Characteristic cycles of local cohomology modules of monomial ideals II

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Abstract

Let $R = k[x_1, \ldots, x_n]$ be the polynomial ring in $n$ independent variables, where $k$ is a field of characteristic zero. In this work, we will describe the multiplicities of the characteristic cycle of the local cohomology modules $H^r_I(R)$ supported on a squarefree monomial ideal $I \subseteq R$ in terms of the Betti numbers of the Alexander dual ideal $I^\vee$. From this description we deduce a Gorensteinness criterion for the quotient ring $R/I$. On the other side, we give a formula for the characteristic cycle of the local cohomology modules $H^p_p(H^r_I(R))$, where $p$ is any homogeneous prime ideal of $R$. This allows us to compute the Bass numbers of $H^r_I(R)$ with respect to any prime ideal and describe its associated primes.

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1. Introduction

Let $R = k[x_1, \ldots, x_n]$ be the polynomial ring in $n$ independent variables, where $k$ is a field of characteristic zero. In [1] we studied the local cohomology modules of $R$ supported on a squarefree monomial ideal $I \subseteq R$ using the fact that they have a natural $\mathcal{D}$-module structure (see also [12,19]). More precisely, we computed the characteristic cycle of $H^r_I(R)$ and $H^m_m(H^r_I(R))$, where $m$ is the homogeneous maximal ideal of $R$. As a consequence we gave a criterion to decide when the local cohomology module $H^r_I(R)$ vanishes and to compute the cohomological dimension $\text{cd}(R, I)$ in terms of the minimal primary decomposition of the monomial ideal $I$. We also gave a
Cohen–Macaulayness criterion for the local ring $R/I$ and computed the Lyubeznik numbers $\lambda_{p,n}(R/I) = \dim_k \Ext^p_k(k,H^n_{R_p}(R))$.

Our first aim in this work is to give some new consequences of the main result in [1]. We also want to generalize the formula of the characteristic cycle of $H^n_{m}(H^r_{I}(R))$ [1, Theorem 4.4] to the case of any homogeneous prime ideal. This new formula will allow us to study the Bass numbers of local cohomology modules supported on squarefree monomial ideals and describe its associated primes.

In Section 2, we introduce the notation we will use throughout this work. In Section 3, we first recall the main result of [1], i.e. a closed formula for the characteristic cycle of any local cohomology module supported on a squarefree monomial ideal in terms of sums of face ideals appearing in the minimal primary decomposition.

Let $I^{\vee} \subseteq R$ be the Alexander dual ideal of the ideal $I \subseteq R$. The algorithm we use to compute the characteristic cycle of $H^r_{I}(R)$ may be interpreted as an algorithm that allows to minimize the free resolution of $I^{\vee}$ given by the Taylor complex. In particular, we can describe the multiplicities of the characteristic cycle in terms of the Betti numbers of $I^{\vee}$. Then, we can easily recover a fundamental result of Eagon and Reiner [8] on the Cohen–Macaulayness of $R/I$ as well, a generalization of this result given by Terai [18]. By using this description of the multiplicities we can also give a criteria for the Gorensteinness of $R/I$ in terms of the minimal primary decomposition of $I$. Moreover, we can describe the type of a Cohen–Macaulay ring $R/I$.

In Section 4, we obtain a closed formula for the characteristic cycle of the local cohomology modules $H^n_{p}(H^r_{I}(R))$, where $p$ is any homogeneous prime ideal of $R$, again in terms of the minimal primary decomposition of $I$. The ideas we use to give this formula are a natural extension of those used in [1] to compute the characteristic cycle of $H^n_{m}(H^r_{I}(R))$. From this formula we are able to compute the Bass numbers $\mu_{p}(p,H^n_{m-1}(R)) := \dim_{k(p)} \Ext^p_{k(p)}(k(p),H^n_{R_p}(R_p))$ of the local cohomology modules $H^r_{I}(R)$ with respect to any prime ideal $p \subseteq R$. In particular, we can give a description of the injective dimension of these modules as well we can describe their associated primes. This will be done in Section 5.

2. Preliminaries

Let $R = k[x_1, \ldots, x_n]$ be the polynomial ring in $n$ independent variables, where $k$ is a field of characteristic zero. In this section, we fix the notation we will use throughout this work.

A squarefree monomial in $R$ is a product $x^\alpha := x_1^{\alpha_1} \cdots x_n^{\alpha_n}$, where $\alpha \in \{0,1\}^n$. An ideal $I \subseteq R$ is said to be a squarefree monomial ideal if it may be generated by squarefree monomials. Its minimal primary decomposition is then given in terms of face ideals $I_\alpha := (x_i \mid \alpha_i \neq 0)$, where $\alpha \in \{0,1\}^n$.

Set $1 = (1, \ldots, 1) \in \{0,1\}^n$. The Alexander dual ideal of $I$ is the ideal $I^{\vee} = (x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n} \mid x_1^{1-\alpha} \not\in I)$.

The minimal primary decomposition of $I^{\vee}$ can be easily described from the ideal $I$. Namely, let $\{x^{\alpha_1}, \ldots, x^{\alpha_k}\}$ be a minimal system of generators of $I$. Then, the minimal
primary decomposition of $I^\lor$ is of the form $I^\lor = I_{x_1} \cap \cdots \cap I_{x_n}$, and we have $I^{\lor \lor} = I$.

The reader might see [16], among many others, for more information.

Let $I = I_{x_1} \cap \cdots \cap I_{x_n}$ be the minimal primary decomposition of a squarefree monomial ideal $I \subseteq R$. In our work it will be very useful to use the following poset in order to encode the information given by the minimal primary decomposition.

The poset $\mathcal{I}$: We consider $\mathcal{I} = \{I_1, I_2, \ldots, I_m\}$, where

$\mathcal{I}_1 = \{I_{x_1} \mid 1 \leq i_1 \leq m\}$,

$\mathcal{I}_2 = \{I_{x_1} + I_{x_2} \mid 1 \leq i_1 < i_2 \leq m\}$,

$\vdots$

$\mathcal{I}_m = \{I_{x_1} + I_{x_2} + \cdots + I_{x_n}\}$.

Namely, the sets $\mathcal{I}_j$ consist of the sums of $j$ face ideals in the minimal primary decomposition of $I$. Notice that the sums of face ideals in the poset are treated as different elements even if they describe the same ideal.

The Taylor complex: Let $\{x^{\beta_1}, \ldots, x^{\beta_n}\}$ be a minimal system of generators of a squarefree monomial ideal $I^\lor \subseteq R$. Let $F$ be the free $R$-module of rank $m$ generated by $e_1, \ldots, e_m$. By using the Taylor complex (see [17]) we have a resolution of the ideal $I^\lor$:

$$\mathbb{T}_*(I^\lor): 0 \rightarrow F_m \xrightarrow{d_m} \cdots \rightarrow F_1 \xrightarrow{d_1} I^\lor \rightarrow 0,$$

where $F_j = \bigwedge^j F$ is the $j$th exterior power of $F$ and the differentials $d_j$ are defined as

$$d_j(e_{i_1} \wedge \cdots \wedge e_{i_j}) = \sum_{1 \leq k \leq j} (-1)^k \frac{\text{LCM}(x^{\beta_{i_1}}, \ldots, x^{\beta_{i_j}})}{\text{LCM}(x^{\beta_{i_1}}, \ldots, x^{\beta_{i_j}})} \times e_{i_1} \wedge \cdots \wedge \hat{e}_{i_k} \wedge \cdots \wedge e_{i_j}.$$

The Taylor complex $\mathbb{T}_*(I^\lor)$ is a cellular free resolution supported on the simplex $\Delta$ whose vertices are labelled by the generators of the ideal. More precisely, it constitutes a nonaugmented oriented chain complex up to a homological shift for $\Delta$. By using Alexander duality it is easy to see that the information given by the Taylor complex is also encoded by the poset $\mathcal{I}$. To this purpose, we only have to point out that $\text{LCM}(x^{\beta_{i_1}}, \ldots, x^{\beta_{i_j}})$ corresponds to the sum of face ideals $I_{x_{i_1}} + \cdots + I_{x_{i_j}} \in \mathcal{I}_j$.

