On the Number of Conjugacy Classes in a Finite Group*

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INTRODUCTION

In the following, G will denote a finite group and we will use the standard notation of the Theory of Groups. We say that G is a \( D_\pi \)-group, if G contains a unique conjugacy class of maximal \( \pi \)-subgroups. In addition, a \( D_\pi \)-group G is said to be a \( L_\pi \)-group, if all epimorphic images of subgroups of G are \( D_\pi \)-groups. For example, if G is \( \pi \)-separable, then G is a \( L_\pi \)-group. In particular, all \( \pi \)-solvable groups are \( L_\pi \)-groups.

In this work, we get new results relative to the conjugacy classes of a finite group G.

Let N be a normal subgroup of G, \( \pi \) a set of prime numbers, \( r(G) \) (resp. \( r^\pi(G) \)) the number of conjugacy classes of elements (resp. \( \pi \)-elements) of G, \( \pi(G) \) the set of all different primes dividing \( |G| \), \( G_\pi \) the set of all \( \pi \)-elements of G; and \( G = G/N \).

In this paper, we analyze the number \( r_\pi(G) \) of conjugacy classes of \( \pi \)-elements of G, through the local analysis of the number \( r_\pi^\pi(gN) \) of conjugacy classes of \( \pi \)-elements of G which intersect the coset \( gN \). Our aims are threefold: to obtain upper and lower bounds of the number \( r_\pi(G) \) of conjugacy classes of \( \pi \)-elements of G, in terms of the numbers \( r_\pi(G/N) \), \( r^\pi(N) \), and \( |G'| \), where \( G' \) denotes the derived subgroup of G; to get the residue class of \( r_\pi(G) \), modulo the "best" number, given in terms of the primes dividing \( |G| \); and finally, to analyze the conjugacy-vector

\[ A_G = (|C_G(g_1)|, ..., |C_G(g_r)|) \]

of G, assuming that G is the disjoint union of the classes \( C_G(g_i), \; i = 1, ..., r = r(G), \) and \( |C_G(g_1)| \geq \cdots \geq |C_G(g_r)| \). The results obtained are useful both for the calculation of the conjugacy-vector of a finite group and for the classification of finite groups according to the number of conjugacy classes (see Examples 11–13 in Section 4). Moreover, they enable us to obtain the following inequalities:

\[ (A) \quad r_\pi(G) \leq r_\pi(G/N) \cdot r^\pi(N), \]

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and equality holds if and only if \( C_{G/N}(gN) = C_G(g) N/N \) for each \( \pi \)-element \( g \) of \( G \).

(B) If \( G \) is a \( L_x \)-group and \( H \) a Hall \( \pi \)-subgroup of \( G \) such that \( HN/N \) is abelian, then

\[
 r^\pi(G) \leq r^\pi(G/N) \cdot r_H(H \cap N),
\]

where \( r_H(H \cap N) \) denotes the number of conjugacy \( H \)-classes that compose the normal subgroup \( H \cap N \) of \( H \).

(C) If \( s \) is the number of elements of any maximal \( \pi(G) \)-partition of \( G \) (see definitions previous to Lemma (3.18)), then

\[
 r(G) \leq r(G/N) \cdot r(N) - (s - 1) \cdot d^2_{|G|},
\]

where \( d_{|G|} = \text{g.c.d.} \ (p - 1 \mid p \) is a prime dividing \( |G| \). In particular, if all elements of \( G \) have primary power orders, then the following is true:

\[
 r(G) \leq r(G/N) \cdot r(N) - (|\pi(G)| - 1) \cdot d^2_{|G|}.
\]

(D) The following inequality holds:

\[
 r^\pi_{G}(gN) \leq r^\pi_{N_G(gN)}(N),
\]

assuming that \( C_G(\bar{g}) \) is a \( \pi \)-group. In addition, equality holds iff

\[
 \bar{g} \in \bigcap_{z \in N_G(gN)_x} \overline{C_G(z)}.
\]

Inequalities (1) and (2) generalize Gallagher's inequality,

\[
 r(G) \leq r(G/N) \cdot r(N) \quad \text{(cf. [5])},
\]

which follows from them as a special case for \( \pi = \pi(G) \). Further, (3) improves the \( r(G/N) \cdot r(N) \) bound. Inequality (2) improves Sah's inequality,

\[
 r(G) \leq |G/N| \cdot r(N),
\]

when \( G/N \) is an abelian group (cf. [12]). In fact, (2) yields

\[
 r(G) \leq |G/N| \cdot r_G(N),
\]

where \( r_G(N) \) denotes the number of conjugacy classes of elements of \( G \) that make up \( N \). Even more, if \( G/N \) is an abelian \( \pi \)-group, then (D) yields

\[
 r^\pi(G) \leq |G/N| \cdot r^\pi_G(N).
\]
In general, we obtain
\[ r(G) \leq (r(G/N) - |Z(\overline{G})|) \cdot r(N) + |Z(\overline{G})| \cdot r_G(N). \]

(E) If \( G \) is a \( \mu \)-group and \( H \) a Hall \( \pi \)-subgroup of \( G \), then
\[ r^\pi(G) \geq (r^\pi(G/N) - 1) \cdot (|N \cap H|/|H, G| \cap N|) + r^\pi_G(N). \quad (4) \]

In particular, putting \( \pi = \pi(G) \) into (4), we get
\[ r(G) \geq (r(G/N) - 1) \cdot |NG'/G'| + r_G(N), \]
an inequality that includes as a special case the one given by E. A. Bertram (cf. [1, Th. 3; or 2, Lemma (2.1)]).

In addition, the following equalities and congruences are proved:

(I) For each element \( g \) of \( G \), there exists a non-negative integer number \( k \) such that
\[ r_G(gN) = |N/(N \cap N_G(gN))| + k \cdot d_{\{N_G(gN)\}}. \quad (5) \]

(II) If \( s^\pi_g \) denotes the number of conjugacy \( N \)-classes of \( \pi \)-elements of \( N \) fixed by the automorphism \( f_g : N \to N \) defined by \( f_g(x) = x^g \) for each \( x \in N \), then
\[ s^\pi_g \equiv r^\pi_G(gN) \pmod{d_{\{N_G(gN)\}.}} \]

(III) If \( G \) is a \( \mu \)-group and \( H \) a Hall \( \pi \)-subgroup of \( G \) satisfying \( N \cap [H, G] = 1 \), then
\[ r^\pi(G) = r^\pi(G/N) \cdot |H \cap N| \quad (6) \]

(putting \( \pi = \pi(G) \), (6) yields Rusin’s result given in [11 Prop. 3]).

(IV) The following is true:

(a) \( r^\pi_G(N) \equiv r^\pi(N) \pmod{d_{|G|} \cdot d_{|G|}}. \)

(b) \( r^\pi(G) \equiv |G| \pmod{d_{|G|} \cdot d_{|G|}} \) with \( d_{|G|} \) = g.c.d. (\( p-1 \mid p \) is a prime dividing \( |G| \) and \( p \in \pi \).

(c) \( r^\pi(G) \equiv |G| \pmod{\delta^\pi_{|G|}} \), where \( \delta^\pi_{|G|} \) is defined by \( \delta^\pi_{|G|} \) = Lo. \( q_1, q_2, \ldots, q_{\pi - 1}, \ldots, q_s - 1 \), assuming that \( \pi(G) = \{ q_1, \ldots, q_s \} \) and \( \pi(G) \cap \pi = \{ q_1, \ldots, q_e \}. \)

(d) \( r^\pi(G) \equiv r^\pi(G/N) \cdot r^\pi(N) \pmod{\text{l.c.m.}(\delta^\pi_{|G|}, d_{|G|} \cdot d_{|G|})}. \)

Congruences of the type (a) or (b) for \( \pi = \pi(G) \) were obtained in a different way by Poland (cf. [10, Prop. (3.9)]) and Mann (cf. [8, Eq. (16)])). Congruence (c) generalizes Hirsch’s congruence given in [7] for \( \pi = \pi(G) \).
Congruence (d), determines \( r^n(G) \), in terms of \( r^n(G/N) \) and \( r^n(N) \), modulo the least common multiple of \( \delta_{|G|}^n \) and \( d_{|G|} \). Congruence given into (II) relates the number \( r^*_n(gN) \) with the number \( s^*_n \); naturally, the number \( s^*_n \) is easier to calculate than \( r^*_n(gN) \).

Finally, if \( G \) is a semidirect product of \( N \) by \( K \), we prove that \( r_n(hN) \geq r_n(N) \) for each \( h \in K \), where \( E = (NC_K(h))/M, M = \langle [hN, N] \rangle \) and \( \bar{N} = N/M \). Furthermore, the bound is reached when \( N \) is abelian (cf. [16, Lemma (2.10)]). In this case, if \( \{ \bar{g}_1 = 1, \bar{g}_2, ..., \bar{g}_t \} \) is a complete system of representatives from distinct conjugacy classes of \( G \), we notice that getting \( A_G \) through the local tuples \( A^G_{\bar{g}_iN} \), \( i = 1, ..., t \), also allows us to tabulate directly the tuples \( A_{G/L} \) for each normal subgroup \( L \) of \( G \) contained in \( N \).

1. Notation

If \( S \) is a non-empty subset of \( G \) and \( \pi \) a set of primes, we define \( r^n(S) = |\{Cl_G(x)|Cl_G(x) \cap S \neq \emptyset \text{ and } x \text{ is a } \pi\text{-element} \}|, S_\pi \) denotes the set of all \( \pi\text{-elements of } S \), and \( S'_\pi = S - S_\pi \), namely, the complementary set of \( S_\pi \) with respect to \( S \). Evidently, \( r^n(S) \) is the number of conjugacy classes of \( \pi\text{-elements that contains the normal set } \bigcup_{x \in G} S^x \). In particular, \( r^n(G) = r^n(S) \) (resp. \( r(G) = r^n(G) \)) denotes the number of conjugacy classes of \( \pi\text{-elements of } G \). If \( S \) is a non-empty subset of \( G \) and \( (\bigcup_{x \in G} S^x) = \bigcup_{i=1}^r Cl_G(z_i) \) with \( |C_G(z_1)| \geq \cdots \geq |C_G(z_r)| \), we define \( A_G = (|C_G(z_1)|, ..., |C_G(z_r)|) \) and the \( r^n(G)\)-tuple \( A_G = A_G \) will be called the conjugacy-\( \pi \)-vector of \( G \). For each normal subgroup \( N \) of \( G \) and each \( g \in G \) we define \( B^g \) = \( \{ n \in N \mid Cl_N(n)^g = Cl_N(n) \} \). We will use the following convention: when \( \pi = \pi(G) \), the symbol \( \pi \) will be dropped from all the above notations. Further, if \( S_1 \) and \( S_2 \) are two non-empty subsets of \( G \), we define \( T_{S_1, S_2} = \{ (x, y) \in S_1 \times S_2 \mid xy = yx \} \) and \( C_{S_1, S_2}(g) = \{ x \in S_1 \mid xg = gx \} \). Every element \( g \) of \( G \) has a unique decomposition \( g = g_n \cdot g_\pi = g_n \cdot g_\pi \) into a \( \pi\text{-element } g_n \) and a \( \pi'\text{-element } g_\pi ', \) where \( \pi' \) denotes the complementary set of primes with respect to \( \pi \). Further, \( g_n \) and \( g_\pi \) are powers of \( g \). For each positive integer number \( t = q_1^{a_1} \cdots q_s^{a_s} \), with \( q_i \) prime for each \( i = 1, ..., s \) and \( q_i \neq q_j \) for each \( i \neq j \), we define \( d = \text{g.c.d.}(q_1 - 1, ..., q_s - 1) \) and \( \delta^n = \text{g.c.d.}(q_1^{a_1} - 1, ..., q_s^{a_s} - 1, q_{s+1} - 1, ..., q_t - 1) \), where \( \{ q_1, ..., q_s \} = \pi \cap \{ q_1, ..., q_t \} \). \( G = Nx_\lambda K \) will denote a semidirect product of the normal \( N \) by the subgroup \( K \) with associated homomorphism \( \lambda: K \rightarrow \text{Aut}(N) \). If we write \( f \), instead of \( \lambda \), we will understand that \( f(h) \) is an automorphism of \( N \) without fixed points for each \( h \in K^* = K \setminus \{ 1 \} \), that is, \( Nx_\lambda K \) is a Frobenius group of kernel \( N \) and complement \( K \); in this case, we will say that \( K \) acts f.p.f. over \( N \). We will also use the following notation: \( C_m \) will designate a direct product of \( t \) copies of the cyclic group of order \( m \), and,
finally, $N$ always will denote a normal subgroup of $G$, $\bar{G} = G/N$, and $\bar{x} = xN$ for every $x \in G$.

