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Quantum b-Functions of Prehomogeneous Vector Spaces of Commutative Parabolic Type

Atsushi Kamita[1]

Department of Mathematics, Graduate School of Science,
Hiroshima University, Higashi-Hiroshima, 739-8526, Japan
E-mail: kamita@hiroshima-u.ac.jp

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We show that there exist natural q-analogues of the b-functions for the prehomogeneous vector spaces of commutative parabolic type and calculate them explicitly in each case. Our method of calculating the b-functions seems to be new even for the original case $q = 1$. © 2001 Academic Press

1. INTRODUCTION

Among prehomogeneous vector spaces those of commutative parabolic type have special features since they have additional information coming from their realization inside simple Lie algebras. In [9] we constructed a quantum analogue $A_q(V)$ of the coordinate algebra $A(V)$ for a prehomogeneous vector space (L, V) of commutative parabolic type. If (L, V) is regular, then there exists a basic relative invariant $f \in A(V)$. In this case a quantum analogue $f_q \in A_q(V)$ of f is also implicitly constructed in [9]. The aim of this paper is to give an explicit form of quantum analogue of the b-function of f.

Let ${}^t f(\partial)$ be the constant coefficient differential operator on V corresponding to the relative invariant ${}^t f$ of the dual space (L, V^*). Then the b-function $b(s)$ of f is given by ${}^t f(\partial) f^{s+1} = b(s) f^s$. See [4, 10, 15] for the explicit form of $b(s)$.

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For $g \in A_q(V)$ we can also define a (sort of q-difference) operator ${}^t g(\partial)$ by

$$\langle {}^t g(\partial)h, h' \rangle = \langle h, gh' \rangle \qquad (h, h' \in A_q(V)),$$

where $\langle\ ,\ \rangle$ is a natural nondegenerate symmetric bilinear form on $A_q(V)$ (see Section 7). We can show that there exists some $b_q(s) \in \mathbb{C}(q)[q^s]$ satisfying

$${}^t f_q(\partial) f_q^{s+1} = b_q(s) f_q^s \qquad (s \in \mathbb{Z}_{\geq 0}).$$

Our main result is the following.

THEOREM 1.1. *If we have* $b(s) = \prod_i (s + a_i)$, *then we have*

$$b_q(s) = \prod_i q_0^{s+a_i-1} [s + a_i]_{q_0} \qquad (\textit{up to a constant multiple}),$$

where $q_0 = q^2$ (*type* B, C) *or* q (*otherwise*), *and* $[n]_t = (t^n - t^{-n})/(t - t^{-1})$.

We shall prove this theorem using an induction on the rank of the corresponding simple Lie algebra. We remark that a quantum analogue of b-function for type A was already obtained in Noumi *et al.* [16] using a quantum analogue of the Capelli identity. The analogues of differential operators in [16] are different from ours (see Remark 7.4 below).

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2. PREHOMOGENEOUS VECTOR SPACES

Let G be a connected linear algebraic group over the complex number field $\mathbb{C}$. A finite dimensional G-module V is called a prehomogeneous vector space if there exists a Zariski open orbit O in V. We denote the ring of polynomial functions on V by $\mathbb{C}[V]$. A nonzero element $f \in \mathbb{C}[V]$ is called a relative invariant of a prehomogeneous vector space (G, V) if there exists a character χ of G such that $f(gv) = \chi(g)f(v)$ for any $g \in G$ and $v \in V$. Let $S_i = \{v \in V | f_i(v) = 0\}$ $(1 \leq i \leq l)$ be the one-codimensional irreducible components of $S = V \backslash O$. Then f_i $(1 \leq i \leq l)$ are algebraically independent relative invariants, and for any relative invariant f there exist $c \in \mathbb{C}$ and $m_i \in \mathbb{Z}$ such that $f = c f_1^{m_1} \cdots f_l^{m_l}$ (see Sato and Kimura [19]). These functions $f_1, \ldots, f_l$ are called basic relative invariants.

A prehomogeneous vector space is called regular if there exists a relative invariant f such that the Hessian $H_f = \det(\partial^2 f / \partial x_i \partial x_j)$ is not identically zero, where $\{x_i\}$ is a coordinate system of V. Let (G, V) be a prehomogeneous vector space with a reductive group G. Then it is regular if and only if S is a hypersurface (see [19]).

3. COMMUTATIVE PARABOLIC TYPE

Let $\mathfrak{g}$ be a simple Lie algebra over the complex number field $\mathbb{C}$ with Cartan subalgebra $\mathfrak{h}$. Let $\Delta \subset \mathfrak{h}^*$ be the root system and $W \subset GL(\mathfrak{h})$ the Weyl group. For $\alpha \in \Delta$ we denote the corresponding root space by $\mathfrak{g}_\alpha$. We denote the set of positive roots by Δ^+ and the set of simple roots by $\{\alpha_i\}_{i\in I_0}$, where I_0 is an index set. For $i \in I_0$ let $h_i \in \mathfrak{h}, \varpi_i \in \mathfrak{h}^*, s_i \in W$ be the simple coroot, the fundamental weight, and the simple reflection corresponding to i, respectively. We denote the longest element of W by w_0. Let $(\ ,\ ) : \mathfrak{g} \times \mathfrak{g} \to \mathbb{C}$ be the invariant symmetric bilinear form such that $(\alpha, \alpha) = 2$ for short roots α. For $i, j \in I_0$ we set

$$d_i = \frac{(\alpha_i, \alpha_i)}{2}, \qquad a_{ij} = \frac{2(\alpha_i, \alpha_j)}{(\alpha_i, \alpha_i)}.$$

We define the antiautomorphism $x \mapsto {}^t x$ of the enveloping algebra $U(\mathfrak{g})$ of $\mathfrak{g}$ by ${}^t x_\alpha = x_{-\alpha}$ and ${}^t h_i = h_i$, where $\{x_\alpha | \alpha \in \Delta\}$ is a Chevalley basis of $\mathfrak{g}$.

For a subset I of I_0 we set

$$\Delta_I = \Delta \cap \sum_{i\in I} \mathbb{Z}\alpha_i, \qquad \mathfrak{l}_I = \mathfrak{h} \oplus \left(\bigoplus_{\alpha\in\Delta_I} \mathfrak{g}_\alpha \right),$$

$$\mathfrak{n}_I^\pm = \bigoplus_{\alpha\in\Delta^+\setminus\Delta_I} \mathfrak{g}_{\pm\alpha}, \qquad W_I = \langle s_i \mid i \in I \rangle.$$

Let L_I be the algebraic group corresponding to $\mathfrak{l}_I$. Assume that $\mathfrak{n}_I^+ \neq 0$ and $[\mathfrak{n}_I^+, \mathfrak{n}_I^+] = 0$. Then it is known that $I = I_0 \setminus \{i_0\}$ for some $i_0 \in I_0$ and $(L_I, \mathfrak{n}_I^+)$ is a prehomogeneous vector space, which is called of commutative parabolic type. Since $\mathfrak{n}_I^-$ is identified with the dual space of $\mathfrak{n}_I^+$ via the Killing form, the symmetric algebra $S(\mathfrak{n}_I^-)$ is isomorphic to $\mathbb{C}[\mathfrak{n}_I^+]$. By the commutativity of $\mathfrak{n}_I^-$ we have $S(\mathfrak{n}_I^-) = U(\mathfrak{n}_I^-)$. Hence $\mathbb{C}[\mathfrak{n}_I^+]$ is identified with $U(\mathfrak{n}_I^-)$. Under this identification the locally finite left $U(\mathfrak{l}_I)$-module structure on $\mathbb{C}[\mathfrak{n}_I^+]$ obtained from the adjoint action of L_I on $\mathfrak{n}_I^+$ corresponds to the $\mathrm{ad}(U(\mathfrak{l}_I))$-module structure on $U(\mathfrak{n}_I^-)$. There exists finitely many L_I-orbits $C_1, C_2, \dots, C_r, C_{r+1}$ in $\mathfrak{n}_I^+$ satisfying the closure relation $\{0\} = C_1 \subset \overline{C_2} \subset \cdots \subset \overline{C_r} \subset \overline{C_{r+1}} = \mathfrak{n}_I^+$. In the remainder of this paper we denote by r the number of nonopen orbits in $\mathfrak{n}_I^+$. For $p \leq r$ we set $\mathcal{I}(\overline{C_p}) = \{f \in \mathbb{C}[\mathfrak{n}_I^+] \mid f(\overline{C_p}) = 0\}$. We denote by $\mathcal{I}^m(\overline{C_p})$ the subspace of $\mathcal{I}(\overline{C_p})$ consisting of homogeneous elements with degree m. It is known that $\mathcal{I}^p(\overline{C_p})$ is an irreducible $\mathfrak{l}_I$-module and $\mathcal{I}(\overline{C_p}) = \mathbb{C}[\mathfrak{n}_I^+]\mathcal{I}^p(\overline{C_p})$. Let f_p be the highest weight vector of $\mathcal{I}^p(\overline{C_p})$, and let λ_p be the weight of f_p. We have the irreducible decomposition

$$\mathbb{C}[\mathfrak{n}_I^+] = \bigoplus_{\mu\in\sum_{p=1}^r \mathbb{Z}_{\geq 0}\lambda_p} V(\mu),$$

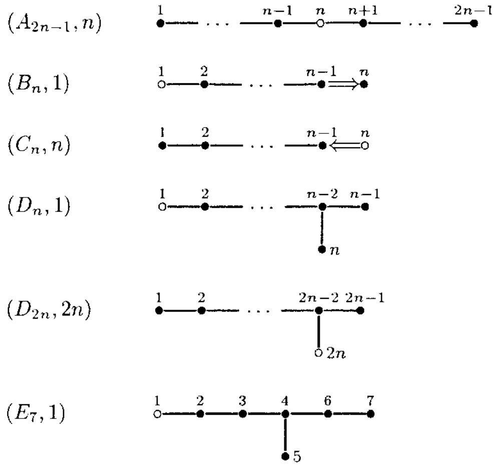

FIG. 1.

where $V(\mu)$ is an irreducible highest weight module with highest weight μ and $V(\lambda_p) = \mathscr{I}^p(\overline{C_p})$ (see Schmid [20] and Wachi [24]).

If the prehomogeneous vector space $(L_I, \mathfrak{n}_I^+)$ is regular, there exists a one-codimensional orbit C_r. Then it is known that $\mathscr{I}^r(\overline{C_r}) = \mathbb{C}f_r$, f_r is the basic relative invariant of $(L_I, \mathfrak{n}_I^+)$ and $\lambda_r = -2\varpi_{i_0}$, where $I = I_0 \backslash \{i_0\}$. The pairs $(\mathfrak{g}, i_0)$ where $(L_I, \mathfrak{n}_I^+)$ are regular are given by the Dynkin diagrams of Fig. 1. Here the white vertex corresponds to i_0.