The ring of differential operators: In the sequel, $\mathcal{D}$ will denote the ring of differential operators corresponding to $R$. For details and unexplained terminology we refer to [5–7]. The ring $\mathcal{D}$ has a natural increasing filtration given by the order, such that the corresponding associated graded ring $\text{gr}(\mathcal{D})$ is isomorphic to the polynomial ring $R[\xi_1, \ldots, \xi_n]$.

Let $M$ be a finitely generated $\mathcal{D}$-module equipped with a good filtration, i.e. an increasing sequence of finitely generated $R$-submodules, such that the associated graded module $\text{gr}(M)$ is a finitely generated $\text{gr}(\mathcal{D})$-module. The characteristic ideal of $M$ is the ideal in $\text{gr}(\mathcal{D}) = R[\xi_1, \ldots, \xi_n]$ given by $J(M) := \text{rad(Ann}_{\text{gr}(\mathcal{D})}(\text{gr}(M)))$. One may prove that $J(M)$ is independent of the good filtration on $M$. The characteristic variety
of $M$ is the closed algebraic set given by
\[ C(M) := V(J(M)) \subseteq \text{Spec}(\text{gr}(\mathcal{D})) = \text{Spec}(R[\xi_1, \ldots, \xi_n]). \]

The characteristic variety allows us to describe the support of a finitely generated $\mathcal{D}$-module as an $R$-module. Let $\pi : \text{Spec}(R[\xi_1, \ldots, \xi_n]) \to \text{Spec}(R)$ be the map defined by $\pi(x, \xi) = x$. Then $\text{Supp}_R(M) = \pi(C(M))$.

The characteristic cycle of $M$ is defined as
\[ \text{CC}(M) = \sum m_i V_i, \]
where the sum is taken over all the irreducible components $V_i = V(p_i)$ of the characteristic variety $C(M)$, where $p_i \in \text{Spec}(\text{gr}(\mathcal{D}))$ and $m_i$ is the multiplicity of the module $\text{gr}(M)_{p_i}$.

The varieties that appear in the characteristic cycle can be described in terms of conormal bundles (see [14] for details). If $X_{f_{VT}} \subseteq X = \text{Spec}(R)$ is defined by the face ideal $I_{f_{VT}} \subseteq k[x_1, \ldots, x_n]$, $f_{VT} \in \{0, 1\}^n$. The conormal bundle to $X_{f_{VT}}$ in $X$ is the subvariety $T^*_X X_{f_{VT}} \subseteq \text{Spec}(R[\xi_1, \ldots, \xi_n])$ defined by the equations $\{x_i = 0 \mid f_{VT_i} = 1\}$ and $\{\xi_i = 0 \mid z_i = 0\}$.

3. The characteristic cycle of $H^*_I(R)$

Let $I = I_{x_1} \cap \cdots \cap I_{x_n}$ be the minimal primary decomposition of a squarefree monomial ideal $I \subseteq R$. In [1] we computed the characteristic cycle of $H^*_I(R)$ by using the local cohomology modules supported on sums of face ideals included in the poset $\mathcal{J} = \{\mathcal{J}_1, \mathcal{J}_2, \ldots, \mathcal{J}_m\}$. We briefly recall the main steps of the process used in that computation. First we introduced the following:

**Definition 3.1.** We say that $I_{x_1} + \cdots + I_{x_i} \in \mathcal{J}_j$ and $I_{x_1} + \cdots + I_{x_j} + I_{x_{j+1}} \in \mathcal{J}_{j+1}$ are paired if $I_{x_1} + \cdots + I_{x_j} = I_{x_1} + \cdots + I_{x_j} + I_{x_{j+1}}$.

Then, a poset of nonpaired sums of face ideals in the minimal primary decomposition of $I$ was constructed by means of the following:

**Algorithm.** Let $m$ be the number of face ideals in the minimal primary decomposition $I = I_{x_1} \cap \cdots \cap I_{x_n}$.

- For $j$ from 1 to $m - 1$, incrementing by 1.
- For all $k \leq j$ **COMPARE** the ideals $I_{x_j} + (I_{x_1} + \cdots + I_{x_k})$ and $I_{x_k} + I_{x_{j+1}} + (I_{x_1} + \cdots + I_{x_k})$.
- For all $k \leq j$ **COMPARE** the ideals $I_{x_k} + (I_{x_1} + \cdots + I_{x_k})$ and $I_{x_k} + I_{x_{j+1}} + (I_{x_1} + \cdots + I_{x_k})$.

More precisely, the poset of nonpaired sums of face ideals was obtained considering:

**Input:** The set $\mathcal{J}$ of all sums of face ideals obtained as a sum of face ideals in the minimal primary decomposition of $I$. 

We apply the algorithm where COMPARE means remove both ideals in case they are paired.

Output: The set \( \mathcal{P} \) of all nonpaired sums of face ideals obtained as a sum of face ideals in the minimal primary decomposition of \( I \).

We ordered \( \mathcal{P} \) by the number of summands \( \mathcal{P} = \{ \mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_m \} \), in such a way that no sum in \( \mathcal{P}_j \) is paired with a sum in \( \mathcal{P}_{j+1} \). Observe that some of these sets can be empty. Finally we defined the sets of nonpaired sums of face ideals with a given height

\[
\mathcal{P}_{j; r} := \{ \text{If}_{VTi_1} + \cdots + \text{If}_{VTi_j} \in \mathcal{P}_j \mid \text{ht}(\text{If}_{VTi_1} + \cdots + \text{If}_{VTi_j}) = r + (j - 1) \}.
\]

The formula given in the main result of [1] is expressed in terms of these sets of nonpaired sums of face ideals.

**Theorem 3.2** (Álvarez Montaner [1]). Let \( I \subseteq R \) be an ideal generated by squarefree monomials and let \( I = \text{If}_{VT_1} \cap \cdots \cap \text{If}_{VT_m} \) be its minimal primary decomposition. Then

\[
\text{CC}(H^r_I(R)) = \sum_{I_{x_i} \in \mathcal{P}_{j; r}} \text{CC}(H^r_{I_{x_i}}(R)) + \sum_{I_{x_i} + I_{x_j} \in \mathcal{P}_{j+1}} \text{CC}(H^r_{I_{x_i} + I_{x_j}}(R)) + \cdots
\]

\[
+ \sum_{I_{x_1} + \cdots + I_{x_m} \in \mathcal{P}_m} \text{CC}(H^r_{I_{x_1} + \cdots + I_{x_m}}(R)).
\]

### 3.1. Betti numbers and multiplicities

Our aim in this section is to explain how the computation of the characteristic cycle of the local cohomology modules \( H^r_I(R) \) is related to the process of minimizing the free resolution of the ideal \( I^\vee \) given by the Taylor complex.

In general, the free resolution of \( I^\vee \) given by the Taylor complex is not minimal. Precisely, it is not minimal when there exist \( j \) and \( k \) such that

\[
\frac{\text{LCM}(x^{a_{i_1}}, \ldots, x^{a_{i_j}})}{\text{LCM}(x^{a_{i_1}}, \ldots, x^{a_{i_k}})} = 1.
\]

By the Alexander duality correspondence this is equivalent to the equality \( I_{x_i_1} + \cdots + I_{x_i_j} = I_{x_i_1} + \cdots + \widehat{I_{x_i_k}} + \cdots + I_{x_i_j} \) of sums of face ideals in the minimal primary decomposition of \( I \), i.e. it is equivalent to say that

\( I_{x_i_1} + \cdots + I_{x_i_j} \) and \( I_{x_i_1} + \cdots + \widehat{I_{x_i_k}} + \cdots + I_{x_i_j} \) are paired ideals.

We can interpret our algorithm for cancelling paired sums of face ideals as an algorithm to minimize the free resolution of the Alexander dual ideal \( I^\vee \) given by the Taylor complex. More precisely, the minimal free resolution we obtain is labelled by the elements in the poset \( \mathcal{P} \), i.e. it is in the form

\[ F_\bullet: 0 \to F_m \xrightarrow{d_m} \cdots \to F_1 \xrightarrow{d_1} I^\vee \to 0, \]

where

\[
F_j = \bigoplus_{\{x \in \{0, 1\}^n \mid I_x \in \mathcal{P} \}} R(-x)^{d_{j+1} - \text{ht}(I_x^\vee)}.
\]
We can easily describe the Betti numbers of \( I^\vee \) from the multiplicities of the characteristic cycle of the local cohomology modules supported on \( I \). However, we should notice that the description of the differentials of this minimal free resolution is more involved than the description of the differentials of the Taylor complex.

