JOURNAL OF ALGEBRA 107, 272–283 (1987)

Representative Functions on Discrete Solvable Groups

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Communicated by Nathan Jacobson

Received June 10, 1985

INTRODUCTION

The topic here is the representation of discrete groups as automorphisms of finite-dimensional vector spaces over a field. The results here are mainly generalizations to the class of solvable groups of finite torsion-free rank of results of Mostow concerning polycyclic groups.

In discussing the finite-dimensional representations of a group Γ over a field k, one is led to consider the "continuous dual" $k[\Gamma]^0$ of the group algebra, where "continuous" refers to the topology on $k[\Gamma]$ in which a fundamental system of neighborhoods of zero is the family of kernels of finite-dimensional representations of Γ over k. $k[\Gamma]^0$ has the structure of a coalgebra—indeed, of a Hopf algebra—and the locally finite-dimensional $k[\Gamma]$-modules correspond to the $k[\Gamma]^0$ comodules. The advantage of looking at $k[\Gamma]^0$ is that any finite-dimensional $k[\Gamma]^0$-comodule is a subcomodule of the direct sum of finitely many copies of $k[\Gamma]^0$. A further advantage is that, when k is algebraically closed, $k[\Gamma]^0$ is the polynomial algebra of a pro-affine algebraic group $G_k(\Gamma)$ whose rational representations correspond exactly to the locally finite-dimensional representations of Γ over k. This brings to bear the structural results on pro-affine algebraic groups, which are strongest when the base field k is of characteristic zero.

In Section 1, we examine the unipotent radical $U_k(\Gamma)$ of the pro-affine group $G_k(\Gamma)$ associated with a discrete group Γ, and we prove that, on the category of solvable groups of finite torsion-free rank (i.e., the groups of type A_1 in Mal'cev's classification [10]), the assignment of $U_k(\Gamma)$ to Γ is an exact functor to the category of unipotent affine algebraic groups. In Section 2, we consider a certain homomorphic image $B_k(\Gamma)$ of $G_k(\Gamma)$ which we call the basic k-group associated with Γ; this group is the "lowest" homomorphic image of $G_k(\Gamma)$ to "preserve" the unipotent radical $U_k(\Gamma)$. The assignment of $B_k(\Gamma)$ is functorial on a subcategory of the category of

0021-8693/87 $3.00

solvable groups (the category contains all solvable groups but its morphisms are only the "subnormal" morphisms). We also determine the kernel of the canonical map from Γ to $B_k(\Gamma)$. Finally, in Section 3, we determine the "unipotent radical" of a solvable group Γ of type A_1 and the kernel of the canonical map from Γ to $G_k(\Gamma)$. This gives a necessary and sufficient condition for a solvable group of type A_1 to have a faithful locally finite-dimensional representation over a field of characteristic zero.

When working over fields of characteristic zero, we can often avoid having to assume that the field is algebraically closed. The price we pay for this is that we must work with Hopf algebras rather than with groups, and this accounts for the rather heavy use in our proofs of the structure theory of commutative Hopf algebras.

1. The Functor of Representative Functions and the Unipotent Radical

Let Γ be a (discrete) group and k a field. We denote by $R_k(\Gamma)$ the Hopf algebra of k-valued *representative functions* on Γ [6, p. 2]; in the notation of [13], $R_k(\Gamma)$ is the Hopf dual $k[\Gamma]^0$ of the group algebra $k[\Gamma]$. The assignment of $R_k(\Gamma)$ to Γ is the object part of a functor from the category of groups to the category of (commutative, reduced) Hopf algebras over k.

The ideas of exact sequences and of extensions carry over from the category of groups to the category of commutative Hopf algebras over a field (see [15] and [1]). Specifically, a sequence $A \to_\alpha B \to_\beta C$ of morphisms of commutative Hopf algebras is called *exact* if the kernel of β is the normal Hopf ideal corresponding to the sub Hopf algebra $\alpha(A)$ of B. If $\Gamma_1 \to \Gamma_2 \to \Gamma_3 \to 1$ is an exact sequence of morphisms of groups, then the corresponding sequence of morphisms of Hopf algebras $R_k(1) \to R_k(\Gamma_3) \to R_k(\Gamma_2) \to R_k(\Gamma_1)$ is exact in this sense.

If the field k is algebraically closed, then $R_k(\Gamma)$ is the polynomial algebra of a pro-affine algebraic group which we denote by $G(R_k(\Gamma))$ (or, simply, $G_k(\Gamma)$). We call $G_k(\Gamma)$ the *universal pro-algebraic hull* of Γ.

