \in \mathbb{R}$ of a diffusion dynamical system with sources f acting in the interior of the domain $\mathcal{D}$ and g on the boundary of $\mathcal{D}$.

In certain applications it is called upon to maintain the state $u(t,x)$ in an interval $\mathbb{I} \subset \mathbb{R}$ for all $x \in \mathcal{D}$ and in an interval $\mathbb{J} \subset \mathbb{R}$ for all $x \in Bd\left(\mathcal{D}\right)$. Practically, these can be realized by adding the supplementary sources $\partial I_{\mathbb{I}}\left(u\left(t,x\right)\right)$ and $\partial I_{\mathbb{J}}\left(u\left(t,x\right)\right)$ to the system. These sources produce "inward pushes" that would keep the state process

$$u(t,x) \text{ in } \mathbb{I}, \quad \forall x \in \mathcal{D} \quad \text{and} \quad u(t,x) \text{ in } \mathbb{J}, \quad \forall x \in Bd\left(\mathcal{D}\right)$$

and do this in a minimal way (i.e. only when $u(t,x)$ arrives on the boundary of $\mathbb{I}$ and respectively $\mathbb{J}$). Hence $\partial I_{\mathbb{I}}\left(u\left(t,x\right)\right)$ and $\partial I_{\mathbb{J}}\left(u\left(t,x\right)\right)$ represent perfect feedback flux controls.

The aim of this paper is to treat the more general case of a parabolic variational inequality with mixed nonlinear multivalued Neumann–Dirichlet boundary condition. This requires the presence of new terms in the associated BSVI under consideration, namely an integral with respect to a continuous increasing process.

The scalar BSDE with one-sided reflection, which provides a probabilistic representation for the unique viscosity solution of an obstacle problem for a nonlinear parabolic PDE, was considered by El Karoui, Kapoudjian, Pardoux, Peng, Quenez in [5]. Pardoux and Răşcanu in [8] (and [9] for the Hilbert spaces framework) studied the case of BSVI and obtained probabilistic representation for the solution of PVI in the whole space. We note that the stochastic results from our article are generalizations of those from [8].

The differences between our paper and [8], are as follows: the backward stochastic variational inequality studied in [8] does not allow us to obtain the representation formula for the solution of the deterministic multivalued Neumann–Dirichlet problem considered here; the multivalued BSDEs presented here generalize the result from [8] by adding a Stieltjes integral which comes from stochastic variational inequalities. Naturally, the additional assumptions are for the supplementary (Stieltjes) term.

The paper is organized as follows: In Section 2 we formulate the Neumann–Dirichlet PVI problem; we present the main results and we prove the uniqueness theorem. For the existence theorem we first study in Section 3 a certain BSVI. The solution of this backward equation gives us, via the Feynman–Kaç representation formula, a viscosity solution for the deterministic multivalued partial differential equation as shown in Section 4.

2. Main results

Let $\mathcal{D}$ be a open connected bounded subset of $\mathbb{R}^d$ of the form

$$\mathcal{D} = \left\{x \in \mathbb{R}^d : \ell(x) < 0\right\}, \qquad Bd(\mathcal{D}) = \left\{x \in \mathbb{R}^d : \ell(x) = 0\right\},$$

where $\ell \in C_b^3\left(\mathbb{R}^d\right)$, $|\nabla \ell(x)| = 1$, for all $x \in Bd(\mathcal{D})$.

We define the outward normal derivative by

$$\frac{\partial v(x)}{\partial n} = \sum_{j=1}^{d} \frac{\partial \ell(x)}{\partial x_j} \frac{\partial v(x)}{\partial x_j} = \langle \nabla \ell(x), \nabla v(x) \rangle, \quad \text{for all } x \in Bd(\mathcal{D}).$$

The aim of this paper is to study the existence and uniqueness of a viscosity solution for the following parabolic variational inequality (PVI) with a mixed nonlinear multivalued Neumann–Dirichlet boundary condition:

$$\begin{cases} \dfrac{\partial u(t,x)}{\partial t} - \mathcal{L}_t u(t,x) + \partial\varphi\big(u(t,x)\big) \ni f\big(t,x,u(t,x),(\nabla u\sigma)(t,x)\big), \\ \quad t > 0,\ x \in \mathcal{D}, \\ \dfrac{\partial u(t,x)}{\partial n} + \partial\psi\big(u(t,x)\big) \ni g\big(t,x,u(t,x)\big), \quad t > 0,\ x \in Bd(\mathcal{D}), \\ u(0,x) = h(x), \quad x \in \overline{\mathcal{D}}, \end{cases} \tag{1}$$

where the operator $\mathcal{L}_t$ is given by

$$\begin{aligned} \mathcal{L}_t v(x) &= \frac{1}{2}\mathrm{Tr}\big[\sigma(t,x)\sigma^*(t,x)D^2 v(x)\big] + \langle b(t,x), \nabla v(x)\rangle \\ &= \frac{1}{2}\sum_{i,j=1}^{d} (\sigma\sigma^*)_{ij}(t,x)\frac{\partial^2 v(x)}{\partial x_i \partial x_j} + \sum_{i=1}^{d} b_i(t,x)\frac{\partial v(x)}{\partial x_i}, \end{aligned}$$

for $v \in C^2\left(\mathbb{R}^d\right)$.

We will make the following assumptions:

(I) Functions

$$\begin{array}{l} b:[0,\infty)\times\mathbb{R}^d\to\mathbb{R}^d,\\ \sigma:[0,\infty)\times\mathbb{R}^d\to\mathbb{R}^{d\times d},\\ f:[0,\infty)\times\overline{\mathcal{D}}\times\mathbb{R}\times\mathbb{R}^d\to\mathbb{R},\\ g:[0,\infty)\times Bd\left(\mathcal{D}\right)\times\mathbb{R}\to\mathbb{R},\\ h:\overline{\mathcal{D}}\to\mathbb{R}\quad\text{are continuous.}\end{array} \tag{2}$$

We assume that for all $T>0$ there exist $\alpha\in\mathbb{R}$ and $L,\beta,\gamma\geq 0$ (which can depend on T) such that $\forall t\in[0,T],\ \forall x,\tilde{x}\in\mathbb{R}^d$,

$$|b(t,x)-b(t,\tilde{x})|+\|\sigma(t,x)-\sigma(t,\tilde{x})\|\leq L\,\|x-\tilde{x}\|\,, \tag{3}$$

and $\forall t\in[0,T], \forall x\in\overline{\mathcal{D}}, u\in Bd\left(\mathcal{D}\right), y,\tilde{y}\in\mathbb{R}, z,\tilde{z}\in\mathbb{R}^d$,

$$\begin{array}{ll} \text{(i)} & (y-\tilde{y})\big(f(t,x,y,z)-f(t,x,\tilde{y},z)\big)\leq\alpha|y-\tilde{y}|^2,\\ \text{(ii)} & |f(t,x,y,z)-f(t,x,y,\tilde{z})|\leq\beta|z-\tilde{z}|,\\ \text{(iii)} & |f(t,x,y,0)|\leq\gamma\big(1+|y|\big),\\ \text{(iv)} & (y-\tilde{y})\big(g(t,u,y)-g(t,u,\tilde{y})\big)\leq\alpha|y-\tilde{y}|^2,\\ \text{(v)} & |g(t,u,y)|\leq\gamma\big(1+|y|\big).\end{array} \tag{4}$$

In fact, conditions ((4)-i and iv) mean that, for all $t\in[0,T], x\in\overline{\mathcal{D}}, u\in Bd\left(\mathcal{D}\right), z\in\mathbb{R}^d$,

$$\begin{array}{l} y\mapsto\alpha y-f\left(t,x,y,z\right):\mathbb{R}\to\mathbb{R},\\ y\mapsto\alpha y-g\left(t,u,y\right):\mathbb{R}\to\mathbb{R}\end{array}$$

are increasing functions.

(II) With respect to the functions φ and ψ we assume

$$\begin{array}{ll} \text{(i)} & \varphi,\psi:\mathbb{R}\to(-\infty,+\infty]\text{are proper convex l.s.c. functions,}\\ \text{(ii)} & \varphi\left(y\right)\geq\varphi\left(0\right)=0\quad\text{and}\quad\psi\left(y\right)\geq\psi\left(0\right)=0,\quad\forall y\in\mathbb{R},\end{array} \tag{5}$$

and there exists a positive constant M such that

$$\begin{array}{ll} \text{(i)} & \left|\varphi\big(h(x)\big)\right|\leq M,\quad\forall x\in\overline{\mathcal{D}},\\ \text{(ii)} & \left|\psi\big(h(x)\big)\right|\leq M,\quad\forall x\in Bd\left(\mathcal{D}\right).\end{array} \tag{6}$$

Remark 1. Condition ((5)-ii) is generally realized by changing problem (1) into an equivalent form. For instance, if $\left(u_0,u_0^*\right)\in\partial\varphi$ we can replace $\varphi\left(u\right)$ by $\varphi\left(u+u_0\right)-\varphi\left(u_0\right)-\left\langle u_0^*,u\right\rangle$; a similar transformation can be made for ψ.

We define

$$\begin{array}{l} Dom\left(\varphi\right)=\left\{u\in\mathbb{R}:\varphi\left(u\right)<\infty\right\},\\ \partial\varphi\left(u\right)=\left\{u^*\in\mathbb{R}:u^*\left(v-u\right)+\varphi\left(u\right)\leq\varphi\left(v\right),\forall v\in\mathbb{R}\right\},\\ Dom\left(\partial\varphi\right)=\left\{u\in\mathbb{R}:\partial\varphi\left(u\right)\neq\emptyset\right\},\\ \left(u,u^*\right)\in\partial\varphi\Leftrightarrow u\in\ Dom\partial\varphi,\quad u^*\in\partial\varphi\left(u\right)\end{array}$$

(for the function ψ we have similar notation).

In every point $y\in Dom\left(\varphi\right)$ we have

$$\partial\varphi(y)=\mathbb{R}\cap\left[\varphi_-'(y),\varphi_+'(y)\right],$$

where $\varphi_-'(y)$ and $\varphi_+'(y)$ are the left derivative and, respectively, the right derivative at point y.

(III) We introduce *compatibility assumptions*: for all $\varepsilon > 0$, $t \geq 0$, $x \in Bd(\mathcal{D})$, $\tilde{x} \in \overline{\mathcal{D}}$, $y \in \mathbb{R}$ and $z \in \mathbb{R}^d$,

$$\begin{aligned} &\text{(i)} \quad \nabla\varphi_\varepsilon(y)\, g(t,x,y) \leq \left[\nabla\psi_\varepsilon(y)\, g(t,x,y)\right]^+, \\ &\text{(ii)} \quad \nabla\psi_\varepsilon(y)\, f(t,\tilde{x},y,z) \leq \left[\nabla\varphi_\varepsilon(y)\, f(t,\tilde{x},y,z)\right]^+, \end{aligned} \tag{7}$$

where $a^+ = \max\{0, a\}$ and $\nabla\varphi_\varepsilon(y)$, $\nabla\psi_\varepsilon(y)$ are the unique solutions U and V, respectively, of equations

$$\partial\varphi(y - \varepsilon U) \ni U \quad \text{and} \quad \partial\psi(y - \varepsilon V) \ni V.$$

Remark 2. (A) Clearly, using the monotonicity of $\nabla\varphi_\varepsilon$, $\nabla\psi_\varepsilon$, we see that, if

$$y \cdot g(t,x,y) \leq 0 \quad \text{and} \quad y \cdot f(t,\tilde{x},y,z) \leq 0,$$

for all $t \geq 0$, $x \in Bd(\mathcal{D})$, $\tilde{x} \in \overline{\mathcal{D}}$, $y \in \mathbb{R}$ and $z \in \mathbb{R}^d$, then the compatibility assumptions (7) are satisfied.

(B) If $\varphi, \psi : \mathbb{R} \to (-\infty, +\infty]$ are convex indicator functions

$$\varphi(y) = I_{[a,\infty)}(y) = \begin{cases} 0, & \text{if } y \in [a,\infty), \\ +\infty, & \text{if } y \notin [a,\infty), \end{cases}$$

and

$$\psi(y) = I_{(-\infty,b]}(y) = \begin{cases} 0, & \text{if } y \in (-\infty,b], \\ +\infty, & \text{if } y \notin (-\infty,b], \end{cases}$$

where $a \leq 0 \leq b$, then

$$\nabla\varphi_\varepsilon(y) = -\frac{1}{\varepsilon}(y-a)^- \quad \text{and} \quad \nabla\psi_\varepsilon(y) = \frac{1}{\varepsilon}(y-b)^+$$

and the compatibility assumptions become

$$\begin{aligned} &g(t,x,y) \geq 0, \quad \text{for } y \leq a, \quad \text{and} \\ &f(t,\tilde{x},y,z) \leq 0, \quad \text{for } y \geq b. \end{aligned}$$

We shall define now the notion of viscosity solution in the language of subjets and superjets; see [2]. $S\mathbb{R}^{d\times d}$ will denote below the set of $d \times d$ symmetric non-negative real matrices.