Our main result in this section is the following:

**Proposition 3.3.** Let \( I^\vee \subseteq R \) be Alexander dual ideal of a squarefree monomial ideal \( I \subseteq R \). Then we have

\[
\beta_{j,z}(I^\vee) = m_{n-|z|+j,z}(R/I).
\]

A different approach to this result is also given in [3, Corollary 2.2].

**Remark 3.4.** Observe that the multiplicities \( m_{i,z} \) of the characteristic cycle of a fixed local cohomology module \( H^i_{\mathfrak{m}}(R) \) describe the modules and the Betti numbers of the \((n-i)\)-linear strand of \( I^\vee \). Namely, it describes the subpieces

\[
F^{(n-i)}_j := \bigoplus_{|z| = n-i+j-1} R(-z)^{\beta_{j-1,z}(I^\vee)}
\]

of the resolution \( \mathcal{F}_\bullet \).

If \( R/I \) is Cohen–Macaulay then there is only one nonvanishing local cohomology module (see [1, Proposition 3.1]), so by the previous remark, we recover the following fundamental result of Eagon and Reiner [8].

**Corollary 3.5.** Let \( I^\vee \subseteq R \) be the Alexander dual ideal of a squarefree monomial ideal \( I \subseteq R \). Then, \( R/I \) is Cohen–Macaulay if and only if \( I^\vee \) has a linear free resolution.

A generalization of that result expressed in terms of the projective dimension of \( R/I \) and the Castelnuovo–Mumford regularity of \( I^\vee \) is given by Terai in [18]. We can also give a different approach by using the previous results.

**Corollary 3.6.** Let \( I^\vee \subseteq R \) be the Alexander dual ideal of a squarefree monomial ideal \( I \subseteq R \). Then we have

\[
\text{pd}(R/I) = \text{reg}(I^\vee).
\]

**Proof.** By using Proposition 3.3 we have

\[
\text{reg}(I^\vee) := \max\{|z| - j \mid \beta_{j,z}(I^\vee) \neq 0\} = \max\{|z| - j \mid m_{n-|z|+j,z}(R/I) \neq 0\}.
\]

Then, by Álvarez Montaner [1, Corollary 3.13] we get the desired result since

\[
\text{pd}(R/I) = \text{cd}(R,I) = \max\{|z| - j \mid m_{n-|z|+j,z}(R/I) \neq 0\},
\]

where the first assertion comes from [11]. \( \square \)
3.2. Arithmetical properties

In [1] we reduce the Cohen–Macaulay property of the quotient ring \( R/I \) to a question on the vanishing of the local cohomology modules \( H^n_{-i}(R) \). Then, by using Theorem 3.2, we give new characterizations of this arithmetical property in terms of the face ideals in the minimal primary decomposition of the monomial ideal \( I \).

The characterization of the Gorenstein property of \( R/I \) is more involved and we point out that apart from the results given in Section 3.1, we will also use the fact that the local cohomology modules \( H^n_{-i}(R) \) supported on squarefree monomial ideals have a natural \( \mathbb{Z}^n \)-graded structure.

Our main result in this section is the following:

**Proposition 3.7.** Let \( I \subseteq R \) be an ideal generated by squarefree monomials and \( I = I_{x_1} \cap \cdots \cap I_{x_n} \) be its minimal primary decomposition. Then, the following are equivalent:

(i) \( R/I \) is Gorenstein such that \( x_i \) is a zero divisor in \( R/I \) for all \( i \).
(ii) \( R/I \) is Cohen–Macaulay and \( m_{n-[\text{ht} I]} = 1 \) for all \( \beta \geq x_j, j = 1, \ldots, m \).

**Remark 3.8.** The condition that \( x_i \) is a zero divisor in \( R/I \) for all \( i \) means that any variable \( x_i \) divides a generator of \( I \).

**Proof.** \( R/I \) is Gorenstein if and only if it is isomorphic to a shifting of its canonical module \( \ast\text{Ext}^R_{R}(R/I, R(-1)) \). Since both modules have a natural structure of squarefree module (see [21] for details), it will be enough to study the graded pieces of these modules in degrees \( \beta \in \{0,1\}^n \) and the morphism of multiplication by the variables \( x_i \) between these pieces. Namely, let \( \varepsilon_1, \ldots, \varepsilon_n \) be the natural basis of \( \mathbb{Z}^n \). Then, we only have to check when the following diagram is commutative for all \( i \):

\[
\begin{array}{ccc}
(R/I)_\beta & \overset{x_i}{\longrightarrow} & (R/I)_{\beta + \varepsilon_i} \\
\ast\text{Ext}^R_{R}(R/I, R(-1))_\beta & \overset{x_i}{\longrightarrow} & \ast\text{Ext}^R_{R}(R/I, R(-1))_{\beta + \varepsilon_i}
\end{array}
\]

First, we have to point out that \( R/I \) is Cohen–Macaulay if and only if the modules \( \ast\text{Ext}^R_{R}(R/I, R(-1)) \) vanish for all \( j \neq \text{ht} I \).

More precisely, by Corollary 3.5, \( R/I \) is Cohen–Macaulay if and only if \( \beta_{j, \beta}(I^\vee) = 0 \) for all \( j \neq |x| - \text{ht} I \), where \( \beta_{j, \beta}(I^\vee) \) denotes the \( j \)th Betti number of the Alexander dual ideal \( I^\vee \). On the other side, by Mustaţă [13, Corollary 3.1] we have

\[
\dim_k \ast\text{Ext}^R_{R}(R/I, R(-1))_{|x| - \beta} = \dim_k \ast\text{Ext}^R_{R}(R/I, R)_{|x| - \beta} - \beta_{j, \beta}(I^\vee)
\]

so, in order to compute the dimension of the graded pieces of \( \ast\text{Ext}^R_{R}(R/I, R(-1)) \) we have to describe the Betti numbers \( \beta_{j, \beta}(I^\vee) \).

Now, we are ready to check when the vertical arrows in the diagram are isomorphisms. First, note that the dimensions of the pieces of \( R/I \) are

\[
\dim_k (R/I)_{\beta} = 1 \quad \text{for all } \beta \leq |x| - x_j, \quad j = 1, \ldots, m.
\]
Then, by using the results of Section 3.1, we will see that the following conditions are equivalent:

(i) \( \dim_k \Ext^i_{R}(R/I, R(-1))_\beta = 1 \) for all \( \beta \leq 1 - \alpha_j \), \( j = 1, \ldots, m \),
(ii) \( m_{n-\height(\lambda, \beta)} = 1 \) for all \( \lambda \geq \alpha_j \), \( j = 1, \ldots, m \).

The proof follows from the fact that the Betti numbers that describe the dimension of the graded pieces are related to the multiplicities \( m_{\lambda, \beta} \) of the characteristic cycle of the local cohomology modules \( H^{n-i}_I(R) \) by means of Proposition 3.3. Namely, we have

\[
\beta|_{\height(\lambda, \beta)}(I) = m_{n-\height(\lambda, \beta)}.
\]

Once we have checked when the graded pieces of both modules coincide we have to study the morphism of multiplication by the variables \( x_i \) between these pieces.

The morphism of multiplication \( x_i : (R/I)_\beta \rightarrow (R/I)_{\beta + \alpha} \) between the graded pieces of the quotient ring \( R/I \) is the identity if both pieces are different from zero and \( \beta_i = 0 \).

