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Real-variables characterization of generalized Stieltjes functions

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Abstract

We obtain a characterization of generalized Stieltjes functions of any order $\lambda > 0$ in terms of inequalities for their derivatives on $(0, \infty)$. When $\lambda = 1$, this provides a new and simple proof of a characterization of Stieltjes functions first obtained by Widder in 1938.

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A real-valued function f defined on an open interval $I \subseteq \mathbb{R}$ is said to be *completely monotone* if it is C^∞ and satisfies $(-1)^n f^{(n)}(x) \geq 0$ for all $x \in I$ and all $n \geq 0$. The most important case is $I = (0, \infty)$, where the Bernstein–Hausdorff–Widder theorem [4,8,9,17,20] states that f is completely monotone on $(0, \infty)$ if and only if it can be written as the Laplace transform of a nonnegative measure supported on $[0, \infty)$, i.e.

$$f(x) = \int_{[0,\infty)} e^{-tx}\, d\mu(t) \tag{1}$$

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with $\mu \geq 0$ and the integral convergent for all $x > 0$.[2] Clearly, any such f has an analytic continuation to the right half-plane $\mathrm{Re}\, x > 0$.

A real-valued function f defined on $(0, \infty)$ is said to be a *Stieltjes function* [15] if it can be written as a nonnegative constant plus the Stieltjes transform [19,20] of a nonnegative measure supported on $[0, \infty)$, i.e.

$$f(x) = C + \int_{[0,\infty)} \frac{d\rho(t)}{x+t} \tag{2}$$

with $C \geq 0$, $\rho \geq 0$ and the integral convergent for some (hence all) $x > 0$. More information on Stieltjes functions can be found in [1, pp. 126–128; 2,3] and the references cited therein. Clearly, every Stieltjes function is completely monotone on $(0, \infty)$, but not every completely monotone function is Stieltjes. It is thus of interest to obtain a characterization of Stieltjes functions in terms of inequalities for the derivatives of f on $(0, \infty)$, analogous to but stronger than the inequalities defining complete monotonicity. Such a characterization was obtained by Widder [19] in 1938 (see also [20, Chapter VIII]), who proved (here $D = d/dx$):

Theorem 1. *Let f be a real-valued function defined on $(0, \infty)$. Then the following are equivalent*:

(a) *f is a Stieltjes function.*
(b) *f is C^∞, and the quantities*

$$F_{n,k}(x) = (-1)^n \sum_{j=0}^{k} \binom{k}{j} \frac{(n+k)!}{(n+j)!} x^j f^{(n+j)}(x) \tag{3a}$$

$$= (-1)^n x^{-n} D^k x^{n+k} D^n f(x) \tag{3b}$$

$$= (-1)^n D^{n+k} x^k f(x) \tag{3c}$$

are nonnegative for all $n, k \geq 0$ and all $x > 0$.
(c) *f is C^∞, and we have $F_{0,0}(x) \geq 0$ and $F_{k-1,k}(x) \geq 0$ for all $k \geq 1$ and all $x > 0$.*

Since $F_{n,0} = (-1)^n f^{(n)}$, condition (b) is manifestly a strengthening of complete monotonicity. The equivalence of the three formulae for $F_{n,k}$ is a straightforward computation.

From (3c) we see that the nonnegativity of $F_{n,k}$ for all $n, k \geq 0$ is equivalent to the assertion that all the functions $F_{0,k} = D^k x^k f$ are completely monotone on $(0, \infty)$.