Definition 3. Let $u : [0,\infty) \times \overline{\mathcal{D}} \to \mathbb{R}$ be a continuous function, and let us have $(t,x) \in [0,\infty) \times \overline{\mathcal{D}}$. We denote by $\mathcal{P}^{2,+}u(t,x)$ (the parabolic superjet of u at (t,x)) the set of triples $(p,q,X) \in \mathbb{R} \times \mathbb{R}^d \times S\mathbb{R}^{d\times d}$ such that for all $(s,y) \in [0,\infty) \times \overline{\mathcal{D}}$ in a neighbourhood of (t,x),

$$\begin{aligned} u(s,y) \leq{}& u(t,x) + p(s-t) + \langle q, y-x\rangle + \frac{1}{2}\langle X(y-x), y-x\rangle \\ &+ o\big(|s-t| + |y-x|^2\big). \end{aligned}$$

Similarly, $\mathcal{P}^{2,-}u(t,x)$ (the parabolic subjet of u at (t,x)) is defined as the set of triples $(p,q,X) \in \mathbb{R} \times \mathbb{R}^d \times S\mathbb{R}^{d\times d}$ such that for all $(s,y) \in [0,\infty) \times \overline{\mathcal{D}}$ in a neighbourhood of (t,x),

$$\begin{aligned} u(s,y) \geq{}& u(t,x) + p(s-t) + \langle q, y-x\rangle + \frac{1}{2}\langle X(y-x), y-x\rangle \\ &+ o\big(|s-t| + |y-x|^2\big), \end{aligned}$$

where $r \to o(r)$ is the Landau function, i.e. $o : [0, \infty[\to \mathbb{R}$ is a continuous function such that $\lim_{r\to 0} \frac{o(r)}{r} = 0$.

We can give now the definition of a viscosity solution of the parabolic variational inequality (1). We define first

$$V(t, x, p, q, X) \overset{def}{=} p - \frac{1}{2}\mathrm{Tr}\big((\sigma\sigma^*)(t, x)X\big) - \langle b(t, x), q\rangle - f\big(t, x, u(t, x), q\sigma(t, x)\big).$$

Definition 4. Let $u : [0, \infty) \times \overline{\mathcal{D}} \to \mathbb{R}$ be a continuous function, which satisfies $u(0, x) = h(x)$, $\forall\, x \in \overline{\mathcal{D}}$.

(a) We say that u is a viscosity subsolution of (1) if

$$\left| \begin{array}{ll} u(t, x) \in Dom(\varphi), & \forall (t, x) \in (0, \infty) \times \overline{\mathcal{D}}, \\ u(t, x) \in Dom(\psi), & \forall (t, x) \in (0, \infty) \times Bd(\mathcal{D}), \end{array} \right.$$

and, at any point $(t, x) \in (0, \infty) \times \overline{\mathcal{D}}$, for any $(p, q, X) \in \mathcal{P}^{2,+}u(t, x)$,

$$\left| \begin{array}{l} V(t, x, p, q, X) + \varphi'_-\big(u(t, x)\big) \leq 0 \quad \text{if } x \in \mathcal{D}, \\ \min\Big\{V(t, x, p, q, X) + \varphi'_-\big(u(t, x)\big), \langle \nabla\ell(x), q\rangle - g\big(t, x, u(t, x)\big) \\ \quad +\psi'_-\big(u(t, x)\big)\Big\} \leq 0 \quad \text{if } x \in Bd(\mathcal{D}). \end{array} \right. \tag{8}$$

(b) The viscosity supersolution of (1) is defined in a similar manner to above, with $\mathcal{P}^{2,+}$ replaced by $\mathcal{P}^{2,-}$, the left derivative replaced by the right derivative, min by max, and the inequalities $\leq$ by $\geq$.

(c) A continuous function $u : [0, \infty) \times \overline{\mathcal{D}}$ is a viscosity solution of (1) if it is both a viscosity subsolution and a viscosity supersolution.

We now present the main results

Theorem 5 (*Existence*). *Let assumptions (2)–(7) be satisfied. Then PVI (1) has a viscosity solution.*

For the proof of the existence we shall study a certain backward stochastic generalized variational inequality (then we use a nonlinear representation Feynman–Kaç type of formula). We present this approach in the following section and finally the proof of Theorem 5 in Section 4.

Theorem 6 (*Uniqueness*). *Let the assumptions of Theorem 5 be satisfied. If the function*

$$r \to g(t, x, r) \quad \textit{is decreasing for } t \geq 0,\ x \in Bd(\mathcal{D}), \tag{9}$$

and there exists a continuous function $\mathbf{m} : [0, \infty) \to [0, \infty)$, $\mathbf{m}(0) = 0$, *such that*

$$|f(t, x, r, p) - f(t, y, r, p)| \leq \mathbf{m}\big(|x - y|\,(1 + |p|)\big), \quad \forall\, t \geq 0,\ x, y \in \overline{\mathcal{D}},\ p \in \mathbb{R}^d, \tag{10}$$

then the viscosity solution is unique.

Proof. It is sufficient to prove the uniqueness on a fixed arbitrary interval $[0, T]$.

Also, it suffices to prove that if u is a subsolution and v is a supersolution such that $u(0, x) = v(0, x) = h(x)$, $x \in \overline{\mathcal{D}}$, then $u \leq v$.

Firstly, from the definition of $\mathcal{D}$, there exists a function $\tilde{\ell} \in C_b^3\left(\mathbb{R}^d\right)$ such that $\tilde{\ell}(x) \geq 0$ on $\overline{\mathcal{D}}$ with $\nabla\tilde{\ell}(x) = \nabla\ell(x)$ for $x \in Bd(\mathcal{D})$ (for example $\tilde{\ell}(x) = \ell(x) + \sup_{y\in\overline{\mathcal{D}}} |\ell(y)|$).

For $\lambda = |\alpha| + 1$ and $\delta, \varepsilon, c > 0$ let

$$\bar{u}(t,x) = e^{\lambda t} u(t,x) - \delta\tilde{\ell}(x) - c$$
$$\bar{v}(t,x) = e^{\lambda t} v(t,x) + \delta\tilde{\ell}(x) + c + \varepsilon/t.$$

Define

$$\tilde{f}(t,x,r,q,X) = \lambda r - \frac{1}{2}\mathrm{Tr}\left[\left(\sigma\sigma^*\right)(t,x)X\right] - \langle b(t,x), q\rangle - e^{\lambda t} f\left(t,x,e^{-\lambda t}r, e^{-\lambda t}q\sigma(t,x)\right) \tag{11}$$

and

$$\tilde{g}(t,x,r) = e^{\lambda t} g(t,x,e^{-\lambda t}r).$$

Clearly $r \to \tilde{f}(t,x,r,q,X)$ is an increasing function for all $(t,x,q,X) \in [0,T]\times\mathbb{R}^d\times\mathbb{R}^d\times S\mathbb{R}^{d\times d}$. Moreover, since

$$M = \sup_{(t,x)\in[0,T]\times\overline{\mathcal{D}}} \left\{|\tilde{\ell}(x)| + |D\tilde{\ell}(x)| + |D^2\tilde{\ell}(x)| + |b(t,x)| + |\sigma(t,x)|\right\} < \infty,$$

then we can choose $c = c(\delta, M) > 0$ such that for $\bar{u} = \bar{u}(t,x)$ and $\tilde{\ell} = \tilde{\ell}(x)$,

$$\tilde{f}\left(t,x,\bar{u},D\bar{u},D^2\bar{u}\right) \leq \tilde{f}(t,x,\bar{u}+\delta\tilde{\ell}+c, D\bar{u}+\delta D\tilde{\ell}, D^2\bar{u}+\delta D^2\tilde{\ell}).$$

Using these properties, assumption (9), and the fact that the left and right derivative of φ,ψ are increasing, we infer that the function $\bar{u}$ satisfies in the viscosity sense

$$\left|\begin{array}{l} \dfrac{\partial\bar{u}}{\partial t}(t,x) + \tilde{f}\left(t,x,\bar{u}(t,x), D\bar{u}(t,x), D^2\bar{u}(t,x)\right) \\ \quad + e^{\lambda t}\varphi'_-\left(e^{-\lambda t}\bar{u}(t,x)\right) \leq 0 \quad \text{if } x \in \mathcal{D}, \\ \min\left\{\dfrac{\partial\bar{u}}{\partial t}(t,x) + \tilde{f}\left(t,x,\bar{u}(t,x), D\bar{u}(t,x), D^2\bar{u}(t,x)\right)\right. \\ \quad + e^{\lambda t}\varphi'_-\left(e^{-\lambda t}\bar{u}(t,x)\right), \left\langle\nabla\tilde{\ell}(x), D\bar{u}(t,x)\right\rangle + \delta \\ \quad \left. - \tilde{g}\left(t,x,\bar{u}(t,x)\right) + e^{\lambda t}\psi'_-\left(e^{-\lambda t}\bar{u}(t,x)\right)\right\} \leq 0 \quad \text{if } x \in Bd(\mathcal{D}). \end{array}\right. \tag{12}$$

Analogously we see that $\bar{v}$ satisfies in the viscosity sense

$$\left|\begin{array}{l} \dfrac{\partial\bar{v}}{\partial t}(t,x) + \tilde{f}\left(t,x,\bar{v}(t,x), D\bar{v}(t,x), D^2\bar{v}(t,x)\right) \\ \quad + e^{\lambda t}\varphi'_+\left(e^{-\lambda t}\bar{v}(t,x)\right) - \varepsilon/t^2 \geq 0 \quad \text{if } x \in \mathcal{D}, \\ \max\left\{\dfrac{\partial\bar{v}}{\partial t}(t,x) + \tilde{f}\left(t,x,\bar{v}(t,x), D\bar{v}(t,x), D^2\bar{v}(t,x)\right)\right. \\ \quad + e^{\lambda t}\varphi'_-\left(e^{-\lambda t}\bar{v}(t,x)\right) - \varepsilon/t^2, \left\langle\nabla\tilde{\ell}(x), D\bar{v}(t,x)\right\rangle - \delta \\ \quad \left. - \tilde{g}\left(t,x,\bar{v}(t,x)\right) + e^{\lambda t}\psi'_+\left(e^{-\lambda t}\bar{v}(t,x)\right)\right\} \geq 0 \quad \text{if } x \in Bd(\mathcal{D}). \end{array}\right. \tag{13}$$

For simplicity of notation we continue to write u, v for $\bar{u}, \bar{v}$ respectively.

We assume now, to the contrary, that

$$\max_{[0,T]\times\overline{\mathcal{D}}} (u-v)^+ > 0. \tag{14}$$

Exactly as in Theorem 4.2 in [8], we have $(\hat{t},\hat{x}) \in [0,T]\times Bd(\mathcal{D})$, where $(\hat{t},\hat{x})$ is the maximum point, i.e.

$$u(\hat{t},\hat{x}) - v(\hat{t},\hat{x}) = \max_{[0,T]\times\overline{\mathcal{D}}} (u-v)^+ > 0.$$

We put now (see also the proof of the Theorem 7.5 in Crandall, Ishii, and Lions [2])

$$\Phi_n(t,x,y) = u(t,x) - v(t,y) - \rho_n(t,x,y), \quad \text{with } (t,x,y) \in [0,T]\times\overline{\mathcal{D}}\times\overline{\mathcal{D}},$$

where

$$\begin{aligned}\rho_n(t,x,y) = {} & \frac{n}{2}|x-y|^2 + \tilde{g}\big(\hat{t},\hat{x},u(\hat{t},\hat{x})\big)\Big\langle \nabla\tilde{\ell}(\hat{x}), x-y\Big\rangle + \big|x-\hat{x}\big|^4 \\ & + |t-\hat{t}|^4 - e^{\lambda\hat{t}}\psi'_-\big(e^{-\lambda\hat{t}}u(\hat{t},\hat{x})\big)\Big\langle\nabla\tilde{\ell}(\hat{x}), x-y\Big\rangle.\end{aligned} \tag{15}$$

Let (t_n,x_n,y_n) be a point of maximum of Φ_n.

We observe that $u(t,x) - v(t,x) - \big|x-\hat{x}\big|^4 - |t-\hat{t}|^4$ has in $(\hat{t},\hat{x})$ a unique maximum point. Then, by Proposition 3.7 in Crandall, Ishii, and Lions [2], we have, as $n\to\infty$,

$$\begin{aligned} & t_n \to \hat{t}, \quad x_n\to\hat{x}, \quad y_n\to\hat{x}, \quad n|x_n-y_n|^2\to 0, \\ & u(t_n,x_n) \to u(\hat{t},\hat{x}), \quad v(t_n,x_n)\to v(\hat{t},\hat{x}).\end{aligned} \tag{16}$$

But domain $\mathcal{D}$ verifies the uniform exterior sphere condition

$$\exists\, r_0>0 \quad \text{such that } S\big(x+r_0\nabla\tilde{\ell}(x), r_0\big)\cap\mathcal{D} = \emptyset, \quad \text{for } x\in Bd(\mathcal{D}),$$

where $S(x,r_0)$ denotes the closed ball of radius r_0 centered at x.