On the other side, a topological interpretation of the multiplication by the variables \( x_i \) between the pieces of the canonical module \( \Ext^i_{R}(R/I, R(-1)) \) has been given in [13]. Namely, let \( \Delta \) be the full simplicial complex whose vertices are labelled by the minimal system of generators \( \{ x_{1}, \ldots, x_{s} \} \) of \( I \). Let

\[
T_x := \{ \sigma_{1-\beta} := \{ x_{1}, \ldots, x_{s} \} \setminus \{ x_i \mid \beta_i = 1 \} \in \Delta \mid \beta \neq \lambda \}
\]

be a simplicial subcomplex of \( \Delta \) (see also [4] for details). Then, the morphism of multiplication is determined by the relative simplicial cohomology morphism

\[ v_i : H^{r-2}(T_{\beta + \alpha}; k) \rightarrow H^{r-2}(T_{\beta}; k), \]

induced by the inclusion \( T_{\beta} \subseteq T_{\beta + \alpha} \). Namely, we have the following commutative diagram:

\[
\begin{array}{c}
\Ext^i_{R}(R/I, R(-1))_\beta \\
\cong \quad \cong \\
\Ext^i_{R}(R/I, R(-1))_{\beta + \alpha}
\end{array}
\]

\[ v_i : H^{r-2}(T_{\beta}; k) \rightarrow H^{r-2}(T_{\beta + \alpha}; k), \]

In the case we are considering, it is easy to check that this morphism is the identity if both pieces are different from zero and \( \beta_i = 0 \).

Remark 3.9. The isomorphism between the \( \mathbb{Z}^n \)-graded pieces of the quotient ring \( R/I \) and the canonical module \( \Ext^i_{R}(R/I, R(-1)) \) states that the corresponding \( \mathbb{Z}^n \)-graded Hilbert series coincide.

Stanley [15] studied the Gorenstein property by using the coarser \( \mathbb{Z} \)-graded Hilbert series. In particular, he gave the following example of a nonGorenstein ring \( R/I \) having the same Hilbert series as its canonical module:

Example 3.10. Let \( R = k[x_1, x_2, x_3, x_4] \). Consider the ideal

\[
I = (x_1, x_2) \cap (x_1, x_4) \cap (x_2, x_4) \cap (x_3, x_4).
\]
The multiplicities of the characteristic cycle of the local cohomology module $H^2_{I}(R)$ allow us to compute the $\mathbb{Z}^n$-graded Hilbert series of the canonical module by means of [3, Proposition 2.1]; [22, Proposition 2.8].

By using Theorem 3.2 we have

$$CC(H^2_{I}(R)) = T^*_{X_{(1,0,0,0)}}X + T^*_{X_{(0,0,1,0)}}X + T^*_{X_{(0,0,0,1)}}X$$
$$+ 2T^*_{X_{(1,0,1,1)}}X + T^*_{X_{(0,1,1,1)}}X + T^*_{X_{(1,1,1,1)}}X.$$ Notice that the corresponding multiplicities do not satisfy $m_{2,\pi} = 1$ for all $\pi$ bigger than the subindices corresponding to the face ideals in the minimal primary decomposition of $I$ so the $\mathbb{Z}^n$-graded Hilbert series of the quotient ring and the canonical module are different. Nevertheless, even though the quotient ring $R/I$ is not Gorenstein, the $\mathbb{Z}$-graded Hilbert series of the quotient ring and the canonical module coincide (see [15]).

Recall that the type of a Cohen–Macaulay quotient ring $R/I$ is the Bass number $r(R/I) = \mu_d(m, R/I)$. By duality (see [9, Corollary 21.16]), it is the nonzero total Betti number of highest homological degree of the ring $R/I$. We will give a description of the type by using the results that appear in Section 3.1. For this purpose let $\mathcal{P}$ (resp. $\mathcal{P}^\vee$) be the poset of nonpaired sums of face ideals obtained from the poset $\mathcal{I}$ (resp. $\mathcal{I}^\vee$) associated to the ideal $I$ (resp. $I^\vee$).

In the sequel, $CC(H^{n-i}_{I^\vee}(R)) = \sum m_{n-|\pi|+h-1,\pi}(R/I^\vee)T^*_{X_{\pi}}X$ will be the characteristic cycle of a local cohomology module $H^{n-i}_{I^\vee}(R)$, where $I^\vee$ is the Alexander dual ideal of a squarefree monomial ideal $I \subseteq R$.

**Proposition 3.11.** Let $I \subseteq R$ be a squarefree monomial ideal of height $\text{ht} I = h$, such that the quotient ring $R/I$ is Cohen–Macaulay. Then, the type of $R/I$ is

$$r(R/I) = \sum_{\{\pi | p_* \in \mathcal{P}^\vee\}} m_{n-|\pi|+h-1,\pi}(R/I^\vee).$$

**Proof.** The projective dimension of $R/I$ is $h = \text{ht} I$ so we have $r(R/I) = \beta_h(R/I)$. This total Betti number is obtained as a sum of the graded Betti numbers $\beta_{h,\pi}(R/I)$ labelled by the set $\mathcal{P}^\vee$ so we have

$$r(R/I) := \mu_d(m, R/I) = \sum_{\{\pi | p_* \in \mathcal{P}^\vee\}} \beta_{h,\pi}(R/I) = \sum_{\{\pi | p_* \in \mathcal{P}^\vee\}} \beta_{h-1,\pi}(I).$$

By Proposition 3.3 we get the desired result. 

A Cohen–Macaulay ring $R/I$ of type 1 is Gorenstein. By using the previous description, we can give another criterion for the Gorenstein property in terms of the Alexander dual ideal $I^\vee$ this time:
Proposition 3.12. Let $I \subseteq R$ be a squarefree monomial ideal of height $\text{ht} I = h$ such that the quotient ring $R/I$ is Cohen–Macaulay. Then, the following are equivalent:

(i) $R/I$ is Gorenstein.
(ii) $\sum_{\{s \mid p_s \in \mathcal{P}_\lambda\}} m_{n-|s|+h-1,s}(R/I^\lambda) = 1$.

4. The characteristic cycle of $H^p_e(H^{n-i}_I(R))$

Let $I \subseteq R$ be a squarefree monomial ideal. In [1], we computed the characteristic cycle of $H^m_p(H^n_{h_1}(R))$, where $m = (x_1, \ldots, x_n)$ is the homogeneous maximal ideal, in order to describe the Lyubeznik numbers $L_{p,s}(R/I)$ defined as the Bass numbers $\mu_p(m, H^{n-i}_I(R)) := \dim_k \operatorname{Ext}^p_k(k, H^{n-i}_I(R))$.

Our aim in this section is to compute the characteristic cycle of $H^p_e(H^n_{h_1}(R))$, where $p_s$ is any face ideal. The techniques we will use throughout this chapter are a natural continuation of those used in [1]. Namely, we will apply the long exact sequence of local cohomology to the sequences we have obtained in the process of computing $\text{CC}(H^{p-n-i}_p(R))$. Roughly speaking, we will divide the local cohomology module $H^p_e(H^n_{h_1}(R))$ into smaller pieces, the modules $H^p_e(H^{n-i}_I(R))$ that are labelled by the poset $\mathcal{P}$ occurring in Theorem 3.2.

To shed some light to the situation, we present the case with two face ideals in the minimal primary decomposition of $I$. First, we have to point out that in the case where there is only one nonvanishing local cohomology module, the spectral sequence $E_2^{p,r} = H^p_{p_s}(H^n_{h_1}(R)) \Rightarrow H^{p+r}_{p_s+i}(R)$ degenerates at the $E_2$-term so we have $H^p_{p_s}(H^n_{h_1}(R)) = H^{p+r}_{p_s+i}(R)$.