It is fairly easy to see that (a) $\Longrightarrow$ (b), while (b) $\Longrightarrow$ (c) is trivial. Widder's proof of (c) $\Longrightarrow$ (a) was, by contrast, fairly long, and was based on explicit construction of a differential

[2] The book of Widder [20] gives several different proofs of the Bernstein–Hausdorff–Widder theorem: one based on the Hausdorff moment problem and Carlson's theorem on analytic functions (pp. 160–161); one based on the Hausdorff moment problem and its uniqueness (pp. 162–163); one based on Laguerre polynomials (pp. 168–177); and one based on a real inversion formula for the Laplace transform (pp. 310–312). See also [7, Chapter I] for a proof based on Newtonian interpolation polynomials, and [6] [12, Chapter 2] for beautiful proofs based on Choquet theory.

operator $L_{k,t}$ that provides a real inversion formula for the Stieltjes transform. Along the way he also gave [19, Lemma 12.52] a direct real-variables proof of (c) $\Longrightarrow$ (b), but he used this only for technical purposes, to guarantee the complete monotonicity and hence the real-analyticity of f on $(0, \infty)$ [19, p. 48].[3]

In addition, Widder [18, Theorem 10.1] proved, two years earlier, a slight variant of Theorem 1(a) $\Longleftrightarrow$ (b) – treating the case in which the measure μ is required to be finite – by applying the Bernstein–Hausdorff–Widder theorem to the functions $F_{0,k}$ and then analyzing the relationship between the representing measures μ_k.

In this paper I would like to give an extremely short and simple proof of Theorem 1, which moreover extends to provide a new characterization of the generalized Stieltjes functions of any order $\lambda > 0$ (see Theorem 2 below). The key idea is to use the well-known solubility conditions for the Hausdorff moment problem to prove (b) $\Longrightarrow$ (a); we then rely on [19, Lemma 12.52] for (c) $\Longrightarrow$ (b). Let us recall that a sequence $\mathbf{c} = (c_n)_{n=0}^{\infty}$ is said to be a *Hausdorff moment sequence* if there exists a finite nonnegative measure ν on $[0, 1]$ such that

$$c_n = \int_{[0,1]} t^n \, d\nu(t) \quad \text{for all } n \geq 0 \tag{4}$$

and it is said to be *completely monotone* if

$$(-1)^k (\Delta^k \mathbf{c})_n \equiv \sum_{j=0}^{k} (-1)^j \binom{k}{j} c_{n+j} \geq 0 \quad \text{for all } n, k \geq 0. \tag{5}$$

Hausdorff [8] proved in 1921 that a sequence $\mathbf{c} = (c_n)_{n=0}^{\infty}$ is a Hausdorff moment sequence if and only if it is completely monotone; furthermore, the representing measure ν is unique.[4] This is obviously a discrete analogue of the Bernstein–Hausdorff–Widder theorem.

Our method also handles, with no extra work, the generalized Stieltjes transform in which the kernel $1/(x+t)$ is replaced by $1/(x+t)^{\lambda}$ for some exponent $\lambda > 0$ [19, Section 8;13,16,5,10,11]. Let us say that a real-valued function f on $(0, \infty)$ is a *generalized Stieltjes function of order* λ (and write $f \in \mathscr{S}_{\lambda}$) if it can be written in the form

$$f(x) = C + \int_{[0,\infty)} \frac{d\rho(t)}{(x+t)^{\lambda}} \tag{6}$$

with $C \geq 0$, $\rho \geq 0$ and the integral convergent for some (hence all) $x > 0$. Since

$$\frac{1}{(x+t)^{\lambda}} = \frac{\Gamma(\lambda')}{\Gamma(\lambda)\Gamma(\lambda'-\lambda)} \int_0^{\infty} u^{\lambda'-\lambda-1} \frac{1}{(x+t+u)^{\lambda'}} \, du \tag{7}$$

whenever $\lambda < \lambda'$, it follows that $\mathscr{S}_{\lambda} \subseteq \mathscr{S}_{\lambda'}$ whenever $\lambda \leq \lambda'$. It is also suggestive that representation (6) tends formally as $\lambda \uparrow \infty$ to representation (1) characteristic of complete monotonicity, in the sense that $\lim_{\lambda \uparrow \infty} (\lambda t)^{\lambda}/(x + \lambda t)^{\lambda} = e^{-x/t}$.

