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Robustness of nonuniform exponential dichotomies in Banach spaces ☆

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Abstract

We give conditions for the robustness of *nonuniform* exponential dichotomies in Banach spaces, in the sense that the existence of an exponential dichotomy for a given linear equation $x' = A(t)x$ persists under a sufficiently small linear perturbation. We also establish the continuous dependence with the perturbation of the constants in the notion of dichotomy and of the "angles" between the stable and unstable subspaces. Our proofs exhibit (implicitly) the exponential dichotomies of the perturbed equations in terms of fixed points of appropriate contractions. We emphasize that we do not need the notion of admissibility (of bounded nonlinear perturbations). We also obtain related robustness results in the case of nonuniform exponential contractions. In addition, we establish an appropriate version of robustness for nonautonomous dynamical systems with discrete time.

MSC: 34D09; 34D10

Keywords: Exponential contractions; Exponential dichotomies; Stability

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1. Introduction

1.1. Exponential dichotomies: Uniform versus nonuniform

The notion of *exponential dichotomy*, introduced by Perron in [25], plays a central role in a substantial part of the theory of differential equations and dynamical systems. In particular, the existence of an exponential dichotomy for a linear equation

$$x' = A(t)x \tag{1}$$

causes the existence of stable and unstable invariant manifolds for the nonlinear differential equation

$$x' = A(t)x + f(t, x), \tag{2}$$

up to mild additional assumptions on the nonlinear part $f(t, x)$ of the vector field. Moreover, the local instability of trajectories caused by the existence of an exponential dichotomy influences the global behavior of the system. In particular, this instability is one of the main mechanisms responsible for the occurrence of stochastic behavior, especially in the presence of nontrivial recurrence caused by the existence of finite invariant measures. The theory of exponential dichotomies and its applications are widely developed. We refer to the books [8,11,13,14,20,32] for details and further references.

Due to the central role played by the notion of exponential dichotomy, particularly in stability theory and perturbation theory, it is crucial to understand how exponential dichotomies vary under perturbations. In this respect it is of interest to consider both linear and nonlinear perturbations. The *linear* case is arguably simpler, and after a period of considerable related research activity it has lacked the same attention, although with several noteworthy nontrivial advances (we refer to Section 1.2 for details and references). The *nonlinear* case has been occupying researchers for several decades, particularly with the following topics:

1. the Grobman–Hartman theorem together with its variants and generalizations that look for topological conjugacies between the solutions of Eqs. (1) and (2), or more generally for Hölder continuous conjugacies;
2. the study of more regular conjugacies and of the corresponding obstructions in terms of resonances (a problem that goes back to the pioneering work of Poincaré; we refer to [3] for details and references), also with the construction of normal forms;
3. the construction of stable and unstable invariant manifolds, in successively more general frameworks and with several variants of the proofs (although always in one way or another related to the methods stemming from the seminal works by Hadamard and Perron).

On the other hand, the existence of exponential dichotomies is a strong requirement and, particularly in view of their central role, it is of considerable interest to look for more general types of hyperbolic behavior. In a series of papers we discussed the more general notion of *nonuniform exponential dichotomy* (see Section 3.1 for the definition), and we studied some of its consequences. In particular, in [2,5] we established the existence and smoothness of stable and unstable invariant manifolds in $\mathbb{R}^n$, assuming a sufficiently small nonuniformity of the dichotomy when compared to the Lyapunov exponents. In [4] we considered the general case of arbitrary Banach spaces, using a different method although with stronger conditions. We also obtained an appropriate version of the Grobman–Hartman theorem in the case of nonuniformly hyperbolic dynamics [3]. In comparison with the notion of (uniform) exponential dichotomy, the notion of nonuniform exponential dichotomy is a much weaker requirement. In particular, in the case of $\mathbb{R}^n$ essentially *any* linear equation $x' = A(t)x$ with global solutions and nonzero Lyapunov exponents possesses a nonuniform exponential dichotomy [6]. On the other hand, as a consequence of Oseledets' multiplicative ergodic theorem, at least from the point of view of ergodic theory the nonuniformity in the dichotomies of "most" of these equations is arbitrarily small (we refer to [6] for details). Our work is also a contribution to the theory of nonuniformly hyperbolic dynamics (we refer to [1,17] for detailed expositions of parts of the theory). We refer to [2] for a detailed discussion of the relation and novelty of our work with respect to the theory of nonuniformly hyperbolic dynamics.