The case $m=2$: Let $I = I_{s_1} \cap I_{s_2}$ be the minimal primary decomposition of a squarefree monomial ideal. Denote $h_i := \text{ht} I_{s_i}$ for $i = 1, 2$ and $h_{12} := \text{ht}(I_{s_1} + I_{s_2})$ and suppose $h_1 \leq h_2$. Then we get the following cases:

(1) If $h_1 < h_2 < h_{12} - 1$ then
\[
\begin{align*}
\text{CC}(H^p_{p_s}(H^{h_1}_{h_1}(R))) &= \text{CC}(H^{p+h_1}_{p_s+i_{s_1}}(R)), \\
\text{CC}(H^p_{p_s}(H^{h_2}_{h_2}(R))) &= \text{CC}(H^{p+h_2}_{p_s+i_{s_2}}(R))
\end{align*}
\]
and
\[
\text{CC}(H^p_{p_s}(H^{h_{12}-1}_{h_{12}}(R))) = \text{CC}(H^{p+h_{12}}_{p_s+i_{s_1}+i_{s_2}}(R)).
\]

(2) If $h_1 = h_2 < h_{12} - 1$ then
\[
\begin{align*}
\text{CC}(H^p_{p_s}(H^{h_1}_{h_1}(R))) &= \text{CC}(H^{p+h_1}_{p_s+i_{s_1}}(R)) \oplus \text{CC}(H^{p+h_2}_{p_s+i_{s_2}}(R))
\end{align*}
\]
and
\[
\text{CC}(H^p_{p_s}(H^{h_{12}-1}_{h_{12}}(R))) = \text{CC}(H^{p+h_{12}}_{p_s+i_{s_1}+i_{s_2}}(R)).
\]
(3) If \( h_1 < h_2 = h_{12} - 1 \) then
\[
CC(H^p_{H^1_1}(H^h_{11}(R))) = CC(H^p_{p+1,11}(R))
\]
and from the exact sequence
\[
\cdots \rightarrow H^p_{p,1}(H^{h_2}_{12}(R)) \rightarrow H^p_{p,1}(H^{h_1}_{11}(R)) \rightarrow H^p_{p}(H^{h_{12}}_{11+12}(R)) \rightarrow \cdots ,
\]
we have to determine the characteristic cycle of \( H^p_{11}(H^{h_{12}}_{11}(R)) \).

Let \( Z_{p-1} \) be the kernel of the morphism \( H^p_{p,1}(H^{h_2}_{12}(R)) \rightarrow H^p_{p,1}(H^{h_1}_{11}(R)) \). Since \( CC(Z_{p-1}) \subseteq CC(H^p_{11,11}(H^{h_{12}}_{11+12}(R))) \),
\[
CC(Z_{p-1}) \subseteq CC(H^p_{p,1}(H^{h_{12}}_{11}(R))),
\]
we have to study the equality between the initial pieces \( H^p_{p,1}(H^{h_1}_{11}(R)) \) and \( H^p_{p,1}(H^{h_2}_{12}(R)) \) in order to compute these modules. Notice that this is equivalent to studying the equality between the sums of face ideals \( p_1 + I_{21} + I_{22} \) and \( p_1 + I_{22} \). We have to consider the following cases:

(i) \( p_1 + I_{21} \neq p_1 + I_{21} + I_{22} \): Let \( ht(p_1 + I_{21}) = p + h_2 \). In this case \( ht(p_1 + I_{21} + I_{22}) = p + h_{12} \) so we get the short exact sequence
\[
0 \rightarrow H^p_{p,1}(H^{h_1}_{11}(R)) \rightarrow H^p_{p,1}(H^{h_{12}}_{11+12}(R)) \rightarrow H^p_{p}(H^{h_{12}}_{11+12}(R)) \rightarrow 0.
\]

By the additivity of the characteristic cycle:
\[
CC(H^p_{p,1}(H^{h_1}_{11}(R))) = CC(H^p_{p,1}(H^{h_{12}}_{11+12}(R))) + CC(H^p_{p,1+12}(R)).
\]

(ii) \( p_1 + I_{22} = p_1 + I_{21} + I_{22} \): Let \( ht(p_1 + I_{21}) = p + h_2 \). We have an exact sequence
\[
0 \rightarrow H^p_{p,1}(H^{h_1}_{11}(R)) \rightarrow H^p_{p,1}(H^{h_{12}}_{11+12}(R)) \rightarrow H^p_{p}(H^{h_{12}}_{11+12}(R)) \rightarrow 0.
\]

We will prove that the local cohomology modules \( H^p_{p,1}(H^{h_1}_{11}(R)) \) vanish for all \( p \), so that \( Z_{p-1} = H^p_{p,1}(H^{h_{12}}_{11+12}(R)) = 0 \).

Let \( x_{i_1} \in I_{21} \) be an independent variable such that we have the equality \( I_{21} + I_{22} = I_{22} + (x_{i_1}) \). Since \( p_1 + I_{22} = p_1 + I_{21} + I_{22} \), we have \( x_{i_1} \in p_1 \). So let \( p_1 = (x_{i_1}, x_{i_2}, \ldots, x_{i_k}) \) be the face ideal.

First we will compute \( H^p_{p_1}(H^{h_1}_{11}(R)) \) by using the spectral sequence
\[
E^2_{p,r} = H^p_{p_1}(H^{h_1}_{11}(R)) \Rightarrow H^{p+r}_{I+(x_{i_1})}(R).
\]

The spectral sequence degenerates at the \( E_2 \)-term because we have
- \( H^p_{p_1}(H^{h_1}_{11}(R)) = 0 \) \( \forall p \geq 2 \).
- \( H^p_{p_1}(H^{h_1}_{11}(R)) = 0 \) \( \forall p \geq 1 \), due to the fact that \( H^{h_1}_{11}(R) \cong H^{h_1}_{11}(R) \) and \( x_{i_1} \in I_{21} \).

As a consequence we have
\[
H^p_{p_1}(H^{h_1}_{11}(R)) = H^{p+h_2}_{I+(x_{i_1})}(R) = H^{p+h_2}_{I+(x_{i_1})}(R) = 0 \quad \forall p
\]
because \( I + (x_{i_1}) = I_{21} \) and \( ht I_{21} < h_2 \).
Now by using Brodmann’s long exact sequence
\[ \cdots \to H_{(x_1, x_2)}^p(H_{I_1}^R) \to H_{(x_1)}^p(H_{I_1}^R) \to (H_{(x_1)}^p(H_{I_2}^R))_{x_2} \to \cdots, \]
we get \( H_{(x_1, x_2)}^p(H_{I_1}^R) = 0 \) \( \forall p \), and iterating this procedure on the generators of \( p \), we get \( H_{(x_1, x_2)}^p(H_{I_1}^R) = 0 \) \( \forall p \).

(4) If \( h_1 = h_2 = h_{12} - 1 \), on use of the Mayer–Vietoris sequence and as \( H_{I_{x_1} + I_{x_2}}^R(R) = H_{I_{x_1}}^R(R) = 0 \), the module \( H_{I_1}^R \) is the only nonvanishing local cohomology module. So, we have
\[ CC(H_{p_1}^p(H_{I_1}^R(R))) = CC(H_{p_1}^{p+h_1}(R)). \]

This last characteristic cycle can be computed by using the results of Section 3. Namely:

(i) If the primary decomposition \( p_\gamma + I = (p_\gamma + I_{x_1}) \cap (p_\gamma + I_{x_2}) \) is minimal then
\[ CC(H_{p_\gamma}^p(H_{I_1}^R(R))) = CC(H_{p_\gamma}^{p+h_1}(R)) + CC(H_{p_\gamma}^{p+h_2}(R)) \]
\[ + CC(H_{p_\gamma}^{p+h_12}(R)). \]

In particular, the corresponding modules \( Z_p \) vanish for all \( p \).

(ii) If the primary decomposition \( p_\gamma + I = (p_\gamma + I_{x_1}) \cap (p_\gamma + I_{x_2}) \) is not minimal then we have \( p_\gamma + I = p_\gamma + I_{x_1} \) or \( p_\gamma + I = p_\gamma + I_{x_2} \) so
\[ CC(H_{p_\gamma}^p(H_{I_1}^R(R))) = CC(H_{p_\gamma}^{p+h_1}(R)), \]
or
\[ CC(H_{p_\gamma}^p(H_{I_1}^R(R))) = CC(H_{p_\gamma}^{p+h_2}(R)), \]

In particular \( Z_p = H_{p_\gamma + I_{x_1} + I_{x_2}}(R) \neq 0 \) for \( p = h(p_\gamma + I_{x_1} + I_{x_2}) - h_{12} \).

