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On Some Nonlinear Sturm–Liouville Problems*

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1. INTRODUCTION

In the study of nonlinear eigenvalue problems, an important role is played, when it exists, by the linearization about zero of the problem under consideration, i.e., its Fréchet derivative at the origin (cf. [4]). In this context of linearizability, a nonlinear version of the classical results for linear Sturm–Liouville second order ordinary differential equations has been given by Rabinowitz [7]. This was shown (in [8, 9]) to be a consequence of a more general global bifurcation theorem. An existence theorem for positive solutions of quasi-linear elliptic partial differential equations was also derived, in the same spirit, in [8, 9], from this global difurcation theorem.

The purpose of this paper is to study nonlinear Sturm–Liouville problems for some second order ordinary differential equations and a class of quasi-linear elliptic partial differential equations. The problems we consider need not have any linearization at the origin, but still can be related to some linear problems. The general idea is to approximate these equations by linearizable ones, for which we apply the results of Rabinowitz. Then, we pass to the limit using a priori bounds which are obtained with the aid of the Sturm comparison theorem, or by a positivity argument in the partial differential equation case.

The main result for ordinary differential equations is proved in Section 2. We consider the problem $\mathscr{L}u = \lambda au + F(x, u, u', \lambda)$ with separated boundary conditions on $[0, \pi]$, where $\mathscr{L}u = \lambda au$ is a classical linear Sturm–Liouville second order ordinary differential equation. We assume that the nonlinear term F is of the form $F = f + g$, f and g being continuous, with g satisfying a $o(|u| + |u'|)$ condition (like the nonlinear term in [7, 8]), and $|f(x, u, u', \lambda)| \leqslant M|u|$ in a neighborhood of $u = u' = 0$, uniformly in x and in λ. For such an equation, we show the existence of two families of continua of solutions, $\mathscr{C}_k^+$ and $\mathscr{C}_k^-$, corresponding to the usual nodal properties and bifurcating from the line of trivial solutions. In general, one can only prove that bifurcation occurs in each interval of a sequence of bounded intervals. Indeed, we give an example of an

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equation illustrating this fact, where all the points of an interval are actually bifurcation points.

As a particular type of problem in the preceding class, we study in Section 3 "half-linear" equations, i.e., equations of the form $\mathscr{L}u = \lambda au + r \mid u \mid$. We obtain the existence of two sequences of "half-eigenvalues" λ_k^+ and λ_k^-, corresponding to the usual nodal properties but differentiated according to the sign of the eigenfunctions in a neighborhood of 0. It is also shown that for a problem possessing different linearizations as $u \to 0^+$ and $u \to 0^-$, these half-eigenvalues correspond to bifurcation points in a global sense.

In the last section, we consider elliptic partial differential equations with a nonlinear term $F(x, u, \nabla u, \lambda)$ satisfying assumptions analogous to that of Section 2. In the same spirit, we show the existence of an unbounded continuum of nontrivial positive solutions (i.e., (λ, u) with $u \geqslant 0$ and $u \not\equiv 0$) bifurcating from points which lie in a bounded interval of the line of trivial solutions.

In [11; 12, Theorems 2–6], Turner has proved an abstract theorem which is related to the type of results we obtain. However, this general theorem does not seem to yield the results presented here.

2. Global Bifurcation for a Class of Nonlinearizable Sturm–Liouville Problems for Second Order Ordinary Differential Equations

Let $\mathscr{L}$ be the Sturm–Liouville differential operator defined by $\mathscr{L}u = -(pu')' + qu$, where p is a positive, continuously differentiable function, and q is a continuous function on $[0, \pi]$. We denote by (b.c.) the set of separated boundary conditions

$$\begin{aligned} b_0 u(0) + c_0 u'(0) &= 0, \\ b_1 u(\pi) + c_1 u'(\pi) &= 0, \end{aligned} \tag{b.c.}$$

where b_i, c_i are real numbers such that $\mid b_i \mid + \mid c_i \mid \neq 0$, $i = 0, 1$.

Let a be a positive continuous function on $[0, \pi]$. It is a classical result (cf. [2]) that the linear Sturm–Liouville problem

$$\begin{aligned} &\mathscr{L}u = \mu au, \qquad \text{in } (0, \pi), \\ &\text{(b.c.)} \end{aligned} \tag{2.1}$$

possesses infinitely many eigenvalues $\mu_1 < \mu_2 < \cdots < \mu_k < \cdots$, all of which are simple, and $\lim_{k \to \infty} \mu_k = +\infty$. The zeros in $[0, \pi]$ of any eigenfunction v_k corresponding to μ_k are nodes (i.e., points ξ where $v_k(\xi) = 0$, $v_k'(\xi) \neq 0$), and v_k has exactly $k - 1$ zeros in $(0, \pi)$.

Let E be the Banach space of all continuously differentiable functions on $[0, \pi]$ which satisfy the conditions (b.c.). E is equipped with its usual norm

$\| u \|_1 = \| u \|_0 + \| u' \|_0$, where $\| u \|_0 = \mathrm{Max}_{x\in[0,\pi]} | u(x)|$. S_k^+ will denote the set of functions $u \in E$ having exactly $k - 1$ zeros in $(0, \pi)$, all zeros of u in $[0, \pi]$ being nodal, and which are positive in a deleted neighborhood of 0. The sets S_k^+, $S_k^- = -S_k^+$, and $S_k = S_k^+ \cup S_k^-$ are open sets in E. In the following, we will denote by v_k^+ the unique eigenfunction of (2.1) associated to μ_k such that $v_k^+ \in S_k^+$ and $\| v_k^+ \|_1 = 1$; we also let $v_k^- = -v_k^+$.

We consider the equation

$$\begin{aligned} &\mathscr{L}u = \lambda a u + F(x, u, u', \lambda), \qquad \text{for} \quad x \in (0, \pi), \\ &(\text{b.c.}). \end{aligned} \tag{2.2}$$

We assume that the nonlinear term F has the form $F = f + g$, where f and g are continuous functions on $[0, \pi] \times \mathbb{R}^3$, satisfying the conditions:

$$\left| \frac{f(x, u, s, \lambda)}{u} \right| \leqslant M; \qquad \forall x \in [0, \pi]; \quad \forall u, s \in \mathbb{R}, \quad 0 < | u | \leqslant 1, \quad | s | \leqslant 1, \quad \text{and} \quad \forall \lambda \in \mathbb{R}, \tag{2.3}$$

where M is a positive constant;

$$g(x, u, s, \lambda) = o(| u | + | s |), \text{ near } (u, s) = (0, 0), \text{ uniformly in } x \in [0, \pi] \text{ and in } \lambda \in \Lambda, \text{ for every bounded interval } \Lambda. \tag{2.4}$$

Because of the presence of the term f, Eq. (2.2) does not in general have a linearization about $u = 0$. For this reason, the set of bifurcation points for (2.2) with respect to the line of trivial solutions need not be discrete (cf. the example at the end of this section). Therefore, to investigate the question of bifurcation for (2.2), one has to consider bifurcation from intervals rather than bifurcation points. We say that bifurcation occurs from an interval if this interval contains at least one bifurcation point. It is possible, in this framework, to extend the results of Rabinowitz to Eq. (2.2).