2. Basic Lemmas

**Lemma (2.1).** Let $g \in G$. Then we have the following assertions:

(i) $r^n_G(gN) = (1/|N|) \sum_{g^n \in (gN)_e} |C_G(gn)|/|C_G(g)|$, assuming that $(gN)_e$ is non-empty.

(ii) $r^n_N(|N|) = (1/|N|) \sum_{x \in N_e} |C_N(n)|$.

(iii) If $s^g_\pi$ denotes the number of conjugacy $N$-classes of $\pi$-elements of $N$ fixed by the automorphism $f_g : N \to N$ defined by $f_g(x) = x^g = g^{-1}xg$ for every $x \in N$, then $s^g_\pi = (1/|N|) \cdot \sum_{n \in s} b^n_\pi |C_N(n)|$.

(iv) If $\{\text{Cl}_G(gm_1), \ldots, \text{Cl}_G(gm_v)\}$ is the set of all conjugacy classes that intersect $(gN)_\pi (\neq \emptyset)$, ordered so that $|C_G(gm_i)| \geq |C_G(gm_{i+1})|$ for each $i = 1, \ldots, v - 1$, then the following is true:

$$\pi \Delta^G_N = \left(\frac{1}{|N|} \cdot \sum_{g \in (gN)_e} |C_G(g)|/|C_G(g)|\right) \cdot \left(\frac{1}{|N|} \cdot \sum_{n \in N_e} |C_N(n)|\right).$$

**Proof.** (i) Let $\{gm_1, \ldots, gm_v\}$ be a complete set of representatives for the conjugacy classes which intersect $(gN)_\pi$. We have $v = r^n_G(gN)$ and

$$\left(\frac{1}{|N|} \cdot \sum_{g \in (gN)_e} |C_G(gn)|/|C_G(g)|\right) = \left(\frac{1}{|N|} \cdot \sum_{i=1}^v (|C_G(gm_i)| \cdot |\text{Cl}_G(gm_i) \cap gN|)/|C_G(g)|\right).$$

In addition, it is not difficult to prove that

$$|\text{Cl}_G(gn) \cap gN| = \left(\frac{|N|}{|C_G(\bar{g})|}|C_G(g)|\right),$$

for each $n \in N$. Thus, by taking this value into (7), we get the desired equality.

(ii) Follows from (i) putting $g = 1$ and $G = N$.

(iii) Follows arguing as in (i).

(iv) Follows directly from the following equality: $|\text{Cl}_G(gn)| = |\text{Cl}_G(\bar{g})| \cdot |[gn, N]| \cdot |N_G(gN) : C_G(gn)N|$, for each $n \in N$ (cf. [15, Th. (3.1)]).
**Lemma (2.2).** For each element \( g \) of \( G \), there exists \( g_n \in (g N) \) such that

\[
r_p^r_N(g N) \geq |(g N) _n|/|[[g_n, N_G(g N)]]|.
\]

**Proof.** Suppose that \((gn_1)_z = gn_2 \) for some \( z \in N_G(g N) \) and \( gn_1, gn_2 \in (g N) _n \). Then \( gn_2 = gn_1[g_n_1, z] \) and consequently \(|\text{Cl}_{N_G(g N)}(gn_1)| = |[gn_1, N_G(g N)]|\). Further, by using the above notation, we have \( (g N)_n = \text{Cl}_{N_G(g N)}(gm_1) \cup \cdots \cup \text{Cl}_{N_G(g N)}(gm_v) \): therefore, \(|(g N)_n| = \sum_{i=1}^v |\text{Cl}_{N_G(g N)}(gm_i)| \leq |[[gn, N_G(g N)]]| \cdot v \), where \( gn \in (g N) _n \) is chosen so that \(|[gm, N_G(g N)]| gm \in (g N) _n \).

**Lemma (2.3).** Let \( g \in G \) and let \( g = g_\pi \cdot g_\eta = g_\eta \cdot g_\pi \) be its decomposition into \( \pi \)-singular and \( \pi \)-regular parts. If \( g = x \cdot y \), with \( x, y \) elements of \( G \) satisfying \( o(x) = o(g_\pi), o(y) = o(g_\eta) \), and \([x, g_\pi] = [y, g_\eta] = 1\), then \( x = g_\pi \) and \( y = g_\eta \).

**Proof.** It is a well-known result.

**Lemma (2.4).** For any non-empty subsets \( S_1 \) and \( S_2 \) of \( G \), we have

\[
\sum_{x \in S_1} |C_G(x) \cap S_2| = \sum_{y \in S_2} |C_G(y) \cap S_1|.
\]

**Proof.** Counting the elements of the set \( T_{S_1, S_2} \) in two different ways and equating the answers we directly obtain the desired result.

**Lemma 2.5.** For any \( \pi \)-elements \( \bar{x} \) and \( \bar{y} \) of \( G \), the following equality holds:

\[
|T_{(xN)_{\eta}, (yN)_{\eta}}| = |T_{(xN)_{\eta}, (yN)_{\eta}}|.
\]

**Proof.** We consider the map \( f: T_{(xN)_{\eta}, (yN)_{\eta}} \rightarrow T_{(xN)_{\eta}, (yN)_{\eta}} \) defined by \( f((xn, ym)) = ((xn)_{\eta}, ym(xn)_{\eta}) \) and we will prove that \( f \) is a bijective map. First, we notice that \((xn)_{\eta}\) is an element of \( N \) for each \( n \in N \), since \((xn)_{o(x)} \in N \) and \( \langle ((xn)_{o(x)})_{\eta} \rangle = \langle (xn)_{\eta} \rangle \). Moreover \( [[(xn)_{\eta}, ym(xn)_{\eta}] = 1 \) and if \( p \) is a prime divisor of \( o(xn) \) for some \( p \in \pi \), then \( p \) divides \( o(ym(xn)_{\eta}) \). Therefore \( f \) is a well-defined map. Now, we observe that if \((xn, ym)\) is an element of \( T_{(xN)_{\eta}, (yN)_{\eta}} \), then Lemma (2.3) yields \((ym(xn)_{\eta})_\eta = ym \) and \((ym(xn)_{\eta})_\eta = (xn)_{\eta} \). Assume then that \( f((xn, ym)) = f((xn_1, ym_1)), \) that is, \((xn)_\eta = (xn_1)_\eta \) and \( ym(xn)_\eta = ym_1(xn_1)_\eta \). Consequently \( ym = ym_1 \), \( (xn)_\eta = (xn_1)_\eta \), and we have \( xn = (xn)_{\eta}(xn)_{\pi} = (xn_1)_{\eta}(xn_1)_{\pi} = xn_1 \) hence \( f \) is injective.

Finally, we prove that \( f \) is surjective. In fact, for each element \((xn_2, ym_2)\) of the set \( T_{(xN)_{\eta}, (yN)_{\eta}} \) there exists \((xn_2(ym_2)_{\pi}, (ym_2)_{\pi})\) in \( T_{(xN)_{\eta}, (yN)_{\eta}} \), the image of which under \( f \) is the element \(((xn_2(ym_2)_{\pi}))_{\eta}, \).
LEMMA (2.6). For each \( \pi \)-element \( \bar{x} \) of \( \overline{G} \) the following is true:

\[
|T_{(x N)_{\pi}}| = |T_{x N, N_{\pi}}|.
\]

Proof. Using \( N = N_{\pi} \cup N \) and \( x N = (x N_{\pi} \cup (x N))_{\pi} \) we have the following equalities: \( |T_{(x N)_{\pi}}| = |T_{(x N\pi, N_{\pi})} + |T_{(x N)_{\pi}}, N_{\pi}| = (applying \ (2.5) \ with \ \bar{y} = 1) = |T_{(x N)}, N_{\pi}|. \)

LEMMA (2.7). For each element \( g \) of \( G \), we have \( r_{\pi}^g(g N) = 0 \) iff \( g \) is not a \( \pi \)-element of \( \overline{G} \).

Proof If \( p \in \pi \) is a prime dividing \( o(g) \), then \( p \) divides \( o(g^{\bar{N}}) \) for any elements \( x \in G \) and \( n \in N \), hence the coset \( g N \) cannot have \( \pi \)-elements and consequently \( r_{\pi}^g(g N) = 0 \). Conversely, if \( g \) is a \( \pi \)-element, then \( g_{\pi} \in N \) and \( g N = g_{\pi} g_{\pi} N = g_{\pi} N \) hence \( g_{\pi} \) is a \( \pi \)-element of the set \( (g N)_{\pi} \) and therefore \( r_{\pi}^g(g N) \geq 1 \).

LEMMA (2.8). Let \( H \) be a subgroup of \( G \) and \( g, x \in G \). Then \( |C_{G}(x) \cap g H| \neq 0 \) iff \( x^{g} \) is \( H \)-conjugate to \( x \). In this case, \( |C_{G}(x) \cap g H| \) is equal to \( |C_{H}(x)| \).

Proof. If \( x^{g} \) is \( H \)-conjugate to \( x \), then there exists \( h_{1} \in H \) such that \( g h_{1} = y \in C_{G}(x) \), hence \( g H = y H \) and \( |C_{G}(x) \cap g H| = |C_{G}(x) \cap y H| = |y^{-1} C_{G}(x) \cap H| = |C_{H}(x)| \). Conversely, if \( |C_{G}(x) \cap g H| \neq 0 \), then \( H \) contains an element \( h' \) satisfying \( (g h')^{y} = g H \); therefore, \( x^{g} = x^{h'^{-1}} \).

LEMMA (2.9). Let \( \bar{x} \) be a \( \pi \)-element of \( \overline{G} \) and \( y \) an element of \( G \). Then

\[
|T_{(x N)_{\pi}, y N}| \leq |T_{x N, N_{\pi}}|
\]

and equality holds iff \( y \in \bigcap_{z \in (x N)_{\pi}} \overline{C_{\pi}(z)} \).

Proof. We have \( |C_{G}(x) \cap y N| \leq |C_{N}(x n)| \), and equality holds iff \( y \in \overline{C_{G}(x n)} \). Therefore \( |T_{(x N)_{\pi}, y N}| = \sum_{x n \in (x N)_{\pi}} |C_{G}(x n) \cap y N| \leq \sum_{x n \in (x N)_{\pi}} |C_{N}(x n)| = |T_{(x N)_{\pi}, N_{\pi}}| = |T_{x N, N_{\pi}}| \) using Lemma (2.6).

In the following, \( H \) and \( K \) are two subgroups of \( G \) such that \( H \trianglelefteq K \), and for each \( g \in G \), \( r_{K}(g H) \) denotes the number of conjugacy \( K \)-classes of \( G \) which intersect the coset \( g H \), that is, \( r_{K}(g H) = |\{Cl_{K}(x) | x \in G \text{ and } Cl_{K}(x) \cap g H \neq \emptyset \}| \). Furthermore, if \( \bigcup_{K \in K} g H \) is the disjoint union of the \( K \)-classes \( Cl_{K}(g h_{i}) \) for all \( i = 1, \ldots, e \), ordered so that \( |C_{K}(g h_{1})| \geq \cdots \geq |C_{K}(g h_{e})| \), then we define \( \Delta_{g H}^{K} = (|C_{K}(g h_{1})|, \ldots, |C_{K}(g h_{e})|) \). We also define \( E_{g}^{K} = \bigcup_{h \in H} C_{K}(g h) = \{ k \in K | Cl_{H}(k^{g}) = Cl_{H}(k) \} \).
LEMMA (2.10). For each element $g$ of $G$, the following statements are true:

(i) $r_K(gH) = r_{N_K(gH)}(gH)$ and $\Delta^K_{gH} = \Delta_{N_K(gH)}^{gH}$.

(ii) $r_K(gH) = (1/|N_K(gH)|) \sum_{k \in E^K_{gH}} |C_H(k)|$.

(iii) $r_K(gH) = (1/|N_K(gH)|) \sum_{h \in H} |C_{N_K(gH)}(gh)|$.

(iv) $r_K(gH) \equiv |H| (\text{mod } d_{|N_K(gH)|})$.

Proof. Suppose that $gh_1$ is $K$-conjugate to $gh_2$, that is, $g^k h_1^k = gh_2$ for some $k \in K$. Then we have $g^k H = gH$, so $k$ normalizes $gH$. Thus, $gh_1$ is $K$-conjugate to $gh_2$ iff $gh_1$ is $N_K(gH)$-conjugate to $gh_2$ and also $C_k(gh) = C_{N_K(gH)}(gh)$ for every $h \in H$, concluding the first statement. On the other hand, counting the number of elements in $T_{gH}$, (ii) and (iii) follow directly from Lemmas (2.4) and (2.8). Finally, (iv) is an immediate consequence from (iii).

Remark. Straightforward arguments show that the following properties are true:

1. $N_G(gN)/N = C_G(\tilde{g})$ and $N_G(gN) \cap \{z \in G \mid [g, z] \in N\}$.