Then

$$\Big|y - x - r_0\nabla\tilde{\ell}(x)\Big|^2 > r_0^2, \quad \text{for } x\in Bd(\mathcal{D}),\ y\in\overline{\mathcal{D}},$$

or equivalently

$$\Big\langle\nabla\tilde{\ell}(x), y-x\Big\rangle < \frac{1}{2r_0}|y-x|^2 \quad \text{for } x\in Bd(\mathcal{D}),\ y\in\overline{\mathcal{D}}. \tag{17}$$

If we define

$$B(t,x,r,q) = \Big\langle\nabla\tilde{\ell}(x), q\Big\rangle - \tilde{g}(t,x,r),$$

then, if $x_n\in Bd(\mathcal{D})$, we have, using the form of ρ_n given by (15) and (17), that

$$\begin{aligned} B\big(t_n,x_n,u(t_n,x_n), D_x\rho_n(t_n,x_n,y_n)\big) = {} & B\Big(t_n,x_n,u(t_n,x_n), n(x_n-y_n) \\ & + \tilde{g}\big(\hat{t},\hat{x},u(\hat{t},\hat{x})\big)\nabla\tilde{\ell}(\hat{x}) + 4\big|x_n-\hat{x}\big|^2(x_n-\hat{x}) - e^{\lambda\hat{t}}\psi'_-\big(e^{-\lambda\hat{t}}u(\hat{t},\hat{x})\big)\nabla\tilde{\ell}(\hat{x})\Big) \\ \geq {} & -\frac{n}{2r_0}|x_n-y_n|^2 + \tilde{g}\big(\hat{t},\hat{x},u(\hat{t},\hat{x})\big)\Big\langle\nabla\tilde{\ell}(\hat{x}),\nabla\tilde{\ell}(x_n)\Big\rangle - \tilde{g}\big(t_n,x_n,u(t_n,x_n)\big) \\ & + 4\big|x_n-\hat{x}\big|^2\Big\langle\nabla\tilde{\ell}(x_n), x_n-\hat{x}\Big\rangle - e^{\lambda\hat{t}}\psi'_-\big(e^{-\lambda\hat{t}}u(\hat{t},\hat{x})\big)\Big\langle\nabla\tilde{\ell}(\hat{x}),\nabla\tilde{\ell}(x_n)\Big\rangle.\end{aligned}$$

Then (16) implies for $x_n \in Bd(\mathcal{D})$

$$\liminf_{n\to\infty}\Big[B\left(t_n, x_n, u\left(t_n, x_n\right), D_x\rho_n\left(t_n, x_n, y_n\right)\right) + \delta + \mathrm{e}^{\lambda t_n}\psi'_-\big(\mathrm{e}^{-\lambda t_n}u(t_n, x_n)\big)\Big] > 0.$$

Analogously if $y_n \in Bd(\mathcal{D})$ we infer

$$\limsup_{n\to\infty}\Big[B\left(t_n, y_n, v\left(t_n, y_n\right), -D_y\rho_n\left(t_n, x_n, y_n\right)\right) - \delta + \mathrm{e}^{\lambda t_n}\psi'_+\big(\mathrm{e}^{-\lambda t_n}v(t_n, x_n)\big)\Big] < 0.$$

Then from (12), (13) we conclude that

$$p + \tilde{f}\big(t_n, x_n, u\left(t_n, x_n\right), D_x\rho_n\left(t_n, x_n, y_n\right), X\big) + \mathrm{e}^{\lambda t_n}\varphi'_-\big(\mathrm{e}^{-\lambda t_n}u(t_n, x_n)\big) \le 0,$$
$$\text{for } \left(p, D_x\rho_n\left(t_n, x_n, y_n\right), X\right) \in \overline{\mathcal{P}}^{2,+}u(t_n, x_n), \tag{18}$$

and

$$p + \tilde{f}\big(t_n, y_n, v\left(t_n, y_n\right), -D_y\rho_n\left(t_n, x_n, y_n\right), Y\big) + \mathrm{e}^{\lambda t_n}\varphi'_+\big(\mathrm{e}^{-\lambda t_n}v(t_n, y_n)\big) \ge \frac{\varepsilon}{t^2},$$
$$\text{for } \big(p, -D_y\rho_n\left(t_n, x_n, y_n\right), Y\big) \in \overline{\mathcal{P}}^{2,-}v(t_n, y_n). \tag{19}$$

From Theorem 8.3 in Crandall, Ishii, and Lions [2] (apply, with $k = 2$, $\mathcal{O}_1 = \mathcal{O}_2 = \overline{\mathcal{D}}$, $u_1 = u$, $u_2 = -v$, $b_1 = p$, $b_2 = -p$) we deduce that there exists

$$(p, X, Y) \in \mathbb{R} \times S\mathbb{R}^{d\times d} \times S\mathbb{R}^{d\times d},$$

such that

$$\big(p, D_x\rho_n\left(t_n, x_n, y_n\right), X\big) \in \overline{\mathcal{P}}^{2,+}u(t_n, x_n),$$
$$\big(p, -D_y\rho_n\left(t_n, x_n, y_n\right), Y\big) \in \overline{\mathcal{P}}^{2,-}v(t_n, y_n),$$

and

$$-\left(n + \|A\|\right)\begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \le \begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} \le A + \frac{1}{n}A^2, \tag{20}$$

where $A = D^2_{x,y}\rho_n\left(t_n, x_n, y_n\right)$. From (15) we have

$$A = n\begin{pmatrix} I & -I \\ -I & I \end{pmatrix} + O\big(\left|x_n - \hat{x}\right|^2\big),$$
$$A^2 = 2n^2\begin{pmatrix} I & -I \\ -I & I \end{pmatrix} + O\big(n\left|x_n - \hat{x}\right|^2 + \left|x_n - \hat{x}\right|^4\big),$$

where $|O(h)| \le C|h|$ (the Landau symbol). Then (20) becomes

$$-\left(3n + \kappa_n\right)\begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \le \begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} \le 3n\begin{pmatrix} I & -I \\ -I & I \end{pmatrix} + \kappa_n\begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}, \tag{21}$$

where $\kappa_n \to 0$. Now from (18) and (19),

$$\frac{\varepsilon}{t^2} \le \tilde{f}\big(t_n, y_n, v\left(t_n, y_n\right), -D_y\rho_n\left(t_n, x_n, y_n\right), Y\big) + \mathrm{e}^{\lambda t_n}\varphi'_+\big(\mathrm{e}^{-\lambda t_n}v(t_n, y_n)\big)$$
$$- \tilde{f}\big(t_n, x_n, u\left(t_n, x_n\right), D_x\rho_n\left(t_n, x_n, y_n\right), X\big) - \mathrm{e}^{\lambda t_n}\varphi'_-\big(\mathrm{e}^{-\lambda t_n}u(t_n, x_n)\big).$$

By (14) and (16) there exists $N \ge 1$ such that

$$u(t_n, x_n) > v(t_n, y_n), \quad \forall\, n \ge N,$$

and consequently

$$e^{\lambda t_n}\varphi'_-\big(e^{-\lambda t}u(t_n,x_n)\big)\geq e^{\lambda t_n}\varphi'_+\big(e^{-\lambda t}v(t_n,y_n)\big)$$

and

$$\begin{aligned}&\tilde f\big(t_n,y_n,u\left(t_n,x_n\right),-D_y\rho_n\left(t_n,x_n,y_n\right),Y\big)\\ &\quad\geq\tilde f\big(t_n,y_n,v\left(t_n,y_n\right),-D_y\rho_n\left(t_n,x_n,y_n\right),Y\big).\end{aligned}$$

Then, by definition (11) of $\tilde f$ and assumption (10), we have

$$\begin{aligned}\frac{\varepsilon}{\hat t^2}&\leq\liminf_{n\to+\infty}\Big[\tilde f\big(t_n,y_n,u\left(t_n,x_n\right),-D_y\rho_n\left(t_n,x_n,y_n\right),Y\big)\\ &\qquad-\tilde f\big(t_n,x_n,u\left(t_n,x_n\right),D_x\rho_n\left(t_n,x_n,y_n\right),X\big)\Big]\\ &\leq\frac12\mathrm{Tr}\big[\big(\sigma\sigma^*\big)\left(t_n,x_n\right)X-\big(\sigma\sigma^*\big)\left(t_n,y_n\right)Y\big].\end{aligned}$$

But from (21), $\forall q,\tilde q\in\mathbb{R}^d$,

$$\langle Xq,q\rangle-\langle Y\tilde q,\tilde q\rangle\leq 3n\left|q-\tilde q\right|^2+\big(|q|^2+|\tilde q|^2\big)\kappa_n.$$

Hence

$$\begin{aligned}&\mathrm{Tr}\big[\big(\sigma\sigma^*\big)\left(t_n,x_n\right)X-\big(\sigma\sigma^*\big)\left(t_n,y_n\right)Y\big]\\ &\quad=\sum_{i=1}^d\big(X\sigma(t_n,x_n)e_i,\sigma(t_n,x_n)e_i\big)-\big(Y\sigma(t_n,y_n)e_i,\sigma(t_n,y_n)e_i\big)\\ &\quad\leq 3C\,n\left|x_n-y_n\right|^2+\big(|\sigma(t_n,x_n)|^2+|\sigma(t_n,y_n)|^2\big)\kappa_n,\end{aligned}$$

and consequently

$$\frac{\varepsilon}{\hat t^2}\leq 0,$$

which is a contradiction.

Then

$$u\left(t,x\right)\leq v\left(t,x\right),\quad\forall\left(t,x\right)\in[0,T]\times\overline{\mathcal{D}}.\quad\blacksquare$$

3. Backward stochastic variational inequalities

Let $\{W_t : t\geq 0\}$ be a d-dimensional standard Brownian motion defined on some complete probability space $(\Omega,\mathcal{F},\mathbb{P})$. We denote by $\{\mathcal{F}_t : t\geq 0\}$ the natural filtration generated by $\{W_t : t\geq 0\}$ and augmented by $\mathcal{N}$, the set of $\mathbb{P}$-null events of $\mathcal{F}$:

$$\mathcal{F}_t=\sigma\{W_r : 0\leq r\leq t\}\vee\mathcal{N}.$$

Let $\tau:\Omega\to[0,\infty)$ be an a.s. $\mathcal{F}_t$-stopping time and let $\{A_t : t\geq 0\}$ be a continuous one-dimensional increasing progressively measurable stochastic process (p.m.s.p.) satisfying $A_0=0$.

We shall study the existence and uniqueness of a solution (Y,Z) of the following backward stochastic variational inequality (BSVI):

$$\begin{cases}\mathrm{d}Y_t+F\left(t,Y_t,Z_t\right)\mathrm{d}t+G\left(t,Y_t\right)\mathrm{d}A_t\in\partial\varphi\left(Y_t\right)\mathrm{d}t+\partial\psi\left(Y_t\right)\mathrm{d}A_t+Z_t\mathrm{d}W_t,\\ \qquad 0\leq t\leq\tau,\\ Y_\tau=\xi.\end{cases}\tag{22}$$

3.1. Assumptions and results

Let $\lambda, \mu \geq 0$.
Let

$$\mathcal{H}_k^{\lambda,\mu} \subset L^2(\mathbb{R}_+ \times \Omega, \mathrm{e}^{\lambda s+\mu A_s} \mathbf{1}_{[0,\tau]}(s)\, \mathrm{d}s \otimes \mathrm{d}\mathbb{P}; \mathbb{R}^k)$$

be the Hilbert space of p.m.s.p. $f : \Omega \times [0,\infty) \to \mathbb{R}^k$ such that

$$\|f\|_{\mathcal{H}} = \left[\mathbb{E}\Big(\int_0^\tau \mathrm{e}^{\lambda s+\mu A_s} |f(s)|^2\, \mathrm{d}s\Big)\right]^{1/2} < \infty,$$

and

$$\tilde{\mathcal{H}}_k^{\lambda,\mu} \subset L^2(\mathbb{R}_+ \times \Omega, \mathrm{e}^{\lambda s+\mu A_s} \mathbf{1}_{[0,\tau]}(s)\, \mathrm{d}A_s \otimes \mathrm{d}\mathbb{P}; \mathbb{R}^k)$$

the Hilbert space of p.m.s.p. $f : \Omega \times [0,\infty) \to \mathbb{R}^k$ such that

$$\|f\|_{\tilde{\mathcal{H}}} = \left[\mathbb{E}\Big(\int_0^\tau \mathrm{e}^{\lambda s+\mu A_s} |f(s)|^2\, \mathrm{d}A_s\Big)\right]^{1/2} < \infty.$$

We also introduce the notation $\mathcal{S}_k^{\lambda,\mu}$ for the Banach space of p.m.s.p. $f : \Omega \times [0,\infty) \to \mathbb{R}^k$ such that

$$\|f\|_{\mathcal{S}} = \left[\mathbb{E}\Big(\sup_{0\leq t\leq\tau} \mathrm{e}^{\lambda t+\mu A_t} |f(t)|^2\Big)\right]^{1/2} < \infty.$$

With respect to BSVI (22) we formulate the following assumptions:

- $F : \Omega \times [0,\infty) \times \mathbb{R}^k \times \mathbb{R}^{k\times d} \to \mathbb{R}^k$, $G : \Omega \times [0,\infty) \times \mathbb{R}^k \to \mathbb{R}^k$ satisfy that there exist $\alpha, \beta \in \mathbb{R}$, $L \geq 0$ and $\eta, \gamma : [0,\infty) \times \Omega \to [0,\infty)$ a p.m.s.p. such that for all $t \geq 0$, $y, y' \in \mathbb{R}^k$, $z, z' \in \mathbb{R}^{k\times d}$,

$$\left|\begin{array}{ll} \text{(i)} & F(\cdot,\cdot,y,z) \quad \text{is p.m.s.p.,} \\ \text{(ii)} & y \longrightarrow F(\omega,t,y,z) : \mathbb{R}^k \to \mathbb{R}^k \quad \text{is continuous, a.s.} \\ \text{(iii)} & \langle y-y', F(t,y,z) - F(t,y',z)\rangle \leq \alpha\,|y-y'|^2, \quad \text{a.s.} \\ \text{(iv)} & |F(t,y,z) - F(t,y,z')| \leq L\,\|z-z'\|, \quad \text{a.s.} \\ \text{(v)} & |F(t,y,z)| \leq \eta_t + L(|y| + \|z\|), \quad \text{a.s.} \end{array}\right. \tag{23}$$

and

$$\left|\begin{array}{ll} \text{(i)} & G(\cdot,\cdot,y) \quad \text{is p.m.s.p.,} \\ \text{(ii)} & y \longrightarrow G(\omega,t,y) : \mathbb{R}^k \to \mathbb{R}^k \quad \text{is continuous, a.s.} \\ \text{(iii)} & \langle y-y', G(t,y) - G(t,y')\rangle \leq \beta\,|y-y'|^2, \quad \text{a.s.} \\ \text{(iv)} & |G(t,y)| \leq \gamma_t + L\,|y|, \quad \text{a.s.} \end{array}\right. \tag{24}$$

- The terminal datum ξ is an $\mathbb{R}^k$-valued $\mathcal{F}_\tau$-measurable random variable such that there exist $\lambda > 2\alpha + 2L^2 + 1$, $\mu > 2\beta + 1$,

$$M(\tau) \stackrel{def}{=} \mathbb{E}\mathrm{e}^{\lambda\tau+\mu A_\tau}\big(|\xi|^2 + \varphi(\xi) + \psi(\xi)\big) + \mathbb{E}\int_0^\tau \mathrm{e}^{\lambda s+\mu A_s}\big[|\eta_s|^2\, \mathrm{d}s + |\gamma_s|^2\, \mathrm{d}A_s\big] < \infty. \tag{25}$$

- Let φ, ψ be such that

$$\left| \begin{array}{ll} \text{(i)} & \varphi, \psi : \mathbb{R}^k \to (-\infty, +\infty] \quad \text{are proper convex l.s.c. functions,} \\ \text{(ii)} & \varphi(y) \geq \varphi(0) = 0, \qquad \psi(y) \geq \psi(0) = 0. \end{array} \right. \tag{26}$$

The subdifferentials are defined by

$$\partial\varphi(x) = \left\{ v \in \mathbb{R}^k : \langle v, y - x \rangle + \varphi(x) \leq \varphi(y), \ \forall y \in \mathbb{R}^k \right\}$$

and similarly for ψ.

The existence result for (22) will be obtained via Yosida approximations. Define for $\varepsilon > 0$ the convex C^1-function φ_ε via

$$\varphi_\varepsilon(y) = \inf\left\{ \frac{1}{2\varepsilon} |y - v|^2 + \varphi(v) : v \in \mathbb{R}^k \right\}$$

(and similarly for ψ_ε).

Define

$$J_\varepsilon y = (I + \varepsilon\partial\varphi)^{-1}(y) \quad \text{and} \quad \nabla\varphi_\varepsilon(y) = \frac{y - J_\varepsilon y}{\varepsilon}.$$

Hence $y \to \nabla\varphi_\varepsilon(y)$ is a monotone Lipschitz function and

$$\varphi_\varepsilon(y) = \frac{1}{2\varepsilon} |y - J_\varepsilon y|^2 + \varphi(J_\varepsilon y)$$

(and analogously for ψ_ε).

- We introduce now *compatibility assumptions*:
for all $\varepsilon > 0$, $t \geq 0$, $y \in \mathbb{R}^k$ and $z \in \mathbb{R}^{k\times d}$,

$$\left| \begin{array}{ll} \text{(i)} & \langle \nabla\varphi_\varepsilon(y), \nabla\psi_\varepsilon(y) \rangle \geq 0, \\ \text{(ii)} & \langle \nabla\varphi_\varepsilon(y), G(t, y) \rangle \leq \langle \nabla\psi_\varepsilon(y), G(t, y) \rangle^+, \\ \text{(iii)} & \langle \nabla\psi_\varepsilon(y), F(t, y, z) \rangle \leq \langle \nabla\varphi_\varepsilon(y), F(t, y, z) \rangle^+. \end{array} \right. \tag{27}$$

Definition 7. (Y, Z, U, V) will be called a solution of BSVI (22) if

$$\begin{array}{ll} \text{(a)} & Y \in \mathcal{S}_k^{\lambda,\mu} \cap \mathcal{H}_k^{\lambda,\mu} \cap \tilde{\mathcal{H}}_k^{\lambda,\mu}, \qquad Z \in \mathcal{H}_{k\times d}^{\lambda,\mu}, \\ \text{(b)} & U \in \mathcal{H}_k^{\lambda,\mu}, \qquad V \in \tilde{\mathcal{H}}_k^{\lambda,\mu}, \\ \text{(c)} & \mathbb{E}\displaystyle\int_0^\tau e^{\lambda s + \mu A_s} \big(\varphi(Y_s)\,ds + \psi(Y_s)\,dA_s\big) < \infty, \\ \text{(d)} & (Y_t, U_t) \in \partial\varphi, \qquad \mathbb{P}(d\omega) \otimes dt, \qquad (Y_t, V_t) \in \partial\psi, \qquad \mathbb{P}(d\omega) \otimes A(\omega, dt) \\ & \quad \text{a.e. on } \Omega \times [0, \tau], \\ \text{(e)} & Y_t + \displaystyle\int_{t\wedge\tau}^\tau U_s ds + \int_{t\wedge\tau}^\tau V_s dA_s = \xi + \int_{t\wedge\tau}^\tau F(s, Y_s, Z_s)\,ds \\ & \quad + \displaystyle\int_{t\wedge\tau}^\tau G(s, Y_s)\,dA_s - \int_{t\wedge\tau}^\tau Z_s dW_s, \quad \text{for all } t \geq 0 \text{ a.s.} \end{array} \tag{28}$$

In all that follows, C denotes a constant, which may depend only on μ, α, β and L, which may vary from line to line.

Proposition 8. *Let assumptions (23), (24) and (26) be satisfied. If (Y, Z, U, V) and $(\tilde{Y}, \tilde{Z}, \tilde{U}, \tilde{V})$ are solutions corresponding to ξ and $\tilde{\xi}$ which satisfy (25), then*

$$\begin{aligned}&\mathbb{E}\int_0^{\tau} e^{\lambda s+\mu A_s}\left[|Y_s-\tilde{Y}_s|^2(ds+dA_s)+\|Z_s-\tilde{Z}_s\|^2 ds\right]\\&+\mathbb{E}\sup_{0\le t\le\tau} e^{\lambda t+\mu A_t}|Y_t-\tilde{Y}_t|^2\le C\,\mathbb{E}\left[e^{\lambda\tau+\mu A_\tau}|\xi-\tilde{\xi}|^2\right].\end{aligned}\tag{29}$$

Proof. From Itô's formula we have

$$\begin{aligned}&e^{\lambda(t\wedge\tau)+\mu A_{t\wedge\tau}}|Y_{t\wedge\tau}-\tilde{Y}_{t\wedge\tau}|^2+\int_{t\wedge\tau}^{\tau} e^{\lambda s+\mu A_s}|Y_s-\tilde{Y}_s|^2(\lambda ds+\mu dA_s)\\&\quad+2\int_{t\wedge\tau}^{\tau} e^{\lambda s+\mu A_s}\langle Y_s-\tilde{Y}_s, U_s-\tilde{U}_s\rangle ds+2\int_{t\wedge\tau}^{\tau} e^{\lambda s+\mu A_s}\langle Y_s-\tilde{Y}_s, V_s-\tilde{V}_s\rangle dA_s\\&\quad+\int_{t\wedge\tau}^{\tau} e^{\lambda s+\mu A_s}\|Z_s-\tilde{Z}_s\|^2 ds\\&=e^{\lambda\tau+\mu A_\tau}|\xi-\tilde{\xi}|^2+2\int_{t\wedge\tau}^{\tau} e^{\lambda s+\mu A_s}\left\langle Y_s-\tilde{Y}_s, F(s,Y_s,Z_s)-F(s,\tilde{Y}_s,\tilde{Z}_s)\right\rangle ds\\&\quad+2\int_{t\wedge\tau}^{\tau} e^{\lambda s+\mu A_s}\left\langle Y_s-\tilde{Y}_s, G(s,Y_s)-G(s,\tilde{Y}_s)\right\rangle dA_s\\&\quad-2\int_{t\wedge\tau}^{\tau} e^{\lambda s+\mu A_s}\left\langle Y_s-\tilde{Y}_s,(Z_s-\tilde{Z}_s)dW_s\right\rangle.\end{aligned}$$

Since

$$\begin{aligned}&\langle Y_s-\tilde{Y}_s, U_s-\tilde{U}_s\rangle ds\ge 0,\qquad \langle Y_s-\tilde{Y}_s, V_s-\tilde{V}_s\rangle dA_s\ge 0,\\&2\left\langle Y_s-\tilde{Y}_s, F(s,Y_s,Z_s)-F(s,\tilde{Y}_s,\tilde{Z}_s)\right\rangle\le(2\alpha+2L^2+1)|Y_s-\tilde{Y}_s|^2+\frac{1}{2}\|Z_s-\tilde{Z}_s\|^2\end{aligned}$$

and

$$2\left\langle Y_s-\tilde{Y}_s, G(s,Y_s)-G(s,\tilde{Y}_s)\right\rangle\le(2\beta+1)|Y_s-\tilde{Y}_s|^2,$$

then (using also the Burkholder–Davis–Gundy inequality), inequality (29) follows. ■

The main result of this section is given by:

Theorem 9. *Let assumptions (23)–(27) be satisfied. Then there exists a unique solution (Y, Z, U, V) for (22).*

3.2. BSVI—proof of the existence

Consider the approximating equation

$$\begin{aligned}&Y_t^{\varepsilon}+\int_{t\wedge\tau}^{\tau}\nabla\varphi_{\varepsilon}\left(Y_s^{\varepsilon}\right)ds+\int_{t\wedge\tau}^{\tau}\nabla\psi_{\varepsilon}\left(Y_s^{\varepsilon}\right)dA_s=\xi+\int_{t\wedge\tau}^{\tau}F\left(s,Y_s^{\varepsilon},Z_s^{\varepsilon}\right)ds\\&\quad+\int_{t\wedge\tau}^{\tau}G\left(s,Y_s^{\varepsilon}\right)dA_s-\int_{t\wedge\tau}^{\tau}Z_s^{\varepsilon}dW_s,\quad\forall t\ge 0,\ P\text{-a.s.}\end{aligned}\tag{30}$$

Since $\nabla\varphi_{\varepsilon}, \nabla\psi_{\varepsilon}:\mathbb{R}^k\to\mathbb{R}^k$ are Lipschitz functions then, by a standard argument (the Banach fixed point theorem when $y\to F(t,y,z)$ and $y\to G(t,y)$ are uniformly Lipschitz functions and Lipschitz approximations when $y \to \alpha y - F(t, y, z)$ and $y \to \beta y - G(t, y)$ are continuous monotone functions; see also [11]), Eq. (30) has a unique solution

$$\left(Y^\varepsilon, Z^\varepsilon\right) \in \left(\mathcal{S}_k^{\lambda,\mu} \cap \mathcal{H}_k^{\lambda,\mu} \cap \tilde{\mathcal{H}}_k^{\lambda,\mu}\right) \times \mathcal{H}_{k\times d}^{\mu,\lambda}.$$

Proposition 10. *Let assumptions (23)–(26) be satisfied. Then*

$$\mathbb{E}\left[\sup_{0\le t\le\tau} e^{\lambda t+\mu A_t}\left|Y_t^\varepsilon\right|^2 + \int_0^\tau e^{\lambda s+\mu A_s}\left(\left|Y_s^\varepsilon\right|^2 + \left\|Z_s^\varepsilon\right\|^2\right)ds + \int_0^\tau e^{\lambda s+\mu A_s}\left|Y_s^\varepsilon\right|^2 dA_s\right] \le C\, M(\tau). \tag{31}$$