Assume that $(L_I, \mathfrak{n}_I^+)$ is regular. For $1 \le p \le r = \sharp\{\text{nonopen orbits}\}$ we set $\gamma_p = \lambda_{p-1} - \lambda_p$, where $\lambda_0 = 0$. Then we have $\gamma_p \in \Delta^+ \backslash \Delta_I$. We denote the coroot of γ_p by h_{γ_p}, and set $\mathfrak{h}^- = \sum_{p=1}^r \mathbb{C}h_{\gamma_p}$. We set

$$\begin{aligned} \Delta_{(p)}^+ &= \left\{ \beta \in \Delta^+ \backslash \Delta_I \mid \beta|_{\mathfrak{h}^-} = \frac{\gamma_j + \gamma_k}{2} \quad \text{for some } 1 \le j \le k \le p \right\} \\ &\quad \cup \{\gamma_1, \ldots, \gamma_p\}, \\ \mathfrak{n}_{(p)}^{\pm} &= \sum_{\beta \in \Delta_{(p)}^+} \mathfrak{g}_{\pm\beta}, \\ \mathfrak{l}_{(p)} &= [\mathfrak{n}_{(p)}^+, \mathfrak{n}_{(p)}^-] \end{aligned}$$

(see Wachi [24] and Wallach [25]). Note that $\alpha_{i_0} \in \Delta_{(p)}^+$ for any p and $\Delta_{(r)}^+ = \Delta^+ \backslash \Delta_I$. Then it is known that $(L_{(p)}, \mathfrak{n}_{(p)}^+)$ is a regular prehomogeneous vector space of commutative parabolic type, where $L_{(p)}$ is the subgroup of G corresponding to $\mathfrak{l}_{(p)}$. Moreover $f_j \in \mathbb{C}[\mathfrak{n}_{(p)}^+]$ for $j \le p$, and f_p is a basic relative invariant of $(L_{(p)}, \mathfrak{n}_{(p)}^+)$. The regular prehomogeneous vector space $(L_{(r-1)}, \mathfrak{n}_{(r-1)}^+)$ is described by the following.

LEMMA 3.1.

(i) *For type* (A_{2n-1}, n) *we have* $r = n$, *and* $(L_{(n-1)}, \mathfrak{n}^+_{(n-1)})$ *is of type* $(A_{2n-3}, n-1)$.

(ii) *For type* $(B_n, 1)$ *we have* $r = 2$, *and* $(L_{(1)}, \mathfrak{n}^+_{(1)})$ *is of type* $(A_1, 1)$.

(iii) *For type* $(C_n, n)\,(n \geq 3)$ *we have* $r = n$, *and* $(L_{(n-1)}, \mathfrak{n}^+_{(n-1)})$ *is of type* $(C_{n-1}, n-1)$.

(iv) *For type* $(D_n, 1)$ *we have* $r = 2$, *and* $(L_{(1)}, \mathfrak{n}^+_{(1)})$ *is of type* $(A_1, 1)$.

(v) *For type* $(D_{2n}, 2n)\,(n \geq 3)$ *we have* $r = n$, *and* $(L_{(n-1)}, \mathfrak{n}^+_{(n-1)})$ *is of type* $(D_{2n-2}, 2n-2)$.

(vi) *For type* $(E_7, 1)$ *we have* $r = 3$, *and* $(L_{(2)}, \mathfrak{n}^+_{(2)})$ *is of type* $(D_6, 1)$.

We recall the definition of the b-function. Let $(L_I, \mathfrak{n}_I^+)$ be a regular prehomogeneous vector space with r nonopen orbits in $\mathfrak{n}_I^+$. For $h \in S(\mathfrak{n}_I^+) \simeq \mathbb{C}[\mathfrak{n}_I^-]$, we define the constant coefficient differential operator $h(\partial)$ by

$$h(\partial)\exp B(x, y) = h(y)\exp B(x, y) \qquad x \in \mathfrak{n}_I^+, y \in \mathfrak{n}_I^-,$$

where B is the Killing form on $\mathfrak{g}$ (see [15]). It is known that for the relative invariant f_r there exists a polynomial $b_r(s)$ such that for $s \in \mathbb{C}$

$${}^t f_r(\partial) f_r^{s+1} = b_r(s) f_r^s.$$

This polynomial $b_r(s)$ is called the b-function of f_r. Then we have $\deg b_r = \deg f_r = r$. The explicit description of $b_r(s)$ is given by

$$\begin{aligned}
(A_{2n-1}, n):&\quad b_n(s) = (s+1)(s+2)\cdots(s+n)\\
(B_n, 1):&\quad b_2(s) = (s+1)\left(s+\frac{2n-1}{2}\right)\\
(C_n, n):&\quad b_n(s) = (s+1)\left(s+\frac{3}{2}\right)\left(s+\frac{4}{2}\right)\cdots\left(s+\frac{n+1}{2}\right)\\
(D_n, 1):&\quad b_2(s) = (s+1)\left(s+\frac{2n-2}{2}\right)\\
(D_{2n}, 2n):&\quad b_n(s) = (s+1)(s+3)\cdots(s+2n-1)\\
(E_7, 1):&\quad b_3(s) = (s+1)(s+5)(s+9).
\end{aligned}$$

(see [4, 10, 15]).

Remark 3.2. The b-function of type A can be calculated by the Capelli identity. In general the b-functions of regular prehomogeneous vector spaces are calculated by using the theory of simple holonomic systems of microdifferential equations (see [18]).

We define a symmetric nondegenerate bilinear form $\langle\, , \rangle$ on $S(\mathfrak{n}_I^-) \simeq \mathbb{C}[\mathfrak{n}_I^+]$ by $\langle f, g\rangle = ({}^t g(\partial)f)(0)$.

LEMMA 3.3 (see Wachi [24]). *For $f, g, h \in S(\mathfrak{n}_I^-) \simeq \mathbb{C}[\mathfrak{n}_I^+]$ we have*

(i) $\langle \mathrm{ad}(u)f, g\rangle = \langle f, \mathrm{ad}({}^t u)g\rangle$ *for* $u \in U(\mathfrak{l}_I)$,

(ii) $\langle f, gh\rangle = \langle {}^t g(\partial)f, h\rangle$.

By definition we have

$$\langle x_{-\beta}, x_{-\beta'}\rangle = \delta_{\beta,\beta'}\frac{2}{(\beta,\beta)}$$

for $\beta, \beta' \in \Delta^+\backslash\Delta_I$. The comultiplication Δ of $U(\mathfrak{g})$ is defined by $\Delta(x) = x\otimes 1 + 1\otimes x$ for $x \in \mathfrak{g}$. We define the algebra homomorphism $\widetilde{\Delta}$ by $\widetilde{\Delta}(x) = \tau\Delta({}^t x)$, where $x \in U(\mathfrak{g})$ and $\tau(y_1\otimes y_2) = {}^t y_1 \otimes {}^t y_2$. Since ${}^t x_{-\beta}(\partial)(fg) = {}^t x_{-\beta}(\partial)(f)\mathfrak{g} + f\,{}^t x_{-\beta}(\partial)(\mathfrak{g})$, we have

$$\langle fg, h\rangle = \langle f\otimes g, \widetilde{\Delta}(h)\rangle. \tag{3.1}$$

Remark 3.4. Let $\langle\, , \rangle_0$ be a bilinear form on $U(\mathfrak{n}_I^-) = S(\mathfrak{n}_I^-)$ satisfying Lemma 3.3(i). It is known that $\langle V(\mu), V(\nu)\rangle_0 = 0$ for the different irreducible components $V(\mu)$ and $V(\nu)$ of $U(\mathfrak{n}_I^-)$ and that $\langle u_1, u_2\rangle_0 = 0$ for the weight vectors u_1 and u_2 with the different weights. Moreover $\langle\, , \rangle_0$ on $V(\mu)$ is unique up to constant multiple (see [2] and [5]). Therefore the symmetric bilinear form $\langle\, , \rangle_0$ on $\mathbb{C}[\mathfrak{n}_I^+]$ satisfying (3.1) and Lemma 3.3(i) is uniquely determined by $\langle x_{-\beta}, x_{-\beta}\rangle_0$ ($\beta \in \Delta^+\backslash\Delta_I$). From this bilinear form the differential operator ${}^t\mathfrak{g}(\partial)$ is defined by Lemma 3.3(ii).

4. QUANTIZED ENVELOPING ALGEBRA

The quantized enveloping algebra $U_q(\mathfrak{g})$ of $\mathfrak{g}$ (Drinfel'd [1], Jimbo [7]) is an associative algebra over the rational function field $\mathbb{C}(q)$ generated by the elements $\{E_i, F_i, K_i^{\pm 1}\}_{i\in I_0}$ satisfying the following relations

$$K_iK_j = K_jK_i, \qquad K_iK_i^{-1} = K_i^{-1}K_i = 1,$$

$$K_iE_jK_i^{-1} = q_i^{a_{ij}}E_j, \qquad K_iF_jK_i^{-1} = q_i^{-a_{ij}}F_j,$$

$$E_iF_j - F_jE_i = \delta_{ij}\frac{K_i - K_i^{-1}}{q_i - q_i^{-1}},$$

$$\sum_{k=0}^{1-a_{ij}}(-1)^k\begin{bmatrix}1-a_{ij}\\ k\end{bmatrix}_{q_i}E_i^{1-a_{ij}-k}E_jE_i^k = 0 \qquad (i\neq j),$$

$$\sum_{k=0}^{1-a_{ij}}(-1)^k\begin{bmatrix}1-a_{ij}\\ k\end{bmatrix}_{q_i}F_i^{1-a_{ij}-k}F_jF_i^k = 0 \qquad (i\neq j),$$

where $q_i = q^{d_i}$, and

$$[m]_t = \frac{t^m - t^{-m}}{t - t^{-1}}, \qquad [m]_t! = \prod_{k=1}^{m} [k]_t,$$

$$\begin{bmatrix} m \\ n \end{bmatrix}_t = \frac{[m]_t!}{[n]_t![m-n]_t!} \qquad (m \geq n \geq 0).$$

For $\mu = \sum_{i \in I_0} m_i \alpha_i$ we set $K_\mu = \prod_i K_i^{m_i}$. We can define an algebra antiautomorphism $x \mapsto {}^t x$ of $U_q(\mathfrak{g})$ by

$${}^t K_i = K_i, \qquad {}^t E_i = F_i, \qquad {}^t F_i = E_i.$$

We define subalgebras $U_q(\mathfrak{b}^\pm)$, $U_q(\mathfrak{h})$, and $U_q(\mathfrak{n}^\pm)$ of $U_q(\mathfrak{g})$ by

$$\begin{aligned} U_q(\mathfrak{b}^+) &= \langle K_i^{\pm 1}, E_i \mid i \in I_0 \rangle, \qquad & U_q(\mathfrak{b}^-) &= \langle K_i^{\pm 1}, F_i \mid i \in I_0 \rangle, \\ U_q(\mathfrak{h}) &= \langle K_i^{\pm 1} \mid i \in I_0 \rangle, & & \\ U_q(\mathfrak{n}^+) &= \langle E_i \mid i \in I_0 \rangle, & U_q(\mathfrak{n}^-) &= \langle F_i \mid i \in I_0 \rangle. \end{aligned}$$