2. $C_G(g) N \leq N_G(gN)$ and $\langle g \rangle N$ is a normal subgroup of $N_G(gN)$.

3. If $G = N_{x,K}$, then $N_G(hN) = N_{x,C_K(h)}$ for each $h \in K$.

4. If $\text{g.c.d.}(o(g), |N|) = 1$, then $N_G(gN) = NC_G(g)$. (This result follows from Schur-Zassenhaus's Theorem applied in the group $N\langle g \rangle$.) In particular, $N_G(x)^{\langle |N| \rangle} = NC_G(x^{\langle |N| \rangle})$ for each $x \in G$.

5. $C_G(\tilde{g}) = \langle \tilde{g} \rangle$ iff $N_G(gN) = N\langle g \rangle$, and $|[g, N]| = |N : C_N(g)|$.

6. If $C_G(\tilde{x}) = C_G(\tilde{y})$, then $N_G(xN) = N_G(yN)$. In particular, if $\text{g.c.d.}(j, o(\tilde{g})) = 1$, then $N_G(gN) = \tilde{N}_G(g^jN)$, and if $\tilde{z} \in Z(\tilde{G})$, then $N_G(gN) = N_G(gzN)$.

7. $r_G(gN) \equiv 1 (\text{mod } d_{|N_G(gN)|})$. (This result follows directly from Lemma (2.10), part (iv), because $N$ is contained in $N_G(gN)$.)

LEMMA (2.11). Suppose that $g$ normalizes $H$. Then $H$ is a normal subgroup of $N_K(gH)$ and $r_K(gH) \leq r_{N_K(gH)}(H)$. Further, equality holds iff $N_K(gH) = E^K_{gH}$.

Proof. Evidently $H$ normalizes $gH$ and we have

$$r_K(gH) = (1/|N_K(gH)|) \sum_{x \in E^K_{gH}} |C_H(x)|.$$  

(9)

Thus, the inequality follows from (9), because $E^K_{gH}$ is a subset of $N_K(gH)$.
Arguing as in [8, p. 83], there exists a natural number \( l = l_{|G|} \), such that \( l_{|G|} \) has exactly order \( d_{|G|} \) modulo any divisor \((\neq 1)\) of \(|G|\). In addition, for each \( g \in G - \{1\} \), we have that \( \text{Cl}_G(g^x) \neq \text{Cl}_G(g^y) \) for \( 1 \leq i \neq j \leq d_{|G|} \). Let \( M \) be a normal subgroup of \( G \). If \( x \) is an element of \( M^* \), \( M_\pi \) contains \( \text{Cl}_G(x^i) \) for every \( i = 1, \ldots, d_{|G|} \), and counting the number of \( G \)-classes of \( M_\pi \) according to the decomposition

\[
M^*_\pi = \bigcup_{j=1}^e \left( \bigcup_{i=1}^{d_{|G|}} \text{Cl}_G(x_j^i) \right),
\]

we have \( r^x(G(M)) = 1 + d_{|G|} \cdot e \) and \( |M_\pi| = 1 + \sum_{j=1}^{e} d_{|G|} \cdot \text{Cl}_G(x_j) \) congruent to \( r^x(G(M)) \) modulo \( d_{|G|} \). Further, each \( G \)-class \( \text{Cl}_G(x_j) \) is the disjoint union of \( c_j = |G : C_G(x_j) M| \) \( M \)-classes, being \( c_j \) congruent to 1 (mod \( d_{|G/M|} \)) and \( r^x(M) = 1 + \sum_{j=1}^{e} c_j \cdot d_{|G|} \). Therefore we have shown:

**Lemma (2.12).** Let \( M \) be a normal subgroup of \( G \). Then the following congruences are true:

(i) \( r^x(G(M)) \equiv |M_\pi| (\text{mod } d_{|G|}) \)

(ii) \( r^x(G(M)) \equiv r^x(M) \text{ (mod } d_{|G/M|} \cdot d_{|G|}). \)

(10)

In particular, putting \( M = G \) and \( \pi = \pi(G) \) into (10), we get Poland's result \( r(G) \equiv |G| (\text{mod } d_{|G|}) \) (cf. [10, Prop. (3.9)]). Our proof of (10) is a simplification of the one given by Mann (cf. [8, Eq. (16)].

### 3. Main Theorems

**Theorem (3.1).** If \( g \) is a \( \pi \)-element of \( G \), then \( r^x_\pi(gN) \leq s^x_\pi \leq r^x(N) \). Furthermore, \( r^x_\pi(gN) = r^x(N) \) iff \( C_G(\bar{g}) = \overline{C_G(gn)} \) for each \( gn \in (gN)_\pi \) and each conjugacy \( N \)-class of \( N_\pi \) is fixed by the conjugation of \( g \).

**Proof.** For each \( n \in N \) we have \(|C_G(gn)|/|C_G(\bar{g})| \leq |C_N(gn)|\), therefore

\[
r^x_\pi(gN) = (1/|N|) \cdot \sum_{gn \in (gN)_\pi} |C_G(gn)|/|C_G(\bar{g})| \\
\leq (1/|N|) \cdot \sum_{gn \in (gN)_\pi} |C_N(gn)| = (1/|N|) \cdot |T_{(gN)_\pi, N}| \\
= (1/|N|) \cdot |T_{gN, N_\pi}| \quad \text{(using Lemma (2.6))} \\
= (1/|N|) \cdot \sum_{n \in N_\pi} |C_G(gn) \cap gN| = (1/|N|) \cdot \sum_{n \in \mathcal{B}_N^x} |C_N(n)| = s^x_\pi,
\]

by using Lemma (2.1), part (iii) and Lemma (2.8).
Evidently, $s_k^* \leq r^*(N)$ and equality holds iff $\text{Cl}_\lambda(m)^* = \text{Cl}_\lambda(m)$ for each $m \in \mathcal{N}_\pi$. Therefore equality $r^*_G(gN) = r^*(N)$ holds iff $r^*_G(gN) = s_k^*$ and $s_k^* = r^*(N)$, iff $|C_G(gn)|/|C_G(g)| = |C_N(gn)|$ for each $gn \in (gN)_\pi$ and $\text{Cl}_\lambda(m)^* = \text{Cl}_\lambda(m)$ for each $m \in \mathcal{N}_\pi$. Since $C_G(gn)$ is isomorphic to $C_G(gn)/\text{Cl}_\lambda(gn)$, this completes the proof of the theorem.

**Theorem (3.2).** For each $\pi$-element $\tilde{g}$ of $\tilde{G}$ the following inequality is true: $r^*_G(gN) \geq (o(\tilde{g}) \cdot s_k^*)/|C_G(\tilde{g})|$. Furthermore equality holds iff $C_G(gn) = \langle \tilde{g} \rangle$ for each $gn \in (gN)_\pi$.

**Proof.** Arguing as in Theorem (3.1), we obtain

$$r^*_G(gN) \geq (o(\tilde{g})/|C_G(\tilde{g})|) \cdot (1/|N|) \cdot \sum_{gn \in (gN)_\pi} |C_N(gn)| = (o(\tilde{g}) \cdot s_k^*)/|C_G(\tilde{g})|,$$

since $\langle gn \rangle \subseteq C_N(gn)$ is a subgroup of $C_G(gn)$ with order $o(\tilde{g}) \cdot |C_N(gn)|$.

Naturally, equality holds iff $|C_G(gn)| = o(\tilde{g}) \cdot |C_N(gn)|$ for each $gn \in (gN)_\pi$, namely $C_G(gn) = \langle \tilde{g} \rangle$ for every $gn \in (gN)_\pi$.

**Remarks.** (1) From Theorems (3.1) and (3.2) we deduce $r^*_{N < \pi}(gN) = s_k^*$ for $\tilde{g} \in \tilde{G}_\pi$.

(2) If $\{\text{Cl}_G(gm_1), ..., \text{Cl}_G(gm_v)\}$ is the set of all conjugacy classes that intersect $(gN)_\pi (\neq \emptyset)$, ordered so that $|C_G(gm_i)| \geq |C_G(gm_{i+1})|$ for each $i = 1, ..., v-1$, then we have $s_k^* = \sum_{i=1}^v |N_G(gN) : C_{N_G(gN)}(gm_i) \langle \tilde{g} \rangle|$, and the following congruence is true: $s_k^* = r^*_G(gN) \pmod{d_{[N_G(gN)]}}$.

Indeed, $C_{N_G(gN)}(gn) \langle \tilde{g} \rangle = C_{N_G(gN)}(gn) \langle gn \rangle = C_G(gn) \langle gn \rangle$ for each $n \in N$, and $(gN)_\pi$ is the disjoint union of the sets $gN \cap \text{Cl}_G(gm_i)$ for $i = 1, ..., v$; therefore,

$$s_k^* = r^*_{N < \pi}(gN) = (1/|N|) \cdot \sum_{gn \in (gN)_\pi} |C_N(gn)|$$

$$= (1/|N|) \cdot \sum_{i=1}^v |gN \cap \text{Cl}_G(gm_i)| \cdot |C_N(gm_i)|$$

$$= (1/|N|) \cdot \sum_{i=1}^v \left( (1/|N|) \cdot |C_G(\tilde{g})| \right) \cdot |C_G(gm_i)| \cdot |C_N(gm_i)|$$

$$= \sum_{i=1}^v |N_G(gN) : C_G(gm_i) \langle \tilde{g} \rangle|.$$

Finally the congruence is an immediate consequence of the fact that $|N_G(gN) : C_{N_G(gN)}(gm_i) \langle \tilde{g} \rangle|$ divides $|N_G(gN)/(N \langle \tilde{g} \rangle)|$ for each $i = 1, ..., v$. 
(3) For each $\pi$-element $\bar{g}$ of $\bar{G}$, the following congruences are true:

(a) $s_{\bar{g}}^n = r_{\bar{G}}(gN) \pmod {d_{\bar{G}(\bar{g})}}$ and $s_{\bar{g}}^n = r_{\bar{G}}(gN) \pmod {d_{|G|}}$.

(b) $s_{\bar{g}} = 1 \pmod {d_{|G|}}$ and $r_{\bar{G}}(N) = 1 \pmod {d_{|G|}}$.

(c) $r_{\bar{G}}(gN) = r_{\bar{G}}(N) \equiv 1 \pmod {d_{|G|}}$.

Indeed, (a) follows arguing as in part (2), because $C_G(\bar{g}) = N_G(gN)/N$ and $d_{C_G(\bar{g})}$ is divisible by $d_{|G|}$.

(b) Let $l = l_{|G|}$ and suppose that $C_G(n)$ is contained in $\pi B^n_{\bar{g}}$, then $\pi B^n_{\bar{g}}$ contains $C_G(n)$ for every $i = 0, 1, ..., d_{|G|} - 1$ and consequently $s_{\bar{g}}^n = r_{\bar{G}}(\pi B^n_{\bar{g}})$ and $s_{\bar{g}}^n = r_{\bar{G}}(N) \equiv 1 \pmod {d_{|G|}}$. Similarly, $r_{\bar{G}}(N) = 1 \pmod {d_{|G|}}$.

Finally, (c) follows directly from (a) and (b).

(4) Evidently the number $s_{\bar{g}}^n$ is greater than or equal to $r_{\bar{G}}(C_N(\bar{g}))$. Even more, $s_{\bar{g}}^n$ is congruent to 1 modulo $d_{|G|}$ and $r_{\bar{G}}(C_N(\bar{g}))$ is congruent to 1 modulo $d_{|G|}$; therefore, there exists a non-negative integer number $u$ such that

$$s_{\bar{g}}^n = r_{\bar{G}}(C_N(\bar{g})) + u \cdot d_{|G|}.$$}

**THEOREM** (3.3). If $N$ is a normal subgroup of $G$, then

$$r^n(G) \leq r^n(G/N) \cdot r^n(N),$$

and equality holds iff $C_G(g) = C_G(\bar{g})$ for each $g \in G$.

**Proof.** Suppose that $\{ \bar{g}_1, ..., \bar{g}_t \}$ is a complete system of representatives from distinct conjugacy classes of elements of $\bar{G} = G/N$. Since $G$ is the disjoint union of the normal sets $\bigcup_{x \in G} g_i^N = G$ for every $i = 1, ..., t$, we have

$$r^n(G) = \sum_{i=1}^t r^n(g_iN) = \sum_{\bar{g}_i, \pi\text{-element}} r^n(g_iN) \leq r^n(G/N) \cdot r^n(N),$$

using both Lemma (2.7) and Theorem (3.1). Furthermore equality holds if and only if $r^n(gN) = r^n(N)$ for each $\pi$-element $g$ of $G$. Moreover $r^n(N) = r^n(N)$ is equivalent to $C_G(n) = C_G(n)$ for each $n \in N_\pi$, that is, $C_G(\bar{n}) = C_G(n)$ for each $n \in N_\pi$, and we have $s_{\bar{g}}^n = r^n(N)$ for every $x \in G$, in this case. Therefore, the condition (11) is satisfied if equality holds.