Proof. Itô's formula for $e^{\lambda t+\mu A_t}\left|Y_t^\varepsilon\right|^2$ yields

$$\begin{aligned}
&e^{\lambda(t\wedge\tau)+\mu A_{t\wedge\tau}}\left|Y_{t\wedge\tau}^\varepsilon\right|^2 + \int_{t\wedge\tau}^\tau e^{\lambda s+\mu A_s}\left|Y_s^\varepsilon\right|^2(\lambda ds + \mu dA_s) + \int_{t\wedge\tau}^\tau e^{\lambda s+\mu A_s}\left\|Z_s^\varepsilon\right\|^2 ds\\
&+2\int_{t\wedge\tau}^\tau e^{\lambda s+\mu A_s}\left[\left\langle Y_s^\varepsilon, \nabla\varphi_\varepsilon\left(Y_s^\varepsilon\right)\right\rangle\lambda ds + \left\langle Y_s^\varepsilon, \nabla\psi_\varepsilon\left(Y_s^\varepsilon\right)\right\rangle\mu dA_s\right] = e^{\lambda\tau+\mu A_\tau}|\xi|^2\\
&+2\int_{t\wedge\tau}^\tau e^{\lambda s+\mu A_s}\left\langle Y_s^\varepsilon, F\left(s, Y_s^\varepsilon, Z_s^\varepsilon\right)\right\rangle ds + 2\int_{t\wedge\tau}^\tau e^{\lambda s+\mu A_s}\left\langle Y_s^\varepsilon, G\left(s, Y_s^\varepsilon\right)\right\rangle dA_s\\
&-2\int_{t\wedge\tau}^\tau e^{\lambda s+\mu A_s}\left\langle Y_s^\varepsilon, Z_s^\varepsilon dW_s\right\rangle.
\end{aligned}$$

But from Schwartz's inequality and assumptions (23)–(25) we obtain

$$\begin{aligned}
2\left\langle Y_s, F(s, Y_s, Z_s)\right\rangle &\le 2\alpha|Y_s|^2 + 2L|Y_s|\,\|Z_s\| + 2|Y_s|\,|F(s,0,0)|\\
&\le \left(2\alpha + 2L^2 + 1\right)|Y_s|^2 + \frac{1}{2}\|Z_s\|^2 + |F(s,0,0)|^2
\end{aligned}$$

and

$$2\left\langle Y_s, G(s, Y_s, Z_s)\right\rangle \le 2\beta|Y_s|^2 + 2|Y_s|\,|G(s,0)| \le (2\beta+1)|Y_s|^2 + |G(s,0)|^2.$$

Hence, using also that $\langle y, \nabla\varphi_\varepsilon(y)\rangle \ge 0$ and $\langle y, \nabla\psi_\varepsilon(y)\rangle \ge 0$,

$$\begin{aligned}
&e^{\lambda(t\wedge\tau)+\mu A_{t\wedge\tau}}\left|Y_{t\wedge\tau}^\varepsilon\right|^2 + \int_{t\wedge\tau}^\tau e^{\lambda s+\mu A_s}\left|Y_s^\varepsilon\right|^2(\lambda - 2\alpha - 2L^2 - 1)ds\\
&+\int_{t\wedge\tau}^\tau e^{\lambda s+\mu A_s}\left|Y_s^\varepsilon\right|^2(\mu - 2\beta - 1)dA_s + \frac{1}{2}\int_{t\wedge\tau}^\tau e^{\lambda s+\mu A_s}\left\|Z_s^\varepsilon\right\|^2 ds \le e^{\lambda\tau+\mu A_\tau}|\xi|^2\\
&+\int_{t\wedge\tau}^\tau e^{\lambda s+\mu A_s}\left(|F(s,0,0)|^2 ds + |G(s,0)|^2 dA_s\right) - 2\int_{t\wedge\tau}^\tau e^{\lambda s+\mu A_s}\left\langle Y_s^\varepsilon, Z_s^\varepsilon dW_s\right\rangle,
\end{aligned}$$

which clearly yields (for $\lambda > 2\alpha + 2L^2 + 1$ and $\mu > 2\beta + 1$)

$$\mathbb{E}\int_0^\tau e^{\lambda s+\mu A_s}\left[\left|Y_s^\varepsilon\right|^2(ds + dA_s) + \left\|Z_s^\varepsilon\right\|^2 ds\right] \le C\, M(\tau).$$

Since, by the Burkholder–Davis–Gundy inequality,

$$\mathbb{E}\sup_{0\le t\le\tau}\left|\int_{t\wedge\tau}^{\tau} e^{\lambda s+\mu A_s}\left\langle Y_s^\varepsilon, Z_s^\varepsilon dW_s\right\rangle\right| \le 3\mathbb{E}\left(\int_0^\tau e^{2(\lambda s+\mu A_s)}\left|\left\langle Y_s^\varepsilon, Z_s^\varepsilon\right\rangle\right|^2 ds\right)^{1/2}$$
$$\le \frac{1}{4}\mathbb{E}\sup_{0\le t\le\tau} e^{\lambda t+\mu A_t}\left|Y_t^\varepsilon\right|^2 + C\,\mathbb{E}\int_0^\tau e^{\lambda s+\mu A_s}\left\|Z_s^\varepsilon\right\|^2 ds,$$

then it follows that

$$\mathbb{E}\sup_{0\le t\le\tau} e^{\lambda t+\mu A_t}\left|Y_t^\varepsilon\right|^2 \le C\,M(\tau).$$

The proof is complete. ■

Proposition 11. *Let assumptions (23)–(27) be satisfied. Then there exists a positive constant C such that for any stopping time $\theta \in [0,\tau]$,*

$$\begin{aligned}
&\text{(a)}\quad \mathbb{E}\int_0^\tau e^{\lambda s+\mu A_s}\left(\left|\nabla\varphi_\varepsilon\left(Y_s^\varepsilon\right)\right|^2 ds + \left|\nabla\psi_\varepsilon\left(Y_s^\varepsilon\right)\right|^2 dA_s\right) \le C\,M(\tau),\\
&\text{(b)}\quad \mathbb{E}\int_0^\tau e^{\lambda s+\mu A_s}\left(\varphi\big(J_\varepsilon\left(Y_s^\varepsilon\right)\big)ds + \psi\big(\hat{J}_\varepsilon\left(Y_s^\varepsilon\right)\big)dA_s\right) \le C\,M(\tau),\\
&\text{(c)}\quad \mathbb{E}e^{\lambda\theta+\mu A_\theta}\left(\left|Y_\theta^\varepsilon - J_\varepsilon\left(Y_\theta^\varepsilon\right)\right|^2 + \left|Y_\theta^\varepsilon - \hat{J}_\varepsilon\left(Y_\theta^\varepsilon\right)\right|^2\right) \le \varepsilon\,C\,M(\tau),\\
&\text{(d)}\quad \mathbb{E}e^{\lambda\theta+\mu A_\theta}\left(\varphi\big(J_\varepsilon\left(Y_\theta^\varepsilon\right)\big) + \psi\big(\hat{J}_\varepsilon\left(Y_\theta^\varepsilon\right)\big)\right) \le C\,M(\tau).
\end{aligned} \tag{32}$$

Proof. Essential for the proof is the stochastic subdifferential inequality introduced by Pardoux and Răşcanu in [8], 1998. We will use this inequality for our purpose. First we write the subdifferential inequality

$$e^{\lambda s+\mu A_s}\varphi_\varepsilon\left(Y_s^\varepsilon\right) \ge (e^{\lambda s+\mu A_s} - e^{\lambda r+\mu A_r})\varphi_\varepsilon\left(Y_s^\varepsilon\right) + e^{\lambda r+\mu A_r}\varphi_\varepsilon\left(Y_r^\varepsilon\right)$$
$$+ e^{\lambda r+\mu A_r}\left\langle\nabla\varphi_\varepsilon\left(Y_r^\varepsilon\right), Y_s^\varepsilon - Y_r^\varepsilon\right\rangle,$$

for $s = t_{i+1}\wedge\tau$, $r = t_i\wedge\tau$, where $t = t_0 < t_1 < t_2 < \cdots < t\wedge\tau$ and $t_{i+1} - t_i = \frac{1}{n}$, then summing up over i, and passing to the limit as $n\to\infty$, we deduce

$$e^{\lambda\tau+\mu A_\tau}\varphi_\varepsilon(\xi) \ge e^{\lambda(t\wedge\tau)+\mu A_{t\wedge\tau}}\varphi_\varepsilon\left(Y_{t\wedge\tau}^\varepsilon\right) + \int_{t\wedge\tau}^\tau e^{\lambda s+\mu A_s}\left\langle\nabla\varphi_\varepsilon\left(Y_s^\varepsilon\right), dY_s^\varepsilon\right\rangle$$
$$+ \int_{t\wedge\tau}^\tau \varphi_\varepsilon\left(Y_s^\varepsilon\right) d(e^{\lambda s+\mu A_s}).$$

We have similar inequalities for the function ψ_ε.

If we sum and we use Eq. (30), we infer that for all $t \ge 0$,

$$e^{\lambda(t\wedge\tau)+\mu A_{t\wedge\tau}}\big(\varphi_\varepsilon\left(Y_{t\wedge\tau}^\varepsilon\right) + \psi_\varepsilon\left(Y_{t\wedge\tau}^\varepsilon\right)\big) + \int_{t\wedge\tau}^\tau e^{\lambda s+\mu A_s}\left|\nabla\psi_\varepsilon\left(Y_s^\varepsilon\right)\right|^2 dA_s$$
$$+ \int_{t\wedge\tau}^\tau e^{\lambda s+\mu A_s}\left|\nabla\varphi_\varepsilon\left(Y_s^\varepsilon\right)\right|^2 ds + \int_{t\wedge\tau}^\tau e^{\lambda s+\mu A_s}\big(\varphi_\varepsilon\left(Y_s^\varepsilon\right) + \psi_\varepsilon\left(Y_s^\varepsilon\right)\big)(\lambda ds + \mu dA_s)$$
$$+ \int_{t\wedge\tau}^\tau e^{\lambda s+\mu A_s}\left\langle\nabla\varphi_\varepsilon\left(Y_s^\varepsilon\right), \nabla\psi_\varepsilon\left(Y_s^\varepsilon\right)\right\rangle(ds + dA_s)$$
$$\le e^{\lambda\tau+\mu A_\tau}\big(\varphi_\varepsilon(\xi) + \psi_\varepsilon(\xi)\big) + \int_{t\wedge\tau}^\tau e^{\lambda s+\mu A_s}\left\langle\nabla\varphi_\varepsilon\left(Y_s^\varepsilon\right), F\left(s, Y_s^\varepsilon, Z_s^\varepsilon\right)\right\rangle ds$$

$$
\begin{aligned}
&+\int_{t\wedge\tau}^{\tau} e^{\lambda s+\mu A_s}\left\langle\nabla\varphi_\varepsilon\left(Y_s^\varepsilon\right), G\left(s, Y_s^\varepsilon\right)\right\rangle dA_s\\
&+\int_{t\wedge\tau}^{\tau} e^{\lambda s+\mu A_s}\left\langle\nabla\psi_\varepsilon\left(Y_s^\varepsilon\right), F\left(s, Y_s^\varepsilon, Z_s^\varepsilon\right)\right\rangle ds\\
&+\int_{t\wedge\tau}^{\tau} e^{\lambda s+\mu A_s}\left\langle\nabla\psi_\varepsilon\left(Y_s^\varepsilon\right), G\left(s, Y_s^\varepsilon\right)\right\rangle dA_s\\
&-\int_{t\wedge\tau}^{\tau} e^{\lambda s+\mu A_s}\left\langle\nabla\varphi_\varepsilon\left(Y_s^\varepsilon\right)+\nabla\psi_\varepsilon\left(Y_s^\varepsilon\right), Z_s^\varepsilon dW_s\right\rangle.
\end{aligned}
$$

The result follows by combining this with (31), assumptions (27) and the following inequalities:

$$
\begin{aligned}
&\frac{1}{2\varepsilon}\left|y-J_\varepsilon(y)\right|^2\le\varphi_\varepsilon(y),\qquad \frac{1}{2\varepsilon}\left|y-\hat{J}_\varepsilon(y)\right|^2\le\psi_\varepsilon(y),\\
&\varphi\big(J_\varepsilon(y)\big)\le\varphi_\varepsilon(y),\qquad \psi\big(\hat{J}_\varepsilon(y)\big)\le\psi_\varepsilon(y),\\
&\varphi_\varepsilon(\xi)\le\varphi(\xi),\qquad \psi_\varepsilon(\xi)\le\psi(\xi),\\
&\langle\nabla\varphi_\varepsilon(y), F(s,y,z)\rangle\le\frac{1}{4}\left|\nabla\varphi_\varepsilon(y)\right|^2+3\big(\eta_s^2+L^2|y|^2+L^2||z||^2\big),\\
&\langle\nabla\psi_\varepsilon(y), G(s,y)\rangle\le\frac{1}{4}\left|\nabla\psi_\varepsilon(y)\right|^2+2\big(\gamma_s^2+L^2|y|^2\big),\\
&\langle\nabla\psi_\varepsilon(y), F(s,y,z)\rangle\ \le\ \langle\nabla\varphi_\varepsilon(y), F(s,y,z)\rangle^+\\
&\qquad\qquad\le\ \frac{1}{4}\left|\nabla\varphi_\varepsilon(y)\right|^2+3\big(\eta_s^2+L^2|y|^2+L^2||z||^2\big),\\
&\langle\nabla\varphi_\varepsilon(y), G(s,y)\rangle\ \le\ \langle\nabla\psi_\varepsilon(y), G(s,y)\rangle^+\\
&\qquad\qquad\le\ \frac{1}{4}\left|\nabla\psi_\varepsilon(y)\right|^2+2\big(\gamma_s^2+L^2|y|^2\big).\qquad\blacksquare
\end{aligned}
$$