Remark. $R_k(\Gamma)$ depends crucially on k. For example, if we take Γ to be the quasi-cyclic p-group $\mathbb{Z}(p^\infty)$, then, for $k = \mathbb{Q}$, we get $R_k(\Gamma)$ to consist of just the constant functions, while for k the field obtained by adjoining the primitive p^n roots of unity for $n = 1, 2, 3,\ldots$, $R_k(\Gamma)$ separates the elements of Γ. Thus, for field extensions K of k, it is not generally true that $R_K(\Gamma) \approx R_k(\Gamma) \otimes_k K$. As a further example, let k be any field, and t an element of a larger field such that t is transcendental over k, $\Gamma = \mathbb{Z}$. $K = k(t)$. Then the function $\Gamma \to K$ that sends the element n of Z to t^n is an element of $R_K(\Gamma)$, but is clearly not in $R_k(\Gamma) \otimes_k K$. However, we do get the equality $R_k(\Gamma) \otimes_k K = R_K(\Gamma)$ if Γ is finitely generated and K an algebraic extension of k, as we see from the following lemma.

LEMMA 1. *Let k be a field and K an algebraic extension of k. Let A be a finitely-generated algebra over k, and let A^0 denote the continuous dual of A (with respect to the topology where the two-sided ideals of finite-codimension form a fundamental system of neighborhoods of zero). Then $(A \otimes_k K)^0$ is naturally isomorphic with $A^0 \otimes_k K$.*

Proof. It is easy to see that the canonical injection of $A^* \otimes_k K$ into $(A \otimes_k K)^*$, where * denotes the full dual, maps $A^0 \otimes K$ into $(A \otimes K)^0$. Now, if $f \in (A \otimes K)^0$, and if J is an ideal of finite codimension in $A \otimes K$ on which f vanishes, then let I denote the kernel of the A-module map $A \to A \otimes K/J$. In order to prove that f is in the image of $A^0 \otimes K$, it suffices to show that I is of finite codimension in A. Let $u_1,..., u_r$ be a K-basis of $A \otimes K/J$ with $u_1 = 1$, and let $y_1,..., y_s$ be k-algebra generators of A. Then, since K is algebraic over k, there is an intermediate field F that is finite-dimensional over k and such that $\sum_{i=1}^{i=r} Fu_i$ contains all of the transforms $y_i \cdot u_j$. Now $\sum Fu_i$ is finite-dimensional over k and contains the image of A in $A \otimes K/J$. This proves that the kernel I is of finite codimension in A.

CONVENTION. For the rest of Section 1, we assume that the field k is of *characteristic zero.*

For any commutative Hopf algebra H, we denote by H_u the quotient of H by the normal Hopf ideal that corresponds to the coradical of H. Then, H_u is an irreducible Hopf algebra (see [14]) and, if k is algebraically closed, the quotient map $H \to H_u$ is the polynomial map corresponding to the inclusion of the unipotent radical $U(H)$ into the pro-affine group $G(H)$ associated with H.

Notation. For an abelian group A, we denote by $r_0(A)$ the *torsion-free rank* of A, i.e., the dimension of the $\mathbb{Q}$-space $A \otimes_{\mathbb{Z}} \mathbb{Q}$. We can (uniquely) extend r_0 to solvable groups in such a way that if B and C are solvable and $B \lhd C$, then $r_0(C) = r_0(B) + r_0(B/C)$. We use the terminology of Mal'cev [10] and call a solvable group Γ an A_1 *group* if $r_0(\Gamma)$ is finite.

THEOREM 2. *Let Γ be a solvable A_1 group and k a field (of characteristic zero). Then $(R_k(\Gamma))_u$ is the polynomial algebra of a unipotent affine algebraic k-group $U_k(\Gamma)$ of dimension $r_0(\Gamma)$. Moreover, the assignment of $U_k(\Gamma)$ to Γ is the object part of an exact functor from the category of A_1-groups to the category of unipotent affine algebraic k-groups. Finally, for every extension field K of k, $(R_k(\Gamma))_u \otimes_k K$ is canonically isomorphic with $R_K(\Gamma)_u$.*

Proof. First, we deal with the case in which k is algebraically closed, and show that $U_k(\Gamma)$ is affine of dimension $r_0(\Gamma)$.

As in [11, 3.3], $U_k(\ )$ is a right-exact functor on solvable groups, so we can factor out of Γ the largest normal periodic subgroup $\tau(\Gamma)$. Thus, by [10, Theorem 3], we may assume that Γ is an A_4-group (i.e., has a finite subnormal series with A_1 abelian factors whose periodic parts are finite), and that $\tau(\Gamma)$ is trivial. Thus, by [10, Theorem 5], the *Hirsch–Plotkin radical* $\rho(\Gamma)$ of Γ is nilpotent (and torsion-free).