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[3] See [20, Chapter VIII] for a slightly different proof that does not make use of [19, Lemma 12.52].

[4] See also [14, pp. 8–9], [20, pp. 60–61 and 100–109] or [1, pp. 74–76]. “Only if” is quite easy; proving “if” takes more work.

We shall prove the following real-variables characterization of the generalized Stieltjes functions of order λ:

Theorem 2. *Let $\lambda > 0$, and let f be a real-valued function defined on $(0, \infty)$. Then the following are equivalent*:

(a) *f is a generalized Stieltjes function of order λ.*
(b) *f is C^∞, and the quantities*

$$F^{[\lambda]}_{n,k}(x) = (-1)^n \sum_{j=0}^{k} \binom{k}{j} \frac{\Gamma(n+k+\lambda)}{\Gamma(n+j+\lambda)} x^j f^{(n+j)}(x) \tag{8a}$$

$$= (-1)^n x^{-(n+\lambda-1)} D^k x^{n+k+\lambda-1} D^n f(x) \tag{8b}$$

are nonnegative for all $n, k \geq 0$ and all $x > 0$.

When $\lambda = 1$ this reduces to Theorem 1(a,b).

Since $F^{[\lambda]}_{n,0} = (-1)^n f^{(n)}$, condition (b) is manifestly a strengthening of complete monotonicity. Furthermore, $F^{[\lambda]}_{n,k}(x)$ is a polynomial in λ of degree k, with leading coefficient

$$\lim_{\lambda\to\infty} \frac{F^{[\lambda]}_{n,k}(x)}{\lambda^k} = (-1)^n f^{(n)}(x). \tag{9}$$

So condition (b) tends formally as $\lambda \uparrow \infty$ to the definition of complete monotonicity, and Theorem 2 tends formally to the Bernstein–Hausdorff–Widder theorem. At the other extreme, we have

$$\lim_{\lambda\to 0} F^{[\lambda]}_{0,1}(x) = x f'(x), \tag{10a}$$

$$\lim_{\lambda\to 0} F^{[\lambda]}_{1,0}(x) = -f'(x) \tag{10b}$$

so that the only functions that are generalized Stieltjes of all orders $\lambda > 0$ are the nonnegative constants.

Remarks.

1. The equivalence of the two formulae for $F^{[\lambda]}_{n,k}$ in (8a)/(8b) is a straightforward computation. However, for $\lambda \neq 1$ we do not know any simple rewriting of $F^{[\lambda]}_{n,k}(x)$ analogous to the third formula (3c), nor do we know (except possibly for integer values of λ, see below) any characterization of the generalized Stieltjes functions in terms of a proper subset of the $\{F^{[\lambda]}_{n,k}\}$ analogous to Theorem 1(c). Even when $\lambda = 1$, it is an interesting open question to find other proper subsets of the $\{F^{[\lambda]}_{n,k}\}$, besides the one given in Theorem 1(c), whose nonnegativity is equivalent to that of the whole set.

2. It would also be interesting to show directly that conditions (b) get weaker as λ grows. The most obvious approach would be to write all the derivatives $(\partial^\ell/\partial\lambda^\ell)F_{n,k}^{[\lambda]}$ as nonnegative linear combinations of $\{F_{n',k'}^{[\lambda]}\}$.
3. Some of Widder's results [19, Theorems 8.2 and 8.3] may imply an alternative characterization of the generalized Stieltjes functions of order λ that generalizes that of Theorem 1(c). When λ is an integer, this characterization will apparently involve the condition that $F_{k-\lambda,k}^{[\lambda]}(x) \geq 0$ for all $k \geq \lambda$ and all $x > 0$, probably together with the nonnegativity of a few other $F_{n,k}^{[\lambda]}$ (e.g. $F_{0,0}^{[\lambda]}$). When λ is noninteger, however, this characterization will be nonlocal, involving convolution as well as differentiation.