**COROLLARY** (3.4). Let $N$ be a normal subgroup of $G$. Then

$$r(G) \leq r(G/N) \cdot r(N),$$

and equality holds iff $C_G(\bar{g}) = C_G(g)$ for each $g \in G$ (cf. P. X. Gallagher [5]).
Proof. It is an immediate consequence from Theorem (3.3), taking \( \pi = \pi(G) \) the set of all prime numbers dividing \( |G| \).

Our second application concerns the number \( r^p(G) \) (resp. \( r^{p'}(G) \)) of conjugacy classes of \( p \)-elements (resp. \( p' \)-elements) of a finite group \( G \), where \( p \) is a prime number dividing \( |G| \). If we take \( \pi \) as the set \( \{p\} \) (resp. \( \pi(G) - \{p\} \)) in Theorem (3.3) we get:

**Corollary (3.5).** The following affirmations are true:

(A) \( r^p(G) \leq r^p(G/N) \cdot r^p(N) \) and equality holds iff \( C_G(\bar{\bar{g}}) = C_G(g) \) for each \( p' \)-element \( g \) of \( G \).

(B) \( r^p(G) \leq r^p(G/N) \cdot r^p(N) \) and equality holds iff \( C_G(\bar{\bar{g}}) = C_G(g) \) for each \( p \)-element \( g \) of \( G \).

*Remark.* Let \( P \) be a Sylow \( p \)-subgroup of \( G \). Evidently, part (B) of the above Corollary is equivalent to following inequality:

\[
|G:P| \leq |G/N| |P/N| \cdot |P\cap N|.
\]

In addition, there is equality iff \( C_G(\bar{\bar{g}}) = C_G(g) \) for each \( g \in P \).

**Example.** Obviously, Theorem (3.3) allow us to improve Gallagher's upper bound given in (3.4), assuming that \( N \) and \( G/N \) are known groups. In fact, if \( \pi_1, \ldots, \pi_s \) are sets of prime numbers chosen so that \( \pi(G) \) is the disjoint union of them and every element \( g \) of \( G \) has order \( \pi_i \)-number for some \( i \), then (3.3) yields

\[
r(G) \leq \left( \sum_{i=1}^{s} b_i c_i \right) - (s-1),
\]

where we denote \( b_i = r^{\pi_i}(G/N) \) and \( c_i = r^{\pi_i}(N) \), while (3.4) states that

\[
r(G) \leq \left( \left( \sum_{i=1}^{s} b_i \right) - (s-1) \right) \cdot \left( \left( \sum_{i=1}^{s} c_i \right) - (s-1) \right).
\]

Naturally, (12) improves (13). For instance, it can be considered that \( \pi_i = \{p_i\} \) \( i = 1, \ldots, s \), in case, \( |G| = p_1^{\pi_1} \cdots p_s^{\pi_s} \) and \( G \) has all elements of primary power orders. Also, if \( G \) is isomorphic to a subgroup of \( \Sigma_p \), symmetric group of degree \( p \), and \( |G| \) is divisible by the prime \( p \), then the partition \( \pi_1 = \pi(G) - \{p\} \) and \( \pi_2 = \{p\} \) satisfies the desired conditions, assuming \( |G| \neq p \).

**Corollary (3.6).** If equality holds in (1), then

\[
r^{\pi}(L) = r^{\pi}(L/N) \cdot r^{\pi}(N),
\]
for each subgroup \( L \supseteq N \). Conversely, if (14) holds for each subgroup \( L = N \langle g, x \rangle \) with \( g \) a \( \pi \)-element and \([g, x] \in N\), then equality holds in (1).

**Proof.** The first statement follows from the fact that (11) for \( G \) implies (11) for \( L \).

Let \( g \) be a \( \pi \)-element and \( x \in C_G(g) \). Assuming (14) for \( L = N \langle g, x \rangle \) we get \( x \in C_L(g) = C_L(g) N/N \), hence \( x \in C_L(g) N \). Thus, if (14) holds for all such \( L \), then \( C_G(g) = C_G(g) N/N \) for each \( \pi \)-element \( g \) of \( G \) and equality holds in (1).

**Corollary (3.7).** Let \( \{ y_1, \ldots, y_w \} \) be a complete system of representatives of conjugacy classes of \( \pi \)-elements of \( \bar{G} = G/N \). Then the following statements are equivalent:

(A) \( r^*(G) = r^*(G/N) \cdot r^*(N) \).

(B) \( r^*(N_G(y_iN)) = r^*(C_G(y_i)) \cdot r^*(N) \) for each \( i = 1, \ldots, w \).

**Proof.** Implication (A) \( \Rightarrow \) (B) follows from the fact that \( N_G(y_iN) \) is a subgroup of \( G \) containing \( N \) and satisfying \( N_G(y_iN)/N = C_G(y_i) \). Conversely, if \( g \in G, \) then \( \tilde{g} \) is \( \bar{G} \)-conjugate to \( \bar{y}_i \) for some \( i \) and we have

\[
r^*(N_G(gN)) = r^*(N_G(y_iN)) = r^*(C_G(y_i)) \cdot r^*(N) = r^*(C_G(g)) \cdot r^*(N).
\]

Therefore, (B) yields \( C_{N_G(gN)}(gN) = C_{N_G(gN)}(g) N/N \), that is, \( C_{N_G(gN)}(g) = C_{N_G(gN)}(g) / N \), but \( C_{N_G(gN)}(g) = C_G(g) \), consequently \( C_G(g) = C_G(g) N/N \) from which (A) follows.

**Corollary (3.8).** If \( G \) is a \( L_{\pi} \)-group, \( N \) a normal subgroup of \( G \), and \( H \) a Hall \( \pi \)-subgroup of \( G \) satisfying \( N \cap [H, G] = 1 \), then the following statements are true:

(A) \( r^*_G(gN) = |H \cap N| \) for each \( \pi \)-element \( \tilde{g} \) of \( \bar{G} \).

(B) \( r^*(G) = r^*(G/N) \cdot |H \cap N| \).

(Putting \( \pi = \pi(G) \), (3.8), part (B), improves the one given by Rusin (cf. [11, Prop. 3])).

**Proof.** Evidently \( H \cap N \) is a Hall \( \pi \)-subgroup of \( N \) contained in \( Z(G) \), and (3.1) yields \( r^*_G(gN) \leq s^*_G = |C_{H \cap N}(g)| = |H \cap N| \). Furthermore, if \( gn \) is a \( \pi \)-element of \( gN \), then \( gn \in H^x \) for some \( x \). Therefore for each \( \bar{y} \in C_G(g) \) we have \( [gn, \bar{y}] \in N \cap [H^x, G] = (N \cap [H, G])^x = 1 \), and consequently \( C_G(gn) = C_G(g) \). Thus, \( r^*_G(gN) = s^*_G = |H \cap N| \) and (B) follows directly from (A).

**Corollary (3.9).** If \( G \) is a \( L_{\pi'} \)-group, \( N \) a normal subgroup of \( G \), and \( H \)
a Hall $\pi$-subgroup of $G$, then $r_G(H) \leq r_{G/N}(HN/N) \cdot r_N(H \cap N)$ and there is equality iff $C_G(h) = C_G(h')$ for each element $h$ of $H$.

**Proof.** It follows directly from the fact that $r_G(H) = r^*(G)$, when $G$ is a $D_\pi'$-group.

**Theorem (3.10).** If $G$ is a $L_\pi'$-group, $N$ a normal subgroup of $G$, and $H$ a Hall $\pi$-subgroup of $G$ such that $\overline{H} = HN/N$ is an abelian group, then

$$r^*(G) \leq r^*(G/N) \cdot r_H(H \cap N).$$

**Proof:** Let $g$ be a $\pi$-element of $G$. We claim that $r_G^*(gN) \leq r_H^*(H \cap N)$. In fact, let $H_1$ be a Hall $\pi$-subgroup of $N_G(gN)$ containing $g$. Then there exists one conjugate subgroup $H_2$ of $H_1$ such that $H_2 \leq H_1$. Even more, $H_2 = H_1$, since $\overline{H}$ abelian implies that $\overline{H}$ normalizes the coset $gN$. Therefore,

$$r_G^*(gN) = r_{N_G(gN)}^*(gN) = r_{N_G(gN)}\left(gN \cap \left(\bigcup_{z \in N_G(gN)} H_1\right)\right)$$

$$= r_{N_G(gN)}\left(\bigcup_{z \in N_G(gN)} (gN \cap H_1)\right) = r_{N_G(gN)}(g(N \cap H_1))$$

$$\leq r_{H_1}(g(N \cap H_1)) = r_{H_1}(N \cap H_1) = r_H(N \cap H),$$

using Lemma (2.11). Now, arguing as in Theorem (3.3), we obtain the desired inequality.

**Theorem (3.11).** Suppose that $C_G(\bar{g})$ is a $\pi$-group. Then

$$r_G^*(gN) \leq r_{N_G(gN)}^*(N),$$

and equality holds iff $\bar{g} \in \cap_{z \in N_G(gN)} C_G(z)$.

**Proof.** Suppose that $N_G(gN)$ is the disjoint union of the cosets $y_iN$ for every $i = 1, \ldots, t$. By hypothesis, $\bar{y}_i$ is a $\pi$-element for each $i$ and we have

$$r_G^*(gN) = (1/|N_G(gN)|) \cdot |T_{N_G(gN)}(gN)_\pi|$$

$$- (1/|N_G(gN)|) \cdot \sum_{i=1}^t |T_{y_iN}(gN)_\pi|$$

$$= (1/|N_G(gN)|) \cdot \sum_{i=1}^t (|T_{(y_iN)\pi}(gN)_\pi| + |T_{(y_iN)\pi}(gN)_\pi|)$$

$$= (1/|N_G(gN)|) \cdot \sum_{i=1}^t (|T_{(y_iN)\pi}(gN)_\pi| + |T_{(y_iN)\pi}(gN)_\pi|)$$

$$= (1/|N_G(gN)|) \cdot \sum_{i=1}^t (|T_{(y_iN)\pi}(gN)_\pi| + |T_{(y_iN)\pi}(gN)_\pi|)$$

$$= (1/|N_G(gN)|) \cdot \sum_{i=1}^t (|T_{(y_iN)\pi}(gN)_\pi| + |T_{(y_iN)\pi}(gN)_\pi|)$$
(by using Lemma (2.5))

\[ (1/|N_G(gN)|) \cdot \sum_{i=1}^{t} |T_{(y_iN)_{N_i},gN}| \leq (1/|N_G(gN)|) \cdot \sum_{i=1}^{t} |T_{y_iN,N_i}| \]

(using Lemma (2.9))

\[ (1/|N_G(gN)|) \cdot |T_{N_G(gN),N_N}| = r_{N_G(gN)}^e(N). \]

Moreover, equality holds if and only if \(|T_{(y_iN)_{N_i},gN}| = |T_{y_iN,N_i}|\) for each \(i\), if and only if \(\bar{g} \in \bigcap_{z \in N_G(gN)} C_G(z)\), inasmuch as \(N_G(gN)_{\pi} = \bigcup_{i=1}^{t} (y_iN)_{\pi}\).

Remarks. (1) Equation (3.10) yields \(r(G) \leq |G/N| \cdot r_G(N)\), when \(G/N\) is an abelian group and \(\pi = \pi(G)\). The last inequality improves Sah's inequality: \(r(G) \leq |G/N| \cdot r(N)\), in this case. In general, if \(\bar{g}\) is a central element of \(\overline{G}\), then

\[ r_G(gN) \leq r_{N_G(gN)}(N) = r_G(N), \]

and consequently

\[ r(G) \leq (r(G/N) - |Z(G/N)|) \cdot r(N) + |Z(G/N)| \cdot r_G(N). \]

(2) Suppose that \(G/N\) is an abelian \(\pi\)-group. Then

\[ r^\pi(G) \leq |G/N| \cdot r_G^e(N), \]

and equality holds if and only if \(\bar{G} = \bigcap_{x \in G} C_G(x)\). This result follows from (3.11).

**Theorem (3.12).** Let \(N\) be a normal subgroup of \(G\) and \(j\) an integer number coprime to \(o(\bar{g})\). Then the following equalities hold:

\[ r^\pi_G(gN) = r_G^e(g^jN) \quad \text{and} \quad \pi A^N_{gN} = \pi A^G_{g^jN}. \quad (14) \]

**Proof.** By Dirichlet's Theorem, one can choose a prime \(q\) greater than \(|G|\) and congruent to \(j \mod o(\bar{g})\), then it is easy to see that \(x \mapsto x^q\) is a conjugacy-preserving bijection of the coset \(gN\) onto the coset \(g^jN\). Further \(|C_G(x)| = |C_G(x^q)|\) for each \(x \in gN\), therefore \(r^\pi_G(gN) = r_G^e(g^jN)\) and \(\pi A^N_{gN} = \pi A^G_{g^jN}\). (We owe this argument to K. G. Kovacs.)