Next, we gather some results on torsion-free nilpotent A_1-groups. First, for any nilpotent group N, the group $G_k(N)$ is a pro-affine nilpotent group (since every finite-dimensional image of $G_k(N)$ is nilpotent, as can be seen by [3, Chap. V, p. 119]) so that its unipotent radical $U_k(N)$ is a direct factor. This means that the map $R_k(N) \to (R_k(N))_u$ restricts to an isomorphism of the sub-Hopf algebra $V_k(N)$ of *unipotent representative functions* (i.e., representative functions g such that N acts unipotently on the module $[N \rightharpoonup g]$ spanned by the left-translates of g) onto $(R_k(N))_u$. Now, N embeds into its *Mal'cev completion* (or, 0-*localization*) $\bar{N}$ (see, e.g., [16, 8.9]), which is a torsion-free radicable nilpotent group such that for each $x \in \bar{N}$, some positive power of x is in N. Since every unipotent representation of N is the restriction of a unipotent representation of $\bar{N}$ (as easily follows from the exactness of the localization functor, [16, 8.11]), we see that, for any field F (of characteristic zero), the restriction map $V_F(\bar{N}) \to V_F(N)$ is surjective, and that the kernel of the restriction map is a sub $\bar{N}$-module of $V_F(\bar{N})$. This implies easily that the kernel of the restriction map is trivial, so that we have $V_F(\bar{N}) \approx V_F(N)$. Moreover, $r_0(\bar{N}) = r_0(N)$ (this is true in the abelian case (see, e.g., [5, p. 107]) and, thus, by exactness of localization, for all nilpotent A_1-groups).

Now, functorially associated with $\bar{N}$ is an $r_0(\bar{N})$-dimensional nilpotent Lie algebra $L(\bar{N})$ over $\mathbb{Q}$ (see, e.g., [12, Theorem 2.4.2]). A unipotent representation of $\bar{N}$ on a finite-dimensional F-space V defines a nilpotent representation of $L(\bar{N}) \otimes_{\mathbb{Q}} F$ on V, and vice versa. This defines a Hopf algebra isomorphism between $V_F(\bar{N})$ and the Hopf algebra $\mathscr{B}(L(\bar{N}) \otimes F)$ of nilpotent representative functions of $L(\bar{N}) \otimes F$ whence, by [6, pp. 231 and 208], we see that $V_F(\bar{N})$ is generated as an algebra by $r_0(N) = n$ algebraically independent elements $f_1, ..., f_n$ such that $\Delta(f_i) \equiv (1 \otimes f_i + f_i \otimes 1) \bmod F[f_1, ..., f_{i-1}] \otimes F[f_1, ..., f_{i-1}]$ (where Δ is the comultiplication map). Now, let F' be an algebraic closure of F and let us look at the following commutative diagram:

$$\begin{array}{ccc} V_F(N) \otimes F' & \longrightarrow & (R_F(N))_u \otimes F' \\ \| & & \downarrow{\scriptstyle \rho} \\ V_{F'}(N) & ===== & (R_{F'}(N))_u \end{array}$$

To see that the upper horizontal map (which is formed by tensoring with F' the restriction to $V_F(N)$ of the quotient map j) is an injection, we argue as follows. From [14], there is an algebra isomorphism $\nu\colon R_F(N) \to R_F(N)_u \otimes (R_F(N))_0$, where $(R_F(N))_0$ is the coradical of $R_F(N)$; specifically, ν is the map $(q \otimes \pi) \circ \Delta$ where q is the quotient map and π a projection onto the coradical. Applying ν to the generating elements f_i of $V_F(N)$, and using the fact that π acts trivially on $V_F(N)$, we see that, on $V_F(N)$, ν is just the map $f \to j(f) \otimes 1$, whence j is injective on $V_F(N)$. The map p comes from the inclusion $R_F(N) \otimes F' \to R_{F'}(N)$ which induces a surjection of the corresponding algebraic groups which, in turn, induces a surjection of the unipotent radicals of those groups. Since $(R_F(N))_u \otimes F'$ is easily seen to equal $(R_F(N) \otimes F')_u$ (we need just to note that the coradical of $R_F(N) \otimes F'$ is exactly the tensor product with F' of the coradical of $R_F(N)$ [14]), the map p is then the polynomial map corresponding to a surjection of groups, so is injective. It follows that all of the maps in the above diagram are isomorphisms.

Thus, from the known structure of $V_F(N)$ and the above discussion, we conclude that, for every field F of characteristic zero and every torsion-free nilpotent A_1-group N, $(R_F(N))_u$ is the polynomial algebra of a $r_0(N)$-dimensional unipotent affine algebraic F-group (i.e., the algebra homomorphisms to F separate the elements of $(R_F(N))_u$) and that, for every extension field K of F, one has $(R_K(N))_u \approx (R_F(N))_u \otimes K$.