Proposition 12. *Let assumptions (23)–(27) be satisfied. Then*

$$
\begin{aligned}
&\mathbb{E}\int_0^{\tau} e^{\lambda t+\mu A_t}\left(\left|Y_s^\varepsilon-Y_s^\delta\right|^2(ds+dA_s)+\left\|Z_s^\varepsilon-Z_s^\delta\right\|^2 ds\right)\\
&\quad+\mathbb{E}\sup_{0\le t\le\tau} e^{\lambda t+\mu A_t}\left|Y_t^\varepsilon-Y_t^\delta\right|^2\le C(\varepsilon+\delta)\ M(\tau).
\end{aligned}\tag{33}
$$

Proof. By Itô's formula

$$
\begin{aligned}
&e^{\lambda(t\wedge\tau)+\mu A_{t\wedge\tau}}\left|Y_{t\wedge\tau}^\varepsilon-Y_{t\wedge\tau}^\delta\right|^2+\int_{t\wedge\tau}^{\tau} e^{\lambda s+\mu A_s}\left|Y_s^\varepsilon-Y_s^\delta\right|^2(\lambda ds+\mu dA_s)\\
&\quad+2\int_{t\wedge\tau}^{\tau} e^{\lambda s+\mu A_s}\left\langle Y_s^\varepsilon-Y_s^\delta, \nabla\varphi_\varepsilon\left(Y_s^\varepsilon\right)-\nabla\varphi_\delta(Y_s^\delta)\right\rangle ds\\
&\quad+2\int_{t\wedge\tau}^{\tau} e^{\lambda s+\mu A_s}\left\langle Y_s^\varepsilon-Y_s^\delta, \nabla\psi_\varepsilon\left(Y_s^\varepsilon\right)-\nabla\psi_\delta(Y_s^\delta)\right\rangle dA_s\\
&=2\int_{t\wedge\tau}^{\tau} e^{\lambda s+\mu A_s}\left\langle Y_s^\varepsilon-Y_s^\delta, F\left(s, Y_s^\varepsilon, Z_s^\varepsilon\right)-F(s, Y_s^\delta, Z_s^\delta)\right\rangle ds\\
&\quad+2\int_{t\wedge\tau}^{\tau} e^{\lambda s+\mu A_s}\left\langle Y_s^\varepsilon-Y_s^\delta, G\left(s, Y_s^\varepsilon\right)-G(s, Y_s^\delta)\right\rangle dA_s\\
&\quad-\int_{t\wedge\tau}^{\tau} e^{\lambda s+\mu A_s}\left\|Z_s^\varepsilon-Z_s^\delta\right\|^2 ds-2\int_{t\wedge\tau}^{\tau} e^{\lambda s+\mu A_s}\left\langle Y_s^\varepsilon-Y_s^\delta, (Z_s^\varepsilon-Z_s^\delta)dW_s\right\rangle.
\end{aligned}
$$

We have moreover,

$$2\left\langle Y_s^\varepsilon - Y_s^\delta, F\left(s, Y_s^\varepsilon, Z_s^\varepsilon\right) - F(s, Y_s^\delta, Z_s^\delta)\right\rangle \le \left(2\alpha + 2L^2\right)\left|Y_s^\varepsilon - Y_s^\delta\right|^2 + \frac{1}{2}\left\|Z_s^\varepsilon - Z_s^\delta\right\|^2,$$
$$2\left\langle Y_s^\varepsilon - Y_s^\delta, G\left(s, Y_s^\varepsilon\right) - G(s, Y_s^\delta)\right\rangle \le 2\beta\left|Y_s^\varepsilon - Y_s^\delta\right|^2.$$

But from the definition of φ_ε and the monotonicity of operator $\partial\varphi$ we have

$$\begin{aligned}0 &\le \left\langle \nabla\varphi_\varepsilon\left(Y_s^\varepsilon\right) - \nabla\varphi_\delta(Y_s^\delta), J_\varepsilon\left(Y_s^\varepsilon\right) - J_\delta(Y_s^\delta)\right\rangle \\ &= \left\langle \nabla\varphi_\varepsilon\left(Y_s^\varepsilon\right) - \nabla\varphi_\delta(Y_s^\delta), Y_s^\varepsilon - Y_s^\delta\right\rangle - \varepsilon\left|\nabla\varphi_\varepsilon\left(Y_s^\varepsilon\right)\right|^2 - \delta\left|\nabla\varphi_\delta(Y_s^\delta)\right|^2 \\ &\quad + (\varepsilon+\delta)\left\langle \nabla\varphi_\varepsilon\left(Y_s^\varepsilon\right), \nabla\varphi_\delta(Y_s^\delta)\right\rangle.\end{aligned}$$

Then

$$\left\langle \nabla\varphi_\varepsilon\left(Y_s^\varepsilon\right) - \nabla\varphi_\delta(Y_s^\delta), Y_s^\varepsilon - Y_s^\delta\right\rangle \ge -(\varepsilon+\delta)\left\langle \nabla\varphi_\varepsilon\left(Y_s^\varepsilon\right), \nabla\varphi_\delta(Y_s^\delta)\right\rangle$$

and in the same manner

$$\left\langle \nabla\psi_\varepsilon\left(Y_s^\varepsilon\right) - \nabla\psi_\delta(Y_s^\delta), Y_s^\varepsilon - Y_s^\delta\right\rangle \ge -(\varepsilon+\delta)\left\langle \nabla\psi_\varepsilon\left(Y_s^\varepsilon\right), \nabla\psi_\delta(Y_s^\delta)\right\rangle,$$

and consequently

$$\begin{aligned}&e^{\lambda(t\wedge\tau)+\mu A_{t\wedge\tau}}|Y_{t\wedge\tau}^\varepsilon - Y_{t\wedge\tau}^\delta|^2 + \int_{t\wedge\tau}^\tau e^{\lambda s+\mu A_s}|Y_s^\varepsilon - Y_s^\delta|^2(\lambda - 2\alpha - 2L^2)ds \\ &\quad + \int_{t\wedge\tau}^\tau e^{\lambda s+\mu A_s}|Y_s^\varepsilon - Y_s^\delta|^2(\mu - 2\beta)\,dA_s + \frac{1}{2}\int_{t\wedge\tau}^\tau e^{\lambda s+\mu A_s}\|Z_s^\varepsilon - Z_s^\delta\|^2 ds \\ &\le 2(\varepsilon+\delta)\int_{t\wedge\tau}^\tau e^{\lambda s+\mu A_s}\Big[\left\langle \nabla\varphi_\varepsilon\left(Y_s^\varepsilon\right), \nabla\varphi_\delta(Y_s^\delta)\right\rangle ds \\ &\quad + \left\langle \nabla\psi_\varepsilon\left(Y_s^\varepsilon\right), \nabla\psi_\delta(Y_s^\delta)\right\rangle dA_s\Big] - 2\int_{t\wedge\tau}^\tau e^{\lambda s+\mu A_s}\left\langle Y_s^\varepsilon - Y_s^\delta, (Z_s^\varepsilon - Z_s^\delta)dW_s\right\rangle. \qquad (34)\end{aligned}$$

Now, from ((32)-a),

$$\begin{aligned}&2(\varepsilon+\delta)\mathbb{E}\int_{t\wedge\tau}^\tau e^{\lambda s+\mu A_s}\Big[\left\langle \nabla\varphi_\varepsilon\left(Y_s^\varepsilon\right), \nabla\varphi_\delta(Y_s^\delta)\right\rangle ds \\ &\quad + \left\langle \nabla\psi_\varepsilon\left(Y_s^\varepsilon\right), \nabla\psi_\delta(Y_s^\delta)\right\rangle dA_s\Big] \le C\ (\varepsilon+\delta)\ M(\tau)\end{aligned}$$

and clearly by standard calculus, inequality (33) follows. ■

We give now the proof of Theorem 9.

Proof. Uniqueness is a consequence of Proposition 8. The existence of the solution (Y, Z, U, V) is obtained as the limit of $\left(Y_s^\varepsilon, Z_s^\varepsilon, \nabla\varphi_\varepsilon\left(Y_s^\varepsilon\right), \nabla\psi_\varepsilon\left(Y_s^\varepsilon\right)\right)$.

From Proposition 12 we have

$$\left|\begin{array}{l}\exists\, Y \in \mathcal{S}_k^{\lambda,\mu} \cap \mathcal{H}_k^{\lambda,\mu} \cap \tilde{\mathcal{H}}_k^{\lambda,\mu}, \quad \exists\, Z \in \mathcal{H}_{k\times d}^{\lambda,\mu}, \\ \lim\limits_{\varepsilon\searrow 0} Y^\varepsilon = Y \quad \text{in } \mathcal{S}_k^{\lambda,\mu} \cap \mathcal{H}_k^{\lambda,\mu} \cap \tilde{\mathcal{H}}_k^{\lambda,\mu}, \\ \lim\limits_{\varepsilon\searrow 0} Z^\varepsilon = Z \quad \text{in } \mathcal{H}_{k\times d}^{\lambda,\mu}.\end{array}\right.$$

Also, from ((32)-a) and ((32)-c) we have

$$\lim_{\varepsilon\searrow 0} J_\varepsilon(Y^\varepsilon) = Y \quad \text{in } \mathcal{H}_k^{\lambda,\mu}, \qquad \lim_{\varepsilon\searrow 0} \hat{J}_\varepsilon(Y^\varepsilon) = Y \quad \text{in } \tilde{\mathcal{H}}_k^{\lambda,\mu},$$

$$\lim_{\varepsilon\searrow 0} \mathbb{E}e^{\lambda\theta+\mu A_\theta} \left|J_\varepsilon(Y_\theta^\varepsilon) - Y_\theta\right|^2 = 0, \qquad \lim_{\varepsilon\searrow 0} \mathbb{E}e^{\lambda\theta+\mu A_\theta} \left|\hat{J}_\varepsilon(Y_\theta^\varepsilon) - Y_\theta\right|^2 = 0,$$

for any stopping time θ, $0 \le \theta \le \tau$.

Using Fatou's Lemma, from ((32)-b), ((32)-d) and the fact that φ is l.s.c. we obtained ((28)-c).

Defining $U^\varepsilon = \nabla\varphi_\varepsilon(Y^\varepsilon)$, $V^\varepsilon = \nabla\psi_\varepsilon(Y^\varepsilon)$, from ((32)-a) it follows that

$$\mathbb{E}\left[\int_\theta^\tau e^{\lambda s+\mu A_s}\left(|U^\varepsilon|^2 ds + |V^\varepsilon|^2 dA_s\right)\right] \le C\ M(\tau).$$

Hence there exists $U \in \mathcal{H}_k^{\lambda,\mu}$ and $V \in \tilde{\mathcal{H}}_k^{\lambda,\mu}$ such that for a subsequence $\varepsilon_n \searrow 0$

$$U^{\varepsilon_n} \rightharpoonup U, \quad \text{weakly in Hilbert space } \mathcal{H}_k^{\lambda,\mu},$$
$$V^{\varepsilon_n} \rightharpoonup V, \quad \text{weakly in Hilbert space } \tilde{\mathcal{H}}_k^{\lambda,\mu},$$

and then

$$\begin{aligned}&\mathbb{E}\left[\int_\theta^\tau e^{\lambda s+\mu A_s}\left(|U|^2\, ds + |V|^2\, dA_s\right)\right]\\ &\quad\le \liminf_{n\to\infty}\mathbb{E}\left[\int_\theta^\tau e^{\lambda s+\mu A_s}\left(|U^{\varepsilon_n}|^2 ds + |V^{\varepsilon_n}|^2 dA_s\right)\right] \le C\ M_2(\theta,\tau).\end{aligned}$$

Passing now to lim in (30) we obtain ((28)-e).