We set $\mathfrak{h}^*_{\mathbf{Z}} = \oplus_{i \in I_0} \mathbb{Z} \varpi_i$. For a $U_q(\mathfrak{h})$-module M we define the weight space M_μ with weight $\mu \in \mathfrak{h}^*_{\mathbf{Z}}$ by

$$M_\mu = \left\{ m \in M \mid K_i m = q_i^{\mu(h_i)} m \ (i \in I_0) \right\}.$$

The Hopf algebra structure on $U_q(\mathfrak{g})$ is defined as follows. The comultiplication $\Delta\colon U_q(\mathfrak{g}) \to U_q(\mathfrak{g}) \otimes U_q(\mathfrak{g})$ is the algebra homomorphism satisfying

$$\begin{aligned} \Delta(K_i) &= K_i \otimes K_i, \\ \Delta(E_i) &= E_i \otimes K_i^{-1} + 1 \otimes E_i, \\ \Delta(F_i) &= F_i \otimes 1 + K_i \otimes F_i. \end{aligned}$$

The counit $\epsilon\colon U_q(\mathfrak{g}) \to \mathbb{C}(q)$ is the algebra homomorphism satisfying

$$\epsilon(K_i) = 1, \qquad \epsilon(E_i) = \epsilon(F_i) = 0.$$

The antipode $S\colon U_q(\mathfrak{g}) \to U_q(\mathfrak{g})$ is the algebra antiautomorphism satisfying

$$S(K_i) = K_i^{-1}, \qquad S(E_i) = -E_i K_i, \qquad S(F_i) = -K_i^{-1} F_i.$$

The adjoint action of $U_q(\mathfrak{g})$ on $U_q(\mathfrak{g})$ is defined as follows. For $x, y \in U_q(\mathfrak{g})$ write $\Delta(x) = \sum_k x_k^{(1)} \otimes x_k^{(2)}$ and set $\mathrm{ad}(x)(y) = \sum_k x_k^{(1)} y S(x_k^{(2)})$. Then $\mathrm{ad}\colon U_q(\mathfrak{g}) \to \mathrm{End}_{\mathbb{C}(q)}(U_q(\mathfrak{g}))$ is an algebra homomorphism.

For $i \in I_0$ we define an algebra automorphism T_i of $U_q(\mathfrak{g})$ (see Lusztig [12]) by

$$T_i(K_j) = K_j K_i^{-a_{ij}},$$

$$T_i(E_j) = \begin{cases} -F_i K_i & (i = j) \\ \sum_{k=0}^{-a_{ij}} (-q_i)^{-k} E_i^{(-a_{ij}-k)} E_j E_i^{(k)} & (i \neq j), \end{cases}$$

$$T_i(F_j) = \begin{cases} -K_i^{-1} E_i & (i = j) \\ \sum_{k=0}^{-a_{ij}} (-q_i)^{k} F_i^{(k)} F_j F_i^{(-a_{ij}-k)} & (i \neq j), \end{cases}$$

where

$$E_i^{(k)} = \frac{1}{[k]_{q_i}!} E_i^k, \qquad F_i^{(k)} = \frac{1}{[k]_{q_i}!} F_i^k.$$

For $w \in W$ we choose a reduced expression $w = s_{i_1} \cdots s_{i_k}$ and set $T_w = T_{i_1} \cdots T_{i_k}$. It does not depend on the choice of the reduced expression by Lusztig [13].

It is known that there exists a unique bilinear form $(\ ,\ ): U_q(\mathfrak{b}^-) \times U_q(\mathfrak{b}^+) \to \mathbb{C}(q)$ such that for any $x, x' \in U_q(\mathfrak{b}^+)$, $y, y' \in U_q(\mathfrak{b}^-)$, and $i, j \in I_0$

$$(y, xx') = (\Delta(y), x' \otimes x), \qquad (yy', x) = (y \otimes y', \Delta(x)),$$

$$(K_i, K_j) = q^{-(\alpha_i, \alpha_j)}, \qquad (F_i, E_j) = -\delta_{ij}(q_i - q_i^{-1})^{-1},$$

$$(F_i, K_j) = 0, \qquad (K_i, E_j) = 0$$

(see Jantzen [6], Tanisaki [22]). Note that for $\mu \in \sum_{i \in I_0} \mathbb{Z}_{\geq 0}\alpha_i$ the restriction $(\ ,\ )|_{U_q(\mathfrak{n}^-)_{-\mu} \times U_q(\mathfrak{n}^+)_{\mu}}$ is nondegenerate. Here $U_q(\mathfrak{n}^\pm)_{\pm\mu}$ are weight spaces with weight $\pm\mu$ relative to the adjoint action of $U_q(\mathfrak{h})$.

Let $y \in U_q(\mathfrak{n}^-)_{-\mu}$ for $\mu \in \sum_{i \in I_0} \mathbb{Z}_{\geq 0}\alpha_i$. For any $i \in I_0$ the elements $r_i(y)$ and $r_i'(y)$ of $U_q(\mathfrak{n}^-)_{-(\mu-\alpha_i)}$ are defined by

$$\Delta(y) \in y \otimes 1 + \sum_{i \in I_0} K_i r_i(y) \otimes F_i + \left(\bigoplus_{\substack{0 < \nu \leq \mu \\ \nu \neq \alpha_i}} K_\nu U_q(\mathfrak{n}^-)_{-(\mu-\nu)} \otimes U_q(\mathfrak{n}^-)_{-\nu} \right),$$

$$\Delta(y) \in K_\mu \otimes y + \sum_{i \in I_0} K_{\mu-\alpha_i} F_i \otimes r_i'(y) + \left(\bigoplus_{\substack{0 < \nu \leq \mu \\ \nu \neq \alpha_i}} K_{\mu-\nu} U_q(\mathfrak{n}^-)_{-\nu} \otimes U_q(\mathfrak{n}^-)_{-(\mu-\nu)} \right).$$

LEMMA 4.1 (see Jantzen [6]).

(i) *We have* $r_i(1) = r'_i(1) = 0$ *and* $r_i(F_j) = r'_i(F_j) = \delta_{ij}$ *for* $j \in I_0$.

(ii) *We have for* $y_1 \in U_q(\mathfrak{n}^-)_{-\mu_1}$ *and* $y_2 \in U_q(\mathfrak{n}^-)_{-\mu_2}$

$$r_i(y_1 y_2) = q_i^{\mu_1(h_i)} y_1 r_i(y_2) + r_i(y_1) y_2,$$
$$r'_i(y_1 y_2) = y_1 r'_i(y_2) + q_i^{\mu_2(h_i)} r'_i(y_1) y_2.$$

(iii) *We have for* $x \in U_q(\mathfrak{n}^+)$ *and* $y \in U_q(\mathfrak{n}^-)_{-\mu}$

$$(y, E_i x) = (F_i, E_i)(r_i(y), x), \qquad (y, x E_i) = (F_i, E_i)(r'_i(y), x).$$

(iv) *We have for* $y \in U_q(\mathfrak{n}^-)_{-\mu}$

$$\mathrm{ad}(E_i) y = (q_i - q_i^{-1})^{-1}(K_i r_i(y) K_i - r'_i(y)).$$

From Lemma 4.1(ii) we have $r_i(F_i^n) = r'_i(F_i^n) = q_i^{n-1}[n]_{q_i} F_i^{n-1}$.

5. QUANTUM DEFORMATIONS OF COORDINATE ALGEBRAS

In this section we recall basic properties of the quantum analogue of the coordinate algebra $\mathbb{C}[\mathfrak{n}_I^+]$ of $\mathfrak{n}_I^+$ satisfying $[\mathfrak{n}_I^+, \mathfrak{n}_I^+] = 0$ (see [9]). We do not assume that $(L_I, \mathfrak{n}_I^+)$ is regular. We take $i_0 \in I_0$ as in Section 3.

We define a subalgebra $U_q(\mathfrak{l}_I)$ by $U_q(\mathfrak{l}_I) = \langle K_i^{\pm 1}, E_j, F_j | i \in I_0, j \in I \rangle$. Let w_I be the longest element of W_I, and set

$$U_q(\mathfrak{n}_I^-) = U_q(\mathfrak{n}^-) \cap T_{w_I}^{-1} U_q(\mathfrak{n}^-).$$

We take a reduced expression $w_I w_0 = s_{i_1} \cdots s_{i_k}$ and set

$$\beta_t = s_{i_1} \cdots s_{i_{t-1}}(\alpha_{i_t}), \qquad Y_{\beta_t} = T_{i_1} \cdots T_{i_{t-1}}(F_{i_t})$$

for $t = 1, \dots, k$. In particular $Y_{\beta_1} = F_{i_0}$. We have $\{\beta_t | 1 \le t \le k\} = \Delta^+ \setminus \Delta_I$ and $Y_{\beta_t} \in U_q(\mathfrak{n}^-)_{-\beta_t}$. The set $\{Y_{\beta_1}^{n_1} \cdots Y_{\beta_k}^{n_k} | n_1, \dots, n_k \in \mathbb{Z}_{\ge 0}\}$ is a basis of $U_q(\mathfrak{n}_I^-)$.

PROPOSITION 5.1 (see [9]).

(i) *We have* $\mathrm{ad}(U_q(\mathfrak{l}_I)) U_q(\mathfrak{n}_I^-) \subset U_q(\mathfrak{n}_I^-)$.

(ii) *The elements* $Y_\beta \in U_q(\mathfrak{n}_I^-)$ *for* $\beta \in \Delta^+ \setminus \Delta_I$ *do not depend on the choice of a reduced expression of* $w_I w_0$*, and they satisfy quadratic fundamental relations as generators of the algebra* $U_q(\mathfrak{n}_I^-)$.