Now we return to the situation at the beginning of the proof—viz. that Γ is an A_4-group whose torsion radical $\tau(\Gamma)$ is trivial, and k is an algebraically closed field.

By a result of Charin [2, Theorem 5], Γ has a subnormal series $1 \subset \rho(\Gamma) \subset M \subset \Gamma$ such that Γ/M is finite and $M/\rho(\Gamma)$ is free abelian. Thus, there is a subnormal series

$$\rho(\Gamma) = M_0 < M_1 < \cdots < M_t = M$$

between $\rho(\Gamma)$ and M such that each M_i is the semidirect product of M_{i-1} and an abelian subgroup of M_i. Thus, by a theorem of Mostow on extensions of representations [11, Theorem 3.2],

$$1 \to U_k(\rho(\Gamma)) \to U_k(M)$$

is exact. Since $U_k(\ )$ is a right-exact functor, we see then that $\dim U_k(M) = \dim U_k(\Gamma) = r_0(\Gamma)$ (see [11, 3.3]). This is enough to prove that the functor $U_k(\ )$ is exact on A_1-groups.

Now, if k is any field of characteristic zero (not necessarily algebraically closed) and if Γ is any A_1-group, then $R_k(\Gamma)_u \approx R_k(\Gamma/\tau(\Gamma))_u$. (This is seen by looking at the exact sequence

$$R_k(1) \to R_k(\Gamma/\tau(\Gamma)) \to R_k(\Gamma) \to R_k(\tau(\Gamma))$$

Tensoring with an algebraically closed field, then passing to the unipotent radicals of the associated groups yields the result.) Moreover, one sees similarly that, for any normal subgroup Γ' of finite index in Γ, $R_k(\Gamma')_u \approx R_k(\Gamma)_u$.

From the above-quoted result of Charin, then, we may assume that Γ is built up from a nilpotent torsion-free group Γ_0 by a succession of semidirect products with free abelian groups. Let us write the intermediate groups as Γ_i, where $\Gamma_{i+1} = \Gamma_i \rtimes A_{i+1}$. We show that each of the sequences

$$k \to (R_k(A_i))_u \to R_k(\Gamma_i)_u \to R_k(\Gamma_{i-1})_u \to k \qquad (*)$$

is exact and is a *split extension* of commutative Hopf algebras (see [1]).

In order to prove that each of the sequences is exact at the right end, we need only show that the map $R_k(\Gamma)_u \to R_k(\Gamma_0)_u$ is surjective. Since $R_k(\Gamma_0)_u$ is isomorphic with the sub Hopf algebra $V_k(\Gamma_0)$ of unipotent representative functions on Γ_0, it suffices to prove that the image of the restriction map $R_k(\Gamma) \to R_k(\Gamma_0)$ contains $V_k(\Gamma_0)$ or, equivalently, that every finite-dimensional unipotent Γ_0-module over k embeds into a finite-dimensional Γ-module. If the field k is algebraically closed, this is a consequence of Mostow's extension theorem [11, Theorem 3.2]. However, since each A_i is finitely-generated, we can apply the considerations in the proof of Lemma 1 to show that we get the result for general fields from the result in the algebraically closed case. This shows then that each sequence $(*)$ is exact and, using the isomorphism $R_k(A_i)_u \approx V_k(A_i)$, we see that it represents a split extension. This implies that $(R_k(\Gamma_i)_u$ is the *co-smash product* of the Hopf algebras $R_k(\Gamma_{i-1})$ and $R_k(A_i)_u$ (see [1, p. 209]). In particular, as an algebra, $(R_k(\Gamma_i))_u$ is isomorphic with the tensor product of $R_k(\Gamma_{i-1})_u$ with $(R_k(A_i))_u$. Thus, we see inductively that the elements of $R_k(\Gamma)_u$ are separated by algebra homomorphisms to the base field, and that $R_k(\Gamma)_u$ is the polynomial algebra of a unipotent affine algebraic k-group $U_k(\Gamma)$ of dimension $r_0(\Gamma)$. For an extension field K of k, since $R_K(\Gamma)_u$ will be built from co-smash products of $R_K(\Gamma_i)_u$'s, with $R_K(A_i)_u$'s we see that $R_K(\Gamma)_u \approx R_k(\Gamma)_u \otimes K$. This concludes the proof of Theorem 2.