We denote by $\mathscr{S}$ the closure in $\mathbb{R} \times E$ of the set of nontrivial solutions of (2.2), and by $\mathscr{S}_k^\nu$ the closure in $\mathbb{R} \times E$ of the set of all solutions (λ, u) of (2.2) with $u \in S_k^\nu$ (ν denotes $+$ or $-$). Our main result for (2.2) is:

THEOREM 1. *Let $d = M/a_0$, where $a_0 = \mathrm{Min}_{x\in[0,\pi]}\, a(x)$, and let $I_k = [\mu_k - d, \mu_k + d]$, μ_k being the kth eigenvalue of* (2.1). *For every $k \in \mathbb{N}$ and $\nu = +$ or $-$, the connected component $\mathscr{D}_k^\nu$ of $\mathscr{S}_k^\nu \cup (I_k \times \{0\})$, containing $I_k \times \{0\}$ is unbounded and lies in $(\mathbb{R} \times S_k^\nu) \cup (I_k \times \{0\})$.*

We first remark that the theorem shows in particular the existence for $k \in \mathbb{N}$ and $\nu = +$ or $-$, of at least one unbounded continuum of $\mathscr{S}$, $\mathscr{C}_k^\nu$, bifurcating from $I_k \times \{0\}$, i.e., $\mathscr{C}_k^\nu \cap (I_k \times \{0\}) \neq \varnothing$, and such that $\mathscr{C}_k^\nu \subset (\mathbb{R} \times S_k^\nu) \cup (I_k \times \{0\})$. $\mathscr{C}_k^\nu \subset \mathscr{D}_k^\nu$; however, it should be noted that one does not necessarily have $\mathscr{C}_k{}^\nu = \mathscr{D}_k{}^\nu \cap \mathscr{S}$.[1] In fact, $\mathscr{D}_k{}^\nu$ is the union of all such components $\mathscr{C}_k{}^\nu$ and of $I_k \times \{0\}$.

To prove Theorem 1, we introduce the approximate equation

$$\begin{aligned} &\mathscr{L}u = \lambda a u + f(x, u \mid u \mid^\epsilon, u', \lambda) + g(x, u, u', \lambda), \qquad x \in (0, \pi), \\ &\text{(b.c.)}. \end{aligned} \tag{2.5}$$

The next lemma will provide uniform a priori bounds for the solutions of (2.5) near the trivial solutions and will also ensure that $\mathscr{S}_k^\nu \cap (\mathbb{R} \times \{0\}) \subset (I_k \times \{0\})$.

Lemma 1. *Let ϵ_n, $0 \leqslant \epsilon_n \leqslant 1$, be a sequence converging to 0. If there exists a sequence $(\lambda_n, u_n) \in \mathbb{R} \times S_k{}^\nu$ such that (λ_n, u_n) is a solution of* (2.5) *corresponding to $\epsilon = \epsilon_n$, and (λ_n, u_n) converges to $(\lambda, 0)$ in $\mathbb{R} \times E$, then $\lambda \in I_k$.*

In the proof of this lemma, we require the following very simple observation.

Lemma 2. *Let j and k be integers, $j \geqslant k \geqslant 2$. Suppose there exist two families of real numbers*

$$\xi_0 = 0 < \xi_1 < \xi_2 < \cdots < \xi_{k-1} < \xi_k = \pi,$$
$$\eta_0 = 0 < \eta_1 < \eta_2 < \cdots < \eta_{j-1} < \eta_j = \pi.$$

Then, if $\xi_1 \leqslant \eta_1$, there exist integers p and q having the same parity, $1 \leqslant p \leqslant k-1$, $1 \leqslant q \leqslant j-1$, such that $\xi_p \leqslant \eta_q < \eta_{q+1} \leqslant \xi_{p+1}$.

The proof of Lemma 2 is by induction on j. The result is obvious for $j = k = 2$. Suppose the property holds up to the order $j-1$, and consider two families as in the lemma. If $\eta_2 \leqslant \xi_2$, then the conclusion of the lemma is true with $p = q = 1$. We assume therefore $\xi_2 \leqslant \eta_2$, and we define $\hat{\xi}_0 = \hat{\eta}_0 = 0$, $\hat{\xi}_p = \xi_{p+1}$, $\hat{\eta}_q = \eta_{q+1}$ for $1 \leqslant p \leqslant k-1$, $1 \leqslant q \leqslant j-1$. Since $\hat{\xi}_1 \leqslant \hat{\eta}_1$, applying the induction hypothesis yields the result in general.

Proof of Lemma 1. Let $w_n = u_n / \| u_n \|_1$. Setting

$$g_n(x) = \frac{g(x, u_n(x), u_n{}'(x), \lambda_n)}{\| u_n \|_1}$$

and

$$f_n(x) = \frac{f(x, u_n(x) \mid u_n(x) \mid^{\epsilon_n}, u_n{}'(x), \lambda_n)}{\| u_n \|_1},$$

[1] Indeed, the intersection of $\mathscr{S}_k{}^\nu$ and $I_k \times \{0\}$ need not be connected. In a similar fashion, for the equation considered in Section 3, $\mathscr{S}$ has two distinct connected components bifurcating from $I_k \times \{0\}$, when $\lambda_k{}^+ \neq \lambda_k{}^-$; one half-line in $\{\lambda_k{}^+\} \times S_k{}^+$ and one half-line in $\{\lambda_k{}^-\} \times S_k{}^-$ (cf. Section 3).

w_n satisfies the equation

$$\begin{aligned} &\mathscr{L}w_n = \lambda_n a w_n + f_n(x) + g_n(x), \qquad x \in (0, \pi), \\ &\text{(b.c.)}. \end{aligned} \tag{2.6}$$

Since w_n is bounded in C^1, f_n is bounded in C^0, and $g_n \to 0$ in C^0, it follows from (2.6) that w_n is bounded in C^2. Therefore, by the Arzela–Ascoli theorem, we may assume that $w_n \to w$ in C^1, $\| w \|_1 = 1$. For all n, $w_n \in S_k^\nu$, hence w lies in the closure of S_k^ν. Let us prove that in fact $w \in S_k^\nu$.

If $w \notin S_k^\nu$, then w has at least one double zero in $[0, \pi]$. Hence there exists $\tau \in [0, \pi]$ such that $w_n(\tau) \to 0$ and $w_n'(\tau) \to 0$, as $n \to \infty$. We may assume that $\| u_n \|_1 \leqslant 1$ so that $| f_n(x)| \leqslant M \mid w_n(x)|$. From (2.6), one has

$$| w_n'' | \leqslant K(| w_n | + | w_n' | + \rho_n), \tag{2.7}$$

where K is a positive constant and $\rho_n = \operatorname{Max}_{x\in[0,\pi]} | g_n(x)|$, $\lim_{n\to\infty} \rho_n = 0$. Let $y_n = \binom{w_n}{w_n'}$, with the norm in $\mathbb{R}^2$ given by $| y_n | = | w_n | + | w_n' |$. From (2.7) we have

$$| y_n' | \leqslant (K + 1)(| y_n | + \rho_n). \tag{2.8}$$

Letting $\theta_n = | y_n(\tau)| + (K + 1)\, \pi \rho_n$, integration of (2.8) leads to

$$| y_n(x)| \leqslant \theta_n + (K + 1) \left| \int_\tau^x | y_n(t) | \, dt \right|. \tag{2.9}$$

Using Gronwall's inequality, we conclude from (2.9) that $| y_n(x)| \leqslant K'\theta_n$, for all $x \in [0, \pi]$, where K' denotes a positive constant. Since $\lim_{n\to\infty} \theta_n = 0$, this means that $w_n \to 0$ in C^1, which is a contradiction. Hence $w \in S_k^\nu$.

