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ON AN APPLICATION OF CONVEXITY TO DISCRETE SYSTEMS

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We prove the following result: Let A be a symmetric matrix, f be a gradient (or certain subgradient) of a convex function, and $\{y_i\}$ be a sequence defined by $y_{i+1}=f(Ay_i)$, y_0 arbitrary. Then the only possible periods of $\{y_i\}$ are 1 or 2.

Introduction

The paper is motivated by the study of discrete systems with a transition mapping $g: Q\to Q$ independent on time. Such systems appear in various applications, for a survey see [2]. We prove the following result:

Theorem 1. *Let $u: R^m\to R$ be a convex differentiable function and A be a (real) symmetric matrix of size m. Let $g: R^m\to R^m$ be a mapping defined by $g(x)=\nabla u(Ax)$, where ∇u is the gradient of u. If the sequence $y, g(y),\dots,g^k(y),\dots$ is periodical for some $y\in R^m$, then its period is either 1 or 2.*

The proof of Theorem 1 is not too complicated. We prove a slightly more general statement (Theorem 2), where the gradient mapping ∇u is replaced by the acyclic subgradient f defined below. Theorem 2 covers a lot of previous results, where the mapping f were of special types like threshold or majority functions. This result is in some sense best possible with respect to proof techniques used in [3] or [4], and provides a geometrical insight on this area.

Further, we show some applications to periodical behaviour of discrete systems. The final section deals with a generalization of cyclically monotone mappings.

1.

We briefly recall some notions of convex analysis [14]. A vector ξ is a subgradient of a convex function u in x if

$$u(y)-u(x)\geq\xi(y-x)\quad\text{for every } y.$$

Each subgradient ξ determines a supporting hyperplane $H(x,\xi)$ to the graph of u,

$$H(x,\xi)=\{(y,z): z-u(x)=\xi(y-x),\ y\in R^m,\ z\in R\}.$$

We say that $f: R^m\to R^m$ is a subgradient function of a convex function u, if $f(x)$ is a subgradient of u in x for every x.

Definition. A subgradient function f is *acyclic* if the following condition holds for every $n\in N$ and every $x_1,x_2,\ldots,x_n,x_{n+1}=x_1\in R^m$: If $(x_{i+1},u(x_{i+1}))$ lies on the hyperplane $H(x_i,f(x_i))$ for all $i=1,\ldots,n$, then $H(x_1,f(x_1))=\cdots=H(x_n,f(x_n))$. It means that all the supporting hyperplanes coincide.

Let $g: Q\to Q$ be a mapping and $y\in Q$. The trajectory $\mathrm{Traj}(y,g)$ is the sequence

$$\mathrm{Traj}(y,g)=(y,g(y),\ldots,g^k(y),\ldots)$$

where $g^{k+1}(y)=g(g^k(y))$. An integer $T>0$ is the period of $\mathrm{Traj}(y,g)$ if $g^{t+T}(y)=g^t(y)$ for some t and T is minimal with this property. Clearly, if Q is a finite set, then period T must exist, and it need not exist for Q infinite.

Theorem 2. *Let u be a convex function on R^m and A be a symmetric matrix of size m. Let f be an acyclic subgradient of u. Let T be a period of* $\mathrm{Traj}(y,fA)$ *for some $y\in R^m$. Then $T\leq 2$. Moreover, $T=1$ provided the matrix A is positive semidefinite.*

Proof. Consider $y_1,\ldots,y_T\in R^m$ such that $fA(y_t)=y_{t+1}$ for every $t=1,\ldots,T$. All indices are taken mod T. Let the expression E be given by

$$E=\sum_{t=1}^{T}((Ay_t)y_{t+1}-(Ay_t)y_{t-1}). \tag{1}$$

Then by the symmetry of A

$$E=\sum_{t=1}^{T}(y_tAy_{t+1}-y_{t-1}Ay_t)=0.$$

Substitute $x_t=Ay_t$ and $y_{t+1}=f(x_t)$ into (1). Then

$$\begin{aligned}E&=\sum_{t=1}^{T}(x_tf(x_t)-x_tf(x_{t-2}))\\&=\sum_{t=1}^{T}f(x_t)(x_t-x_{t-2})\end{aligned}$$

$$\geq \sum_{t=1}^{T} (u(x_t) - u(x_{t-2})) = 0.$$

Hence, as $E=0$, $u(x_t)-u(x_{t-2})=f(x_t)(x_t-x_{t-2})$ for $t=1,\dots,T$. By the definition of acyclic subgradient we have $f(x_1)=f(x_3)=\cdots$ and $f(x_2)=f(x_4)=\cdots$. Thus, $T\leq 2$.

Let A be positive semidefinite. We have, using $T\leq 2$,

$$\begin{aligned}0 &= (u(x_2)-u(x_1))+(u(x_1)-u(x_2))\\ &= f(x_1)(x_2-x_1)+f(x_2)(x_1-x_2)\\ &= y_2A(y_2-y_1)+y_1A(y_1-y_2)\\ &= (y_2-y_1)A(y_2-y_1).\end{aligned}$$

Then $y_2=y_1$ as A is positive semidefinite. □

Theorem 1 immediately follows from Theorem 2 as every gradient u is an acyclic subgradient.

2.

In this section we show an application to discrete systems. We derive two previous results. The applications are derived in detail in [9] and [11].

Goles and Olives [4] considered models with $f: R^m \to R^m$, $f=f_1\times\cdots\times f_m$, where each $f_i: R\to R$ was a nondecreasing function. As each such f_i is an acyclic subgradient of some convex $u_i: R\to R$, $i=1,\dots,m$, the function f is an acyclic subgradient of $u: R^m\to R$, where $u(x_1,\dots,x_m)=u_1(x_1)+\cdots+u_m(x_m)$.