Proof of Theorem 2. (a) $\Longrightarrow$ (b): Suppose that

$$f(x) = C + \int_{[0,\infty)} \frac{d\rho(t)}{(x+t)^\lambda} \tag{11}$$

with $C \geq 0$, $\rho \geq 0$ and $\int d\rho(t)/(1+t)^\lambda < \infty$. Then f is infinitely differentiable on $(0,\infty)$, with

$$f^{(n)}(x) = C\delta_{n,0} + (-1)^n \frac{\Gamma(n+\lambda)}{\Gamma(\lambda)} \int_{[0,\infty)} \frac{d\rho(t)}{(x+t)^{n+\lambda}} \quad \text{for all } n \geq 0. \tag{12}$$

It follows that

$$f_n^{[\lambda]}(x) \equiv (-1)^n \frac{\Gamma(\lambda)}{\Gamma(n+\lambda)} x^n f^{(n)}(x) = \int_{[0,1]} u^n \, d\nu_x(u), \tag{13}$$

where $d\nu_x(u)$ is the image of the measure $d\rho(t)/(x+t)^\lambda$ under the map $u = (1+t/x)^{-1}$ together with a point mass C at $u=0$. In other words, for each $x > 0$ the sequence $\boldsymbol{f}^{[\lambda]}(x) = (f_n^{[\lambda]}(x))_{n=0}^\infty$ is a Hausdorff moment sequence; therefore, by (the easy half of) Hausdorff's theorem, the sequence $\boldsymbol{f}^{[\lambda]}(x)$ is completely monotone, i.e. the functions

$$f_{n,k}^{[\lambda]}(x) \equiv (-1)^k [\Delta^k \boldsymbol{f}^{[\lambda]}(x)]_n = (-1)^n x^n \sum_{j=0}^{k} \binom{k}{j} \frac{\Gamma(\lambda)}{\Gamma(n+j+\lambda)} x^j f^{(n+j)}(x) \tag{14}$$

are nonnegative for all $n, k \geq 0$ and all $x > 0$. The same is therefore true of the functions

$$F_{n,k}^{[\lambda]}(x) \equiv \frac{\Gamma(n+k+\lambda)}{\Gamma(\lambda)} \frac{f_{n,k}^{[\lambda]}(x)}{x^n}. \tag{15}$$

This proves (a) $\Longrightarrow$ (b).

(b) $\Longrightarrow$ (a): Now we use the sufficiency half of Hausdorff's theorem: it follows that, for each $x > 0$, there exists a finite nonnegative measure ν_x on $[0,1]$ such that

$$(-1)^n \frac{\Gamma(\lambda)}{\Gamma(n+\lambda)} x^n f^{(n)}(x) = \int_{[0,1]} u^n \, d\nu_x(u) \quad \text{for all } n \geq 0. \tag{16}$$

Changing variables back to $t = x(u^{-1} - 1)$, we see that there exists a nonnegative measure ρ_x on $[0, \infty)$ satisfying $\int d\rho_x(t)/(x+t)^\lambda < \infty$, and a constant $C_x \geq 0$, such that

$$f^{(n)}(x) = C_x \delta_{n,0} + (-1)^n \frac{\Gamma(n+\lambda)}{\Gamma(\lambda)} \int_{[0,\infty)} \frac{d\rho_x(t)}{(x+t)^{n+\lambda}} \quad \text{for all } n \geq 0 \tag{17}$$