To obtain the bound on λ, we will now compare w and v_k^ν in the spirit of the Sturm comparison theorem (cf. Section 3, Lemma 3). Since $f_n \cdot w_n^{-1}$ is not known to converge, this comparison is not readily derived from Lemma 3. Nevertheless, the proof of this lemma can be adapted to the present situation. Let $[\zeta, \eta] \subset [0, \pi]$. Integrating by parts

$$\int_\zeta^\eta v_k^\nu \mathscr{L} w_n - w_n \mathscr{L} v_k^\nu,$$

and taking the limit as $n \to \infty$, one has

$$[p(w(v_k^\nu)' - v_k^\nu w')]_\zeta^\eta = \int_\zeta^\eta (\lambda - \mu_k)\, a w v_k^\nu + \lim_{n\to\infty} \int_\zeta^\eta f_n(x)\, v_k^\nu. \tag{2.10}$$

v_k^ν and w are both in S_k^ν. Thus by Lemma 2, there are two intervals (ζ_1, η_1) and (ζ_2, η_2) in $(0, \pi)$ where w and v_k^ν do not vanish and have the same sign and such that either (α) $w(\zeta_1) = w(\eta_1) = 0$, or ($\beta$) $\zeta_1 = 0$ and $w(\eta_1) = 0$, or (γ) $w(\zeta_1) = 0$ and $\eta_1 = \pi$, or (δ) $\zeta_1 = 0$ and $\eta_1 = \pi$ (this case occurs when $k = 1$), and the same for $[\zeta_2, \eta_2]$ with w replaced by v_k^ν. In all cases, one has

$$\begin{aligned} &[p(w(v_k^\nu)' - v_k^\nu w')]_{\zeta_1}^{\eta_1} \geqslant 0, \\ &[p(w(v_k^\nu)' - v_k^\nu w')]_{\zeta_2}^{\eta_2} \leqslant 0. \end{aligned} \tag{2.11}$$

Assuming $\| u_n \|_1 \leqslant 1$, $| f_n(x) | \leqslant M \mid w_n(x) |$. Hence, if w and v_k^ν have the same sign in (ζ, η), one has

$$\left| \lim_{n\to\infty} \int_\zeta^\eta f_n(x)\, v_k^\nu \right| \leqslant M \int_\zeta^\eta w v_k^\nu.$$

From (2.10) and (2.11) we obtain

$$\text{if} \quad \lambda \geqslant \mu_k, \qquad \int_{\zeta_2}^{\eta_2} [(\lambda - \mu_k)\, a_0 - M]\, w v_k^\nu \leqslant 0, \qquad \text{hence} \quad \lambda \leqslant \mu_k + d, \tag{2.12}$$

$$\text{if} \quad \lambda \leqslant \mu_k, \qquad \int_{\zeta_1}^{\eta_1} [(\lambda - \mu_k)\, a_0 + M]\, w v_k^\nu \geqslant 0, \qquad \text{hence} \quad \lambda \geqslant \mu_k - d. \tag{2.13}$$

Thus $\lambda \in I_k$.

Proof of Theorem 1. $\mathscr{D}_k^{\nu}$ is the connected component of $\mathscr{S}_k^\nu \cup (I_k \times \{0\})$ containing $I_k \times \{0\}$. Let $(\lambda, u) \in \mathscr{S}_k^\nu$ with $u \in \partial S_k^\nu$. Then u has at least one double zero in $[0, \pi]$. Since from the equation one can find a constant K such that $| u'' | \leqslant K(| u | + | u' |)$, it follows that $u = 0$. Thus $(\lambda, u) \in \mathscr{S}_k^\nu \cap (\mathbb{R} \times \{0\})$, and by Lemma 1 (taking $\epsilon_n = 0$, for all n), $\mathscr{S}_k^\nu \cap (\mathbb{R} \times \{0\}) \subset I_k \times \{0\}$. Hence $\mathscr{D}_k^\nu \subset (\mathbb{R} \times S_k^\nu) \cup (I_k \times \{0\})$.

To complete the proof, it remains to show that $\mathscr{D}_k^\nu$ is unbounded in $\mathbb{R} \times E$. Let us suppose that $\mathscr{D}_k^\nu$ is bounded. Then $\mathscr{D}_k^\nu$ is compact in $\mathbb{R} \times E$ since Eq. (2.2) shows that solutions which are bounded in $\mathbb{R} \times C^1$ are also bounded in $\mathbb{R} \times C^2$. Following [8, Lemma 1–2], we can find a neighborhood $\mathscr{O}$ of $\mathscr{D}_k^\nu$ such that $\partial\mathscr{O} \cap \mathscr{S}_k^\nu = \varnothing$. Indeed, let $\mathscr{U}$ be a uniform neighborhood of $\mathscr{D}_k^\nu$ in $\mathbb{R} \times E$. If $\partial\mathscr{U} \cap \mathscr{S}_k^\nu \neq \varnothing$, since $\bar{\mathscr{U}} \cap \mathscr{S}_k^\nu$ is compact, it is possible to find (cf. [13]) two disjoint compact subsets K_1, K_2 of $\bar{\mathscr{U}} \cap \mathscr{S}_k^\nu = K_1 \cup K_2$ such that $\partial\mathscr{U} \cap \mathscr{S}_k^\nu \subset K_2$ and $\mathscr{D}_k^\nu \subset K_1$. Define $r > 0$ to be the smallest of the distances in $\mathbb{R} \times E$ between K_1 and K_2 and between K_1 and $\partial\mathscr{U}$. Then $\mathscr{O}$, the $r/2$ uniform neighborhood of K_1 is a neighborhood of $\mathscr{D}_k^\nu$ such that $\partial\mathscr{O} \cap \mathscr{S}_k^\nu = \varnothing$. If $\partial\mathscr{U} \cap \mathscr{S}_k^\nu = \varnothing$, we just take $\mathscr{O} = \mathscr{U}$.