Remark 4.1. From the previous example, we can observe that the computation of the characteristic cycle of \( H_{p_\gamma}^p(H_{I_1}^R(R)) \) is more involved than in the case where \( p_\gamma = m \) is the homogeneous maximal ideal considered in [1]. The difficulty lies in the fact that when we add the face ideal \( p_\gamma \) to different sums of face ideals in the poset \( \mathcal{P} \) we can obtain different face ideals. This contrasts with the case considered in [1] where we always obtain the homogeneous maximal ideal \( m \) when we add \( m \) to any sum in \( \mathcal{P} \).

The general case: Let \( I = I_{x_1} \cap \cdots \cap I_{x_n} \) be the minimal primary decomposition of a squarefree monomial ideal \( I \subseteq R \) and \( p_\gamma \subseteq R \) a face ideal. To compute \( CC(H_{p_\gamma}^p(H_{I}^R(R))) \) we consider the sums of face ideals in the poset \( \mathcal{P} \). For any face ideal \( I_{x} \subseteq R \) we define the subsets
\[ \mathcal{P}_{\gamma, x} := \{ I_{x_1} + \cdots + I_{x_i} \in \mathcal{P} | p_\gamma + I_{x_1} + \cdots + I_{x_i} = I_{x} \}, \]
and we introduce the following:
Definition 4.2. We say that \( I_{x_1} + \cdots + I_{x_j} \in \mathcal{P}_{j,x} \) and \( I_{x_1} + \cdots + I_{x_j} + I_{x_{j+1}} \in \mathcal{P}_{j+1,x} \) are almost paired if \( \text{ht}(I_{x_1} + \cdots + I_{x_j}) + 1 = \text{ht}(I_{x_1} + \cdots + I_{x_j} + I_{x_{j+1}}) \).

To get the formula in Theorem 4.3 below, we will consider the poset of nonpaired and nonalmost paired sums of face ideals in the minimal primary decomposition of \( I \).

Input: The sets \( \mathcal{P}_{j,x} \) of all nonpaired sums of face ideals obtained as a sum of face ideals in the minimal primary decomposition of \( I \) which give the face ideal \( I_{x} \) when added to the ideal \( p_{j} \).

We apply the algorithm where \( \text{COMPARE} \) means remove both ideals in case they are almost paired.

Output: The sets \( Q_{j,x} \) of all nonpaired and nonalmost paired sums of face ideals obtained as a sum of face ideals in the minimal primary decomposition of \( I \) which give the face ideal \( I_{x} \) when added to the ideal \( p_{j} \).

We order \( Q_{j,x} = \{ Q_{j,x,1}, Q_{j,x,2}, \ldots, Q_{j,x,m} \} \) by the number of summands, in such a way that no sum in \( Q_{j,x,j} \) is almost paired with a sum in \( Q_{j,x,j+1} \). Observe that some of the sets \( Q_{j,x,j+1} \) can be empty. Finally, we define the sets of nonpaired and nonalmost paired sums of face ideals with a given height

\[
Q_{j,x,j,r} := \{ I_{x_1} + \cdots + I_{x_j} \in Q_{j,x,j} \mid \text{ht}(I_{x_1} + \cdots + I_{x_j}) = r + (j - 1) \}.
\]

The formula we will give in Theorem 4.3 is stated in terms of these sets of nonpaired and nonalmost paired sums of face ideals.

Theorem 4.3. Let \( I = I_{x_1} \cap \cdots \cap I_{x_n} \) be the minimal primary decomposition of a square-free monomial ideal \( I \subset R \) and \( p_j \subset R \) a face ideal. Then

\[
\text{CC}(H^p_{\mathcal{P}_{j}}(H^f_j(R))) = \sum_{\lambda \in \{0,1\}^{n}} \lambda_{\tau,p,n-r} \text{CC}(H^{[\lambda]}_{\mathcal{P}_{\tau}}(R)),
\]

where \( \lambda_{\tau,p,n-r} = \# Q_{j,r,j,x} \) such that \( |\lambda| = p + (r + j - 1) \).

The following remark will be very useful for the proof of the theorem.

Remark 4.4. From the formula it is easy to see the following:

If \( \text{CC}(H^p_{\mathcal{P}_{j}}(H^{r+1,(j-1)}_{I_{x_1} + \cdots + I_{x_j}}(R))) \in \text{CC}(H^p_{\mathcal{P}_{j}}(H^f_j(R))) \), then

\[
\text{CC}(H^p_{\mathcal{P}_{j+1}}(H^{r+1,(j-1)}_{I_{x_1} + \cdots + I_{x_j}}(R))) \notin \text{CC}(H^p_{\mathcal{P}_{j+1}}(H^f_j(R))), \quad \forall q \neq p.
\]

Proof. We are going to use similar ideas as in the proof of [1, Theorem 4.4]. We proceed by induction on \( m \), the number of ideals in the minimal primary decomposition. The case \( m = 1 \) is trivial. So, let \( m > 1 \).

We start with the Mayer–Vietoris sequence

\[
\cdots \to H^U_{U+V}(R) \to H^U_{U}(R) \oplus H^V_{V}(R) \to H^U_{U \cap V}(R) \to H^{r+1}_{U+V}(R) \to \cdots,
\]
where
\[ U = I_{x_1} \cap \cdots \cap I_{x_{n-1}}, \quad U \cap V = I = I_{x_1} \cap \cdots \cap I_{x_n}, \]
\[ V = I_{x_n}, \quad U + V = (I_{x_1} \cap \cdots \cap I_{x_{n-1}}) + I_{x_n}. \]

First, we split the Mayer–Vietoris sequence into short exact sequences of kernels and cokernels:
\[ 0 \to B_r \to H^r_U(R) \oplus H^r_V(R) \to C_r \to 0, \]
\[ 0 \to C_r \to H^r_{U \cap V}(R) \to A_{r+1} \to 0, \]
\[ 0 \to A_{r+1} \to H^{r+1}_{U \cap V}(R) \rightarrow B_{r+1} \to 0. \]

Applying the long exact sequence of local cohomology we get
\[ \cdots \to H^p_p(C_r) \rightarrow H^p_p(H^r_{U \cap V}(R)) \rightarrow H^{p+1}_p(A_{r+1}) \rightarrow \cdots. \]

Assume we have proved the formula for ideals with less than \( m \) terms in the minimal primary decomposition. This allows to compute the characteristic cycles of \( H^p_p(C_r) \) and \( H^{p+1}_p(A_{r+1}) \).

To describe \( \text{CC}(H^p_p(C_r)) \) we will denote by \( \mathcal{P}(C) \), the set of face ideals that give the initial pieces which allow us to compute the characteristic cycle of \( C_r \). Applying the algorithm of cancellation to \( \mathcal{P}(C) \) we obtain the poset \( \mathcal{P}(C) := \{ \mathcal{P}_1(C), \ldots, \mathcal{P}_{m-1}(C) \} \).

Ordering the ideals by heights and checking the sums with the ideal \( p_j \), we obtain the sets
\[ \mathcal{P}_{j,r,z}(C) := \{ I_{x_{i_1}} + \cdots + I_{x_{j-1}} \in \mathcal{P}_{j,r,z}(C) \mid \text{ht}(I_{x_{i_1}} + \cdots + I_{x_{j-1}}) = r + (j - 1) \}. \]

By induction we have
\[ \text{CC}(H^p_p(C_r)) = \sum_{x \in \{0,1\}^n} \hat{\lambda}_{j,r,z}(C) \text{CC}(H^{1|z|}_x(R)), \]
where \( \hat{\lambda}_{j,r,z}(C) = \# \mathcal{P}_{j,r,z}(C) \) such that \( |z| = p + (r + (j - 1)) \).

To describe \( \text{CC}(H^p_p(A_{r+1})) \), we will denote by \( \mathcal{P}(A) \) the set of face ideals that give the initial pieces which allow us to compute the characteristic cycle of \( A_{r+1} \).

Applying the algorithm of cancellation to \( \mathcal{P}(A) \) we obtain the poset \( \mathcal{P}(A) := \{ \mathcal{P}_1(A), \ldots, \mathcal{P}_{m-1}(A) \} \).