2. The Basic Group Functor

Let k be a field of characteristic zero, and let H be a commutative Hopf algebra over k. For the moment, let us denote the comultiplication, multiplication, and antipode maps on H by Δ, μ, and η, respectively. As in Section 1, we let H_0 denote the coradical of H and H_u the quotient of H by the normal Hopf ideal corresponding to H_0. Let π: $H \to H_0$ be a Hopf algebra projection [14, Theorem 1], q: $H \to H_u$ the quotient map, and j: $H_0 \to H$

the inclusion. Then, as in [14, Corollary 7], the map $f=(q\otimes\pi)\circ\Delta$ is an isomorphism of algebras from H to $H_u\otimes H_0$. Moreover, from [14] we see that the inverse of f is a map $g: H_u\otimes H_0\to H$ which factors into the composite of a map $\bar{t}\otimes j$: $H_u\otimes H_0\to H\otimes H$ followed by the multiplication map μ. Specifically, $\bar{t}$ comes from factoring through H_u the map $t=\mu\circ(i\otimes(\eta\circ j\circ\pi))\circ\Delta: H\to H$. One verifies that, in the notation of [13],

$$\Delta\circ t(h)=\sum h_{(1)}(\eta\circ j\circ\pi)(h_{(3)})\otimes t(h_{(2)}).$$

We see then, that $\bar{t}$ maps H_u isomorphically as an algebra onto a left-coideal subalgebra $\bar{t}(H_u)$ of H such that the multiplication map $\bar{t}(H_u)\otimes H_0\to H$ is an isomorphism of algebras. Since it is a tensor factor, $\bar{t}(H_u)$ is a direct summand of H and, thus, is injective as a left H-comodule. Since $\bar{t}(H_u)\cap H_0=k$, and since every sub comodule of $\bar{t}(H_u)$ must meet H_0 nontrivially, we see that $\bar{t}(H_u)$ is an essential extension of the trivial H-comodule.

Now, switching left to right, we can summarize.

LEMMA 3 (Takeuchi). *If k is a field of characteristic zero and H a commutative Hopf algebra over k, then there is a right-coideal subalgebra J of H such that*

(i) *J is isomorphic as an algebra with H_u,*

(ii) *J is an injective hull of the trivial H-comodule,*

(iii) *the multiplication map on H yields an isomorphism of algebras $J\otimes_k H_0\to H$.*

In particular, if Γ is an A_1-solvable group and $H=R_k(\Gamma)$, then J is an injective hull (in the cateory of locally finite-dimensional Γ-modules over k) of the trivial Γ-module, and, by Theorem 2 and the proof of Lemma 3, for any field extension K of k, $J_K=J\otimes_k K$ is a right-coideal subalgebra of the Hopf algebra $R_K(\Gamma)$ that satisfies the properties of Lemma 3.

Remark. If C is a co-algebra and V a right-coideal of C, then the smallest sub co-algebra of C to contain V is exactly the space $cf(V)$ of *coefficient functions* of V (in the notation of [4]). In particular, the smallest sub coalgebra of C to contain V will contain every right-coideal of C that is isomorphic as a C-comodule with V.

Notation. A pro-affine algebraic group G over a field of characteristic zero has a largest normal (pro-) reductive subgroup, which we denote by $Q(G)$. The group $Q(G)$ centralizes the unipotent radical $U(G)$ of G (see [9, Lemma 4] or [11, Lemma 4.6]).

Now, let k be a field of characteristic zero, and let K be an algebraic closure of k. Let Γ be a solvable A_1-group, let $B_K(\Gamma)=G_K(\Gamma)/Q(G_K(\Gamma))$ and let $\mathscr{B}_K(\Gamma)$ denote the algebra of polynomial functions on $B_k(\Gamma)$.

THEOREM 4. *In the above notation, $B_K(\Gamma)$ is an affine algebraic K-group, and its polynomial algebra $\mathscr{B}_K(\Gamma)$ has a k-form $\mathscr{B}_k(\Gamma)$ which is the smallest sub-Hopf algebra of $R_k(\Gamma)$ to contain an injective hull of the trivial $R_k(\Gamma)$-comodule.*

Proof. Write Q for $Q(G_K(\Gamma))$ and G for $G_K(\Gamma)$. Let H_k be the smallest sub Hopf algebra of $R_k(\Gamma)$ to contain an injective hull of the trivial $R_k(\Gamma)$-comodule. Then, by the discussion following Lemma 3, $H_k \otimes K = H_K$. Moreover, by Lemma 3, $H_K(R_K(\Gamma))_0 = R_K(\Gamma)$.