For $\epsilon > 0$, $f(x, u \mid u |^\epsilon, s, \lambda)$ and $g(x, u, s, \lambda)$ are $o(| u | + | s |)$ near $(u, s) = (0, 0)$ in the uniform sense of condition (2.4). The linearization of (2.5) at

$u = 0$ is given by (2.1). Hence by a theorem of Rabinowitz [7, 8], there exists an unbounded continuum of solutions of (2.5), $\mathscr{D}^{\nu}_{k,\epsilon}$ such that

$$(\mu_k, 0) \in \mathscr{D}^{\nu}_{k,\epsilon} \subset (\mathbb{R} \times S_k^{\nu}) \cup \{(\mu_k, 0)\}.$$

$\mathscr{D}^{\nu}_{k,\epsilon}$ being connected, there exists $(\lambda_\epsilon, u_\epsilon) \in \mathscr{D}^{\nu}_{k,\epsilon} \cap \partial\mathscr{O}$ for all $\epsilon > 0$. Since $\mathscr{O}$ is bounded in $\mathbb{R} \times E$, Eq. (2.5) shows that $(\lambda_\epsilon, u_\epsilon)$ is bounded in $\mathbb{R} \times C^2$ independently of ϵ. Therefore, one can find a sequence $\epsilon_n \searrow 0$ such that $(\lambda_{\epsilon_n}, u_{\epsilon_n})$ converges to a solution (λ, u) of (2.2). u lies in the closure of S_k^{ν}. But if $u \in \partial S_k^{\nu}$, then (as we have seen) $u = 0$, and by Lemma 1, $\lambda \in I_k$, which is impossible ($\mathscr{O}$ is a neighborhood of $I_k \times \{0\}$). Hence $\partial\mathscr{O} \cap \mathscr{S}_k^{\nu} \neq \varnothing$, which contradicts the assumption that $\mathscr{D}_k^{\nu}$ was bounded. Q.E.D.

An Example of Bifurcation from a Whole Interval. It is actually possible for an equation of type (2.2) to have a whole interval of bifurcation points. To illustrate this situation, consider the equation

$$\begin{aligned} -u'' &= \lambda u + u \sin(u^2 + u'^2)^{-1/2} \qquad \text{in } (0, \pi), \\ u(0) &= u(\pi) = 0. \end{aligned} \tag{2.14}$$

This equation possesses the family of solutions $(\lambda(\gamma), u(\gamma))$ where $u(\gamma)(x) = \gamma \sin x$ and $\lambda(\gamma) = 1 - \sin |\gamma|^{-1}$, $\gamma \neq 0$. It is clear from the graph of $\lambda(\gamma) = 1 - \sin |\gamma|^{-1}$, that all the points of $[0, 2] \times \{0\} = I_1 \times \{0\}$ are bifurcation points for (2.14).

Remark. Aside from the case when the equation is linearizable or half-linearizable (in the sense of Section 3), the structure of the set of bifurcation points within $I_k \times \{0\}$ is not clear. The proof of Theorem 1 remains valid if we choose to define M by

$$M = \inf_{\eta > 0} \sup_{\substack{x \in [0,\pi] \\ 0 < |u| + |s| < \eta, \lambda \in \mathbb{R}}} \left| \frac{f(x, u, s, \lambda)}{u} \right|.$$

It would be interesting to have more information about the set of bifurcation points in $I_k \times \{0\}$, e.g,. under what conditions is this set finite? Or when does it contain an interval?, etc.

3. A Particular Class of Nonlinearizable Problems: "Half-Linear" and "Half-Linearizable" Equations

Let α and β be two continuous functions on $[0, \pi]$. We consider the "half-linear"[2] problem

$$\begin{aligned} \mathscr{L}u &= \lambda a u + \alpha u^+ + \beta u^-, \qquad \text{in } (0, \pi), \\ &(\text{b.c.}), \end{aligned} \tag{3.1}$$

[2] Equation (3.1) is called half-linear because it is positively homogeneous and linear in the cones $u > 0$ and $u < 0$.

where $u^+ = \mathrm{Max}(u, 0)$, $u^- = (-u)^+$, and $\mathscr{L}$, a, (b.c.) are as in Section 2. We say that λ is a "half-eigenvalue" of (3.1) if there exists a nontrivial solution (λ, u_λ) of (3.1). In this situation, $\{(\lambda, tu_\lambda), t > 0\}$ is a half-line of nontrivial solutions of (3.1). λ is said to be simple if all solutions (λ, v) of (3.1), with v and u_λ having the same sign on a deleted neighborhood of 0, are on this half-line. There may exist another half-line of solutions $\{(\lambda, tv_\lambda), t > 0\}$, but then we say that λ is simple if v_λ and u_λ have different signs on a deleted neighborhood of 0, and all solutions (λ, v) of (3.1) lie on these two half-lines. Equation (3.1) belongs to the class of equations we have investigated in the preceding section. In the case of this equation, Theorem 1 can be improved to give the following result.

THEOREM 2. *There exist two sequences of simple half-eigenvalues for* (3.1), $\lambda_1^+ < \lambda_2^+ < \cdots < \lambda_k^+ < \cdots$, *and* $\lambda_1^- < \lambda_2^- < \cdots < \lambda_k^- < \cdots$. *The corresponding half-lines of solutions are in* $\{\lambda_k^+\} \times S_k^+$ *and* $\{\lambda_k^-\} \times S_k^-$. *Furthermore, aside from these solutions and the trivial ones, there are no other solutions of* (3.1).

By Theorem 1, we know that there exists at least one solution of (3.1), $(\lambda_k^\nu, u_k^\nu) \in \mathbb{R} \times S_k^\nu$, for every $k = 1, 2, \ldots$, $\nu = +$ and $\nu = -$. The positive homogeneity of (3.1) then implies that $\{(\lambda_k^\nu, tu_k^\nu), t > 0\}$ are half-lines of solutions in $\{\lambda_k^\nu\} \times S_k^\nu$. To prove the remaining assertions, we recall the Sturm comparison theorem.

LEMMA 3. *Let* $[\xi, \eta] \subset [0, \pi]$ *and* w_1, w_2 $(w_2 \not\equiv 0)$ *be two* C^2 *functions on* $[\xi, \eta]$ *satisfying*

$$\left.\begin{aligned} \mathscr{L}w_1 &= a_1 w_1 \\ \mathscr{L}w_2 &= a_2 w_2 \end{aligned}\right\} \quad in\ (\xi, \eta),$$

where a_1, a_2 *are continuous on* $[\xi, \eta]$ *and* $a_1 > a_2$ *on* (ξ, η). *Suppose, moreover, that either*

(i) $w_2(\xi) = w_2(\eta) = 0$, *or*

(ii) $b_0 w_i(\xi) + c_0 w_i'(\xi) = 0$, $i = 1, 2$ *and* $w_2(\eta) = 0$, *or*

(iii) $b_1 w_i(\eta) + c_1 w_i'(\eta) = 0$, $i = 1, 2$ *and* $w_2(\xi) = 0$, *or*

(iv) $b_0 w_i(\xi) + c_0 w_i'(\xi) = 0$, $i = 1, 2$, *and*
$b_1 w_i(\eta) + c_1 w_i'(\eta) = 0$, $i = 1, 2$,

where $|b_i| + |c_i| \neq 0$, $i = 0, 1$. *Then there exists* $\zeta \in (\xi, \eta)$ *such that* $w_1(\zeta) = 0$.

Proof of Lemma 3 (cf. [2]). If the conclusion does not hold, we may assume without loss of generality that $w_1 > 0$ and $w_2 > 0$ in (ξ, η). But then we have

$$\int^{\eta} w_2 \mathscr{L} w_1 - w_1 \mathscr{L} w_2 = \int_{\xi}^{\eta} (a_1 - a_2)\, w_1 w_2 > 0.$$

On the other hand

$$\int_\xi^\eta w_2 \mathscr{L} w_1 - w_1 \mathscr{L} w_2 = [p(w_1 w_2' - w_2 w_1')]_\xi^\eta .$$

The last expression is nonpositive in all the four cases, a contradiction.