**Theorem (3.13).** Let \(G\) be a \(L_\pi\)-group and \(H\) a Hall \(\pi\)-subgroup of \(G\). Then the following inequality holds:

\[ r^\pi(G) \geq (r(G/N) - 1) \cdot (|H \cap N|/|\lbrack H,G \rbrack \cap N|) + r_G(N). \quad (15) \]

In particular, \(r(G) \geq (r(G/N) - 1) \cdot |NG'/G'| + r_G(N)\). \quad (16)
Proof. From Lemma (2.2), there exists $g \in (gN)_\pi$ such that
$r^*_{gN}(gN) \geq |(gN)_\pi|/[gN, N\pi(gN)]$. Let $H^\pi$ be a Hall $\pi$-subgroup of $G$ containing $gN$. Then, $(gN)_\pi = (gN)_\pi$ contains $gN(H^\pi \cap N)$ and $[H^\pi, G] \cap N$ contains $[gN, N\pi(gN)]$. Consequently,

$$r^*_{gN}(gN) \geq |H^\pi \cap N|/[H^\pi, G] \cap N] = |H \cap N|/[H, G] \cap N|. \quad (17)$$

Now, (15) follows directly from (17).

Remarks. (1) From Lemma (2.2) we get $r_{gN}(gN) \geq |N/(N \cap N\pi(gN))| \geq |N/(N \cap G^\pi)|$ for each element $g$ of $G$. Therefore, $r_{gN}(gN) = 1$ implies that $N\pi(gN)$ contains $N$.

(2) Suppose that $G$ is a $L_\pi$-group. If both $N$ and $C\pi(g)$ are $\pi$-groups, then $N\pi(gN)$ is a $\pi$-group and there exists $H$ a Hall $\pi$-subgroup of $G$ such that $N\pi(gN) \leq H$, and consequently, $N\pi(gN) \leq H^\pi$. Thus, if $r_{gN}(gN) = 1$, then $N$ is contained in $\bigcap_{x \in G}(H^\pi)^x$. In particular, taking $\pi = \{p\}$, we get (1.7) of [9].

(3) We have $r_{gN}(gN) = 1$ iff $gN = Cl_{N\pi(gN)}(gN)$ iff $N = [gN, N\pi(gN)]$. In this case, $N$ consists entirely of commutators of $G$. Therefore, if $N$ contains a non-commutator of $G$ then $r(G) \geq (1 + d_{G\pi}) \cdot r(G/N)$.

Theorem (3.14). Let $N$ be a normal subgroup of $G$ and $g$ an element of $G$. Then $s^*_{gN} \geq r^*_{gN}(C\pi(g))$ and equality holds iff $\pi B^N_{gN} = (\bigcup_{x \in G} C\pi(g)^x)_\pi$. In addition if $o(g)$ is coprime to $|N|$, then the following is true:

(i) $r^*_{gN}(gN) \leq r^*_{gN}(C\pi(g)) = s^*_{gN}$.

(ii) If $\{m_1, \ldots, m_e\} \subseteq C\pi(g)$ is a complete system of representatives from distinct conjugacy $N$-classes that compose the set $(\bigcup_{x \in G} C\pi(g)^x)_\pi$, ordered so that $|C\pi(g^x)(gm_i)| \geq |C\pi(g^x)(gm_{i+1})|$ for each $i = 1, \ldots, e-1$ and $g$ is a $\pi$-element, then $\pi A^N_{g^x} = (|C\pi(g^x)(gm_1)|, \ldots, |C\pi(g^x)(gm_e)|)$ and there exists $\{m_1', \ldots, m_e'\}$ subset of $\{m_1, \ldots, m_e\}$ such that $\pi A^N_{g^x} = (|C\pi(gm_1')|, \ldots, |C\pi(gm_e')|)$.

Proof. (i) Suppose that $o(g)$ is coprime to $|N|$. For each $m \in \pi B^N_{gN}$ we have $C\pi(g^x)(m)/C\pi(m) \simeq \langle g^x \rangle$, therefore there exists $g_1 \in C\pi(g^x)(m)$ such that $o(g_1) = o(g)$ and $C\pi(g^x)(m)$ is a semidirect product of $C\pi(m)$ by $\langle g_1 \rangle$. Now, applying Schur-Zassenhaus's Theorem, there is $g'^x \in N \langle g \rangle = \langle g \rangle N$ such that $\langle g_1 \rangle = \langle g \rangle^{x'} = \langle g \rangle^{x''}$. Consequently $g'^x \in C\pi(g^x)(m)$, that is, $m \in C\pi(g^x)$. Thus, $\pi B^N_{gN} = (\bigcup_{x \in G} C\pi(g^x)_\pi$ and $s^*_{gN} = r^*_{N\pi(gN)}(gN)$.

(ii) Suppose that $g m_i$ is $\langle g \rangle N$-conjugate to $g m_j$. Since $g$ commutes with each $m_i$, $g m_i$ is $N$-conjugate to $g m_j$ and consequently $m_i$ is $N$-conjugate to $m_j$, because $o(g)$ is coprime to $o(m_i)$, so, $i = j$. Thus, (ii) follows directly from the following equality: $e = r^*_{N\pi(gN)}(C\pi(g)) = s^*_{gN} = r^*_{N\pi(gN)}(gN)$. 


Corollary (3.15). If g.c.d. \((o(g), |N|) = 1\), then we have
\[ r_G(gN) \leq |C_N(g)|, \]
and there is equality iff \(\Delta^G_{gN} = ([C_G(g)], |C_N(g)|, |C_P(g)|)\).

Proof. For each \(n \in N\), \((gn)^{|N|}\) is \(G\)-conjugate to \(g^{|N|}\) by Schur-Zassenhaus’s Theorem (in fact, these have orders equal to \(o(g)/\text{g.c.d.}(o(g), |N|)\), which is a number coprime to \(|N|\)), therefore \(C_G(g) = C_G(g)\), \(|C_G(gn)| \leq |C_G((gn)^{|N|})| = |C_G(g)|\), and
\[
r_G(gN) = (1/|N|) \cdot \sum_{n \in N} |C_G(gn)|/|C_G(g)| \leq (1/|N|) \cdot \sum_{n \in N} |C_G(g)|/|C_G(g)| = |C_N(g)|.
\]
Finally, equality holds iff \(|C_X(gn)| = |C_G(g)|\) for each \(n \in N\) iff \(\Delta^G_{gN} = ([C_G(g)], |C_N(g)|, |C_P(g)|)\).

Next, we analyze the residue class of \(r^n(G)\) modulo the “best” number, given in terms of the primes which divide the group order.

Theorem (3.16). For each finite group \(G\), the following congruence holds:
\[ r^n(G) \equiv |G| (mod \delta_{|G|}^n). \] (17)

Proof. Consider the set \(T_{G^*, G} = \{(y, z) \in G^* \times G | yz = zy\}\). Since \((y, z)\) is an element of \(\langle y, z \rangle \times \langle y, z \rangle\) for every \((y, z) \in T_{G^*, G}\) and \(T_{G^*, G}\) contains \(\langle y, z \rangle \times \langle y, z \rangle\), we deduce the following decomposition:
\[ T_{G^*, G} = \bigcup \left\{ \langle y, z \rangle \times \langle y, z \rangle \mid (y, z) \in T_{G^*, G} \right\}. \]

Set \(A_{y, z} = \langle y, z \rangle \times \langle y, z \rangle - \{(1, 1)\}\) and \(\{A_{y, z} \mid (y, z) \in T_{G^*, G}\} = \{B_1, \ldots, B_w\}\), with \(B_i = A_{y_i, z_i}\). Evidently, \(T_{G^*, G} - \{(1, 1)\}\) is the union of the sets \(B_i, i = 1, \ldots, w\), and we have
\[
r^n(G) \cdot |G| - 1 = |T_{G^*, G}| - 1 = \left| \bigcup_{i=1}^w B_i \right| = \sum_{j=1}^w \sum_{1 \leq i_1 < \cdots < i_j \leq w} (-1)^{j+1}|B_{i_1} \cap \cdots \cap B_{i_j}|, \] (18)
furthermore, 

\[ |B_n \cap \cdots \cap B_k| = \left| \left( \bigcap_{k=1}^{J} \langle y_{k}, z_{ik} \rangle \right) \times \left( \bigcap_{k=1}^{J} \langle y_{k}, z_{ik} \rangle \right) \left| -1 \right. \]

\[ = \left| \left( \bigcap_{k=1}^{J} \langle y_{k}, z_{ik} \rangle \right) \pi \right| 2 \cdot \left( \bigcap_{k=1}^{J} \langle y_{k}, z_{ik} \rangle \pi \right) \pi - 1 \]

\[ = 0 \text{(mod } \delta_{[G]}^n \text{)}, \]

because \( \cap_{k=1}^{J} \langle y_{k}, z_{ik} \rangle \) is an abelian subgroup of \( G \). Thus, (18) yields

\[ r^n(G) \cdot |G| - 1 \equiv 0 \text{(mod } \delta_{[G]}^n \text{)}, \]

but \( |G|^2 = 1 \) (mod \( \delta_{[G]}^n \)), therefore \( r^n(G) \equiv |G| \text{(mod } \delta_{[G]}^n \text{)}. \)

The congruence (17) was obtained by Hirsch, in case \( r = \pi(G) \) (cf. [7]). A different proof was given by van der Waall in [13]. On the other hand, by taking \( M = G \) and \( \pi = \pi(G) \) in Lemma (2.12), (i), we deduce

\[ r(G) \equiv |G| \text{ (mod } d_{[G]}^2 \text{),} \]

and using (17) we conclude

\[ r(G) \equiv |G| \text{ (mod l.c.m.}(\delta_{[G]}, d_{[G]}^2)) \]

If \( |G| \) is an odd number, then l.c.m.\((\delta_{[G]}, d_{[G]}^2) = (\delta_{[G]} \cdot d_{[G]})/2\), and we have

\[ r(G) \equiv |G| \text{ (mod } (\delta_{[G]} \cdot d_{[G]})/2). \] (19)

In 1978, A. Mann (cf. [8]), using Brauer's Lemma (e.g., [4, Eq. (12.1)]), showed that

\[ r(G) \equiv |G| \text{ (mod } \delta_{[G]} \cdot d_{[G]}) \] (20)

and the above congruence was obtained by A. Vera (cf. [14]) for solvable groups without using character theory.

Next, let \( N \) be a normal subgroup of \( G \) such that \( |G/N| \neq 1 \) and \( I = l_{[G]} \).

Since \( \cap \) is coprime to \( o(\tilde{g}) \) for each \( \tilde{g} \in \tilde{G}_* \) and each \( i = 0, 1, 2, \ldots, d_{[G]} - 1 \)
(3.12) yields \( r^n(G^-1 N) = r^n(gN) \) and \( r^n(\bigcup_{i=1}^{d_{[G]}} g^i N) = d_{[G]} \cdot r^n(gN) \). Thus, \( r^n(G) = r^n(G^-1 N) + d_{[G]} \cdot \sum_{i=1}^{d_{[G]}} r^n(g^i N) \) for some subset \( \{ \tilde{g}_1, \ldots, \tilde{g}_u \} \) of a system of representatives from distinct conjugacy classes of \( \pi \)-elements of \( \tilde{G}_* - \{ 1 \} \). Furthermore, \( r^n(G \cdot N) \equiv r^n(N) \) (mod \( d_{[G]} \cdot d_{[G]}^2 \)) and \( r^n(gN) \equiv r^n(N) \) (mod \( d_{[G]} \)). Therefore

\[ r^n(G) \equiv r^n(G \cdot N) + r^n(N) - 1 \cdot r^n(N) \text{ (mod } d_{[G]} \cdot d_{[G]}^2) \]

\[ r^n(G) \equiv r^n(G/N) \cdot r^n(N) \text{ (mod } d_{[G]} \cdot d_{[G]}). \] (21)
On the other hand, \( r^n(G/N) \equiv |G/N| \quad (\text{mod } \delta^n_{|G|}) \) and \( r^n(N) \equiv |N| \quad (\text{mod } \delta^n_{|N|}) \) imply

\[
r^n(G/N) \cdot r^n(N) \equiv |G| \equiv r^n(G) \quad (\text{mod } \delta^n_{|G|}).
\] (22)

Finally, (21) and (22) yields

\[
r^n(G) \equiv r^n(G/N) \cdot r^n(N) \quad (\text{mod l.c.m.}(d_{|G|}, d_{|G|}, \delta^n_{|G|})).
\]

Thus, we have shown:

**Theorem (3.17).** For each \( N \) normal subgroup of \( G \), the following congruence is true:

\[
r^n(G) \equiv r^n(G/N) \cdot r^n(N) \quad (\text{mod l.c.m.}(d_{|G|}, d_{|G|}, \delta^n_{|G|})).
\]

Now, we improve Gallagher's inequality. Before stating our next result, we introduce the following notation. Set \( |G| = p_1^{\alpha_1} \cdots p_r^{\alpha_r} \) and \( \pi(G) = \{p_1, \ldots, p_r\} \). We say that \( \{\pi_1, \ldots, \pi_e\} \) is a \( \pi(G) \)-partition, if \( \pi(G) \) is the disjoint union of the sets \( \pi_i, i = 1, \ldots, e \), and each element \( g \) of \( G \) is a \( \pi_i \)-element for some \( i \). If \( \{\pi_1, \ldots, \pi_s\} \) is a \( \pi(G) \)-partition of maximal cardinality, we say that this is a maximal \( \pi(G) \)-partition. Evidently, \( 1 \leq s \leq t \); \( s = 1 \) if \( G \) has elements of order \( p_1 \cdots p_r \); and \( s = t \) iff all elements of \( G \) have primary power orders. Also, for each set \( \pi \) contained in \( \pi(G) \) and each proper normal subgroup \( N \) of \( G \), the following assertions hold: \( r^n(G) \equiv 1 \quad (\text{mod } d_{|G|}), \quad (r^n(G/N), r^n(N)) \neq (1, 1), \quad \text{and} \quad r^n(G/N) = 1 \quad \text{iff} \quad |G|_\pi \) divides \( |N| \).