The coordinate algebra $\mathbb{C}[\mathfrak{n}_I^+]$ of $\mathfrak{n}_I^+$ is identified with the enveloping algebra $U(\mathfrak{n}_I^-)$, and the action of $U(\mathfrak{l}_I)$ on $\mathbb{C}[\mathfrak{n}_I^+]$ corresponds to the adjoint action on $U(\mathfrak{n}_I^-)$. Hence we can regard the subalgebra $U_q(\mathfrak{n}_I^-)$ of $U_q(\mathfrak{n}^-)$ as a quantum analogue of the coordinate algebra $\mathbb{C}[\mathfrak{n}_I^+]$. For example in the case of type A we have $\mathfrak{l}_I \simeq \{(l_1, l_2) \in \mathfrak{gl}_m \times \mathfrak{gl}_n | \mathrm{trace}(l_1) + \mathrm{trace}(l_2) = 0\}$ and $\mathfrak{n}_I^+ \simeq \mathrm{Mat}_{m,n}(\mathbb{C})$ for some integers m and n. Then $U_q(\mathfrak{n}_I^-)$ is generated by $Y_{ij}\,(1 \le i \le m, 1 \le j \le n)$ satisfying the fundamental relations

$$Y_{ij}Y_{kl} = \begin{cases} qY_{kl}Y_{ij} & (i = k, j < l \text{ or } i < k, j = l) \\ Y_{kl}Y_{ij} & (i < k, j > l) \\ Y_{kl}Y_{ij} + (q - q^{-1})Y_{kj}Y_{il} & (i < k, j < l). \end{cases}$$

(See [8] for the action of $U_q(\mathfrak{l}_I)$.)

We label the L_I-orbits $C_p\,(1 \le p \le r+1)$ in $\mathfrak{n}_I^+$ as in Section 3. Since $\mathbb{C}[\mathfrak{n}_I^+]$ is a multiplicity free $\mathfrak{l}_I$-module, for the orbit C_p in $\mathfrak{n}_I^+$ there exist unique $U_q(\mathfrak{l}_I)$-submodules $\mathcal{I}_q(\overline{C_p})$ and $\mathcal{I}_q^p(\overline{C_p})$ of $U_q(\mathfrak{n}_I^-)$ satisfying

$$\mathcal{I}_q(\overline{C_p})|_{q=1} = \mathcal{I}(\overline{C_p}), \qquad \mathcal{I}_q^p(\overline{C_p})|_{q=1} = \mathcal{I}^p(\overline{C_p})$$

(see [9]).

PROPOSITION 5.2 (see [9]).

$$\mathcal{I}_q(\overline{C_p}) = U_q(\mathfrak{n}_I^-)\mathcal{I}_q^p(\overline{C_p}) = \mathcal{I}_q^p(\overline{C_p})U_q(\mathfrak{n}_I^-).$$

Let $f_{q,p}$ be the highest weight vector of $\mathcal{I}_q^p(\overline{C_p})$. We have the irreducible decomposition

$$U_q(\mathfrak{n}_I^-) = \bigoplus_{\mu \in \sum_p \mathbb{Z}_{\ge 0}\lambda_p} V_q(\mu),$$

where $V_q(\mu)$ is an irreducible highest weight module with highest weight μ and $V_q(\lambda_p) = \mathcal{I}_q^p(\overline{C_p})$. Explicit descriptions of $U_q(\mathfrak{n}_I^-)$, $\mathcal{I}_q^p(\overline{C_p})$, and $f_{q,p}$ are given in [8] in the case where $\mathfrak{g}$ is classical and in [14] for the exceptional cases.

Let f be a weight vector of $U_q(\mathfrak{n}_I^-)$ with the weight $-\mu$. If $\mu \in m\alpha_{i_0} + \sum_{i \in I} \mathbb{Z}_{\ge 0}\alpha_i$, then f is an element of $\sum_{\beta_{i_1},\dots,\beta_{i_m} \in \Delta^+ \setminus \Delta_I} \mathbb{C}(q)Y_{\beta_{i_1}} \cdots Y_{\beta_{i_m}}$. So we can define the degree of f by $\deg f = m$. In particular $\deg f_{q,p} = p$.

6. QUANTUM DEFORMATIONS OF RELATIVE INVARIANTS

In the remainder of this paper we assume that $(L_I, \mathfrak{n}_I^+)$ is regular, $\{i_0\} = I_0 \setminus I$, and L_I-orbits $C_1, \dots, C_r, C_{r+1}$ satisfy $\{0\} = C_1 \subset \overline{C_2} \subset \cdots \subset \overline{C_r} \subset \overline{C_{r+1}} = \mathfrak{n}_I^+$. Then we regard the highest weight vector $f_{q,r}$ of $\mathcal{I}_q^r(\overline{C_r})$ as the quantum analogue of the basic relative invariant. We give some properties of $f_{q,r}$ in this section.

By $\mathcal{I}_q^r(\overline{C_r}) = \mathbb{C}(q)f_{q,r}$ and $\lambda_r = -2\varpi_{i_0}$, we have the following.

PROPOSITION 6.1. *We have*

$$\mathrm{ad}(K_i)f_{q,r} = f_{q,r}, \qquad \mathrm{ad}(E_i)f_{q,r} = 0 \qquad \textit{and} \qquad \mathrm{ad}(F_i)f_{q,r} = 0.$$

for any $i \in I$, *and* $\mathrm{ad}(K_{i_0})f_{q,r} = q_{i_0}^{-2} f_{q,r}$.

LEMMA 6.2.

(i) *For* $i \in I$ *we have* $r_i(U_q(\mathfrak{n}_I^-)) = 0$.

(ii) *For* $\beta \in \Delta^+ \backslash \Delta_I$ *we have* $r'_{i_0}(Y_\beta) = \delta_{\alpha_{i_0},\beta}$.

Proof. (i) By Jantzen [6] we have

$$\{y \in U_q(\mathfrak{n}^-) | r_i(y) = 0\} = U_q(\mathfrak{n}^-) \cap T_i^{-1} U_q(\mathfrak{n}^-).$$

For any $i \in I$ we have $U_q(\mathfrak{n}_I^-) \subset U_q(\mathfrak{n}^-) \cap T_i^{-1} U_q(\mathfrak{n}^-)$. Hence we have $r_i(U_q(\mathfrak{n}_I^-)) = 0$ for $i \in I$.

(ii) We show the formula by the induction on β. By the definition of r'_{i_0}, it is clear that $r'_{i_0}(Y_{\alpha_{i_0}}) = r'_{i_0}(F_{i_0}) = 1$. Assume that $\beta > \alpha_{i_0}$ and the statement is proved for any root β_1 in $\Delta^+ \backslash \Delta_I$ satisfying $\beta_1 < \beta$. For some $i \in I$ we can write

$$Y_\beta = c\,\mathrm{ad}(F_i) Y_{\beta'} = c\,(F_i Y_{\beta'} - q^{-(\alpha_i, \beta')} Y_{\beta'} F_i),$$

where $\beta' = \beta - \alpha_i$ and $c \in \mathbb{C}(q)$. Hence we have

$$r'_{i_0}(Y_\beta) = c(F_i r'_{i_0}(Y_{\beta'}) - q^{(\alpha_i, \alpha_{i_0} - \beta')} r'_{i_0}(Y_{\beta'}) F_i).$$

If $\beta' = \alpha_{i_0}$, then we have $r'_{i_0}(Y_\beta) = c(F_i - F_i) = 0$ since $r'_{i_0}(Y_{\beta'}) = 1$. If $\beta' \neq \alpha_{i_0}$, then $r'_{i_0}(Y_\beta) = 0$ since $r'_{i_0}(Y_{\beta'}) = 0$. ∎

PROPOSITION 6.3. *The quantum analogue* $f_{q,r}$ *is a central element of* $U_q(\mathfrak{n}^-)$.

Proof. For $i \in I$ we have $[F_i, f_{q,r}] = \mathrm{ad}(F_i) f_{q,r}$. By Proposition 6.1 we have to show $[F_{i_0}, f_{q,r}] = 0$. The quantum analogue $f_{q,r}$ is a linear combination of $Y_{\beta_{j_1}} \cdots Y_{\beta_{j_r}}$ satisfying $\sharp\{j_k | \beta_{j_k} = \alpha_{i_0}\} \leq 1$ (see [8] and [14]). By using Lemma 6.2 it is easy to show that $r'_{i_0}(f_{q,r}) \neq 0$ and ${r'_{i_0}}^2(f_{q,r}) = 0$. Hence we have

$${r'_{i_0}}^2(F_{i_0} f_{q,r}) = {r'_{i_0}}^2(f_{q,r} F_{i_0}) = (q_{i_0}^2 + 1) r'_{i_0}(f_{q,r}).$$

By Proposition 5.2 there exists $c \in \mathbb{C}(q)$ such that $F_{i_0} f_{q,r} = c f_{q,r} F_{i_0}$; hence we have $(q_{i_0}^2 + 1) r'_{i_0}(f_{q,r}) = c(q_{i_0}^2 + 1) r'_{i_0}(f_{q,r})$. Therefore we obtain $c = 1$. ∎

The explicit description of the quantum analogue $f_{q,r}$ in the case where $\mathfrak{g}$ is classical is given as follows.

LEMMA 6.4 (see [8]). *We label the vertices of the Dynkin diagram as in Fig.* 1.

(i) *Type*(A_{2n-1}, n) $(r = n)$. *For* $1 \leq i, j \leq n$ *we set* $\beta_{i,j} = \alpha_{n-i+1} + \cdots + \alpha_n + \cdots + \alpha_{n+j-1} \in \Delta^+ \backslash \Delta_I$ *and* $Y_{i,j} = Y_{\beta_{i,j}}$. *Then we have the quantum analogue*

$$f_{q,n} = \sum_{\sigma \in \mathfrak{S}_n} (-q)^{\ell(\sigma)} Y_{1,\sigma(1)} \cdots Y_{n,\sigma(n)},$$

where $\mathfrak{S}_n$ *is the symmetric group and* $\ell(\sigma) = \sharp\{(i,j) | i < j, \sigma(i) > \sigma(j)\}$. *This is a quantum determinant.*

(ii) *Type* $(B_n, 1)$ $(r = 2)$. *For* $1 \leq i \leq 2n-1$ *we set*

$$\beta_i = \begin{cases} \alpha_1 + \cdots + \alpha_i & (1 \leq i \leq n) \\ \alpha_1 + \cdots + \alpha_{2n-i} + 2\alpha_{2n-i+1} + \cdots + 2\alpha_n & (n+1 \leq i \leq 2n-1) \end{cases}$$

and $Y_i = Y_{\beta_i}$. *Then we have the quantum analogue*

$$f_{q,2} = \sum_{i=1}^{n-1} (-q^2)^{i+1-n} Y_{n+i} Y_{n-i} + (q + q^{-1})^{-2} q^{-1} (-q^2)^{1-n} Y_n^2.$$

(iii) *Type* (C_n, n) $(r = n)$. *For* $1 \leq i \leq j \leq n$ *we set* $\beta_{i,j} = \alpha_i + \cdots + \alpha_{j-1} + 2\alpha_j + \cdots + 2\alpha_{n-1} + \alpha_n$ *and* $Y_{i,j} = c_{i,j} Y_{\beta_{i,j}}$, *where* $c_{i,j} = q + q^{-1}$ *if* $i = j$ *and* 1 *if* $i \neq j$. *For* $i < j$ *we define* $Y_{j,i}$ *by* $Y_{j,i} = q^{-2} Y_{i,j}$. *Then we have the quantum analogue*

$$f_{q,n} = \sum_{\sigma \in \mathfrak{S}_n} (-q)^{-\ell(\sigma)} Y_{1,\sigma(1)} \cdots Y_{n,\sigma(n)}.$$