Proof of Theorem 2. A nontrivial solution (λ, u) of (3.1) is such that u has only nodal zeros in $[0, \pi]$ by the uniqueness of the solution of the initial value problem (the right-hand side of (3.1) being Lipschitz continuous in u). Hence u lies in some S_k^ν. Now suppose we have two solutions (λ, u) and (μ, v) of (3.1) with $u \in S_k^\nu$ and $v \in S_k^\nu$. We may assume without loss of generality that the first zero of uv to occur in $(0, \pi]$ is a zero of u. That is, there exists $\zeta \in (0, \pi]$ such that $u(\zeta) = 0$, u and v do not vanish and have the same sign in $(0, \zeta)$. By Lemma 3 applied to u and v in $(0, \zeta)$, one has $\mu \leqslant \lambda$. On the other hand, by Lemma 2, there must exist an interval $[\xi, \eta] \subset [0, \pi]$ such that u and v do not vanish and have the same sign in (ξ, η), and either $v(\xi) = v(\eta) = 0$, or $v(\xi) = 0$ and $\eta = \pi$, or $\xi = 0$ and $\eta = \zeta = \pi$ (the latter occurring when $k = 1$). Again by Lemma 3, $\lambda \leqslant \mu$; hence $\lambda = \mu$. But then, the uniqueness in the initial value problem implies the existence of a positive constant c such that $v = cu$. Thus the λ_k are simple half-eigenvalues and aside from the trivial solutions and the half-lines $\{(\lambda_k^\nu, tu_k^\nu), t > 0\}$, there are no other solutions of (3.1).

To show that the sequences λ_k^ν, $\nu = +$ or $-$ are increasing, we observe that, given solutions (λ_k^ν, u) and (λ_j^ν, v) with $u \in S_k^\nu$, $v \in S_j^\nu$, and $k < j$, the first zero of uv to occur in $(0, \pi)$ is a zero of v. Indeed, if this were not the case, using the same argument as above, Lemma 2 (since $k < j$) and Lemma 3 would imply $\lambda_j^\nu = \lambda_k^\nu$, which is impossible, since the half-eigenvalues were shown to be simple. Therefore, by Lemma 3, $\lambda_k^\nu < \lambda_j^\nu$.

The preceding result for Eq. (3.1) leads naturally to investigation of another particular class of problems of type (2.2). We now consider equations which possess "half-linearizations" about $u = 0$. This occurs when $F(x, u, u', \lambda) = \alpha u^+ + \beta u^- + g(x, u, u', \lambda)$ and g satisfies condition (2.4) as in Section 2 (α and β are continuous functions). Then (3.1) is a "half-linearization" of

$$\begin{aligned} &\mathscr{L}u = \lambda a u + \alpha u^+ + \beta u^- + g(x, u, u', \lambda), \qquad x \in (0, \pi) \\ &(\text{b.c.}). \end{aligned} \tag{3.2}$$

The next result describes the bifurcation structure for Eq. (3.2).

THEOREM 3. *For each* $k \in \mathbb{N}$, $\nu = +$ *or* $\nu = -$, $(\lambda_k^\nu, 0)$ *is a bifurcation point for* (3, 2). *Moreover, there exists an unbounded continuum of solutions of* (3.2), $\mathscr{D}_k^\nu$ *such that* $(\lambda_k^\nu, 0) \in \mathscr{D}_k^\nu \subset (\mathbb{R} \times S_k^\nu) \cup \{(\lambda_k^\nu, 0)\}$.

To derive this result from Theorem 1, one observes that the only possible bifurcation points for (3.2) are the points $(\lambda_k^\nu, 0)$. Indeed, let (λ_n, u_n), $u_n \not\equiv 0$ be a sequence of solutions of (3.2) converging to $(\lambda, 0)$. Then dividing (3.2) by $\| u_n \|_1$, the equation shows that $u_n/\| u_n \|_1$ is bounded in C^2. Hence there exists a subsequence of $u_n/\| u_n \|_1$ converging to u in C^1 and thus also in C^2 by the equation. (λ, u) is a solution of (3.1) with $\| u \|_1 = 1$. By Theorem 2, λ must be one of the half-eigenvalues of (3.1). Furthermore if $u_n \in S_k^\nu$ for all n, then u is in the closure of S_k^ν and in fact $u \in S_k^\nu$, whence $\lambda = \lambda_k^\nu$. Denoting by $\mathscr{S}_k^\nu$ the closure in $\mathbb{R} \times E$ of the set of solutions (μ, v) of (3.2) with $v \in S_k^\nu$, we have $\mathscr{S}_k^\nu \cap (\mathbb{R} \times \{0\}) \subset \{(\lambda_k^\nu, 0)\}$. If $\tilde{\mathscr{D}}_k^\nu$ is the component given by Theorem 1, we define $\mathscr{D}_k^\nu = \tilde{\mathscr{D}}_k^\nu \cap \mathscr{S}_k^\nu$. It is then readily verified that $\mathscr{D}_k^\nu$ is an unbounded component of $\mathscr{S}_k^\nu$ and $(\lambda_k^\nu, 0) \in \mathscr{D}_k^\nu \subset (\mathbb{R} \times S_k^\nu) \cup \{(\lambda_k^\nu, 0)\}$.

4. Existence of Positive Solutions for Nonlinearizable Elliptic Partial Differential Equations

In this section we study nonlinear elliptic partial differential equations corresponding to the Sturm–Liouville problems of Section 2. We extend the result of Rabinowitz concerning the existence of a branch of positive solutions, [8, 9], to this class of nonlinearizable equations. As in [8, 9], the positivity plays here the same role as nodal properties in Section 2.

Let Ω be a bounded domain in $\mathbb{R}^N$ with a smooth boundary $\partial\Omega = \Gamma$, and let L be the divergence type differential operator in Ω defined by

$$Lu = -\sum_{i,j=1}^{N} \partial/\partial x_i(a_{ij}(x)\partial u/\partial x_j).$$

We assume that L is uniformly elliptic in $\bar{\Omega}$ and that the a_{ij} are in $C^1(\bar{\Omega})$. Let $a(x)$ be a continuous function on $\bar{\Omega}$ such that $a(x) > 0$, for all $x \in \bar{\Omega}$.