Poljak and Sůra [8] considered models with $f=g\times\cdots\times g$, where $g: R^p\to R^p$ was a majority function. They defined the majority function g by $g(x_1,\dots,x_p)=(y_1,\dots,y_p)$, where

$$\begin{aligned}y_s&=1 \quad \text{if } x_s>x_r \text{ for } r<s \text{ and } x_s\geq x_r \text{ for } r>s\\ y_s&=0 \quad \text{otherwise, } s=1,\dots,p.\end{aligned}$$

Such majority function is an acyclic subgradient of the convex function $u(x_1,\dots,x_p)=\max(x_1,\dots,x_p)$. Hence, the function f is also an acyclic subgradient as it is the product of acyclic subgradients.

3.

The purpose of this section is to support our statement in the introduction that Theorem 2 is 'best possible' for Euclidean spaces. In [9] we axiomized the proof techniques of a series of papers, e.g. [3], [4] and [5]. We introduced the notions of a cyclically monotone mapping and a symmetric mapping with respect to a binary operation $*$, and proved Theorem 3. In this paper we introduce the notion of a potential which provides a dual characterization of c.m. mappings.

Definition. Let S and Q be finite or infinite sets, and $*$ be a real-valued mapping on $S\times Q$. We will write $x*y$ for $x\in S$ and $y\in Q$. A mapping $f:S\to Q$ is cyclically monotone (abbreviated as c.m.) if

$$\forall n\,\forall x_1,\dots,x_n\in S:\quad \sum_{i=1}^{n} x_i*f(x_i)\geq \sum_{i=1}^{n} x_i*f(x_{i+1}).$$

The subscripts are taken mod n. We say that a c.m. mapping $f:S\to Q$ is *strongly c.m.* if the condition

$$\sum_{i=1}^{n} x_i*f(x_i)=\sum_{i=1}^{n} x_i*f(x_{i+1}) \;\Rightarrow\; f(x_1)=\cdots=f(x_n)$$

holds for every $n\in N$.

Let us say that a mapping $a:Q\to S$ is *symmetric* if $a(x)*y=a(y)*x$ for every $x,y\in Q$.

Theorem 3 [9]. *Let S and Q be two sets and $*$ be a real-valued function on $S\times Q$. Let $f:S\to Q$ be a strongly c.m. mapping, and $a:Q\to S$ be a symmetric mapping. If* $\mathrm{Traj}(y,fa)$ *has a period T for some $y\in Q$, then $T\leq 2$.* □

Theorem 2 is a special case of Theorem 3 if the operation $*$ is the scalar product, which follows from Propositions 4 and 5 below.

We say that a mapping $u:S\to R$ is a potential of a mapping $f:S\to Q$ if

$$\forall x,y\in S:\quad u(x)-y(y)\geq x*f(y)-y*f(y).$$

Proposition 4. *A mapping $f:S\to Q$ is c.m. if and only if there exists a potential u of f.*

If $S=Q=R^m$ and $*$ is the scalar product, then Proposition 4 says that f is c.m. iff it is a subgradient function of some convex function u. This was proved by Rockafeller in [13]. In fact, Proposition 4 may be proved in the same way. Namely, for a given c.m. mapping f, a potential u may be defined as

$$u(x)=\sup\left(\sum_{i=0}^{k}(x_{i+1}*f(x_i)-x_i*f(x_i))\right),$$

where x_0 is a fixed element and supremum is taken over all finite sequences $x_0,x_1,\dots,x_{k+1}=x_0\in S$. Another proof of Proposition 4, which uses duality in the assignment problem and compactness principle, will appear in [12]. The differences between our and Rockafellar's approach are discussed in detail in [6].

Let us turn to the symmetric mappings. Clearly, if A is a symmetric matrix of size m, then the linear mapping $a(x)=Ax$ is symmetric. On the other hand the following holds.

Proposition 5. *Let Q be a subset of R^m and $a:Q\to Q$ be symmetric. Then a is linear.*

Proof. Let $x_1,\ldots,x_k$ be a base in Q (i.e. the maximum number of linearly independent vectors in Q), and let $x\in Q$. then there exist unique coefficients $c_1,\ldots,c_k\in R$ so that $a(x)=\sum_{j=1}^{k} c_j x_j$. We have $x_i a(x)=xa(x_i)$ for $i=1,\ldots,k$ by the symmetry of a. This is equivalent to

$$\sum_{j=1}^{k} c_j(x_i x_j)=xa(x_i) \quad \text{for } i=1,\ldots,k. \tag{2}$$

We can look at (2) as at a system of k linear equations for variables $c_1,\ldots,c_k$. As the matrix $Z=(z_{ij})$, $z_{ij}=x_i x_j$, is regular the system has a unique solution determined by values $a(x_1),\ldots,a(x_k)$. It follows immediately that a is a linear mapping. □

As there is an increasing amount of literature related to the topic of the paper, and some later papers may appear sooner, we would like to survey briefly our contributions.

The area was first studied by Goles and Olivos who were motivated by cellular automata and obtained a lot of results for threshold and multithreshold functions. A similar statement but for a different type of functions, was independently obtained in [8]. The immediate predecessor of this paper was [9] quoted here on several places. In the meantime after finishing the first and before writing this revised version, we considered generalizations in some further directions: The limit behaviour for systems with an infinite number of states in [7], the number of steps before a system falls into a period in [10], applications to social systems in [11], and the connection to convexity in [6].

The role of convexity in discrete systems stimulated also research of other authors, e.g. [1].

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