(iv) *Type* $(D_n, 1)$ $(r = 2)$. *For* $1 \leq i \leq 2n-2$ *we set*

$$\beta_i = \begin{cases} \alpha_1 + \cdots + \alpha_i & (1 \leq i \leq n-1) \\ \alpha_1 + \cdots + \alpha_{n-2} + \alpha_n & (i = n) \\ \alpha_1 + \cdots + \alpha_{2n-i} & \\ \quad + 2\alpha_{2n-i+1} + \cdots + 2\alpha_{n-2} + \alpha_{n-1} + \alpha_n & (n+1 \leq i \leq 2n-2) \end{cases}$$

and $Y_i = Y_{\beta_i}$. *Then we have the quantum analogue*

$$f_{q,2} = \sum_{i=1}^{n-1} (-q)^{i+1-n} Y_{n+i-1} Y_{n-i}.$$

(v) *Type* $(D_{2n}, 2n)$ $(r = n)$. *For* $1 \le i < j \le 2n$ *we set*

$$\beta_{i,j} = \begin{cases} \alpha_i + \cdots + \alpha_{j-1} + 2\alpha_j \\ \quad + \cdots + 2\alpha_{2n-2} + \alpha_{2n-1} + \alpha_{2n} & (j < 2n) \\ \alpha_i + \cdots + \alpha_{2n-2} + \alpha_{2n} & (j = 2n), \end{cases}$$

and $Y_{i,j} = Y_{\beta_{i,j}}$. *Then we have the quantum analogue*

$$f_{q,n} = \sum_{\sigma \in S_{2n}} (-q^{-1})^{\ell(\sigma)} Y_{\sigma(1),\sigma(2)} \cdots Y_{\sigma(2n-1),\sigma(2n)},$$

where $S_{2n} = \{\sigma \in \mathfrak{S}_{2n} \mid \sigma(2k-1) < \sigma(2k+1), \sigma(2k-1) < \sigma(2k)$ *for all* $k\}$. *This is a quantum analogue of a Pfaffian.*

Remark 6.5. In the case of type $(E_7, 1)$ there exist three nonopen orbits C_1, C_2, C_3 satisfying $\{0\} = C_1 \subset \overline{C_2} \subset \overline{C_3}$. Then we have the quantum analogue $f_{q,3} = \sum_{j=1}^{27} (-q)^{|\beta_j|-1} Y_j \psi_j$, where $\{\beta_1, \ldots, \beta_{27}\} = \Delta^+ \setminus \Delta_I$, $|\beta| = \sum_{i=1}^{7} m_i$ for $\beta = \sum_{i=1}^{7} m_i \alpha_i$, and $\psi_1, \ldots, \psi_{27}$ are generators of $\mathcal{I}_q^2(\overline{C_2})$. See Morita [14] for the explicit descriptions of β_j and ψ_j. Note that $\{\beta_1 = \alpha_{i_0}, \beta_2, \ldots, \beta_{10}\} = \Delta^+_{(2)}$ and ψ_{27} is a highest weight vector of $\mathcal{I}_q^2(\overline{C_2})$.

7. QUANTUM ANALOGUES OF b-FUNCTIONS

In Section 3 the symmetric bilinear form $\langle\, , \rangle$ on $\mathbb{C}[\mathfrak{n}_I^+]$ determines the differential operator ${}^t g(\partial)$ for $g \in \mathbb{C}[\mathfrak{n}_I^+]$ by $\langle {}^t g(\partial) f, h\rangle = \langle f, gh\rangle$. For the purpose of constructing a quantum analogue of ${}^t g(\partial)$ we use a symmetric bilinear form $\langle\, , \rangle$ on $U_q(\mathfrak{n}_I^-)$ satisfying

$$\langle \mathrm{ad}(u) f, g\rangle = \langle f, \mathrm{ad}({}^t u) g\rangle \ (u \in U_q(\mathfrak{l}_I), f, g \in U_q(\mathfrak{n}_I^-)) \tag{7.1}$$

and

$$\langle fg, h\rangle = \langle f \otimes g, \widetilde{\Delta}(h)\rangle \ (f, g, h \in U_q(\mathfrak{n}_I^-)), \tag{7.2}$$

where $\widetilde{\Delta}(x) = \tau\Delta({}^t x)$ and $\tau(y_1 \otimes y_2) = {}^t y_1 \otimes {}^t y_2$. This bilinear form is uniquely determined by the restriction on the irreducible component $V_q(-\alpha_{i_0}) = \sum_{\beta \in \Delta^+ \setminus \Delta_I} \mathbb{C}(q) Y_\beta$ of $U_q(\mathfrak{n}_I^-)$ from the condition (7.2). Similarly to the classical case $q = 1$, the symmetric form on $V_q(-\alpha_{i_0})$ satisfying (7.1) is unique up to constant multiple. Hence the symmetric bilinear form on $U_q(\mathfrak{n}_I^-)$ satisfying (7.1) and (7.2) is unique up to constant multiple if it exists. By using the natural paring $(\ ,\ )$ in Section 4 such a symmetric bilinear form is explicitly constructed as follows. We set

$$\langle f, g\rangle = (q^{-1} - q)^{\deg f} (f, {}^t g),$$

for the weight vectors f, g of $U_q(\mathfrak{n}_I^-)$. It is easy to show that this bilinear form $\langle\, , \rangle$ is symmetric.

PROPOSITION 7.1. *Let* $f, g, h \in U_q(\mathfrak{n}_I^-)$.

(i) $\langle fg, h\rangle = \langle f \otimes g, \widetilde{\Delta}(h)\rangle$, *where* $\widetilde{\Delta}(h) = \tau\Delta({}^t h)$ *and* $\tau(h_1 \otimes h_2) = {}^t h_1 \otimes {}^t h_2$.

(ii) *For* $u \in U_q(\mathfrak{l}_I)$ *we have*

$$\langle \mathrm{ad}(u)f, g\rangle = \langle f, \mathrm{ad}({}^t u)g\rangle.$$

(iii) *The bilinear form* $\langle\ ,\ \rangle$ *is nondegenerate.*

Proof. (i) It is clear from the definition.

(ii) It is sufficient to show that the statement holds for the weight vectors f, g and the canonical generator u of $U_q(\mathfrak{l}_I)$. If $u = K_i$ for $i \in I_0$, then the assertion is obvious. Let $u = E_i$ for $i \in I$. By Lemmas 4.1 and 6.2 we have

$$(\mathrm{ad}(E_i)f, {}^t g) = (q_i^{-1} - q_i)^{-1}(r_i'(f), {}^t g) = (f, {}^t g E_i).$$

Since $(U_q(\mathfrak{n}_I^-), E_i U_q(\mathfrak{n}^+)) = 0$ by Lemmas 4.1 and 6.2, we have

$$(f, {}^t g E_i) = (f, {}^t g E_i - q_i^{-\mu(h_i)} E_i\ {}^t g) = (f, {}^t(\mathrm{ad}(F_i)g)),$$

where $-\mu$ is the weight of g. We have $\deg f = \deg(\mathrm{ad}(F_i)f)$, and hence the statement for $u = E_i$ holds. By the symmetry of $\langle\ ,\ \rangle$ it also holds for $u = F_i$.

(iii) We take the reduced expression $w_0 = s_{i_1} \cdots s_{i_k} s_{i_{k+1}} \cdots s_{i_l}$ such that $w_I w_0 = s_{i_1} \cdots s_{i_k}$. We define Y_{β_j} $(1 \le j \le l)$ as in Section 5. Then $\{Y_{\beta_1}^{n_1} \cdots Y_{\beta_k}^{n_k} Y_{\beta_{k+1}}^{n_{k+1}} \cdots Y_{\beta_l}^{n_l}\}$ is a basis of $U_q(\mathfrak{n}^-)$, and for $j > k$ we have $Y_{\beta_j} \in U_q(\mathfrak{n}^-) \cap U_q(\mathfrak{l}_I)$. Hence we have $U_q(\mathfrak{n}^-) = U_q(\mathfrak{n}_I^-) + \sum_{i\in I} U_q(\mathfrak{n}^-)F_i$. Since ${}^t U_q(\mathfrak{n}^-) = U_q(\mathfrak{n}^+)$, we have $U_q(\mathfrak{n}^+) = {}^t U_q(\mathfrak{n}_I^-) + \sum_{i\in I} E_i U_q(\mathfrak{n}^+)$. Moreover, we have $(U_q(\mathfrak{n}_I^-), E_i U_q(\mathfrak{n}^+)) = 0$ for $i \in I$. Hence if $\langle f, g\rangle = 0$ for any $g \in U_q(\mathfrak{n}_I^-)$, then $(f, u) = 0$ for any $u \in U_q(\mathfrak{n}^+)$. Thus the assertion follows from the nondegeneracy of $(\ ,\ )$. ∎

Moreover we have the following.