We next claim:

**Lemma (3.18).** Let \( b_1, \ldots, b_s, c_1, \ldots, c_s, d \) be natural numbers satisfying all the following conditions:

(A) \( b_i \equiv 1 (\text{mod } d), \quad c_i \equiv 1 (\text{mod } d), \quad (b_i, c_i) \neq (1, 1) \) for each \( i = 1, \ldots, s \).

(B) \( (b_1, \ldots, b_s) \neq (1, \ldots, 1) \neq (c_1, \ldots, c_s) \).

If \( f(s) = \left( \sum_{i=1}^{s} b_i \right) - (s - 1) \), \( \left( \sum_{i=1}^{s} c_i \right) - (s - 1) \), \( \left( \sum_{i=1}^{s} b_i c_i \right) - (s - 1) \), then the following inequality holds:

\[
f(s) \geq (s - 1) d^2.
\]

**Proof.** We proceed by induction on \( s \). For \( s = 2 \), we have

\[
f(2) = (b_1 - 1)(c_2 - 1) + (b_2 - 1)(c_1 - 1).
\]

Suppose \( (b_1 - 1)(c_2 - 1) = 0 = (b_2 - 1)(c_1 - 1) \). If \( b_1 = 1 \), then \( c_1 > 1 \) and \( b_2 = 1 \), which is impossible, so, \( b_1 > 1 \) and consequently \( c_2 = 1 \). Hence \( c_1 > 1 \).
and $b_2 = 1$, therefore $(c_2, b_2) = (1, 1)$, which is impossible by our hypothesis. Thus, $f(2) \geq d^2$.

Assume that $f(s - 1) \geq (s - 2) d^2$ for any sets $\{b_i\}$, $\{c_i\}$ of $s - 1$ elements satisfying the conditions (A) and (B).

If there exists $i$ such that $b_i = 1$, without loss we can suppose that $i = 1$. Then $c_i$ is greater than 1. If $c_i = 1$ for all $i = 2, \ldots, s$, then necessarily $b_i - 1 \geq d$ for all $i = 2, \ldots, s$ and

$$f(s) = \left( \sum_{i=2}^{s} (b_i - 1) \right) (c_1 - 1) \geq (s - 1) d^2,$$

in this case. On the other hand, if $(c_2, \ldots, c_s) \neq (1, \ldots, 1)$, we conclude by induction that

$$w = \left( \sum_{i=2}^{s} b_i \right) - (s - 2) \cdot \left( \sum_{i=2}^{s} c_i \right) - (s - 2) \cdot \left( \sum_{i=2}^{s} (b_i - 1) \right) (c_1 - 1) \geq (s - 2) d^2.$$

Furthermore,

$$f(s) = (b_1 - 1) \left( \sum_{i=2}^{s} (c_i - 1) \right) + \left( \sum_{i=2}^{s} (b_i - 1) \right) (c_1 - 1) + w \quad (23)$$

with $(\sum_{i=2}^{s} (b_i - 1))(c_1 - 1) \geq d^2$, therefore $f(s) \geq d^2 + (s - 2) d^2 = (s - 1) d^2$. Thus we can assume $b_i \neq 1$ for each $i = 1, \ldots, s$. Arguing with $c_i$ in a similar way to $b_i$, we can assume $c_i \neq 1$ for all $i$. Finally (23) implies $f(s) \geq (s - 1) d^2$, in this case, and the lemma is proved.

**Theorem (3.19).** Let $\{\pi_1, \ldots, \pi_s\}$ be a maximal $\pi(G)$-partition for $G$ and $N$ a proper normal subgroup of $G$. Then the following is true:

$$r(G) \leq r(G/N) \cdot r(N) - (s - 1) d_{|G|}^2.$$

In particular, if all elements of $G$ have primary power orders, then

$$r(G) \leq r(G/N) \cdot r(N) - (t - 1) d_{|G|}^2,$$

where $t$ denotes the number of different prime numbers dividing $|G|$.

**Proof.** We notice that $r(G) = \sum_{i=1}^{s} r^{\pi_i}(G) - (s - 1)$, $r(N) = \sum_{i=1}^{s} r^{\pi_i}(N) - (s - 1)$, and $r(G/N) = \sum_{i=1}^{s} r^{\pi_i}(G/N) - (s - 1)$. Therefore (3.19) follows directly from the previous lemma and Theorem (3.3), setting $b_i = r^{\pi_i}(G/N)$ and $c_i = r^{\pi_i}(N)$, and observing that the conditions given into Lemma (3.18) are satisfied with $d = d_{|G|}$. 
Remarks. (1) Using Mann’s congruence (cf. [8, Eq. (20)]), there exists a non-positive integer number \( k \) such that \( r(G) = r(G/N) \cdot r(N) + k \delta_{(G)} \cdot d_G \), and using (3.19) the number \( k \) is less than or equal to \( -(s-1)((d_{(G)})/\delta_{(G)})). \]

(2) For each element \( g \) of \( G \), set \( U_g = \bigcup \{ Cl_{\sigma}(g) \mid \text{g.c.d.}(j, o(g)) = 1 \} \). Evidently, \( |U_g| = \phi(o(g)) \cdot |Cl_{\sigma}(g)|/|N\sigma(\langle g \rangle)/C_{\sigma}(g)| = \phi(o(g)) \cdot |G; N_G(\langle g \rangle)| \), where \( \phi \) denotes Euler’s function. Further, the equivalence relation \( x \sim y \iff \langle x \rangle \) is \( G \)-conjugate to \( \langle y \rangle \) yields one partition into \( G_{\pi} \), whose equivalence classes are precisely the sets \( U_g \) with \( g \in G_{\pi} \). Set \( G_{\pi} = U_{z_1} \cup \cdots \cup U_{z_s} \) for some \( \pi \)-elements \( z_i, i = 1, \ldots, s \). We have \( |C_G(x)| = |C_G(z_i)| \) for every \( x \in U_{z_i} \) and the following equalities are true:

\[
|G_{\pi}| |G| - r^\sigma(G) |G| = |G_{\pi} \times G - T_{G_{\pi}, G}| = \sum_{g \in G_{\pi}} (|G| - |C_G(g)|)
\]

\[
= \sum_{z = 2}^s \phi(o(z_i)) \cdot |G; N_G(\langle z_i \rangle)| \cdot (|G| - |C_G(z_i)|),
\]

assuming \( z_1 = 1 \). But, \( \phi(o(z_i)) \equiv 0 \pmod{d_{G_{\pi}}} \) and \( |G| - |C_G(z_i)| \equiv 1 - 1 = 0 \pmod{d_{G_{\pi}}} \), therefore

\[
|G|(|G_{\pi}| - r^\sigma(G)) \equiv 0 \pmod{d_{G_{\pi}} \cdot d_G},
\]

and consequently

\[
r^\sigma(G) \equiv |G_{\pi}|(\text{mod} (d_{G_{\pi}} \cdot d_G)/(\text{g.c.d.}(|G|, d_{G_{\pi}}))). \tag{24}
\]

For example, if \( G \) has a normal Hall \( \pi \)-subgroup \( H \), then

\[
r^\sigma(G) \equiv |H|(\text{mod} (d_H \cdot d_G)/(\text{g.c.d.}(|G|, d_H))).
\]

In the following, we are interested in obtaining precise information about \( r^\sigma_G(gN) \) for \( \pi = \pi(G) \).

**THEOREM (3.20).** Let \( \{n_1, \ldots, n_e\} \) be a complete system of representatives from distinct cosets of \( N \cap N_G(gN)^{\prime} \) in \( N \). Then

\[
r^\sigma_G(gN) = \sum_{i = 1}^c r_{N_G(gN)}(g_{n_i}(N \cap N_G(gN)^{\prime})) \tag{25}
\]

and

\[
\Delta^G_{RN} = (\Delta^{N_G(gN)}_{gN(N \cap N_G(gN)^{\prime})}, \ldots, \Delta^{N_G(gN)}_{g_{n_e}N(N \cap N_G(gN)^{\prime})}). \tag{26}
\]

**Proof.** If \( (gn)^{\prime} = gn' \) for some \( n, n' \in N \) and \( z \in N_G(gN) \), then \( n^{-1}n' = (gn)^{-1}(gn') = [gn, z] \), hence \( n(N \cap N_G(gN)^{\prime}) = n'(N \cap N_G(gN)^{\prime}) \); in other
words, two different cosets of $N \cap N_G(gN)'$ in $N$, do not intersect the same set of the type $g^{-1} \cdot \text{Cl}_{N_G(gN)}(gn)$, with $n \in N$. Consequently, since $gN = \bigcup_{i=1}^{r} gn_i(N \cap N_G(gN)')$ the equalities $(25)$ and $(26)$ hold.

**Corollary (3.21).** For each $g \in G$ there exists a non-negative integer number $k'$ such that $r_G(gN) = |N/(N \cap N_G(gN)')| + k' \cdot d_{|N_G(gN)|}$.

**Proof.** The number $r_G(gN)$ is greater than or equal to $|N/(N \cap N_G(gN)')|$ and $r_G(gn(N \cap N_G(gN)'))$ is congruent to 1 modulo $d_{|N_G(gN)|}$ for each $n \in N$, therefore we get the sought equality.

In the following, we will use the previous results in the particular case, in which $G$ is a semidirect product of $N$ by $K$. Set $G = N \times K$ and let $\{h_1 = 1, h_2, \ldots, h_t\}$ a complete system of representatives from conjugacy classes of $K$. Then $r(G) = r_G(N) + \sum_{i=2}^{t} r_G(h_i N) = r_G(N) + \sum_{i=2}^{t} r_{NC_K(h_i)}(h_i N)$.

Furthermore:

**Theorem (3.22).** Let $G = N \times K$ and $h$ an element of $K$. Then we have the following assertions:

(i) $A_h^G \equiv A_h^{NC_K(h)}$ and $s_h \equiv r_G(hN) \pmod{d_{|C_K(h)|}}$.

(ii) If $M = [[hN, N]]$, then $M$ is a normal subgroup of $N_G(hN) = NC_K(h)$ contained in $N$ and satisfying $r_G(hN) \geq r_E(\tilde{N}) \geq |N/(N \cap N_G(hN)')|$, with $E = NC_K(h)/M$ and $\tilde{N} = N/M$. Furthermore, if $N$ is an abelian group, then $r_G(hN) = r_E(\tilde{N})$, $M = [hN]$, and $A_{hN}^G = (((|C_K(h)|/|\text{Cl}_E(\tilde{m}_1)|, \ldots, (|C_K(h)|/|\text{Cl}_E(\tilde{m}_v)|)/|\text{Cl}_E(\tilde{m}_v)|))$, where $\{\tilde{m}_1, \ldots, \tilde{m}_v\}$ is a complete system of representatives from conjugacy classes of $E$ that make up the normal subgroup $\tilde{N}$ ordered such that $|\text{Cl}_E(\tilde{m}_i)| \leq |\text{Cl}_E(\tilde{m}_{i+1})|$ for every $i = 1, \ldots, v - 1$.