[namely, $d\rho_x(t) = (x+t)^\lambda d\varphi_x(\nu_x)(t)$ where $\varphi_x(u) = x(u^{-1} - 1)$, and $C_x = \nu_x(\{0\})$]. We now use the fact that (b) implies the complete monotonicity of f, hence the existence of an analytic continuation of f to the right half-plane; in particular, the Taylor series for f or any of its derivatives around the point x must have radius of convergence at least x. So let us take (17) with n replaced by $n+k$, multiply it by $\xi^k/k!$, and sum over $k \geq 0$: for $|\xi| < x$ the series is absolutely convergent, and we obtain

$$f^{(n)}(x+\xi) = C_x \delta_{n,0} + (-1)^n \frac{\Gamma(n+\lambda)}{\Gamma(\lambda)} \int_{[0,\infty)} \frac{d\rho_x(t)}{(x+\xi+t)^{n+\lambda}} \quad \text{for all } n \geq 0 \tag{18}$$

whenever $\xi \in (-x, x)$, or in other words

$$f^{(n)}(y) = C_x \delta_{n,0} + (-1)^n \frac{\Gamma(n+\lambda)}{\Gamma(\lambda)} \int_{[0,\infty)} \frac{d\rho_x(t)}{(y+t)^{n+\lambda}} \quad \text{for all } n \geq 0 \tag{19}$$

whenever $y \in (0, 2x)$, or equivalently

$$(-1)^n \frac{\Gamma(\lambda)}{\Gamma(n+\lambda)} y^n f^{(n)}(y) = \int_{[0,1]} u^n d\nu'_{x,y}(u) \quad \text{for all } n \geq 0, \tag{20}$$

where $d\nu'_{x,y}(u)$ is the image of the measure $d\rho_x(t)/(y+t)^\lambda$ under the map $u = (1+t/y)^{-1}$ together with a point mass C_x at $u = 0$. On the other hand, we already know from (16) that

$$(-1)^n \frac{\Gamma(\lambda)}{\Gamma(n+\lambda)} y^n f^{(n)}(y) = \int_{[0,1]} u^n d\nu_y(u) \quad \text{for all } n \geq 0. \tag{21}$$

Comparing (20)/(21), we see that the measures $\nu'_{x,y}$ and ν_y have the same moments whenever $0 < y < 2x$; so by the uniqueness in the Hausdorff moment problem, we conclude that $\nu'_{x,y} = \nu_y$ and hence $C_x = C_y$ and $\rho_x = \rho_y$ whenever $0 < y < 2x$. In particular, $C_x = C_y$ and $\rho_x = \rho_y$ whenever $0 < y < x$ and this implies, using the symmetry $x \leftrightarrow y$, that $C_x = C_y$ and $\rho_x = \rho_y$ for all $x, y > 0$. This proves (b) $\Longrightarrow$ (a). $\square$

Remark. Here is an alternate proof of (a) $\Longrightarrow$ (b): since [10, p. 299]

$$(-1)^n x^{-(n+\lambda-1)} \frac{d^k}{dx^k} x^{n+k+\lambda-1} \frac{d^n}{dx^n} \frac{1}{(x+t)^\lambda} = \frac{\Gamma(n+k+\lambda)}{\Gamma(\lambda)} \frac{t^k}{(x+t)^{n+k+\lambda}} \tag{22}$$

representation (11) implies that

$$F^{[\lambda]}_{n,k}(x) = \frac{\Gamma(n+k+\lambda)}{\Gamma(\lambda)} \left[C\delta_{n,0} + \int_{[0,\infty)} \frac{t^k}{(x+t)^{n+k+\lambda}} d\rho(t) \right] \geq 0. \quad \square \tag{23}$$

Let us conclude by remarking that the Stieltjes functions also have a beautiful complex-analysis characterization: a function $f : (0, \infty) \to \mathbb{R}$ is Stieltjes if and only if it is the restriction to $(0, \infty)$ of an analytic function on the cut plane $\mathbb{C}\backslash(-\infty, 0]$ satisfying $f(z) \geq 0$ for $z > 0$ and $\operatorname{Im} f(z) \leq 0$ for $\operatorname{Im} z > 0$. See e.g. [1, p. 127] or [3]. It would be interesting to know whether the generalized Stieltjes functions of order λ have an analogous complex-analysis characterization for some (or all) $\lambda \neq 1$.

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