PROPOSITION 7.2. *For* $\beta, \beta' \in \Delta^+ \backslash \Delta_I$ *we have*

$$\langle Y_\beta, Y_{\beta'}\rangle = \delta_{\beta,\beta'}\left[\frac{(\beta, \beta)}{2}\right]_q^{-1}.$$

Proof. By the definition it is clear that $\langle Y_\beta, Y_{\beta'}\rangle = 0$ if $\beta \neq \beta'$. In the case where $\beta = \beta'$ we shall show the statement by the induction on β. Since $Y_{\alpha_{i_0}} = F_{i_0}$, we obtain $\langle Y_{\alpha_{i_0}}, Y_{\alpha_{i_0}}\rangle = \left[(\alpha_{i_0}, \alpha_{i_0})/2\right]_q^{-1}$. Assume that $\beta > \alpha_{i_0}$

and the statement holds for any root β_1 in $\Delta^+\backslash\Delta_I$ satisfying $\beta_1 < \beta$. Then there exists a root γ $(< \beta)$ in $\Delta^+\backslash\Delta_I$ such that

$$Y_\beta = c_{\gamma,\beta}\ \mathrm{ad}(F_i)Y_\gamma, \qquad Y_\gamma = c'_{\gamma,\beta}\ \mathrm{ad}(E_i)Y_\beta,$$

where $i \in I$ satisfying $\beta = \gamma + \alpha_i$ and $c_{\gamma,\beta}, c'_{\gamma,\beta} \in \mathbb{C}(q)\backslash\{0\}$. We denote by R the set of the pairs $\{\gamma, \beta\}$ as above. By Proposition 7.1 we have for $\{\gamma, \beta\} \in R$

$$\begin{aligned}\langle Y_\beta, Y_\beta\rangle &= \langle Y_\beta, c_{\gamma,\beta}\ \mathrm{ad}(F_i)Y_\gamma\rangle = c_{\gamma,\beta}\langle\ \mathrm{ad}(E_i)Y_\beta, Y_\gamma\rangle \\ &= \frac{c_{\gamma,\beta}}{c'_{\gamma,\beta}}\langle Y_\gamma, Y_\gamma\rangle = \frac{c_{\gamma,\beta}}{c'_{\gamma,\beta}}\left[\frac{(\gamma,\gamma)}{2}\right]_q^{-1}.\end{aligned}$$

Here we have for $\{\gamma, \beta\} \in R$

$$\begin{aligned}
&c_{\gamma,\beta} = c'_{\gamma,\beta} = 1 && \text{if } (\beta,\beta) = (\gamma,\gamma),\\
&c_{\gamma,\beta} = (q+q^{-1})^{-1},\ c'_{\gamma,\beta} = 1 && \text{if } 4 = (\beta,\beta) > (\gamma,\gamma) = 2,\\
&c_{\gamma,\beta} = 1,\ c'_{\gamma,\beta} = (q+q^{-1})^{-1} && \text{if } 2 = (\beta,\beta) < (\gamma,\gamma) = 4
\end{aligned}$$

(see [8] and [14]). Hence we obtain $\langle Y_\beta, Y_\beta\rangle = [(\beta,\beta)/2]_q^{-1}$. ∎

By Propositions 7.1 and 7.2 this bilinear form on $U_q(\mathfrak{n}_I^-)$ can be regarded as the q-analogue of the symmetric bilinear form on $\mathbb{C}[\mathfrak{n}_I^+]$ defined in Section 3.

PROPOSITION 7.3.

(i) *For any* $g \in U_q(\mathfrak{n}_I^-)$ *there exists a unique* ${}^t g(\partial) \in \mathrm{End}_{\mathbb{C}(q)}(U_q(\mathfrak{n}_I^-))$ *such that* $\langle {}^t g(\partial)f, h\rangle = \langle f, gh\rangle$ *for any* $f, h \in U_q(\mathfrak{n}_I^-)$. *In particular we have*

$${}^tY_{\alpha_{i_0}}(\partial) = [d_{i_0}]_q^{-1} r'_{i_0},$$

and for $\beta > \alpha_{i_0}$

$${}^tY_\beta(\partial) = c_{\beta',\beta}\big({}^tY_{\beta'}(\partial)\mathrm{ad}(E_i) - q_i^{-\beta'(h_i)}\mathrm{ad}(E_i){}^tY_{\beta'}(\partial)\big),$$

where $Y_\beta = c_{\beta',\beta}\ \mathrm{ad}(F_i)Y_{\beta'}$.

(ii) *For* $f \in U_q(\mathfrak{n}_I^-)_{-\mu}$ *and* $g \in U_q(\mathfrak{n}_I^-)_{-\nu}$ *we have* ${}^t g(\partial)f \in U_q(\mathfrak{n}_I^-)_{-(\mu-\nu)}$.

Proof. (i) The uniqueness follows from the nondegeneracy of $\langle\, ,\rangle$. If there exist ${}^t g(\partial)$ and ${}^t g'(\partial)$, then we have ${}^t(gg')(\partial) = {}^t g'(\partial){}^t g(\partial)$. Therefore we have only to show the existence of ${}^tY_\beta(\partial)$ for any $\beta \in \Delta^+\backslash\Delta_I$. By Lemma 4.1 we have ${}^tY_{\alpha_{i_0}}(\partial) = [d_{i_0}]_q^{-1} r'_{i_0}$. Let $\beta > \alpha_{i_0}$. Then there exists a root $\beta'(<\beta)$ such that $Y_\beta = c_{\beta',\beta}\ \mathrm{ad}(F_i)Y_{\beta'}$ $(c_{\beta',\beta} \in \mathbb{C}(q))$. By Proposition 7.1 we can show that ${}^tY_\beta(\partial) = c_{\beta',\beta}({}^tY_{\beta'}(\partial)\mathrm{ad}(E_i) - q_i^{-\beta'(h_i)}\mathrm{ad}(E_i){}^tY_{\beta'}(\partial))$ easily.

(ii) The assertion follows from (i). ∎

This linear map ${}^t g(\partial)$ is regarded as a quantum analogue of a differential operator on $\mathbb{C}[\mathfrak{n}_I^+]$.

Remark 7.4. Let $(L_I, \mathfrak{n}_I^+)$ be the regular prehomogeneous vector space of type (A_{2n-1}, n). We define the root vectors $Y_{i,j}$ $(1 \le i, j \le n)$ as in Lemma 6.4. For $1 \le i \le n$ let U_i be the subalgebra of $U_q(\mathfrak{n}_I^-)$ generated by $Y_{i,1}, \dots, Y_{i,n}$. Note that $Y_{i,j} Y_{i,k} = q Y_{i,k} Y_{i,j}$ for $j < k$. Then we have

$$ {}^t Y_{k,l}(\partial)(Y_{i,1}^{a_1} \cdots Y_{i,n}^{a_n}) Y_{k,l} = \delta_{k,i} q^{a_{l-1}} [a_l]_q Y_{i,1}^{a_1} \cdots Y_{i,n}^{a_n}. $$

Therefore ${}^t Y_{k,l}(\partial)|_{U_i}$ is a sort of q-difference operator different from the operator in Noumi *et al.* [16] (cf. [16, Propositions 2.2 and 5.2]).

LEMMA 7.5. *For $i \in I$*

$$ \mathrm{ad}(E_i)^t f_{q,r}(\partial) = {}^t f_{q,r}(\partial) \mathrm{ad}(E_i), \ \mathrm{ad}(F_i)^t f_{q,r}(\partial) = {}^t f_{q,r}(\partial) \mathrm{ad}(F_i). $$

Proof. Let $y_1, y_2 \in U_q(n_I^-)$. Since $\mathrm{ad}(F_i) f_{q,r} = 0$ for $i \in I$, we have $\mathrm{ad}(F_i)(f_{q,r} y_2) = f_{q,r}\ \mathrm{ad}(F_i) y_2$. Hence we obtain

$$ \begin{aligned} \langle \mathrm{ad}(E_i)^t f_{q,r}(\partial)(y_1), y_2 \rangle &= \langle y_1, f_{q,r}\ \mathrm{ad}(F_i) y_2 \rangle = \langle y_1, \mathrm{ad}(F_i)(f_{q,r} y_2) \rangle \\ &= \langle {}^t f_{q,r}(\partial) \mathrm{ad}(E_i)(y_1), y_2 \rangle. \end{aligned} $$

Similarly we obtain $\mathrm{ad}(F_i)^t f_{q,r}(\partial) = {}^t f_{q,r}(\partial) \mathrm{ad}(F_i)$. ∎

By Proposition 7.3 and Lemma 7.5 the element ${}^t f_{q,r}(\partial)(f_{q,r}^{s+1})$ $(s \in \mathbb{Z}_{\ge 0})$ is the highest weight vector with highest weight $s\lambda_r = -2s\varpi_{i_0}$. Since $U_q(\mathfrak{n}_I^-)$ is a multiplicity free $U_q(\mathfrak{l}_I)$-module, there exists $\tilde{b}_{q,r,s} \in \mathbb{C}(q)$ such that

$$ {}^t f_{q,r}(\partial)(f_{q,r}^{s+1}) = \tilde{b}_{q,r,s} f_{q,r}^s. $$

PROPOSITION 7.6. *There exists a polynomial $\tilde{b}_{q,r}(t) \in \mathbb{C}(q)[t]$ such that $\tilde{b}_{q,r,s} = \tilde{b}_{q,r}(q_{i_0}^s)$ for any $s \in \mathbb{Z}_{\ge 0}$.*

Proof. Let $\varphi = \varphi_1 \cdots \varphi_m$, where $\varphi_j = r'_{i_0}$ or $\mathrm{ad}(E_i)$ for some $i \in I$. Set $n = n(\varphi) = \sharp\{j | \varphi_j = r'_{i_0}\}$. For $k \in \mathbb{Z}_{\ge 0}$ and $y \in U_q(\mathfrak{n}_I^-)_{-\mu}$ we have

$$ r'_{i_0}(f_{q,r}^k y) = q_{i_0}^{k-1+\mu(h_{i_0})} [k]_{q_{i_0}} f_{q,r}^{k-1} r'_{i_0}(f_{q,r}) y + f_{q,r}^k r'_{i_0}(y) $$

by the induction on k. Moreover $\mathrm{ad}(E_i)(f_{q,r}^k y) = f_{q,r}^k\, \mathrm{ad}(E_i) y$ for $i \in I$. Hence we have

$$ \varphi(f_{q,r}^{s+1}) = \sum_{p=1}^{n} c_p(q_{i_0}^s) f_{q,r}^{s+1-p} y_p, $$

where $c_p \in \mathbb{C}(q)[t]$ and $y_p \in U_q(\mathfrak{n}_I^-)$ does not depend on s. By Proposition 7.3 ${}^t f_{q,r}(\partial)$ is a linear combination of such φ satisfying $n(\varphi) = r$. The assertion is proved. ∎

We set $b_{q,r}(s) = \tilde{b}_{q,r}(q_{i_0}^s)$ for simplicity. We call $b_{q,r}(s)$ a quantum analogue of the b-function. By definition we have

$$ \langle f_{q,r}^{s+1}, f_{q,r}^{s+1} \rangle = b_{q,r}(s) b_{q,r}(s-1) \cdots b_{q,r}(0). $$

8. EXPLICIT FORMS OF QUANTUM b-FUNCTIONS

Our main results is the following.

THEOREM 8.1. *Let $b_r(s) = \prod_{i=1}^{r}(s + a_i)$ be a b-function of the basic relative invariant of the regular prehomogeneous vector space $(L_I, \mathfrak{n}_I^+)$. Then the quantum analogue $b_{q,r}(s)$ of $b_r(s)$ is given by*

$$b_{q,r}(s) = \prod_{i=1}^{r} q_{i_0}^{s+a_i-1}[s + a_i]_{q_{i_0}} \qquad (\textit{up to a constant multiple}),$$

where $\{i_0\} = I_0 \backslash I$.