Now, let T denote the reductive pro-affine algebraic group corresponding to the coradical $R_K(\Gamma)_0$ of $R_K(\Gamma)$, and G_H that corresponding to H_K. Then, we get a commutative diagram:

$$\begin{tikzcd} G_H & & G_H \times T \arrow[ll, "p_1"'] \\ & G \arrow[ul, "\rho"] \arrow[ur, "\sigma"'] & \end{tikzcd}$$

where σ dualizes the multiplication map $H_K \otimes (R_K(\Gamma))_0 \to R_K(\Gamma)$, p_1 is the projection onto the first factor, and ρ is the restriction map. Thus, $\ker \rho$ is a subgroup of G that maps injectively into T. Since Γ is solvable, the connected component T_1 of the identity in T is a pro-toroid, so that $(\ker(\rho))_1$ is a pro-toroid, whence $\ker \rho$ is reductive. Thus, $\ker \rho \subset Q$, and $H_K \supset B_K(\Gamma)$.

On the other hand, if P is any maximal reductive subgroup of G, V a G-module, and U the Q-fixed part of V, then U is a sub G-module of V, and so has a P-module complement, W say, in V. Now, Q is contained in P, so that W is the sum of all simple sub Q-modules of V on which Q acts non-trivially. Q centralizes the unipotent radical $U(G)$ of G, so that the translate $g \cdot W$ of W by any element g of $U(G)$ is isomorphic as a Q-module with W, hence *is* W. Thus, W is a sub G-module of V and U is a direct G-module summand in V. It follows that an injective hull of the trivial G-module is acted upon trivially by Q, which shows that $H_K \subset B_K(\Gamma)$.

This finishes the proof of Theorem 4 except for the assertion that $B_K(\Gamma)$ is affine, which is in [9, p. 152].

Remarks. We call $B_K(\Gamma)$ the *basic K-group* associated with Γ. The above result shows that the groups discussed by Donkin [4] and Magid [9] in connection with polycyclic groups exist in a more general situation and are, in fact, identical.

If Γ is a group and $\Gamma' \lhd \Gamma$, then the image in $G_K(\Gamma)$ of $G_K(\Gamma')$ will be a normal subgroup (see, e.g., [3, Chap. V p. 117]) so that $Q(G_K(\Gamma'))$ will map into $Q(G_K(\Gamma))$. Moreover, if Γ is a solvable A_1-group, then, by Theorem 2, $U_K(\Gamma')$ injects into $U_K(\Gamma)$, whence the kernel of the map $G_K(\Gamma') \to G_K(\Gamma)$ is reductive. Thus, we see that, on the category of A_1 solvable groups with *subnormal morphisms* (i.e., morphisms whose image is a subnormal subgroup of the codomain) the assignment $\Gamma \to B_K(\Gamma)$ is the object part of a functor that is semi left-exact and semi right-exact (but not "exact-in-the-middle," see [4, p. 120]).

For any group Γ, we consider the canonical group homomorphism from Γ to the basic K-group associated with Γ.

PROPOSITION 5. *If Γ is a solvable A_1-group, then the kernel of the canonical map $\Gamma \to B_K(\Gamma)$ is the largest periodic normal subgroup $\tau(\Gamma)$ of Γ.*

Proof. First, suppose that $\tau(\Gamma)$ is trivial. Then, by the result of Charin quoted in the proof of Theorem 2, Γ has a torsion-free nilpotent normal subroup Γ_0 and a subnormal series $\Gamma_0 \lhd \Gamma_1 \lhd \cdots \lhd \Gamma_t$, with Γ_t normal and of finite index in Γ and each Γ_{i+1} the semidirect product of Γ_i with a free abelian group A_{i+1}. Now, Γ_0 and each A_i have faithful finite-dimensional unipotent representations over K (this is a well-known result of Charin's, and can be seen from the discussion of nilpotent torsion-free groups in the proof of Theorem 2). Further, as in the proof of Theorem 2, we can apply Mostow's extension theorem to show that Γ has a faithful finite-dimensional representation over K and that (since $\tau(\Gamma)$ is trivial) *no* element of Γ other than the neutral element acts semisimply in all representations of Γ.

That is enough to show that the natural map $\Gamma \to G_K(\Gamma)$ is injective, and that its image has trivial intersection with $Q(G_K(\Gamma))$. Thus, the factored map $\Gamma \to B_K(\Gamma)$ is injective.

Now, in the general case, let π denote the map $\Gamma \to \Gamma/\tau(\Gamma)$. From the fact that $U_K(\Gamma) \approx U_K(\Gamma/\tau(\Gamma))$, we see that the kernel of the map $B_K(\pi)$: $B_K(\Gamma) \to B_K(\Gamma/\tau(\Gamma))$ is reductive, and thus trivial. Thus, $B_K(\pi)$ is an isomorphism, and the conclusion of the proposition follows from the earlier part of the proof.