We consider the nonlinear boundary value problem

$$\begin{aligned} Lu &= \lambda a u + F(x, u, \nabla u, \lambda), \qquad \text{in } \Omega, \\ u &= 0, \qquad \text{on } \Gamma. \end{aligned} \tag{4.1}$$

Here $\nabla u = (\partial u/\partial x_1, ..., \partial u/\partial x_N)$, λ is a real parameter, and the nonlinear term F is of the form $F = f + g$, with f and g continuous functions on $\bar{\Omega} \times \mathbb{R} \times \mathbb{R}^N \times \mathbb{R}$ such that

$$\left| \frac{f(x, u, s, \lambda)}{u} \right| \leqslant M; \qquad \begin{aligned} &\forall x \in \bar{\Omega}; \quad \forall u \in \mathbb{R}, \quad 0 < | u | \leqslant 1; \\ &\forall s \in \mathbb{R}^N, \quad | s | \leqslant 1; \quad \forall \lambda \in \mathbb{R} \end{aligned} \tag{4.2}$$

(M is a positive constant);

$$g(x, u, s, \lambda) = o(|u| + |s|), \text{ near } (u, s) = (0, 0), \text{ uniformly in } x \in \bar{\Omega} \text{ and in every bounded interval of } \lambda. \tag{4.3}$$

For $k \in \mathbb{N}$, and $\alpha \in (0, 1)$, $C^{k,\alpha}(\bar{\Omega})$ denotes the Banach space of the functions in $C^k(\bar{\Omega})$ having all their derivatives of order k Hölder continuous with exponent α. $W^{k,p}(\Omega)$ is the Sobolev space of functions $u \in L^p(\Omega)$ such that $D^\beta u \in L^p(\Omega)$, $\forall \beta$, $|\beta| \leqslant k$ (multiindex notation). It is well known (cf, e.g., [6]) that, when $p > N$, there exists a constant χ such that

$$\|u\|_{C^{1,1-N/p}} \leqslant \chi \|u\|_{W^{2,p}}, \qquad \forall u \in W^{2,p}(\Omega).$$

In the following, $\alpha \in (0, 1)$ is given and p will denote a real number such that $p > N$ and $\alpha < 1 - N/p$. Thus $W^{2,p}(\Omega)$ is compactly embedded in $C^{1,\alpha}(\bar{\Omega})$.

Let $E = \{u \in C^{1,\alpha}(\bar{\Omega}); u = 0 \text{ on } \Gamma\}$. E is equipped with the usual norm $\|\cdot\|_{C^{1,\alpha}}$. A couple $(\lambda, u) \in \mathbb{R} \times E$ is said to be a solution of (4.1) if $u \in W^{2,p}(\Omega)$ and (λ, u) satisfies (4.1). We define $P^+ = \{u \in E; u > 0 \text{ in } \Omega, \text{ and } \partial u/\partial \nu < 0 \text{ on } \Gamma\}$, where $\partial u/\partial \nu$ is the outward normal derivative of u on Γ. The sets P^+, $P^- = -P^+$ and $P = P^+ \cup P^-$ are open sets in E. It is a classical consequence of a theorem of Krein–Rutman [5], that the linear eigenvalue problem

$$\begin{aligned} Lv &= \lambda a v, \qquad \text{in } \Omega, \\ v &= 0, \qquad \text{on } \Gamma \end{aligned} \tag{4.4}$$

possesses a smallest positive eigenvalue λ_1, which is simple, and such that the corresponding eigenfunctions are in P. Let v_1 be the unique such eigenfunction satisfying $\|v_1\|_{C^{1,\alpha}} = 1$ and $v_1 \in P^+$.

As in Section 2, we let $a_0 = \operatorname{Min}_{x \in \bar{\Omega}} a(x)$, $d = M/a_0$ and $I = [\lambda_1 - d, \lambda_1 + d]$. We also let $K^\nu = \overline{P^\nu} - \{0\}$, for $\nu = +$ or $-$, $\overline{P^\nu}$ being the closure of P^ν in E. The closure in $\mathbb{R} \times E$ of the set of solutions (λ, u) of (4.1) with $u \in K^\nu$ is denoted by $\mathscr{S}^\nu$.

We have the following result for Eq. (4.1).

THEOREM 4. *The connected component $\mathscr{C}^\nu$ of $\mathscr{S}^\nu \cup (I \times \{0\})$ containing $I \times \{0\}$ is unbounded and lies in $(\mathbb{R} \times K^\nu) \cup (I \times \{0\})$, for $\nu = +$ and $\nu = -$.*

To prove this theorem, we approximate (4.1) by a family of linearizable equations, as in Section 2. However, with a view to applying the result of Rabinowitz [9, 10], we need to approximate (4.1) by equations where all the coefficients and the nonlinear terms are of class C^1. In order to construct such an approximation, we will first prove the following lemma.

LEMMA 4. *There exist two families of C^1 functions on $\bar{\Omega} \times \mathbb{R} \times \mathbb{R}^N \times \mathbb{R}$, f_ϵ and g_ϵ, for $0 < \epsilon \leqslant 1$, converging to f and g, respectively, as $\epsilon \searrow 0$, uniformly on compact subsets of $\Omega \times \mathbb{R} \times \mathbb{R}^N \times \mathbb{R}$, and such that*

$$\left| \frac{f_\epsilon(x, u, s, \lambda)}{u} \right| \leqslant M; \qquad \begin{array}{l} \forall x \in \bar{\Omega}; \quad \forall u \in \mathbb{R}, \quad 0 < |u| \leqslant \frac{1}{2}; \\ \forall s \in \mathbb{R}^N, \quad |s| \leqslant \frac{1}{2}; \quad \forall \lambda \in \mathbb{R}; \\ \forall \epsilon \in (0, 1]; \end{array} \tag{4.5}$$

$$\begin{array}{l} g_\epsilon(x, u, s, \lambda) = o(|u| + |s|) \text{ near } (u, s) = (0, 0), \text{ uniformly in} \\ x \in \bar{\Omega}, \text{ in } \epsilon \in (0, 1], \text{ and in every bounded interval of } \lambda. \end{array} \tag{4.6}$$

Furthermore, f_ϵ and g_ϵ are bounded independently of ϵ on compact sets of $\bar{\Omega} \times \mathbb{R} \times \mathbb{R}^N \times \mathbb{R}$.

The proof of Lemma 4 is by regularization and truncation.

We first construct f_ϵ under the stronger assumption on f:

$$\left| \frac{f(x, u, s, \lambda)}{u} \right| \leqslant M; \qquad \begin{array}{l} \forall x \in \bar{\Omega}; \quad \forall u \in \mathbb{R}, \quad 0 \leqslant |u| \leqslant 1; \\ \forall s \in \mathbb{R}^N; \quad \forall \lambda \in \mathbb{R} \end{array} \tag{4.2$^{\text{bis}}$}$$

(i.e., condition (4.2) is satisfied for all $s \in \mathbb{R}^N$).

Define a function $\tilde{f}$ by $\tilde{f}(x, u, s, \lambda) = f(x, u, s, \lambda)/u$, if $x \in \bar{\Omega}$, $u \neq 0$, and $\tilde{f}(x, u, s, \lambda) = 0$ otherwise. Under condition (4.2$^{\text{bis}}$), $\tilde{f}$ lies in $L^\infty_{\text{loc}}(\mathbb{R}^{2N+2})$. Let ω_ϵ be a family of "mollifiers" in $\mathbb{R}^{2N+2}$. For $0 < \epsilon \leqslant 1$, ω_ϵ is a C^∞ function on $\mathbb{R}^{2N+2}$ whose support lies in the ball $\{X \in \mathbb{R}^{2N+2}; |X| \leqslant \epsilon/2\}$, $\omega_\epsilon \geqslant 0$, and such that

$$\int_{\mathbb{R}^{2N+2}} \omega_\epsilon(X)\, dX = 1.$$

Define

$$f_\epsilon(X) = u \int_{\mathbb{R}^{2N+2}} \omega_\epsilon(X - Y) \tilde{f}(Y)\, dY,$$

where $X = (x, u, s, \lambda)$ and $Y = (y, v, t, \mu)$. f_ϵ is a C^∞ function for all ϵ, $0 < \epsilon \leqslant 1$. It is easy to see that f_ϵ converges to f uniformly on compact subsets of $\Omega \times \mathbb{R} \times \mathbb{R}^N \times \mathbb{R}$. Furthermore, f_ϵ is bounded on compact sets of $\bar{\Omega} \times \mathbb{R} \times \mathbb{R}^N \times \mathbb{R}$ independently of ϵ. For $x \in \bar{\Omega}$, $|u| \leqslant \frac{1}{2}$, $s \in \mathbb{R}^n$, and $\lambda \in \mathbb{R}$, $|X - Y| \leqslant \epsilon/2 \leqslant \frac{1}{2}$ implies $|v| \leqslant 1$; hence $|\tilde{f}(Y)| \leqslant M$ and f_ϵ satisfies (4.5).