Ordering the ideals by heights and checking the sums with the ideal \( p_j \), we obtain the sets
\[ \mathcal{P}_{j,r+1,z}(A) := \{ I_{x_{i_1}} + \cdots + I_{x_{j-1}} + I_{x_n} \in \mathcal{P}_{j,r+1,z}(A) \mid \text{ht}(I_{x_{i_1}} + \cdots + I_{x_{j-1}} + I_{x_n}) = r + 1 + (j - 1) \}. \]

By induction we have
\[ \text{CC}(H^p_p(A_{r+1})) = \sum_{x \in \{0,1\}^n} \hat{\lambda}_{j,p,n-(r+1),z}(A) \text{CC}(H^{1|z|}_x(R)), \]
where \( \hat{\lambda}_{j,p,n-(r+1),z}(A) = \# \mathcal{P}_{j,r+1,z}(A) \) such that \( |z| = p + (r + 1 + (j - 1)) \).
Now we split the long exact sequence into short exact sequences of kernels and cokernels:

\[
0 \to Z_{p-1} \to H_p^r(C_r) \to X_p \to 0,
\]

\[
0 \to X_p \to H_p^r(U \cap V(R)) \to Y_p \to 0,
\]

\[
0 \to Y_p \to H_p^r(A_{r+1}) \to Z_p \to 0.
\]

So, by additivity we have:

\[
CC(H_p^r(U \cap V(R))) = CC(X_p) + CC(Y_p)
\]

\[
= (CC(H_p^r(C_r)) - CC(Z_{p-1})) + (CC(H_p^r(A_{r+1})) - CC(Z_p)).
\]

To get the desired formula, it remains to cancel the possible almost pairs formed by ideals \(I_{z_1} + \cdots + I_{z_j} \in \mathcal{J}_{i,j,r,z}(C)\) coming from the computation of \(CC(H_p^r(C_r))\) and ideals \(I_{z_1} + \cdots + I_{z_j} + I_{z_n} \in \mathcal{J}_{i,j,r+1,z}(A)\) coming from the computation of \(CC(H_p^r(A_{r+1}))\).

It also remains to cancel the possible almost pair formed by the ideal \(I_{z_n} \in \mathcal{J}_{i,j,r+1,z}(C)\) coming from the computation of the cycles \(CC(H_p^r(C_r))\) and some \(I_{z_1} + I_{z_n} \in \mathcal{J}_{i,j,r+1,z}(A)\), coming from the computation of the cycles \(CC(H_p^r(A_{r+1}))\).

So, we only have to prove that the initial pieces coming from these almost paired ideals describe the cycles \(CC(Z_p) \forall p\). This will be done by means of the following:

Claim.

\[
CC(Z_{p-1}) = \sum CC(H_p^{r+1}(R)),
\]

where the sum is taken over the cycles that come from almost pairs of the form

- \(I_{z_1} + \cdots + I_{z_j} \in \mathcal{J}_{i,j,r,z}(C)\) and \(I_{z_1} + \cdots + I_{z_j} + I_{z_n} \in \mathcal{J}_{i,j,r+1,z}(A)\), with \(I_z = p_j + I_{z_1} + \cdots + I_{z_j} + I_{z_n}, \) and \(|z| = p + (r + (j - 1)).\)

- \(I_{z_n} \in \mathcal{J}_{i,j,r,z}(C)\) and \(I_{z_1} + I_{z_n} \in \mathcal{J}_{1,r+1,z}(A),\) with \(I_z = p_j + I_{z_n} = p_j + I_{z_1} + I_{z_n}, \) and \(|z| = p + r.\)

The inclusion \(\subseteq\) is obvious because \(CC(Z_{p-1})\) belongs to \(CC(H_p^{r+1}(A_{r+1}))\) and \(CC(H_p^r(C_r))\). To prove the opposite inclusion, let \(T_{X^*}X = CC(H_p^{r+1}(R))\) be a cycle which comes from an almost pair and suppose that this cycle is not contained in \(CC(Z_{p-1})\).

If \(T_{X^*}X = CC(H_p^r(\{H_{t_{z_1}}^{r+1}(j-1) + I_{z_1} + \cdots + I_{z_j} + I_{z_n}\})(R))) = CC(H_p^{r+1}(H_{t_{z_1}}^{r+1}(j-1) + I_{z_1} + \cdots + I_{z_j} + I_{z_n})(R)))\) is contained in \(CC(H_p^r(C_r))\) and \(CC(H_p^{r+1}(A_{r+1}))\) we set

\[
U' = I_{z_1} + \cdots + I_{z_j}, \quad U' \cap V' = (I_{z_1} + \cdots + I_{z_j}) \cap I_{z_n},
\]

\[
V' = I_{z_n}, \quad U' + V' = I_{z_1} + \cdots + I_{z_j} + I_{z_n}.
\]
If $T^*_X X = CC(H_p^p(\text{I}^m_n(R))) = CC(H_p^{p-1}(\text{I}^{m+1}_{n+1}(R)))$ is contained in $CC(H_p^p(C_r))$ and $CC(H_{i+1}^r(A_{i+1}))$ we set

$U' = I_{Z_i}, \quad U' \cap V' = I_{Z_i} \cap I_{Z_m}$,

$V' = I_{Z_m}, \quad U' + V' = I_{Z_i} + I_{Z_m}$.

Consider the Mayer–Vietoris sequence

$$
\cdots \to H_r^{U \cap V'}(R) \to H_r^U(R) \oplus H_r^V(R) \to H_r^{U \cap V'}(R) \to H_r^{U \cap V'}(R) \to \cdots.
$$

Notice that in the corresponding short exact sequences

$0 \to Z^p_{p-1} \to H^p_p(C'_r) \to X^p_p \to 0,$

$0 \to X^p_p \to H^p_p(H_{U \cap V'}(R)) \to Y^p_p \to 0,$

$0 \to Y^p_p \to H^p_p(A_{r+1}) \to Z^p_p \to 0$

we have

$T^*_X X \in CC(H^p_p(C'_r))$,

$T^*_X X \in CC(H^p_{p-1}(A_{r+1}))$,

$T^*_X X \notin CC(Z^p_{p-1}).$

The last statement is due to the fact that, if the cycle $T^*_X X$ has not been cancelled during the computation of $H^p_p(H^r_I(R))$, i.e., it does not belong to $CC(Z_p)$ then, it cannot be cancelled in the computation of $H^p_p(H^r_{U \cap V'}(R))$, i.e., it does not belong to any $CC(Z^p_p)$ of the corresponding short exact sequences.

Finally, by induction and using Remark 4.4 we get a contradiction as

$T^*_X X \in CC(H^p_{p-1}(H^r_{U \cap V'}(R)))$ and $T^*_X X \in CC(H^p_p(H^r_{U \cap V'}(R)))$.

Applying Theorem 4.3 to the case of the homogeneous maximal ideal $m \subseteq R$ we obtain the result of [1, Theorem 4.4], as for any sum of face ideals $I_{Z_i} + \cdots + I_{Z_i} \in \mathcal{P}_{j,r}$ we have $m = m + I_{Z_i} + \cdots + I_{Z_i}$ so that $\mathcal{P}_{j,r} = \mathcal{P}_{j,m} \cap \mathcal{P}_{j,r} \forall j, \forall r$.

5. Bass numbers of local cohomology modules

Let $CC(H^p_p(H^{n-i}_I(R))) = \sum \lambda_{j; p, i, z} T^*_X X$, be the characteristic cycle of the local cohomology module $H^p_p(H^{n-i}_I(R))$. It provides much information on the modules $H^p_p(H^{n-i}_I(R))$ as well on the ring $R/I$ because the multiplicities $\lambda_{j; p, i, z}$ are invariants of $R/I$ (see [2]). Nevertheless, the aim of this section is to extract information on the modules $H^{n-i}_I(R)$. Namely, we want to compute the Bass numbers of these latter modules.
First, we recall that the multiplicities $\lambda_{\gamma, p, i, z}(R/I)$ of the characteristic cycle of $H^n_p(H^{n-i}_p(R))$ are described in terms of the sets $\mathcal{D}_{\gamma, i, n-i, z}$ of nonpaired and nonalmost paired sums of $j$ face ideals obtained by means of the algorithm. Namely we have:

**Proposition 5.1.** Let $I \subseteq R$ be a squarefree monomial ideal. Let $\mathcal{D}$ be the poset of sums of face ideals obtained from the poset $\mathcal{P}$ by means of the algorithm of cancelling almost paired ideals. Then:

$$\lambda_{\gamma, p, i, z}(R/I) = \# \mathcal{D}_{\gamma, |x|+1-p-(n-i), n-i, z}.$$

This description gives an effective method to compute the Bass numbers of the local cohomology modules $H^n_{p}(H^{n-i}_p(R))$ with respect to a face ideal $p$ by means of [2, Proposition 2.1]. For completeness, we recall the precise statement.