**Proof.** (i) Follows from Lemma (2.10), part (i), and (3.2), Remark (2).

(ii) Evidently, $M$ is a normal subgroup of $N_G(hN)$ contained in $N$. Moreover, if $(hn_1) = hn_2$ for some $n_1, n_2 \in N$ and $z \in N_G(hN) = NC_K(h)$, then $z = h'n'$ for some $h' \in C_K(h)$ and $n' \in N$, so, $\tilde{n}_1 = \tilde{n}_2$, because $[h, n'] = \tilde{1}$, thus $\tilde{n}_1$ is $E$-conjugate to $\tilde{n}_2$, and we conclude that $r_G(hN) = r_{NC_K(hN)}(hN) \geq r_E(\tilde{N})$. Evidently $r_E(\tilde{N}) \geq |N/(N \cap N_G(hN)')|$. Finally, if $N$ is an abelian group, then $M = [hN] \trianglelefteq N_G(hN)$ and the remaining affirmations are proved in [16, Lemma (2.10)].

**Example.** Consider the group $G = C_{A_4}(g)$, where $A_4$ is the alternating group of degree 8 and $g$ is a central involution in a Sylow 2-subgroup of $A_8$. We have $G = (C_2 \times C_2) \times \sum_{i=3}^{12} = (a_1, a_2, a_3) x \langle d, a_2 \rangle = N \times \sum_{i=3}^{12}$ with relations $a_1^p = a_1, a_2^p = a_2, a_3^p = a_3, a_4^p = a_4, a_5^p = a_5, a_6^p = a_6, a_7^p = a_7, a_8^p = a_8, a_9^p = a_9, b^p = b, c^p = c, d^p = d^{-1}, a_i^p = a_i, i = 1, 3, 4, 5, \ldots, 12$.

In this case, there is not $1 \neq M \nleq N_G(a_2 N)$ such that
\[r_G(a_2 N) = r_{NG(a_2 N)/M}(N/M)\]. In fact, we have \(r_G(a_2 N) = 5\) and the non-trivial normal subgroups of \(NG(a_2 N)\) contained in \(N\) satisfy either \(r_{NG(a_2 N)/M}(N/M) \leq 4\) (in case \(|M| \geq 8\)) or \(r_{NG(a_2 N)/M}(N/M) = 6\) (in case \(|M| = 4\)) or \(r_{NG(a_2 N)/M}(N/M) = 10\) (in case \(M = \langle a_3 \rangle\)).

Let \(G = N \times K\), with \(N\) an abelian normal subgroup of \(G\). Suppose that \(M\) is a normal subgroup of \(G\) contained in \(N\). Assume that \(A_G\) is obtained from the local tuples \(A_{G_i}^j, i = 1, \ldots, t\). Now we will carry out the direct calculation of \(A_{G/M}\) using the previous tuples. Thus we have:

**Theorem (3.23).** Assume that \(G = N \times K\) with \(N\) abelian, and that \(M\) is a normal subgroup of \(G\) contained in \(N\). Let \(h \in K\) and consider the following groups: \(E = NC_K(h)/[h, N]\), \(\tilde{N} = N/[h, N]\), and \(D = C_K(h)[h, N]/[h, N]\). Then \(E = \tilde{N} \times D\) and the following assertions hold:

(i) If \(\tilde{M} = M[h, N]/[h, N]\), then \(\tilde{M} \leq E\) and \(r_{G/M}((hM) N/M) = r_{E/\tilde{M}}(\tilde{M}/\tilde{N})\).

(ii) If \(\{\tilde{x}_1, \hat{x}_2, \tilde{x}_3, \ldots, \hat{x}_u\}\) is a complete system of representatives from conjugacy classes of \(E/\tilde{M}\) that make up the normal subgroup \(\tilde{N}/\tilde{M}\), ordered so that \(|C_{E/\tilde{M}}(\tilde{x}_i \tilde{M})| \geq |C_{E/\tilde{M}}(\tilde{x}_{i-1} \tilde{M})|\) for every \(i = 2, \ldots, u\), then we have

\[
A_{G/M}^j N/M = ((|C_{N}(h)| |C_K(h)|)/(|\tilde{M}| |C_{E/\tilde{M}}(\tilde{x}_2 \tilde{M})|), \ldots, (|C_{N}(h)| |C_K(h)|)/(|\tilde{M}| |C_{E/\tilde{M}}(\tilde{x}_u \tilde{M})|)).
\]

**Proof.** Since \(G/M = (N/M) \times (KM/M)\), with \(N/M\) abelian, the Lemma (2.10) from [16] yields \(r_{G/M}(hM) N/M) = r_{E_1}(N_1)\), in which \(N_1 = (N/M)/[hM, N/M]\), \(D_1 = C_{KM/M}(hM)[hM, N/M]/[hM, N/M]\), and \(E_1 = N_1 \times D_1 = ((N/M) C_{KM/M}(hM))/[hM, N/M]\). Moreover, if \(\{y_1, y_2, \ldots, y_u\}\) is a complete system of representatives from conjugacy classes of \(E_1\), that make up the normal subgroup \(N_1\), ordered so that

\[|Cl_{E_1}(y_1)| \leq \ldots \leq |Cl_{E_1}(y_u)|,\]

then we have

\[
A_{G/M}^j (N/M) = ((|C_{N/M}(hM)| |C_{KM/M}(hM)|)/|Cl_{E_1}(y_1)|, \ldots, (|C_{N/M}(hM)| |C_{KM/M}(hM)|)/|Cl_{E_1}(y_u)|).
\]

(27)

Consider now the following isomorphism of groups:

\[\alpha: K \rightarrow KM/M, \beta: (NC_K(h)/M)/(([h, N]/M)/M) \rightarrow (NC_K(h)) / ([h, N]/M),\]

and \(\gamma: ((NC_K(h))/([h, N]) /(([h, N]/M)/[h, N])) \rightarrow (NC_K(h))/([h, N]/M).\)
We have \( \tau = \beta^{-1} \cdot \gamma \colon E/\tilde{M} \to E_1 \) in which \( \tau(\tilde{N}/\tilde{M}) = N_1 \) and \( \tau(D\tilde{M}/\tilde{M}) = D_1 \). Let \( \tilde{x}_i, \tilde{M} \in \tilde{N}/\tilde{M} \) be such that \( \tau(\tilde{x}_i, \tilde{M}) = y_i \), Then

\[
|Cl_{E_i}(y_i)| = |Cl_{E_i}(\tau(\tilde{x}_i, \tilde{M}))| = |\tau(Cl_{E_i}(\tilde{x}_i, \tilde{M}))| = |Cl_{E_i}(\tilde{x}_i, \tilde{M})|, \tag{28}
\]
and evidently \( \{ \tilde{x}_1, \tilde{M}, \ldots, \tilde{x}_u, \tilde{M} \} \) is a complete system of representatives from conjugacy classes of \( E/\tilde{M} \) that make up the normal subgroup \( \tilde{N}/\tilde{M} \).

On the other hand, we have \( \alpha(C_{k}(h)) = C_{\alpha(h)}(\alpha(h)) = C_{KM/M}(hM) \), hence

\[
|C_{KM/M}(hM)| = |C_{K}(h)| \tag{29}
\]
and

\[
|C_{N/M}(hM)| = |(N/M)/[hM, N/M]| = |N/([h, N] M)|
= (|N|/|[h, N]|)/|\tilde{M}| = |C_{N}(h)|/|\tilde{M}|. \tag{30}
\]
Thus, by taking the values (28)-(30) to tuple (27), we obtain the desired tuple.

Remark. Observing the coordinates of \( A_{G/M}^{G/N} \), it is deduced that once the tuple \( A_{G/N}^{G} \) is obtained, we only need to know \( \tilde{M} \) to get the tuple \( A_{G/M}^{G/N} \).

**Corollary (3.24).** Using the notations introduced in Theorem (3.23), we have the following assertions:

(i) If \( M \leq [h, N] \) then \( A_{hN}^{G} = A_{hM}^{G/M} \).

(ii) If \( N = M[h, N] \), then \( r_{G/M}((hM)N/M) = 1 \) and \( A_{(hM)N/M}^{G/M} = (|C_{K}(h)|) \).

(iii) If there exists \( N_0 \leq G \) such that \( N = N_0 \times M[h, N] \) then

\[
|Cl_{E_i}(\tilde{x}_i, \tilde{M})| = |\tau(Cl_{E_i}(\tilde{x}_i, \tilde{M}))| = |Cl_{E}(\tilde{x}_i)| \text{ for every } i = 1, \ldots, u.
\]

**Proof.** (i) and (ii) follow from the fact that, either \( \tilde{M} = \tilde{1} \) or \( \tilde{M} = \tilde{N} \), in these cases, respectively.

(iii) We have \( \tilde{N}/\tilde{M} = (N/[h, N])/(M[h, N]/[h, N]) \simeq N_0 \) and \( \tau_1 : E/\tilde{M} = (\tilde{N}/\tilde{M}) x_1(D\tilde{M}/\tilde{M}) \sim \tilde{N}_0 x_1 D \) with \( \tilde{N}_0 = (N_0[h, N])/[h, N] = \tau_1(\tilde{N}/\tilde{M}) \). Therefore

\[
|Cl_{E_i}(\tilde{x}_i, \tilde{M})| = |D\tilde{M}/\tilde{M} : C_{D\tilde{M}/\tilde{M}}(\tilde{x}_i, \tilde{M})| = |\tau_1(D\tilde{M}/\tilde{M} : C_{1}(D\tilde{M}/\tilde{M})| = |\tau_1(\tilde{x}_i, \tilde{M})| = |Cl_{E}(\tilde{x}_i)|.
\]

**Remark.** Let \( G = Nx_1 K \) with \( N \) abelian and let \( H \leq K \). Suppose that \( A_{N_1 H}^{N_1} \) is known, we are interested in obtaining information about \( A_{G}^{G} \), through the calculus carried out for the group \( N x_1 H \). Of course, \( A_{N_1 H}^{N} = A_{hN}^{N_1 \cdot H} \), hence \( A_{N_1 H}^{N} = A_{hN}^{N_1 \cdot H} \), if \( h \) is an element of \( H \) satisfying
$C_H(h) = C_K(h)$. For example, for the calculus of all conjugacy-vectors of relative holomorphs of $C_4^4$, we shall bear in mind the above observation, to avoid repetitions in the calculus. In this sense, we consider the following equalities,

$$\Delta_{hC_2^4}^{\text{Hol}(C_4^4)} = \Delta_{hC_2^4}^{C_4^4} = \Delta_{hC_2^4}^{C_A(h)}$$

for each $H \leq A_8$ satisfying $h \in H$ and $C_H(h) = C_A(h)$.

4. Some Examples

Through the several examples given in this section, we wish to show that our results are useful in calculating the conjugacy-vector of a finite group and in classifying finite groups according to the number of conjugacy classes.

**Example 1.** If $N$ is a subgroup of the center of $G$, then

$$A_g^N = (|C_C(g)|, \frac{\nu}{|C_C(g)|}, \nu, \cdots, \nu).$$

**Example 2.** If $N \cap C(g) = 1$, then we have

$$A_g^N = (|N|, |C_C(g)|, \cdots, |N|, |C_C(g)|).$$

**Example 3.** If $N$ and $C(g)$ are $p$-groups for some prime number $p$, then there exist non-negative integers $c_1, \cdots, c_v, c$ satisfying $v \equiv 1 (\text{mod } p-1)$, $c_1 \geq \cdots \geq c_v \geq c$, $1 = \sum_{i=1}^{v} (1/p_{c_i} - c)$, $p^c = |C_C(g)|$, and

$$A_g^N = (p^{c_1}, \cdots, p^{c_v}).$$

**Example 4.** Let $G$ be a $p$-group and $N$ a normal subgroup of $G$ such that $G/N$ is abelian. Then the following congruence is true:

$$r_G(N) \equiv |G| - |G/N| + 1 \pmod{(p-1)^2}. \quad (31)$$

Indeed, let $t_i$ be the number of elements of order $p^i$ of $G/N$. If $o(\bar{x}) = p^i$, we have $r_G(x^iN) = r_G(x^iN)$ for each $j$ coprime to $p^i$. As the number of such $j$ is $\phi(p^i) = p^i - 1 - (p-1)$, it follows that we can find elements $x_j \in G$ such that

$$r(G) = r_G(N) + \sum_{i=1}^{e} \sum_{j=1}^{t_i} r_G(x_jN) \phi(p^i),$$

where $r_i = t_i / \phi(p^i)$. Now (31) follows from the following assertions:

$r_G(x_jN) \equiv 1 \pmod{p-1}$, $|G/N| = t_1 + \cdots + t_e + 1$, assuming that
$|G/N| = p^r$ and $|G| \equiv r(G) \pmod{(p - 1)^2}$. For example, if $G$ is a nonabelian $p$-group of order $p^3$ and $|N| = p^2$, then $r_G(N) \equiv p^3 - p + 1 \equiv 2p - 1 \pmod{(p - 1)^2}$. In fact, we have $r_G(N) = 2p - 1$, in this case.