We prove this theorem by calculating $b_{q,r}(s)$ in each case. Let $(L_I, \mathfrak{n}_I^+)$ be a regular prehomogeneous vector space with $r + 1$ L_I-orbits. For $p = 1, \dots, r$ we define $\Delta_{(p)}^+$, $\mathfrak{l}_{(\mathfrak{p})}$, and $\mathfrak{n}_{(p)}^{\pm}$ as in Section 3. Set $I_{(p)} = \{i \in I_0 | \mathfrak{g}_{\alpha_i} \subset \mathfrak{l}_{(\mathfrak{p})}\}$ and $U_q(\mathfrak{l}_{(p)}) = \langle K_i^{\pm}, E_j, F_j | i \in I_{(p)} \cup \{i_0\}, j \in I_{(p)} \rangle$. We define the subalgebra $U_q(\mathfrak{n}_{(p)}^-)$ of $U_q(\mathfrak{n}_I^-)$by

$$U_q(\mathfrak{n}_{(p)}^-) = \langle Y_\beta | \beta \in \Delta_{(p)}^+ \rangle.$$

Then $U_q(\mathfrak{n}_{(p)}^-)$ is q-analogue of $\mathbb{C}[\mathfrak{n}_{(p)}^+]$, and $f_{q,p} \in U_q(\mathfrak{n}_{(p)}^-)$ is a q-analogue of basic relative invariant f_p of the regular prehomogeneous vector space $(L_{(p)}, \mathfrak{n}_{(p)}^+)$. We denote by $b_{q,p}(s)$ the q-analogue of the b-function of f_p.

The regular prehomogeneous vector space $(L_{(1)}, \mathfrak{n}_{(1)}^+)$ is of type $(A_1, 1)$, and we have $U_q(\mathfrak{n}_{(1)}^-) = \langle F_{i_0} \rangle, f_{q,1} = cF_{i_0}$ for $c \in \mathbb{C}(q)\backslash\{0\}$. Since $r'_{i_0}(F_{i_0}^{s+1}) = q_{i_0}^s[s+1]_{q_{i_0}} F_{i_0}^s$ for $s \in \mathbb{Z}_{\geq 0}$, we obtain

$$b_{q,1}(s) = c^2[d_{i_0}]_q^{-1} q_{i_0}^s [s+1]_{q_{i_0}}.$$

If we determine $a_p(s) \in \mathbb{C}(q)$ by

$$\langle f_{q,p}^s, f_{q,p}^s \rangle = a_p(s) \langle f_{q,p-1}^s, f_{q,p-1}^s \rangle,$$

then we have $b_{q,p}(s) = (a_p(s+1)/a_p(s)) b_{q,p-1}(s)$. Therefore we can inductively obtain the explicit form of $b_{q,r}$. The next two lemmas are useful for the calculation of $a_p(s)$.

LEMMA 8.2. *Let $\beta \in \Delta_{(p)}^+$.*

(i) ${}^t Y_\beta(\partial)(f_{q,p}^n y) = {}^t Y_\beta(\partial)(f_{q,p}^n)\mathrm{ad}(K_\beta^{-1})y + f_{q,p}^n {}^t Y_\beta(\partial) y$ $(y \in U_q(\mathfrak{n}_{(p)}^-))$.

(ii) ${}^t Y_\beta(\partial)(f_{q,p}^n) = q_{i_0}^{n-1}[n]_{q_{i_0}} f_{q,p}^{n-1} {}^t Y_\beta(\partial)(f_{q,p})$.

(iii) *If $\beta \notin \Delta_{(p-1)}^+$, then we have ${}^t Y_\beta(\partial)(f_{q,p-1}^n) = 0$.*

Proof. (i) This is proved easily by the induction on β. Note that $\mathrm{ad}(E_i)(f_{q,p}) = 0$ for $i \in I$.

(ii) Since $f_{q,p}$ is a central element of $U_q(\mathfrak{n}^-_{(p)})$, this follows from (i).

(iii) Let $\beta \in \Delta^+_{(p)} \setminus \Delta^+_{(p-1)}$. Then there exists some $j \in I$ such that $\beta \in \mathbb{Z}_{>0}\alpha_j + \sum_{i \neq j} \mathbb{Z}_{\geq 0}\alpha_i$ and $\gamma \in \sum_{i \neq j} \mathbb{Z}_{\geq 0}\alpha_i$ for any $\gamma \in \Delta^+_{(p-1)}$. Hence we have $U_q(\mathfrak{n}^-_{(p-1)})_{-(\lambda_{p-1}-\beta)} = \{0\}$, and the statement follows. ∎

LEMMA 8.3. *For $2 \leq p \leq r$ we have the decomposition*

$$f_{q,p} = \sum_{j=1}^{t_p} Y_{\beta_j^{(p)}} \ \mathrm{ad}(u_j^{(p)}) f_{q,p-1}$$

satisfying the following conditions:

(I) $\beta_1^{(p)}, \dots, \beta_{t_p}^{(p)} \in \Delta^+_{(p)} \setminus \Delta^+_{(p-1)}$, $u_1^{(p)}, \dots, u_{t_p}^{(p)} \in U_q(\mathfrak{l}_{(p)}) \cap U_q(\mathfrak{n}^-)$.

(II) *For any j there exists a scalar $c_j^{(p)} \in \mathbb{C}_{(q)}$ such that ${}^tY_{\beta_j^{(p)}}(\partial) f_{q,p} = c_j^{(p)} \mathrm{ad}(u_j^{(p)}) f_{q,p-1}$.*

Proof. By Lemma 3.1 it is sufficient to show the existence for $p = r$. We take $f_{q,r}$ as in Lemma 6.4 and Remark 6.5. It is easy to show that there exist the following decompositions of $f_{q,r}$ satisfying (I) and (II).

(i) Type $(A_{2n-1,}, n)$ $(r = n)$.

$$f_{q,n-1} = \sum_{\sigma \in \mathfrak{S}_{n-1}} (-q)^{\ell(\sigma)} Y_{1,\sigma(1)} \cdots Y_{n-1,\sigma(n-1)},$$

$$f_{q,n} = \sum_{j=1}^{n} (-q^{-1})^{n-j} Y_{n,j} \mathrm{ad}(F_{n+j}F_{n+j+1} \cdots F_{2n-1}) f_{q,n-1},$$

$${}^tY_{n,j}(\partial) f_{q,n} = (-q)^{n+j-2} \mathrm{ad}(F_{n+j}F_{n+j+1} \cdots F_{2n-1}) f_{q,n-1}.$$

(Note that we have

$$\begin{aligned}(-q^{-1})^{n-j} &\mathrm{ad}(F_{n+j}F_{n+j+1} \cdots F_{2n-1}) f_{q,n-1} \\ &= (-q^{-1})^{n-j} \sum_{\sigma \in \mathfrak{S}_{n-1}} (-q)^{\ell(\sigma)} Y_{1,i_{\sigma(1)}} \cdots Y_{n-1,i_{\sigma(n-1)}},\end{aligned}$$

where $i_1 = 1, \dots, i_{j-1} = j-1, i_j = j+1, \dots, i_{n-1} = n$. This is the quantum analogue of the (n, j)-cofactor.)

(ii) Type $(B_n, 1)$ $(r = 2)$.

$$f_{q,1} = Y_1 = F_1,$$

$$f_{q,2} = \sum_{j=1}^{n-1} (-q^2)^{j+1-n} Y_{n+j} \mathrm{ad}(F_{n-j}F_{n-j-1}\cdots F_2) f_{q,1}$$
$$+(q+q^{-1})^{-2} q^{-1} (-q^2)^{1-n} Y_n \mathrm{ad}(F_n F_{n-1}\cdots F_2) f_{q,1},$$

$${}^tY_{n+j}(\partial) f_{q,2} = \begin{cases} (q+q^{-1})^{-1}(-q^2)^{j-1}\mathrm{ad}(F_n\cdots F_2)f_{q,1} & (j=0) \\ -(q+q^{-1})^{-1}(-q^2)^{j-2} \\ \qquad \times \mathrm{ad}(F_{n-j}\cdots F_2)f_{q,1} & (1\le i\le n-1). \end{cases}$$

(iii) Type (C_n, n) $(r = n)$.

$$f_{q,n-1} = \sum_{\sigma\in\mathfrak{S}_{n-1}} (-q)^{-\ell(\sigma)} Y_{i_1, i_{\sigma(1)}} \cdots Y_{i_{n-1}, i_{\sigma(n-1)}} \qquad (i_k = k+1),$$

$$f_{q,n} = Y_{1,1} f_{q,n-1} + \sum_{j=2}^{n} \frac{(-q)^{-1-j}}{q+q^{-1}} Y_{1,j} \mathrm{ad}(F_{j-1}F_{j-2}\cdots F_1) f_{q,n-1},$$

$${}^tY_{1,j}(\partial) f_{q,n} = \begin{cases} (-q)^{2n-2}(q+q^{-1}) f_{q,n-1} & (j=1) \\ -(-q)^{2n-j}\mathrm{ad}(F_{j-1}\cdots F_1)(f_{q,n-1}) & (j\ge 2). \end{cases}$$

(iv) Type $(D_n, 1)$ $(r = 2)$.

$$f_{q,1} = F_1 = Y_1,$$

$$f_{q,2} = \sum_{j=1}^{n-1} (-q)^{j+1-n} Y_{n+j-1} \mathrm{ad}(F_{n-j}F_{n-j-1}\cdots F_2) f_{q,1},$$

$${}^tY_{n+j-1}(\partial) f_{q,2} = (-q)^{n+j-3} \mathrm{ad}(F_{n-j}\cdots F_2) f_{q,1}.$$

(v) Type $(D_{2n}, 2n)$ $(r = n)$.

$$f_{q,n-1} = \sum_{\sigma\in S_{2n-2}} (-q^{-1})^{\ell(\sigma)} Y_{i_{\sigma(1)}, i_{\sigma(2)}} \cdots Y_{i_{\sigma(2n-3)}, i_{\sigma(2n-2)}} \quad (i_k = k+2),$$

$$f_{q,n} = \sum_{j=2}^{2n} (-q)^{2-j} Y_{1,j} \mathrm{ad}(F_{j-1}F_{j-2}\cdots F_2) f_{q,n-1},$$

$${}^tY_{1,j}(\partial) f_{q,n} = (-q)^{4n-2-j} \mathrm{ad}(F_{j-1}F_{j-2}\cdots F_2) f_{q,n-1}.$$