Let $u \in \mathcal{H}_k^{\lambda,\mu}$, $v \in \tilde{\mathcal{H}}_k^{\lambda,\mu}$. Since $\nabla\varphi_\varepsilon(Y_t^\varepsilon) \in \partial\varphi\big(J_\varepsilon(Y_t^\varepsilon)\big)$ and $\nabla\psi_\varepsilon(Y_t^\varepsilon) \in \partial\psi\big(\hat{J}_\varepsilon(Y_t^\varepsilon)\big)$, $\forall t \ge 0$, then as signed measures on $\Omega \times [0,\tau]$,

$$\begin{aligned}&e^{\lambda s+\mu A_s}\left\langle U_s^\varepsilon, u_s - J_\varepsilon\left(Y_s^\varepsilon\right)\right\rangle \mathbb{P}(d\omega)\otimes ds + e^{\lambda s+\mu A_s}\varphi\big(J_\varepsilon\left(Y_s^\varepsilon\right)\big)\,\mathbb{P}(d\omega)\otimes ds\\ &\quad\le e^{\lambda s+\mu A_s}\varphi(u_s)\,\mathbb{P}(d\omega)\otimes ds\end{aligned}$$

and

$$\begin{aligned}&e^{\lambda s+\mu A_s}\left\langle V_s^\varepsilon, v_s - \hat{J}_\varepsilon\left(Y_s^\varepsilon\right)\right\rangle \mathbb{P}(d\omega)\otimes A(\omega,ds) + e^{\lambda s+\mu A_s}\psi\big(\hat{J}_\varepsilon\left(Y_s^\varepsilon\right)\big)\,\mathbb{P}(d\omega)\otimes A(\omega,ds)\\ &\quad\le e^{\lambda s+\mu A_s}\psi(v_s)\,\mathbb{P}(d\omega)\otimes A(\omega,ds).\end{aligned}$$

Taking the lim inf in these last two inequalities we obtain ((28)-d). The proof is complete. ■

4. PVI — proof of the existence theorem

It follows from a result in [6] that for each $(t,x) \in \mathbb{R}_+ \times \overline{\mathcal{D}}$ there exists a unique pair of continuous $\mathcal{F}_s^t$-p.m.s.p. $(X_s^{t,x}, A_s^{t,x})_{s\ge 0}$, with values in $\overline{\mathcal{D}} \times \mathbb{R}_+$, a solution of the reflected stochastic differential equation

$$\begin{cases} X_s^{t,x} = x + \displaystyle\int_t^{s\vee t} b(r, X_r^{t,x})dr + \int_t^{s\vee t}\sigma(r, X_r^{t,x})dW_r - \int_t^{s\vee t}\nabla\ell(X_r^{t,x})dA_r^{t,x},\\ s \longmapsto A_s^{t,x} \quad \text{is increasing},\\ A_s^{t,x} = \displaystyle\int_t^{s\vee t}\mathbf{1}_{\{X_r^{t,x}\in Bd(\mathcal{D})\}}dA_r^{t,x},\end{cases} \tag{35}$$

where

$$\mathcal{F}_s^t = \sigma\{W_r - W_t : t \le r \le s\} \vee \mathcal{N}.$$

Since $\overline{\mathcal{D}}$ is a bounded set, then

$$\sup_{s\ge 0} |X_s^{t,x}| \le M \tag{36}$$

and with calculus similar to that in [11] we have that for all $\mu, T, p > 0$, there exists a positive constant C such that $\forall\, t, t' \in [0,T],\ x, x' \in \overline{\mathcal{D}}$,

$$\mathbb{E} \sup_{s\in[0,T]} |X_s^{t,x} - X_s^{t',x'}|^p \le C\big(|x-x'|^p + |t-t'|^{\frac{p}{2}}\big), \tag{37}$$

and

$$\mathbb{E}[\mathrm{e}^{\mu A_T^{t,x}}] < \infty. \tag{38}$$

Let $T > 0$ be arbitrary and fixed. Under assumptions (2)–(7), it follows from Theorem 9 with τ replaced by T that for each $(t,x) \in [0,T] \times \overline{\mathcal{D}}$ there exists a unique solution $(Y^{tx}, Z^{tx}, U^{tx}, V^{tx})$ of the p.m.s.p.

$$\begin{aligned} &Y^{tx} \in \mathcal{S}_1^{\lambda,\mu} \cap \mathcal{H}_1^{\lambda,\mu} \cap \tilde{\mathcal{H}}_1^{\lambda,\mu}, \\ &Z^{tx} \in \mathcal{H}_d^{\lambda,\mu}, \qquad U^{tx} \in \mathcal{H}_1^{\lambda,\mu}, \qquad V^{tx} \in \tilde{\mathcal{H}}_1^{\lambda,\mu}, \end{aligned}$$

with $Y_s^{t,x} = Y_t^{t,x}$, $Z_s^{t,x} = 0$, $U_s^{t,x} = 0$, $V_s^{t,x} = 0$, for all $s \in [0,t]$, a solution of the BSDE

$$\begin{aligned} Y_s^{t,x} &+ \int_s^T U_r^{t,x}\mathrm{d}r + \int_s^T V_r^{t,x}\mathrm{d}A_r^{t,x} = h\left(X_T^{t,x}\right) + \int_s^T 1_{[t,T]}(r)\ f\left(r, X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x}\right)\mathrm{d}r \\ &+ \int_s^T 1_{[t,T]}(r)\, g\left(r, X_r^{t,x}, Y_r^{t,x}\right)\mathrm{d}A_r^{t,x} - \int_s^T Z_r^{t,x}\mathrm{d}W_r, \quad \text{for all } s \in [0,T] \text{ a.s.} \end{aligned}$$

such that $\left(Y_s^{t,x}, U_s^{t,x}\right) \in \partial\varphi$, $\mathbb{P}(\mathrm{d}\omega) \otimes \mathrm{d}t$, $\left(Y_s^{t,x}, V_s^{t,x}\right) \in \partial\psi$, $\mathbb{P}(\mathrm{d}\omega) \otimes A(\omega, \mathrm{d}t)$, a.e. on $\Omega \times [t,T]$.

We observe that functions f, g depend on ω only via function $X^{t,x}$.

Proposition 13. *Under assumptions (2)–(7), we have*

$$\mathbb{E} \sup_{s\in[0,T]} \mathrm{e}^{\lambda s + \mu A_s} |Y_s^{t,x}|^2 \le C(T) \tag{39}$$

and

$$\begin{aligned} \mathbb{E} \sup_{s\in[0,T]} \mathrm{e}^{\lambda s+\mu A_s} |Y_s^{t,x} - Y_s^{t',x'}|^2 &\le \mathbb{E}\Big[\mathrm{e}^{\lambda\tau+\mu A_\tau} \left|h(X_T^{tx}) - h(X_T^{t'x'})\right|^2 \\ &+ \int_0^T \mathrm{e}^{\lambda r+\mu A_r} \left|1_{[t,T]}(r) f(r, X_r^{tx}, Y_r^{tx}, Z_r^{tx}) - 1_{[t',T]}(r) f(r, X_r^{t'x'}, Y_r^{tx}, Z_r^{tx})\right|^2 \mathrm{d}r \\ &+ \int_0^T \mathrm{e}^{\lambda r+\mu A_r} \left|1_{[t,T]}(r) g(r, X_r^{tx}, Y_r^{tx}) - 1_{[t',T]}(r) g(r, X_r^{t'x'}, Y_r^{tx})\right|^2 \mathrm{d}A_r^{t,x}\Big]. \end{aligned} \tag{40}$$

Proof. Inequality (39) follows from Theorem 9 using also (36), (38). Inequality (40) follows from (29) in Proposition 8. ■

We define

$$u(t,x) = Y_t^{tx}, \quad (t,x) \in [0,T] \times \overline{\mathcal{D}}, \tag{41}$$

which is a determinist quantity since Y_t^{tx} is $\mathcal{F}_t^t \equiv \mathcal{N}$-measurable.

From the Markov property we have

$$u(s, X_s^{tx}) = Y_s^{tx}. \tag{42}$$

Corollary 14. *Under assumptions (2)–(7), the function u satisfies*

$$\begin{array}{ll} \text{(a)} & u(t,x) \in Dom\,(\varphi)\,, \quad \forall (t,x) \in [0,T] \times \overline{\mathcal{D}}, \\ \text{(b)} & u(t,x) \in Dom\,(\psi)\,, \quad \forall (t,x) \in [0,T] \times Bd\,(\mathcal{D})\,, \\ \text{(c)} & u \in C\big([0,T] \times \overline{\mathcal{D}}\big). \end{array} \tag{43}$$

Proof. Using ((28)-c) we have $\varphi\big(u(t,x)\big) = \mathbb{E}\varphi(Y_t^{tx}) < +\infty$ and similarly for ψ. Hence ((43)-a,b) follows. Let $(t_n, x_n) \to (t,x)$. Then

$$|u(t_n,x_n) - u(t,x)|^2 = \mathbb{E}|Y_{t_n}^{t_n x_n} - Y_t^{tx}|^2 \le 2\mathbb{E} \sup_{s\in[0,T]} |Y_s^{t_n x_n} - Y_s^{tx}|^2 + 2\mathbb{E}|Y_{t_n}^{tx} - Y_t^{tx}|^2.$$

Using (40) and (36)–(38) we obtain $u(t_n,x_n) \to u(t,x)$ as $(t_n,x_n) \to (t,x)$. ∎

We present now the proof of Theorem 5 (existence of the viscosity solutions).

Proof. It suffices to show the existence of the solution of PVI (1) on an arbitrary fixed interval $[0,T]$. Setting

$$\tilde{u}\,(t,x) = u\,(T-t,x)$$

then the existence of a solution for (1) is equivalent to the existence of a solution for (44):

$$\begin{cases} \dfrac{\partial \tilde{u}(t,x)}{\partial t} + \tilde{\mathcal{L}}_t \tilde{u}\,(t,x) + \tilde{f}\big(t,x,\tilde{u}(t,x),(\nabla\tilde{u}\sigma)(t,x)\big) \in \partial\varphi\big(\tilde{u}(t,x)\big), \\ \qquad t \in (0,T)\,,\ x \in \mathcal{D}, \\ -\dfrac{\partial \tilde{u}(t,x)}{\partial n} + \tilde{g}\big(t,x,\tilde{u}(t,x)\big) \in \partial\psi\big(\tilde{u}(t,x)\big), \quad t \in (0,T)\,,\ x \in Bd\,(\mathcal{D})\,, \\ \tilde{u}(T,x) = h(x), \quad x \in \overline{\mathcal{D}}, \end{cases} \tag{44}$$

where

$$\tilde{f}\,(t,x,u,z) = f\,(T-t,x,u,z)\,, \qquad \tilde{g}\,(t,x,u) = g\,(T-t,x,u)\,,$$
$$\tilde{\sigma}\,(t,x) = \sigma\,(T-t,x)\,, \qquad \tilde{b}\,(t,x) = b\,(T-t,x)$$

and

$$\tilde{\mathcal{L}}_t v\,(x) = \frac{1}{2}\sum_{i,j=1}^{d} (\tilde{\sigma}\tilde{\sigma}^*)_{ij}(t,x)\frac{\partial^2 v\,(x)}{\partial x_i \partial x_j} + \sum_{i=1}^{d} \tilde{b}_i(t,x)\frac{\partial v\,(x)}{\partial x_i}.$$

We define also

$$\tilde{V}\,(t,x,p,q,X) \overset{def}{=} -p - \frac{1}{2}\mathrm{Tr}\big((\tilde{\sigma}\tilde{\sigma}^*)(t,x)X\big) - \big\langle \tilde{b}(t,x), q\big\rangle - \tilde{f}\big(t,x,\tilde{u}(t,x),q\tilde{\sigma}(t,x)\big).$$

In the sequel, for simplicity we keep the notation $b, \sigma, u, f, g, \mathcal{L}, V$ instead of $\tilde{b}, \tilde{\sigma}, \tilde{u}, \tilde{f}, \tilde{g}, \tilde{\mathcal{L}}, \tilde{V}$ and we shall prove that function u defined by (41) is a viscosity solution of parabolic variational inequality (44). We show only that u is a viscosity subsolution of (44) (the supersolution case is similar).

Let $(t,x) \in [0,T] \times \overline{\mathcal{D}}$ and $(p,q,X) \in \mathcal{P}^{2,+}u(t,x)$.

1. The proof for the case $x \in \mathcal{D}$ is similar of that from [8].
2. Let $x \in Bd\,(\mathcal{D})$. Suppose, contrary to our claim, that

$$\min\Big\{V\,(t,x,p,q,X) + \varphi'_-\big(u(t,x)\big),\\ \langle \nabla\ell\,(x)\,,q\rangle - g\big(t,x,u(t,x)\big) + \psi'_-\big(u(t,x)\big)\Big\} > 0$$

and we will find a contradiction.