3. THE UNIPOTENT RADICAL AND THE REPRESENTATION KERNEL OF A SOLVABLE GROUP

Notation. For any group Γ, we denote by $N(\Gamma)$ the intersection of the normal subgroups of finite index in Γ, and by $M(\Gamma)$ the intersection of the commutator subgroups of the normal subgroups of finite index in Γ (i.e., $M(\Gamma) = \bigcap \{[\Gamma', \Gamma'] \colon \Gamma' \text{ normal of finite index in } \Gamma\}$).

As before, for any algebraically closed field k, we denote by $G_k(\Gamma)$ the universal pro-algebraic hull of Γ. Let ν denote the natural map $\Gamma \to G_k(\Gamma)$.

PROPOSITION 6. *In the above notation, $N(\Gamma)$ is the inverse image under ν of the connected component of the identity $G_k(\Gamma)_1$ in $G_k(\Gamma)$. Indeed, there is an isomorphism of groups from $G_k(\Gamma)/G_k(\Gamma)_1$ to the pro-finite completion $\hat{\Gamma}$ of Γ that commutes with the canonical maps from Γ.*

Proof. The profinite completion $\hat{\Gamma}$ of Γ is the projective limit of the groups Γ/N, where N runs over the family of normal subgroups of finite index in Γ. The kernel of the canonical map from Γ to $\hat{\Gamma}$ is $N(\Gamma)$.

Let $\mathbf{R}$ denote the family of finitely-generated sub-Hopf algebras of $R_k(\Gamma)$ ordered by inclusion. For $\rho \in \mathbf{R}$ let $G(\rho)$ denote the affine algebraic group associated with ρ and let $G(\rho)_1$ denote the connected component of the identity in $G(\rho)$. Then, as in [7, 2.1], $G_k(\Gamma)/G_k(\Gamma)_1$ is the projective limit of the $G(\rho)/G(\rho)_1$'s for $\rho \in \mathbf{R}$.

For $\rho \in \mathbf{R}$ let N_ρ denote the kernel of the map $\Gamma \to G(\rho)/G(\rho)_1$. Since the image of Γ is dense in $G(\rho)$, we see that the induced map $\tilde{\rho}$: $\Gamma/N_\rho \to G(\rho)/G(\rho)_1$ is an isomorphism of groups. Now, the map $G_k(\Gamma)/G_k(\Gamma)_1 \to \Gamma/N_\rho$ (from inverting $\tilde{\rho}$) depends only on N_ρ, not on ρ. To see this, suppose that $\sigma, \rho \in \mathbf{R}$ and that $N_\sigma = N_\rho$. Let τ denote the tensor product of σ and ρ. Then, $N_\tau = N_\rho$ and it suffices to show that the maps from $G_k(\Gamma)/G_k(\Gamma)_1$ to Γ/N_τ and Γ/N_ρ are the same. This comes from the commutativity of the diagram:

$$
\begin{tikzcd}
G_k(\Gamma)/G_k(\Gamma)_1 \arrow[rr] \arrow[dr] & & G(\rho)/G(\rho)_1 \arrow[rr] & & \Gamma/N_\rho \arrow[dl, equal] \\
& G(\tau)/G(\tau)_1 \arrow[ur] \arrow[rr] & & \Gamma/N_\tau &
\end{tikzcd}
$$

Clearly, all normal subgroups of finite index in Γ occur among the N_ρ's. Thus, for every normal subgroup N of finite index in Γ, there is a morphism from $G_k(\Gamma)/G_k(\Gamma)_1$ to Γ/N, and these maps commute with the restriction maps among the Γ/N's. This defines an injective morphism of groups ψ from $G_k(\Gamma)/G_k(\Gamma)_1$ to $\hat{\Gamma}$ that commutes with the canonical maps from Γ. In particular, the kernel of the canonical map $\Gamma \to G_k(\Gamma)/G_k(\Gamma)_1$ is $N(\Gamma)$, so that, for each normal subgroup N of finite index in Γ, there is a natural map from Γ/N to $G_k(\Gamma)/G_k(\Gamma)_1$, and these maps induce a morphism of groups from $\hat{\Gamma}$ to $G_k(\Gamma)/G_k(\Gamma)_1$ which, when followed by ψ, gives the identity on $\hat{\Gamma}$. Thus, ψ is an isomorphism of groups. This concludes the proof.