(vi) Type $(E_7, 1)$ $(r = 3)$

$$f_{q,2} = \psi_{27},$$

$$f_{q,3} = (1+q^8+q^{16}) Y_{27}\psi_{27}$$
$$+ \frac{q^{-10}+q^{-8}-q^{-4}+1+q^2}{1+q^2} \sum_{j=11}^{26} (-q)^{|\beta_j|-1} Y_j \psi_j,$$

$${}^tY_j(\partial) f_{q,3} = (1+q^8+q^{16})(-q)^{|\beta_j|-1}\psi_j.$$

∎

By Lemmas 8.2 and 8.3 we have

$$\begin{aligned}
&\langle f_{q,p}^{s_1} f_{q,p-1}^{s_2}, f_{q,p}^{s_1} f_{q,p-1}^{s_2} \rangle \\
&\quad = \sum_{j-1}^{t_p} \langle {}^t Y_{\beta_j^{(p)}}(\partial)(f_{q,p}^{s_1} f_{q,p-1}^{s_2}), g_j^{(p)} f_{q,p}^{s_1-1} f_{q,p-1}^{s_2} \rangle \\
&\quad = \sum_{j=1}^{t_p} c_j^{(p)} q_{i_0}^{s_1-1} q^{-s_2(\beta_j^{(p)}, \lambda_{p-1})} [s_1]_{q_{i_0}} \langle f_{q,p}^{s_1-1} g_j^{(p)} f_{q,p-1}^{s_2}, f_{q,p}^{s_1-1} g_j^{(p)} f_{q,p-1}^{s_2} \rangle,
\end{aligned}$$

where $g_j^{(p)} = \mathrm{ad}(u_j^{(p)}) f_{q,p}$. Note that $g_j^{(p)} f_{q,p} = f_{q,p} g_j^{(p)}$ since $f_{q,p}$ is a central element of $U_q(\mathfrak{n}_{(p)}^-)$ and $g_j^{(p)} \in U_q(\mathfrak{n}_{(p)}^-)$. Moreover we can calculate $C_j^{(p)}(s_1, s_2) \in \mathbb{C}(q)$ such that

$$\left\langle f_{q,p}^{s_1-1} g_j^{(p)} f_{q,p-1}^{s_2}, f_{q,p}^{s_1-1} g_j^{(p)} f_{q,p-1}^{s_2} \right\rangle = C_j^{(p)}(s_1, s_2) \left\langle f_{q,p}^{s_1-1} f_{q,p-1}^{s_2+1}, f_{q,p}^{s_1-1} f_{q,p-1}^{s_2+1} \right\rangle;$$

hence we have

$$\langle f_{q,p}^{s_1} f_{q,p-1}^{s_2}, f_{q,p}^{s_1} f_{q,p-1}^{s_2} \rangle = C^{(p)}(s_1, s_2) \langle f_{q,p}^{s_1-1} f_{q,p-1}^{s_2+1}, f_{q,p}^{s_1-1} f_{q,p-1}^{s_2+1} \rangle,$$

where $C^{(p)}(s_1, s_2) = \sum_{j=1}^{t_p} C_j^{(p)}(s_1, s_2) c_j^{(p)} q_{i_0}^{s_1-1} q^{-s_2(\beta_j^{(p)}, \lambda_{p-1})} [s_1]_{q_{i_0}}$. From this formula we obtain

$$a_p(s) = \prod_{t=0}^{s-1} C^{(p)}(s-t, t).$$

For example we calculate $C_j^{(p)}(s_1, s_2)$ of type (A_{2n-1}, n) as follows. We set β_{ij} and Y_{ij} as in Lemma 6.4. The analogue $f_{q,p}$ $(1 \le p \le n)$ is defined by

$$f_{q,p} = \sum_{\sigma \in \mathfrak{S}_p} (-q)^{\ell(\sigma)} Y_{1,\sigma(1)} \cdots Y_{p,\sigma(p)}.$$

Similar to the proof of Lemma 8.3 we have $t_p = p, \beta_j^{(p)} = \beta_{p,j}, u_j^{(p)} = (-q)^{j-p} F_{n+j} \cdots F_{n+p-1}$, and $c_j^{(p)} = q^{2n-2}$. Clearly $C_{t_p}^{(p)}(s_1, s_2) = 1$. For $1 \le j \le t_p - 1$ we have

$$\begin{aligned}
&g_j^{(p)} = -q^{-1} \mathrm{ad}(F_{n+j}) g_{j+1}^{(p)}, && g_{j+1}^{(p)} = -q\, \mathrm{ad}(E_{n+j}) g_j^{(p)}, \\
&\mathrm{ad}(E_{n+j}) f_{q,p} = \mathrm{ad}(E_{n+j}) f_{q \cdot p-1} = 0, && \\
&\mathrm{ad}(F_{n+j}) f_{q,p} = 0, && \mathrm{ad}(F_{n+j}) f_{q,p-1} = -q\, \delta_{j,p-1} g_{p-1}^{(p)}
\end{aligned}$$

(see [8]). Therefore we have

$$\begin{aligned}\mathrm{ad}({}^t u_j^{(p)})(f_{q,p}^{s_1-1} g_j^{(p)} f_{q,p-1}^{s_2}) &= -q\ \mathrm{ad}({}^t u_{j+1}^{(p)})\mathrm{ad}(E_{n+j})(f_{q,p}^{s_1-1} g_j^{(p)} f_{q,p-1}^{s_2})\\ &= \mathrm{ad}({}^t u_{j+1}^{(p)})(f_{q,p}^{s_1-1} g_{j+1}^{(p)} f_{q,p-1}^{s_2}) = \cdots\\ &= \mathrm{ad}({}^t u_{p-1}^{(p)})(f_{q,p}^{s_1-1} g_{p-1}^{(p)} f_{q,p-1}^{s_2})\\ &= q^{-s_2} f_{q,p}^{s_1-1} f_{q,p-1}^{s_2+1},\end{aligned}$$

and

$$\begin{aligned}f_{q,\,p}^{s_1-1} g_j^{(p)} f_{q,\,p-1}^{s_2} &= -q^{-1}\mathrm{ad}(F_{n+j})(f_{q,\,p}^{s_1-1} g_{j+1}^{(p)} f_{q,\,p-1}^{s_2}) = \cdots\\ &= (-q^{-1})^{p-j-1}\mathrm{ad}(F_{n+j}\cdots F_{n+p-2})(f_{q,\,p}^{s_1-1} g_{p-1}^{(p)} f_{q,\,p-1}^{s_2}).\end{aligned}$$

Since $g_{p-1}^{(p)} f_{q,\,p-1} = q^{-1} f_{q,\,p-1} g_{p-1}^{(p)}$, we have hence

$$f_{q,\,p}^{s_1-1} g_j^{(p)} f_{q,\,p-1}^{s_2} = q^{-s_2}[s_2+1]_q^{-1}\mathrm{ad}\big(u_j^{(p)}\big)\big(f_{q,\,p}^{s_1-1} f_{q,\,p-1}^{s_2+1}\big).$$

By Proposition 7.1 we have

$$\begin{aligned}&\langle f_{q,\,p}^{s_1-1} g_j^{(p)} f_{q,\,p-1}^{s_2}, f_{q,\,p}^{s_1-1} g_j^{(p)} f_{q,\,p-1}^{s_2}\rangle\\ &\quad= q^{-s_2}[s_2+1]_q^{-1}\Big\langle f_{q,\,p}^{s_1-1} g_j^{(p)} f_{q,\,p-1}^{s_2}, \mathrm{ad}(u_j^{(p)})(f_{q,\,p}^{s_1-1} f_{q,\,p-1}^{s_2+1})\Big\rangle\\ &\quad= q^{-s_2}[s_2+1]_q^{-1}\Big\langle \mathrm{ad}({}^t u_j^{(p)})(f_{q,\,p}^{s_1-1} g_j^{(p)} f_{q,\,p-1}^{s_2}), f_{q,\,p}^{s_1-1} f_{q,\,p-1}^{s_2+1}\Big\rangle\\ &\quad= q^{-2s_2}[s_2+1]_q^{-1}\Big\langle f_{q,\,p}^{s_1-1} f_{q,\,p-1}^{s_2+1}, f_{q,\,p}^{s_1-1} f_{q,\,p-1}^{s_2+1}\Big\rangle;\end{aligned}$$

hence $C_j^{(p)}(s_1, s_2) = q^{-2s_2}[s_2+1]_q^{-1}$ for $1 \le j < t_p - 1$. Therefore $a_p(s)$ of type (A_{2n-1}, n) is given by

$$a_p(s) = q^{\frac{s(s+2p-3)}{2}} \prod_{i=1}^{s} [i+p-1]_q.$$

Similarly we have the following.

LEMMA 8.4. *We have the explicit descriptions of $a_p(s)$ $(2 \le p \le r)$ (up to constant multiple) as follows.*

$$(A_{2n-1}, n)\colon\ a_p(s) = q^{\frac{s(s+2p-3)}{2}} \prod_{i=1}^{s} [i+p-1]_q \qquad (2 \le p \le r = n),$$

$$(B_n, 1)\colon\ a_p(s) = (q+q^{-1})^{-s} q^{s(s+2n-4)} \prod_{i=1}^{s} \left[i + \frac{2n-3}{2}\right]_{q^2}$$

$$(p = r = 2),$$

$$(C_n, n):\ a_p(s) = (q+q^{-1})^s q^{s(s+p-2)} \prod_{i=1}^{s} \left[i + \frac{p-1}{2} \right]_{q^2}$$
$$(2 \le p \le r = n),$$
$$(D_n, 1):\ a_p(s) = q^{\frac{s(s+2n-5)}{2}} \prod_{i=1}^{s} [i+n-2]_q$$
$$(p = r = 2),$$
$$(D_{2n,} 2n):\ a_p(s) = q^{\frac{s(4p+s-5)}{2}} \prod_{j=1}^{s} [j+2p-2]_q$$
$$(2 \le p \le r = n),$$
$$(E_7, 1):\ a_p(s) = (1+q^8+q^{16})^{2s} q^{\frac{s(s+15)}{2}} \prod_{i=1}^{s} [i+8]_q$$
$$(p = r = 3).$$

We note that $a_2(s)$ of type $(E_7, 1)$ is that of type $(D_6, 1)$ by Lemma 3.1. From Lemma 8.4 we obtain the explicit form of $b_{q,r}(s)$ as follows.

$$(A_{2n-1}, n):\ b_{q,n}(s) = \prod_{p=1}^{n} q^{s+p-1}[s+p]_q$$
$$(B_n, 1):\ b_{q,2}(s) = (q+q^{-1})^{-2} q^{2s}[s+1]_{q^2} q^{2s+2n-3} \left[s + \frac{2n-1}{2} \right]_{q^2}$$
$$(C_n, n):\ b_{q,n}(s) = (q+q^{-1})^n \prod_{p=1}^{n} q^{2s+p-1} \left[s + \frac{p-1}{2} \right]_{q^2}$$
$$(D_n, 1):\ b_{q,2}(s) = q^s[s+1]_q q^{s+n-2}[s+n-1]_q$$
$$(D_{2n}, 2n):\ b_{q,n}(s) = \prod_{p=1}^{n} q^{s+2p-2}[s+2p-1]_q$$
$$(E_7, 1):\ b_{q,3}(s) = (1+q^8+q^{16})^2 q^s[s+1]_q q^{s+4}[s+5]_q q^{s+8}[s+9]_q$$

Note that we have $q_{i_0} = q^2$ (type B, C) or q (otherwise).

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