**Example 5.** Let $N$ be a normal $p$-subgroup of $G$ and $g$ an element of $G$ such that $C_G(g)$ is a $p$-group. Then $r_G(gN) \equiv 1 \pmod{p - 1}$. Further,

$$s_g = r_{N^g}(gN) \geq |Z(N_G(gN)) \cap N|.$$ Therefore, if $C_G(g) = \langle g \rangle$, then $r_G(gN) \geq |Z(N_G(gN)) \cap N| \geq p$.

**Example 6.** If

$$C_G(g) = \overline{C_G(gn)}$$

for each $n \in N$, then we have $r_G(gN) \equiv 1 \pmod{d^i|N|}$. Indeed, $|N| \equiv r_G(gN) = \sum_{n \in N} |C_N(gn)|$ and $|N| \equiv 1 \equiv |C_N(gn)|(\pmod{d^i|N|})$ imply the desired result. The condition (32) is satisfied, for example, in case $C_G(g) = \langle g \rangle$, thus we generalize the lemma of [14].

**Example 7.** The number $r(N) - s_g$ is localized. Indeed, there exist elements $m_i \in N$ such that $r(N) - s_g = t_m + \cdots + t_m$ with $t_m = |N^g(m_i) N| > 1$ for each $i = 1, \ldots, k$. In particular, each $t_m$ is a divisor of $o(g) = |N^g(g)/N|$, and if $o(g)$ is a prime number, we have $r(N) - s_g = u \cdot o(g)$.

**Example 8.** For each $g \in G$, we have $r_G(gN) \leq s_g$ and $r_G(gN) \equiv s_g(\pmod{d^i|G|})$, therefore, if $\{ \tilde{g}_1, \ldots, \tilde{g}_t \}$ is a complete system of representatives from distinct conjugacy classes of $\tilde{G} = G/N$, then

$$r(G) = r_G(N) + \sum_{i=2}^t s_{\tilde{g}_i} - k \cdot d^i|G|,$$

for some non-negative integer number $k$.

**Example 9.** If $C_G(g) = \langle g \rangle$, then we have $r_G(gN) = s_g \geq r_N(C_N(g))$. Furthermore, if $N$ is abelian, then $r_G(gN) = |C_N(g)|$.

**Example 10.** Let $G = N \times \langle g \rangle = N \times C_q$ with $|N| = p^a$ and $p, q$ be two prime numbers. Set $B_N^N = \Cl_N(n_1) \cup \cdots \cup \Cl_N(n_2)$ and $N - B_N^N = \Cl_N(m_1) \cup \cdots \cup \Cl_N(m_s)$, with $t = (r(N) - s)/q$. We have $s \equiv 1(\pmod{p - 1})$, $s \equiv r(N)(\pmod{q})$ and

$$\Delta_G = (q \mid C_N(n_1), \ldots, q \mid C_N(n_s), |C_N(m_1)|, \ldots, |C_N(m_s)|, c_1, \ldots, c_s, \ldots, c_s).$$
where \( \Delta_{G}(N) = (c_1, ..., c_q) \) for each \( i = 1, 2, ..., q - 1 \). Moreover \( s = r_N(C_N(g)) + u(p - 1) \) for some non-negative integer number \( u \).

**Example 11.** The following inequality is true: \( r_G(gN) \leq r_N(C_N(gN))(N) \). But, in general \( r_G(gN) \) is not smaller than \( r_G(N) \). For example, if \( G = N \times A_5 \), with \( A_5 \) acting transitively by conjugation over \( N - \{ 1 \} \), and \( h \in A_5 \) satisfies \( |C_N(h)| = 4 \), then \( \Delta_{hN} = \Delta_{hN}(hN) = \Delta_{hN} = (16, 16, 16, 16) \), hence \( r_G(hN) = 4 \), although \( r_G(N) = 2 \) (notice that \( r_{N_G(hN)}(N) = 7 \), in this case).

**Example 12.** \( G \) is a group satisfying \( G/S(G) \simeq \text{Hol}(C_2, C_7 \times C_3) \) and \( r(G) = 14 \) if and only if \( G = \text{Hol}(P, C_7 \times C_3) \), with \( P \) a 2-group isomorphic to one Sylow 2-subgroup of finite simple group \( Sz(8) \). Furthermore, we have \( r(G) - 14 \) and

\[
\Delta_G = (1344, 192, 48, 48, 12, 8, 12, 7, 7).
\]

Indeed, set \( G/S(G) = G = \langle \tilde{x}_1, \tilde{x}_2, \tilde{x}_3 \rangle \times \langle \langle \tilde{a} \rangle \times \langle \tilde{b} \rangle \rangle \) with \( \tilde{x}_6 = \tilde{x}_1 \). Then, \( \alpha(G) = r_G(G - S(G)) = r_G(x_1S(G)) + 2r_G(aS(G)) + 2r_G(bS(G)) + 2r_G(x_1bS(G)) \). In addition, we can assume that \( a \) is a 7-element and \( b \), a 3-element.

If \( S(G) \) is a non-solvable group, then both \( r_G(aS(G)) \) and \( r_G(bS(G)) \) are greater than or equal to 2, so, \( r_G(S(G)) \leq 3 \), and consequently, \( |S(G)| \) is divisible by at most two prime numbers, which is impossible. Thus, \( S(G) \) is an abelian group and we can write

\[
\alpha(G) = r_G(x_1S(G)) + 2|C_G(a) \cap S(G)| + 2r_G(bS(G)) + 2|C_G(x_1b) \cap S(G)|,
\]

because \( \Delta_G = (168, 24, 7, 7, 6, 6, 6, 6) \) and \( \{ 1, \tilde{x}_1, \tilde{a}, \tilde{a}^2, \tilde{b}, \tilde{b}^{-1}, \tilde{x}_1\tilde{b}, (\tilde{x}_1\tilde{b})^{-1} \} \) is a complete system of representatives from distinct conjugacy classes of \( G \). We have \( \alpha(G) \leq 12 \), hence \( |C_G(a) \cap S(G)| \leq (12 - 5)/2 \) and 7 does not divide \( |S(G)| \). If 3 divides \( |S(G)| \), then \( |O_3(S(G))| \) is greater than or equal to 3, because \( C_7 \) acts f.p.f. on \( O_3(S(G)) \), so 9 divides \( |C_G(b) \cap S(G)| \) and \( r_G(bS(G)) \geq (3.9)/6 \), using both [17, Lemma (1.9)] and Theorem (3.2), which is impossible. Suppose that \( O_p(S(G)) \neq 1 \) with \( p \) an odd prime number different from 3 and 7. Since \( r_G(bS(G)) \leq (12 - 5)/2 \), the only possibility is that \( p = 5 \), where \( |C_G(b) \cap S(G)| = 5 \) and consequently \( |O_5(S(G))| = 5 \). Since \( O_3(S(G)) \) is odd, \( |O_5(S(G))| = 5 \), and consequently \( |C_G(a) \cap S(G)| \) is less than 3. If \( C_G(a) \cap S(G) = \langle z \rangle \cong C_2 \), then \( z^5 = z \) and \( \alpha(G) \geq 1 + 2.2 + 2.2 + 2.2 = 13 \), which is impossible. Therefore, \( \langle a \rangle \) acts f.p.f. on \( S(G) \) and \( |S(G)| = 2^{3k} \). Moreover,
$S(G) \subseteq Z(N)$, setting $G/N \simeq C_7 x_f C_3$. Even more, $r_G(bS(G))$ is greater than 1, since $C_7 x_f C_3$ does not act f.p.f. on $S(G)$. But $r_G(x_1S(G)) \geq (2.2^k)/24$ implies $k = 1$ and $|N| = 2^6$. If $N$ is abelian, then we have $\alpha(G) \geq 3 + 2 + 2.2 + 2.2 = 13$, which is impossible. Obviously the center of $N$ is of order 8, and all involutions are in the center. Taking any element outside of the center and specifying its square, then the action of the element of order 7 gives all squares and also all commutators. Thus $N$ is of type $Sz(8)$ and necessarily $\Delta_G = (1344, 192, 48, 48, 12, 8, 12, 7, 7)$.

**Example 13.** If $G$ is a group satisfying $G/S(G) \simeq A_6$ and $r(G) \leq 14$, then either $G \simeq SL(2, 9)$ or $G$ is isomorphic to the unique split extension of $C_2^4$ by $A_6$. Further,

$$\Delta_{SL(2, 9)} = (720, 720, 18, 18, 18, 18, 10, 10, 10, 8, 8, 8),$$

and $\Delta_{C_2^4 \times A_6} = (5760, 384, 36, 32, 32, 16, 12, 9, 8, 8, 5, 5)$ ($S(G)$ denotes the socle of $G$). Indeed, we have $\Delta_{A_6} = (360, 9, 9, 8, 5, 5) \leq (|C_{A_6}(1)|, |C_{A_6}(x_1)|, |C_{A_6}(x_2)|, |C_{A_6}(\bar{a})|, |C_{A_6}(\bar{c})|, |C_{A_6}(\bar{d})|)$, and

$$\alpha(G) = r_G(G - S(G)) = r_G(x_1S(G)) + r_G(x_2S(G)) + r_G(a^2S(G))$$

$$+ 2r_G(cS(G)) + r_G(aS(G)).$$

Suppose that $S(G)$ is non-solvable. Then $S(G)$ does not have automorphism $\sigma$ satisfying either $|C_{S(G)}(\sigma)| = 2$ and $o(\sigma)$ prime or $C_{S(G)}(\sigma) = 1$ and $o(\sigma) = 4$. Therefore, $r_G(cS(G)) \geq 3$ and $r_G(aS(G)) \geq 2$, hence $\alpha(G) \geq 11$ and $r_G(S(G)) \leq 3$, which is impossible. Thus $S(G)$ is abelian and

$$\alpha(G) = r_G(x_1S(G)) + r_G(x_2S(G)) + r_G(a^2S(G))$$

$$+ 2|C_G(c) \cap S(G)| + |C_G(a) \cap S(G)|,$$

in this case. We have $|C_G(c) \cap S(G)| \leq (12 - 4)/2 = 4$, hence 5 does not divide $|S(G)|$. Suppose that $O_3(S(G)) \neq 1$. If $O_3(S(G)) \simeq C_3$, then $\alpha(G) \geq 2 + 2 + 2 + 2.3 + 1 = 13$, which is impossible. If $|O_3(S(G))| = 3^2$ or $3^3$, then $c$ centralizes $O_3(S(G))$, which is impossible. Therefore, $|O_3(S(G))|$ is greater than $3^3$ and consequently $9$ divides $|C_G(c) \cap S(G)|$. Now, $r_G(x_iS(G)) \geq (3.9)/9 = 3$ for each $i = 1, 2$, and considering the action by conjugation of $N_{A_6}(C_3) \simeq D_{10}$ on $O_3(S(G))$, we have that $r_G(a^2S(G))$ is greater than $(2.3^2)/8$. Our conditions imply that $r_G(x_iS(G)) = 3$ and

$$\Delta_{x_iS(G)} = (27, 27, 27).$$

Thus, $C_3$ acts f.p.f. on $O_3(S(G))$ and necessarily $S(G)$ is a 3-group of order $3^4$. Now, we have $\alpha(G) = 12$, $r_G(S(G)) = 2$, and the non-trivial conjugacy class contained in $S(G)$ has cardinality 80, which is impossible. Therefore, 3 does not divide $|S(G)|$. Furthermore, if
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$O_p(S(G)) \neq 1$ for some $p$ odd prime, then $C_G(x_i) \cap O_p(S(G)) \simeq C_p$, and necessarily $p = 7$, where $|O_7(S(G))| \geq 7^4$ and $r_G(a^2 S(G)) \geq (2.7^2)/8$, which is impossible. We conclude therefore that $S(G)$ is a 2-group.

If $|S(G)| \geq 2^7$, then $r_G(a^2 S(G)) \geq (2.16)/8$, 4 divides $|C_G(a) \cap S(G)|$, and $r_G(x_i S(G)) \geq 2$, which is impossible. If $|S(G)| = 2^6$, then $4 = |C_G(c) \cap S(G)|$ and 4 divides $|C_G(a) \cap S(G)|$, which is impossible. If $|S(G)| = 2^5$, we have $\alpha(G) \geq 2 + 1 + 2 + 2.2 + 4 = 13$, which is impossible. Thus we conclude that $|S(G)|$ is less than $2^5$. If $|S(G)| = 8$, then $S(G) \leq Z(G)$, and necessarily $S(G) \simeq C_2$ and $G \simeq SL(2, 9)$. Finally, if $|S(G)| = 2^4$, then $G$ is isomorphic to the unique split extension of $C_2^4$ by $A_6$.

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