**Proposition 5.2** (Álvarez Montaner [2]). Let $I \subseteq R$ be an ideal generated by squarefree monomials and let $p \subseteq R$ be a face ideal. If

$$\text{CC}(H^n_p(H^{n-i}_p(R))) = \sum \lambda_{\gamma, p, i, z} T_{x_{\gamma}} X$$

is the characteristic cycle of the local cohomology module $H^n_p(H^{n-i}_p(R))$, then

$$\mu_p(p, H^{n-i}_p(R)) = \lambda_{\gamma, p, i, \gamma}.$$

In order to compute the Bass numbers $\mu_p(p, H^{n-i}_p(R))$ of the local cohomology modules $H^{n-i}_p(R)$ with respect to any prime ideal $p \subseteq R$, one may use the following remark:

**Remark 5.3.** For any prime ideal $p \subseteq R$, let $p^*$ denote the largest face ideal contained in $p$. Let $d$ be the Krull dimension of $R_{p/p^*R_p}$. Then, by Goto and Watanabe [10, Theorem 1.2.3], we have

$$\mu_p(p, H^{n-i}_p(R)) = \mu_{p-d}(p^*, H^{n-i}_p(R)), \quad \forall p.$$

### 5.1. Injective dimension of local cohomology modules

The first goal of this section is to give a vanishing criterion for the Bass numbers $\mu_p(p, H^{n-i}_p(R))$ of the local cohomology modules $H^{n-i}_p(R)$. We remark that this criterion will be described in terms of the face ideals in the minimal primary decomposition of the monomial ideal $I$.

**Proposition 5.4.** Let $I \subseteq R$ be an ideal generated by squarefree monomials and let $p \subseteq R$ be a face ideal. The following are equivalent:

(i) $\mu_p(p, H^{n-i}_p(R)) \neq 0$. 

(ii) $T_X^*X \in CC(H^n_p(H^{n-i}_f(R)))$.

(iii) There exists $I_{s_1} + \cdots + I_{s_j} \in \mathcal{Z}_{\gamma,j,\gamma}$ such that $\text{ht}(I_{s_1} + \cdots + I_{s_j}) = n - i + (j - 1) = |\gamma| - p$.

Proof. By Proposition 5.2 the Bass number $\mu_p(p, H^{n-i}_f(R))$ does not vanish if and only if the corresponding face ideal $p = f \subset R$ belongs to the support of the module $H^n_p(H^{n-i}_f(R))$. Then, we conclude by Proposition 5.1. □

As a consequence, we can compute the graded injective dimension of the local cohomology modules $H^n_i(R)$.

Corollary 5.5. Let $I \subseteq R$ be an ideal generated by squarefree monomials. The graded injective dimension of $H^n_i(R)$ is

$$\ast \text{id} H^n_i(R) = \max_{\gamma, p} \{ p \mid \lambda_{\gamma,j,p,i,\gamma} \neq 0 \}. $$

We also have

$$\ast \text{id} H^n_i(R) = \max_{\gamma, j} \{|\gamma| - \text{ht}(I_{s_1} + \cdots + I_{s_j}) \mid I_{s_1} + \cdots + I_{s_j} \in \mathcal{Z}_{\gamma,j}\}. $$

Remark 5.6. The injective dimension $\text{id} H^n_i(R)$ can be computed from the graded injective dimension $\ast \text{id} H^n_i(R)$ by means of Remark 5.3.

5.2. Associated prime ideals of local cohomology modules

Bass numbers allow us to describe the associated primes of a $R$-module $M$. Namely, we have

$$\text{Ass}_R(M) := \{ p \in \text{Spec } R \mid \mu_0(p,M) \neq 0 \}. $$

Recall that the characteristic cycle of the local cohomology modules $H^n_i(R)$ describe the support of these modules (see Section 2). Now, as a consequence of Proposition 5.4, we will distinguish the associated prime ideals among all prime ideals in the support of $H^n_i(R)$.

Proposition 5.7. Let $I \subseteq R$ be an ideal generated by squarefree monomials and let $p = f \subset R$ be a face ideal. The following are equivalent:

(i) $p \in \text{Ass}_R H^n_i(R)$.

(ii) $T_X^*X \in CC(H^n_p(H^{n-i}_f(R)))$.

(iii) There exists $I_{s_1} + \cdots + I_{s_j} \in \mathcal{Z}_{\gamma,j,\gamma}$ such that $\text{ht}(I_{s_1} + \cdots + I_{s_j}) = n - i + (j - 1) = |\gamma|$. 

The minimal primes in the support of any $R$-module are associated primes, i.e. the corresponding 0th Bass number does not vanish. As a consequence of Proposition 5.2, we can compute explicitly all the Bass numbers of local cohomology modules with respect to their minimal primes.
Proposition 5.8. Let $I \subseteq R$ be an ideal generated by squarefree monomials and $\mathfrak{p}_i \subseteq R$ be a face ideal. If $\mathfrak{p}_i$ is a minimal prime in the support of the local cohomology module $H^{n-i}_I(R)$ then

$$\mu_0(\mathfrak{p}_i, H^{n-i}_I(R)) \neq 0,$$

and

$$\mu_p(\mathfrak{p}_i, H^{n-i}_I(R)) = 0 \quad \text{for all } p > 0.$$

Moreover, if $CC(H^{n-i}_I(R)) = \sum m_{i,s} T^{s}_X$ is the corresponding characteristic cycle, then:

$$\mu_0(\mathfrak{p}_i, H^{n-i}_I(R)) = m_{i,\gamma}.$$

The minimal primes of the local cohomology modules are easy to describe by the characteristic cycle. We have to point out that not all the associated primes are minimal as it can be seen in the following example where we prove the existence of embedded prime ideals.

Example 5.9. Let $R = k[x_1,x_2,x_3,x_4,x_5]$. Consider the ideal

- $I = (x_1,x_2,x_3) \cap (x_3,x_4,x_5) \cap (x_1,x_2,x_3,x_4)$.

By using Theorem 3.2 we compute the characteristic cycle of the corresponding local cohomology modules. Namely, we get

$$CC(H^2_I(R)) = T^*_X_{(1,1,0)} X + T^*_X_{(0,0,1,1)} X,$$

$$CC(H^3_I(R)) = T^*_X_{(1,1,1,0)} X + 2 T^*_X_{(1,1,1,1)} X.$$

Collecting the multiplicities by the dimension of the corresponding varieties we obtain the following triangular matrix (see [1] for details)

$$\Gamma(R/I) = \begin{pmatrix} 0 & 2 & 0 \\ 1 & 0 & \end{pmatrix}.$$

If we compute the Bass numbers with respect to the maximal ideal, i.e. the Lyubeznik numbers, we obtain the triangular matrix (see [20] for details)

$$\Lambda(R/I) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & \end{pmatrix}.$$

Then

- For the module $H^2_I(R)$ we have
  - $\text{dim} H^2_I(R) = \dim H^2_I(R) = 2$.
  - $\text{Ass}_R(H^2_I(R)) = \text{Ass}_R(H^2_I(R))$. 
For the module $H^3_I(R)$ we have
- $0 = *\text{id} H^3_I(R) < \dim H^3_I(R) = 1$.
- $\operatorname{Min}_R(H^3_I(R)) \subseteq \operatorname{Ass}_R(H^3_I(R))$.

More precisely, $H^3_I(R) \cong \bigoplus_{R} \operatorname{E}(R/(x_1,x_2,x_3,x_4)) \oplus \operatorname{E}(R/(x_1,x_2,x_3,x_4,x_5))$ so, the maximal ideal $m = (x_1,x_2,x_3,x_4,x_5)$ is an embedded associated prime ideal of this local cohomology module.

References