Consider now a function f as in the lemma, i.e., f satisfies condition (4.2). Let ζ be a continuous function on $\mathbb{R}^N$, $0 \leqslant \zeta \leqslant 1$, such that $\zeta(s) = 0$ for $|s| \geqslant 1$, and $\zeta(s) = 1$ for $|s| \leqslant \frac{2}{3}$. We write $f(X) = \zeta(s) f(X) + (1 - \zeta(s)) f(X)$.

The function $\zeta(s)f(X)$ satisfies condition (4.2$^{\text{bis}}$). Hence by the above construction, there exists a family of C^1 functions h_ϵ on $\bar{\Omega} \times \mathbb{R} \times \mathbb{R}^N \times \mathbb{R}$ converging to $\zeta(s)f(X)$ uniformly on compact subsets of $\Omega \times \mathbb{R} \times \mathbb{R}^N \times \mathbb{R}$. Furthermore, h_ϵ satisfies condition (4.5). Let k_ϵ be a family of C^1 functions on $\bar{\Omega} \times \mathbb{R} \times \mathbb{R}^N \times \mathbb{R}$ converging to the continuous function $(1 - \zeta(s))f(X)$, uniformly on compact subsets of $\bar{\Omega} \times \mathbb{R} \times \mathbb{R}^N \times \mathbb{R}$ and such that k_ϵ is bounded on these subsets independently of ϵ. Let $\eta(s)$ be a C^1 function on $\mathbb{R}^N$ such that $\eta(s) = 0$ for $|s| \leq \frac{1}{2}$, and $\eta(s) = 1$ for $|s| \geqslant \frac{2}{3}$. Define $f_\epsilon(X) = h_\epsilon(X) + \eta(s)\,k_\epsilon(X)$. Since $\eta(1 - \zeta) \equiv 1 - \zeta$, f_ϵ converges to f uniformly on compact subsets of $\Omega \times \mathbb{R} \times \mathbb{R}^N \times \mathbb{R}$ and is bounded on compact subsets of $\bar{\Omega} \times \mathbb{R} \times \mathbb{R}^N \times \mathbb{R}$ independently of ϵ. For $|s| \leqslant \frac{1}{2}$, $f_\epsilon(X) = h_\epsilon(X)$, hence f satisfies (4.5).

The construction of g_ϵ is also by regularization. First extend g by setting $g(X) = 0$ if $x \notin \bar{\Omega}$. Then define

$$g_\epsilon(X) = (|u|^{1+\epsilon} + |s|^{1+\epsilon}) \int_{\mathbb{R}^{2N+2}} \omega_\epsilon(X - Y) \frac{g(Y)}{|v| + |t|}\, dY.$$

It is easily seen that g satisfies the requirements of the lemma.

Proof of Theorem 4. We first observe that if a set B of solutions of (4.1) is bounded in $\mathbb{R} \times E$, then it is relatively compact in $\mathbb{R} \times E$. Indeed for $(\lambda, u) \in B$, the right-hand side of (4.1) is bounded in $C^0(\bar{\Omega})$. Using the L^p estimate (cf. [1]), we obtain a bound for u in $W^{2,p}(\Omega)$. Since $W^{2,p}(\Omega)$ is compactly embedded in $C^{1,\alpha}(\bar{\Omega})$, B is relatively compact. With the aid of this observation, the same argument as in Section 2 applies here (cf. [8, 13]). To prove the theorem, it suffices to show that for every bounded open neighborhood $\mathcal{O}$ of $I \times \{0\}$ in $\mathbb{R} \times E$, there is a solution (λ, u) of (4.1) on $\partial\mathcal{O}$ with $u \in K^\nu$. We also must show that $\mathscr{S}^\nu \subset (\mathbb{R} \times K^\nu) \cup (I \times \{0\})$. These facts will be proved by approximation.

Let f_ϵ and g_ϵ be the functions given by Lemma 4 and let L^ϵ be the differential operator defined by

$$L^\epsilon u = -\sum_{i,j=1}^{N} \partial/\partial x_i(a_{ij}^\epsilon(x)\partial u/\partial x_j),$$

where $a_{ij}^\epsilon \in C^2(\bar{\Omega})$ and a_{ij}^ϵ converge to a_{ij} in $C^1(\bar{\Omega})$ as $\epsilon \searrow 0$. We may choose a_{ij} so that L^ϵ is uniformly elliptic in $\bar{\Omega}$ with an ellipticity constant independent of $\epsilon \in [0, 1]$ (we set $L^0 \equiv L$). Let a_ϵ be a family of C^1 functions converging to a, uniformly on $\bar{\Omega}$ and such that $a_\epsilon > 0$ on $\bar{\Omega}$.

We consider the approximate problem

$$\begin{aligned} L^\epsilon u &= \lambda a_\epsilon u + f_\epsilon(x, u\,|u|^\epsilon, \nabla u, \lambda) + g_\epsilon(x, u, \nabla u, \lambda) \qquad \text{in } \Omega, \\ u &= 0 \qquad \text{on } \Gamma. \end{aligned} \tag{4.7}$$

This equation possesses the linearization near $u = 0$:

$$\begin{aligned} L^\epsilon u &= \lambda a_\epsilon u \qquad \text{in } \Omega, \\ u &= 0 \qquad \text{on } \Gamma. \end{aligned} \tag{4.8}$$

By Krein–Rutman's theorem [5], the problem (4.8) has a least eigenvalue $\lambda_{1,\epsilon} > 0$ which is simple, and a unique eigenfunction $v_{1,\epsilon}$ associated to $\lambda_{1,\epsilon}$, such that $\| v_{1,\epsilon} \|_{C^{1,\alpha}} = 1$ and $v_{1,\epsilon} \in P^+$.