PROPOSITION 7. *Let k be an algebraically closed field of characteristic zero, and Γ a solvable group. Then $M(\Gamma)$ is the inverse image under ν of the unipotent radical $U_k(\Gamma)$ of $G_k(\Gamma)$.*

Proof. That $M(\Gamma)$ is contained in $\nu^{-1}(U_k(\Gamma))$ is a consequence of the Lie–Kolchin–Mal'cev theorem. Conversely, it is clear from Proposition 6 that $\nu^{-1}(U_k(\Gamma))$ is contained in $N(\Gamma)$. If $x \in N(\Gamma)$, and $x \notin M(\Gamma)$, then there is a normal subgroup Γ' of finite index in Γ such that $x \in \Gamma$, $x \notin [\Gamma', \Gamma']$. Now, an abelian group embeds into a divisible abelian group which is a direct sum of $\mathbb{Q}$'s and $\mathbb{Z}(p^\infty)$'s. Thus, for each element y of a divisible group D, there is a one-dimensional representation of D over k on which y acts nontrivially. Thus, there is a representation of Γ' (which, by inducing, gives a representation of Γ) on which x acts non-unipotently. Thus, $\nu^{-1}(U_k(\Gamma)) = M(\Gamma)$.

PROPOSITION 8. *Let Γ be a solvable group of type A_1 and let k be an algebraically closed field of characteristic zero. Then the kernel of ν: $\Gamma \to G_k(\Gamma)$ is the largest normal periodic subgroup $\tau(M(\Gamma))$ of $M(\Gamma)$.*

Proof. By Proposition 7, $M(\Gamma)$ acts unipotently on all Γ-modules over k. On the other hand, it is clear that periodic elements of Γ act semisimply on all Γ-modules. Thus, $\tau(M(\Gamma)) \subset \ker \nu$.

By Theorem 4, $\ker \nu$ is contained in $\tau(\Gamma)$ and, by the proof of Proposition 7 we see that $\ker \nu \subset M(\Gamma)$. Since $M(\Gamma)$ is characteristic in Γ, $M(\Gamma) \cap \tau(\Gamma) = \tau(M(\Gamma))$, which concludes the proof.

COROLLARY. *A necessary and sufficient condition that a solvable group Γ of type A_1 have a faithful locally-finite dimensional representation over a field of characteristic zero is that $M(\Gamma)$ should be torsion-free.*

Remark. An example of Warfield's [16, (5.11)] shows that the condition that Γ be of type A_1 cannot be dispensed with in Theorem 2, in Proposition 5, or in Proposition 8. His example is of a nilpotent torsion-free group Γ such that the kernel of the canonical map ν: $\Gamma \to G_k(\Gamma)$ is the commutator subgroup of Γ which is isomorphic with the additive group of rational numbers.

ACKNOWLEDGMENT

I would like to thank Professor Gerhard Hochschild for his many helpful comments in the course of writing this paper.

References

1. E. Abe, "Hopf Algebras," Cambridge Tracts in Math., Cambridge, 1977.
2. V. S. Charin, On solvable groups of type A_4, *Mat. Sb.* **52** (94). (3) (1960), 895–914.
3. C. Chevalley, "Theorie des Groupes de Lie," Vol. III, Hermann, Paris, 1955.
4. S. Donkin, Polycyclic groups, Lie algebras and algebraic groups, *J. Reine Angew. Math.* **326** (1981), 104–123.
5. L. Fuchs, "Infinite Abelian Groups," Vol. 1, Academic Press, New York, 1970.
6. G. Hochschild, "Basic Theory of Algebraic Groups and Lie Algebras," GTM, Vol. 75, Springer-Verlag, Berlin, 1981.
7. G. Hochschild, Coverings of pro-affine algebraic groups, *Pacific J. Math.* (2) (1970), 399–415.
8. G. Hochschild and G. Mostow, Pro-affine algebraic groups, *Amer. J. Math.* **91** (4) (1969), 1127–1140.
9. A. Magid, Analytic representations of polycyclic groups, *J. Algebra* **74** (1982), 149–158.
10. A. Mal'cev, Some classes of infinite solvable groups, *Mat. Sb.* **28** (1951), 567–588; Trans. in Amer. Math. Soc. Ser. 2, Vol. 2).
11. G. D. Mostow, Representative functions on discrete groups and solvable arithmetic subgroups, *Amer. J. Math.* **92** (1) (1970), 1–32.
12. I. N. Stewart, An algebraic treatment of Mal'cev's theorems concerning nilpotent Lie groups and their Lie algebras, *Compositio Math.* **22** (1970), 289–312.
13. M. Sweedler, "Hopf Algebras," Benjamin, New York, 1969.
14. M. Takeuchi, A semidirect product decomposition of affine algebraic groups over a field of characteristic zero, *Tohoku J. Math.* **24** (1972), 453–456.
15. M. Takeuchi, A correspondence between Hopf ideals and sub-Hopf algebras, *Manuscripta Math.* **7** (1972), 251–270.
16. R. B. Warfield, "Nilpotent Groups," Lecture Notes in Math., Vol. 513, Springer-Verlag, Berlin, 1976.