It follows by continuity of f, g, u, b, σ, ℓ, left continuity and monotonicity of φ'_- and ψ'_- that there exists $\varepsilon > 0$, $\delta > 0$ such that for all $|s-t| \le \delta$, $|y-x| \le \delta$,

$$-\,(p+\varepsilon) - \frac{1}{2}\mathrm{Tr}\left((\sigma\sigma^*)(s,y)\,(X+\varepsilon I)\right) - \langle b(s,y), q + (X+\varepsilon I)\,(y-x)\rangle \\ -\, f\big(s,y,u(s,y),\big(q+(X+\varepsilon I)\,(y-x)\big)\sigma(s,y)\big) + \varphi'_-\big(u(s,y)\big) > 0, \quad \text{if } x \in D \tag{45}$$

and

$$\langle \nabla\ell\,(y)\,,q + (X+\varepsilon I)\,(y-x)\rangle - g\big(s,y,u(s,y)\big) + \psi'_-\big(u(s,y)\big) > 0,\\ \text{if } x \in Bd\,(\mathcal{D})\,. \tag{46}$$

Now since $(p,q,X) \in \mathcal{P}^{2,+}u(t,x)$ there exists $0 < \delta' \le \delta$ such that

$$u(s,y) < \hat{u}(s,y),$$

for all $s \in [0,T]$, $s \neq t$, $y \in \overline{\mathcal{D}}$, $y \neq x$ such that $|s-t| \le \delta'$, $|y-x| \le \delta'$, where

$$\hat{u}(s,y) = u(t,x) + (p+\varepsilon)\,(s-t) + \langle q, y-x\rangle + \frac{1}{2}\,\langle (X+\varepsilon I)\,(y-x), y-x\rangle\,.$$

Let

$$\nu \overset{def}{=} \inf\left\{s > t :\ |X_s^{t,x} - x| \ge \delta'\right\}.$$

We note that

$$(\bar{Y}_s^{t,x}, \bar{Z}_s^{t,x}) = \left(Y_s^{t,x}, Z_s^{t,x}\right), \quad t \le s \le \left(t+\delta'\right) \wedge \nu$$

solves the BSDE

$$\begin{cases} \bar{Y}_s^{t,x} = u\left(\nu, X_\nu^{t,x}\right) + \displaystyle\int_s^\nu \left(f(r, X_r^{t,x}, \bar{Y}_r^{t,x}, \bar{Z}_r^{t,x}) - U_r^{t,x}\right)\mathrm{d}r - \int_s^\nu \bar{Z}_r^{t,x}\,\mathrm{d}W_r \\ \quad + \displaystyle\int_s^\nu \left(g(r, X_r^{t,x}, \bar{Y}_r^{t,x}) - V_r^{t,x}\right)\mathrm{d}A_r^{t,x}, \\ \left(Y_s^{t,x}, U_s^{t,x}\right) \in \partial\varphi, \mathbb{P}\,(\mathrm{d}\omega) \otimes \mathrm{d}t, \qquad \left(Y_s^{t,x}, V_s^{t,x}\right) \in \partial\psi, \mathbb{P}\,(\mathrm{d}\omega) \otimes A\,(\omega, \mathrm{d}t)\,, \\ \quad \text{a.e. on } \Omega \times [t,T]\,. \end{cases}$$

Moreover, it follows from Itô's formula that

$$(\hat{Y}_s^{t,x}, \hat{Z}_s^{t,x}) = \left(\hat{u}(s, X_s^{t,x}), \left(\nabla\hat{u}\sigma\right)(s, X_s^{t,x})\right), \quad t \le s \le t+\delta'$$

satisfies

$$\begin{aligned}\hat{Y}_s^{t,x} &= \hat{u}(\nu, X_\nu^{t,x}) - \int_s^\nu \Big[\frac{\partial \hat{u}(r, X_r^{t,x})}{\partial t} + \mathcal{L}_r \hat{u}(r, X_r^{t,x})\Big]\mathrm{d}r - \int_s^\nu \hat{Z}_r^{t,x}\mathrm{d}W_r \\ &\quad + \int_s^\nu \left\langle \nabla_x \hat{u}\left(r, X_r^{t,x}\right), \nabla\ell(X_r^{t,x})\right\rangle \mathrm{d}A_r^{t,x}.\end{aligned}$$

Let $(\tilde{Y}_s^{t,x}, \tilde{Z}_s^{t,x}) = (\hat{Y}_s^{t,x} - \bar{Y}_s^{t,x}, \hat{Z}_s^{t,x} - \bar{Z}_s^{t,x})$.

We have

$$\begin{aligned}\tilde{Y}_s^{t,x} &= \left[\hat{u}(\nu, X_\nu^{t,x}) - u\left(\nu, X_\nu^{t,x}\right)\right] + \int_s^\nu \Big[-\frac{\partial \hat{u}(r, X_r^{t,x})}{\partial t} - \mathcal{L}_r \hat{u}(r, X_r^{t,x}) \\ &\quad - f(r, X_r^{t,x}, \bar{Y}_r^{t,x}, \bar{Z}_r^{t,x}) + U_r^{t,x}\Big]\mathrm{d}r - \int_s^\nu \tilde{Z}_r^{t,x}\mathrm{d}W_r \\ &\quad + \int_s^\nu \Big[\left\langle \nabla_x \hat{u}(r, X_r^{t,x}), \nabla\ell(X_r^{t,x})\right\rangle - g(r, X_r^{t,x}, \bar{Y}_r^{t,x}) + V_r^{t,x}\Big]\mathrm{d}A_r^{t,x}.\end{aligned}$$

Let

$$\begin{aligned}\overline{\beta}_s &= \mathcal{L}_s \hat{u}(s, X_s^{t,x}) + f(s, X_s^{t,x}, \bar{Y}_s^{t,x}, \bar{Z}_s^{t,x}), \\ \hat{\beta}_s &= \mathcal{L}_s \hat{u}(s, X_s^{t,x}) + f(s, X_s^{t,x}, \bar{Y}_s^{t,x}, \hat{Z}_s^{t,x}).\end{aligned}$$

Since $|\hat{\beta}_s - \bar{\beta}_s| \le C\,|\hat{Z}_s^{t,x} - \bar{Z}_s^{t,x}|$, there exists a bounded d-dimensional p.m.s.p. $\{\zeta_s; 0 \le s \le \nu\}$ such that $\hat{\beta}_s - \bar{\beta}_s = \langle \zeta_s, \tilde{Z}_s^{t,x}\rangle$.

Now

$$\begin{aligned}\tilde{Y}_s^{t,x} &= \left[\hat{u}(\nu, X_\nu^{t,x}) - u(\nu, X_\nu^{t,x})\right] + \int_s^\nu \Big[-\frac{\partial \hat{u}(r, X_r^{t,x})}{\partial t} + \langle \zeta_r, \tilde{Z}_r^{t,x}\rangle - \hat{\beta}_r + U_r^{t,x}\Big]\mathrm{d}r \\ &\quad + \int_s^\nu \Big[\left\langle \nabla_x \hat{u}(r, X_r^{t,x}), \nabla\ell\left(X_r^{t,x}\right)\right\rangle - g\left(r, X_r^{t,x}, \bar{Y}_r^{t,x}\right) + V_r^{t,x}\Big]\mathrm{d}A_r^{t,x} - \int_s^\nu \tilde{Z}_r^{t,x}\mathrm{d}W_r.\end{aligned}$$

It is easy to see that, for the process

$$\Gamma_s^t = \exp\left[-\frac{1}{2}\int_t^s |\zeta_r|^2\,\mathrm{d}r + \int_t^s \langle \zeta_r, \mathrm{d}W_r\rangle\right],$$

we have, from Itô's formula,

$$\Gamma_s^t = \Gamma_t^t + \int_t^s \Gamma_r^t\ \langle \zeta_r, \mathrm{d}W_r\rangle$$

and so

$$\begin{aligned}\mathrm{d}(\tilde{Y}_s^{t,x}\ \Gamma_s^t) &= \Gamma_s^t\Big[\frac{\partial \hat{u}(s, X_s^{t,x})}{\partial t} + \hat{\beta}_s - U_s^{t,x}\Big]\mathrm{d}s + \Gamma_s^t\langle \tilde{Z}_r^{t,x} + \tilde{Y}_s^{t,x}\zeta_s, \mathrm{d}W_s\rangle \\ &\quad + \Gamma_s^t\Big[\left\langle \nabla_x \hat{u}\left(r, X_r^{t,x}\right), \nabla\ell(X_r^{t,x})\right\rangle + g(s, X_s^{t,x}, \bar{Y}_s^{t,x}) - V_s^{t,x}\Big]\mathrm{d}A_s^{t,x}.\end{aligned}$$

Then

$$\begin{aligned}\tilde{Y}_t^{t,x} &= \mathbb{E}\Big[\Gamma_\nu^t\left(\hat{u}(\nu, X_\nu^{t,x}) - u(\nu, X_\nu^{t,x})\right)\Big] - \mathbb{E}\Big[\int_t^\nu \Gamma_r^t\Big[\frac{\partial \hat{u}(r, X_r^{t,x})}{\partial t} + \hat{\beta}_r - U_r^{t,x}\Big]\mathrm{d}r\Big] \\ &\quad - \mathbb{E}\Big[\int_t^\nu \Gamma_r^t\Big(-\left\langle \nabla_x \hat{u}\left(r, X_r^{t,x}\right), \nabla\ell\left(X_r^{t,x}\right)\right\rangle + g(r, X_r^{t,x}, \bar{Y}_r^{t,x}) - V_r^{t,x}\Big)\mathrm{d}A_r^{t,x}\Big].\end{aligned} \tag{47}$$

We first note that $(Y_t, U_t) \in \partial\varphi$ and $(Y_t, V_t) \in \partial\psi$ imply that

$$\varphi'_-\big(u(s, X_s^{t,x})\big)\mathrm{d}s \le U_s^{t,x}\mathrm{d}s, \qquad \psi'_-\big(u(s, X_s^{t,x})\big)\mathrm{d}A_s^{t,x} \le V_s^{t,x}\mathrm{d}A_s^{t,x}.$$

Moreover, the choice of δ' and ν implies that

$$u(\nu, X_\nu^{t,x}) < \hat{u}(\nu, X_\nu^{t,x}).$$

From (45) and (46) it follows that

$$-(p+\varepsilon) - \hat{\beta}_s + \varphi'_-\big(u(s, X_s^{t,x})\big) > 0, \quad \text{if } x \in \mathcal{D}$$

and

$$\frac{\partial \hat{u}\left(s, X_s^{t,x}\right)}{\partial n} - g(s, X_s^{t,x}, \bar{Y}_s^{t,x}) + \psi'_-\big(u(s, X_s^{t,x})\big) > 0, \quad \text{if } x \in Bd\,(\mathcal{D})\,.$$

All these inequalities and Eq. (47) imply that $\tilde{Y}_t^{t,x} > 0$ and equivalently

$$\hat{u}\,(t,x) > u\,(t,x)\,,$$

which is a contradiction with the definition of $\hat{u}$. Hence we have

$$\min\Big\{V\,(t,x,p,q,X) + \varphi'_-\,(u(t,x))\,, \langle\nabla\ell\,(x)\,, q\rangle - g\big(t,x,u(t,x)\big) + \psi'_-\big(u(t,x)\big)\Big\} \le 0.$$

This proves that u is a viscosity subsolution of (44). Symmetric arguments show that u is also a supersolution; hence u is a viscosity solution of PVI (44). ■

Remark 15. If b, σ, f and g do not depend on t then we have a directly a representation formula for the viscosity solution u of PVI (1):

$$u\,(t,x) = Y_0^{0,x;t},$$

where $(Y_s^{0,x;t}, Z_s^{0,x;t}, U_s^{0,x;t}, V_s^{0,x;t})_{0\le s\le t}$ is a solution of the BSVI

$$\begin{aligned} Y_s^{0,x;t} &+ \int_s^t U_r^{0,x;t}\mathrm{d}r + \int_s^t V_r^{0,x;t}\mathrm{d}A_r^{0,x} = h(X_t^{0,x}) + \int_s^t f(X_r^{t,x}, Y_r^{0,x;t}, Z_r^{0,x;t})\mathrm{d}r \\ &+ \int_s^t g(X_r^{0,x}, Y_r^{0,x;t})\mathrm{d}A_r^{0,x} - \int_s^t Z_r^{0,x;t}\mathrm{d}W_r, \quad \text{for all } s \in [0,T] \text{ a.s.} \end{aligned}$$

and $(X_s^{0,x}, A_s^{0,x})_{0\le s\le t}$ solves the SDE

$$\begin{cases} X_s^{0,x} = x + \int_0^s b(X_r^{0,x})\mathrm{d}r + \int_0^s \sigma(X_r^{0,x})\mathrm{d}W_r - \int_0^s \nabla\ell(X_r^{0,x})\mathrm{d}A_r^{0,x}, \\ s \longmapsto A_s^{0,x} \quad \text{is increasing}, \\ A_s^{0,x} = \int_0^s \mathbf{1}_{\{X_r^{0,x} \in Bd(\mathcal{D})\}}\mathrm{d}A_r^{0,x}. \end{cases}$$

Corollary 16. *We have*

$$u(t,x) \in Dom\,(\partial\varphi)\,, \quad \forall (t,x) \in [0,T] \times \mathcal{D}.$$

Proof. Let (t,x) be fixed. We have two cases:

(1) $Dom\,(\partial\varphi) = Dom\,(\varphi)$, and so, from ((43)-a), $u(t,x) \in Dom\,(\partial\varphi)$.
(2) $Dom\,(\partial\varphi) \ne Dom\,(\varphi)$. Let $b \in Dom\,\varphi \setminus Dom\,(\partial\varphi)$.

Then $b = \sup(Dom\,\varphi)$ or $b = \inf Dom\,\varphi$. If $b = \sup(Dom\,\varphi)$ and $u(t,x) = b$, then $(0,0,0) \in \mathcal{P}^{2,+}u(t,x)$ since

$$u(s,y) \le u(t,x) + o\big(|s-t| + |y-x|^2\big)$$

and from (8) it follows that $\varphi'_-(b) = \varphi'_-\big(u(t,x)\big) < \infty$ and consequently $b \in Dom\,(\partial\varphi)$, a contradiction which shows that $u(t,x) < b$. And similarly for $b = \inf(Dom\,\varphi)$. ■

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