Using the variational characterization of the first eigenvalue (cf. [3]),

$$\lambda_{1,\epsilon} = \operatorname*{Inf}_{u \in H_0^1(\Omega)} \left(\int_\Omega \sum_{i,j=1}^N a_{ij}^\epsilon \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j} \Big/ \int_\Omega a_\epsilon u^2 \right),$$

where $H_0^1(\Omega) = \{u \in W^{1,2}(\Omega),\, u = 0 \text{ on } \Gamma\}$, one has $\lim_{\epsilon \searrow 0} \lambda_{1,\epsilon} = \lambda_1$. By the L^p estimate, [1], $\| v_{1,\epsilon} \|_{W^{2,p}}$ is bounded. Hence, there exists a subsequence v_{1,ϵ_n} converging weakly in $W^{2,p}(\Omega)$ and strongly in $C^{1,\alpha}(\bar{\Omega})$ to v. On the other hand, for any such subsequence $v = v_1$, since v must then satisfy

$$\begin{aligned} Lv &= \lambda_1 a v \qquad \text{in } \Omega, \\ v &= 0 \qquad \text{on } \Gamma, \end{aligned} \tag{4.9}$$

and $v \in \overline{P^+}$, $\| v \|_{C^{1,\alpha}} = 1$. Therefore, as $\epsilon \searrow 0$, $v_{1,\epsilon}$ converges to v_1, weakly in $W^{2,p}(\Omega)$ and strongly in $C^{1,\alpha}(\bar{\Omega})$.

For Eq. (4.7), a result of Rabinowitz [8–10],[3] applies. There exist two unbounded connected sets of solutions of (4.7), $\mathscr{C}_\epsilon^\nu$, in $\mathbb{R} \times E$ such that

$$(\lambda_{1,\epsilon}, 0) \in \mathscr{C}_\epsilon^\nu \subset (\mathbb{R} \times P^\nu) \cup \{(\lambda_{1,\epsilon}, 0)\}.$$

Since $\mathcal{O}$ is a given bounded open neighborhood of $I \times \{0\}$ in $\mathbb{R} \times E$, for $\epsilon > 0$ small, $(\lambda_{1,\epsilon}, 0) \in \mathcal{O}$. Thus, $\mathscr{C}_\epsilon^\nu \cap \partial\mathcal{O} \neq \varnothing$. Let ϵ_n be a sequence converging to 0, and let $(\lambda_n, u_n) \in \mathscr{C}_{\epsilon_n}^\nu \cap \partial\mathcal{O}$. Since u_n is bounded in $C^{1,\alpha}(\bar{\Omega})$, the right-hand side of (4.7), where we take $\epsilon = \epsilon_n$ and $u = u_n$, is bounded in $C^0(\bar{\Omega})$ hence in $L^p(\Omega)$. The L^p estimate [1] gives then a bound in $W^{2,p}(\Omega)$ for u_n, independently of $\epsilon_n > 0$. Therefore, one can find a subsequence, denoted again by ϵ_n, $\epsilon_n \searrow 0$, such that $\lambda_n \to \lambda$, $u_n \to u$ in E and $u_n \rightharpoonup u$ in $W^{2,p}(\Omega)$. Clearly (λ, u) is a solution of (4.1) (by Lebesgue's dominated convergence theorem) and $u \in \overline{P^\nu}$. It suffices now to prove that $u \neq 0$.

[3] The result of Rabinowitz we use here was proved in [8, 9] under some additional hypotheses on the nonlinear term. However, it remains valid without these assumptions: In [10, Section VIII.7], the existence of one unbounded continuum in $\mathbb{R} \times P$ is proved in general. It is not difficult then to obtain two unbounded subcontinua, corresponding to P^+ and P^-, using [10, Lemma VIII.10] and an argument similar to [8, Corollary 2.13].

Suppose $u = 0$, and let $w_n = u_n/\| u_n \|_{C^{1,\alpha}}$,

$$\tilde{f}_n(x) = \frac{f_{\epsilon_n}(x, u_n(x) \mid u_n(x)|^{\epsilon_n}, \nabla u_n(x), \lambda_n)}{\| u_n \|_{C^{1,\alpha}}}$$

and

$$\tilde{g}_n(x) = \frac{g_{\epsilon_n}(x, u_n(x), \nabla u_n(x), \lambda_n)}{\| u_n \|_{C^{1,\alpha}}}.$$

Thus (4.7) divided by $\| u_n \|_{C^{1,\alpha}}$ gives

$$\begin{aligned} L^{\epsilon_n} w_n &= \lambda_n a_{\epsilon_n} w_n + \tilde{f}_n(x) + \tilde{g}_n(x), \qquad \text{in } \Omega, \\ w_n &= 0, \qquad \text{on } \Gamma. \end{aligned} \tag{4.10}$$

The terms $\tilde{f}_n$, $\tilde{g}_n$ are bounded in $C^0(\bar{\Omega})$ and $\| w_n \|_{C^{1,\alpha}} = 1$. Again by the L^p estimate, w_n is bounded in $W^{2,p}(\Omega)$. Hence, after extraction of a subsequence, we may assume that $w_n \to w$ in E with $\| w \|_{C^{1,\alpha}} = 1$. Since L^ϵ is self-adjoint, one has

$$\int_\Omega v_{1,\epsilon_n} L^{\epsilon_n} w_n - w_n L^{\epsilon_n} v_{1,\epsilon_n} = 0,$$

which gives

$$\int_\Omega (\lambda_n - \lambda_{1,\epsilon_n})\, a_{\epsilon_n} w_n v_{1,\epsilon_n} + \tilde{f}_n(x)\, v_{1,\epsilon_n} + \tilde{g}_n(x)\, v_{1,\epsilon_n} = 0; \tag{4.11}$$

$\tilde{g}_n$ converges to 0 in $C^0(\bar{\Omega})$. Letting $n \to \infty$, (4.11) yields

$$0 = \int_\Omega (\lambda - \lambda_1)\, a w v_1 + \lim_{n\to\infty} \int_\Omega \tilde{f}_n(x)\, v_{1,\epsilon_n}. \tag{4.12}$$

For n large enough, $\| u_n \|_{C^{1,\alpha}} \leqslant \frac{1}{2}$, so that

$$|\tilde{f}_n(x)| \leqslant M \mid w_n(x)|. \tag{4.13}$$

Using (4.13) and the fact that $w \in K^\nu$, the same discussion as in Section 2 for (4.12) (cf. (2.12) and (2.13)), shows that $\lambda \in I$. But this is a contradiction, since $(\lambda, 0) \in \partial\mathcal{O}$. Hence $u \neq 0$ and $u \in K^\nu$.

Finally, we observe that taking $\epsilon_n = 0$ for all n, in (4.10), the preceding argument also shows that $\mathcal{S}^\nu \subset (\mathbb{R} \times K^\nu) \cup (I \times \{0\})$. Q.E.D.

From Theorem 4, we see in particular that there exist two unbounded continua $\mathcal{D}^+$, $\mathcal{D}^-$, of solutions of (4.1) in $\mathbb{R} \times E$, bifurcating from $I \times \{0\}$, i.e., $\mathcal{D}^\nu \cap (I \times \{0\}) \neq \varnothing$ and $\mathcal{D}^\nu \subset \mathcal{S}^\nu$, for $\nu = +$ and $\nu = -$.

Remark. As in the result of Rabinowitz [8–10], we only obtain the existence of one branch of positive solutions and one branch of negative solutions, since we cannot exploit nodal properties. It would be of interest to know whether the results of Section 3 can be generalized to the analogous type of nonlinear eigenvalue problems for elliptic partial differential equation. For instance, does the equation

$$\begin{aligned} -\Delta u &= \lambda u + \mid u \mid, \qquad \text{in } \Omega \\ u &= 0, \qquad \text{on } \Gamma, \end{aligned} \tag{4.14}$$

possess infinitely many half-eigenvalues?

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