1.2. Robustness: Behavior under linear perturbations

The purpose of our paper is to understand how *nonuniform* exponential dichotomies vary under *linear* perturbations. This is the so-called problem of robustness (also called problem of roughness). Roughly speaking, a nonuniform exponential dichotomy associated to a given linear equation as in (1) is said to be *robust* in a class of linear perturbations if for any arbitrarily small perturbation $B(t)$ in this class, the perturbed equation

$$x' = \big[A(t) + B(t)\big]x \tag{3}$$

still admits a nonuniform exponential dichotomy. We note that in view of the central role played by dichotomies, as well as the fact that the notion of nonuniform exponential dichotomy is much more common than the notion of (uniform) exponential dichotomy (see Section 1.1), it is of interest to consider the problem of robustness of nonuniform dichotomies. To the best of our knowledge, all former related results in the literature consider only the case of uniform exponential dichotomies.

Our main objective is precisely to study the robustness of *nonuniform* exponential dichotomies under linear perturbations in Banach spaces, looking in particular for reasonable conditions that guarantee robustness. The main aspects of our work are the following:

1. We consider the general case of nonuniform exponential dichotomies associated to a nonautonomous linear equation $x' = A(t)x$ in a Banach space (see Section 3.1). In addition, we establish the continuous dependence with the perturbation of the constants in the notion of dichotomy, and of the "angles" between the stable and unstable subspaces (see Section 4.2).
2. Contrarily to several former works on the robustness of (uniform) exponential dichotomies, we do not assume the function $t \mapsto \|A(t)\|$ to be bounded (we only need the information provided by the exponential dichotomy for the associated evolution operator; see (16)–(17)).
3. We also consider the case of nonuniform exponential *contractions* (see Section 2). These essentially correspond to dichotomies whose associated projections are the identity, thus lacking any unstable component. The proofs in the case of dichotomies are elaborations of the ones for contractions.
4. We obtain appropriate versions of the results in the case of a nonautonomous discrete time dynamics in a Banach space, corresponding to the iteration of a sequence $(A_n)_{n\in\mathbb{N}}$ of bounded linear operators with bounded inverse (see Section 6).

Our proofs exhibit the dichotomies of the perturbed equations as explicitly as possible, in terms of fixed points of appropriate contractions. Some of our arguments are inspired by work of Popescu in [27] in the case of uniform exponential dichotomies. Incidentally, we note that he uses the notion that is sometimes called admissibility, while we do not need this notion, of course independently of its interest in other situations. This notion also goes back to Perron and refers to the characterization of the existence of an exponential dichotomy in terms of the existence and uniqueness of bounded solutions of the equation

$$x' = A(t)x + f(t)$$

for $f(t)$ in a certain class of bounded nonlinear perturbations. This property is called the *admissibility* of the pair of spaces in which we respectively take the perturbation and look for the solutions. One can also consider the admissibility of other pairs of spaces. For details and references we refer to the book of Chicone and Latushkin [8] (see in particular the final remarks of Chapters 3 and 4), and for more recent work to Huy [16] (we mention in particular the papers [21,22,28]).

To obtain a criterion for the existence of nonuniform exponential dichotomies one can also use a Fredholm alternative for the nonlinear perturbations. In particular, related work is due to Palmer [24] for ordinary differential equations, Lin [18] for functional differential equations, Blázquez [7], Rodrigues and Silveira [30], Zeng [33] and Zhang [34] for parabolic evolution equations, and Chow and Leiva [9], Sacker and Sell [31] and Rodrigues and Ruas-Filho [29] for abstract evolution equations.

We note that in the case of uniform exponential dichotomies the study of robustness has a long history. In particular, the robustness was discussed by Massera and Schäffer [19] (building on earlier work of Perron [25]; see also [20]), Coppel [10], and in the case of Banach spaces by Dalec'kiĭ and Kreĭn [12], with different approaches and successive generalizations. The continuous dependence of the projections for the exponential dichotomies of the perturbed equations was obtained by Palmer [24]. For more recent works we refer to [9,23,26,27] and the references therein (since we are dealing with nonuniform exponential dichotomies we refrain ourselves to be more detailed on the literature). In particular, Chow and Leiva [9] and Pliss and Sell [26] considered the context of linear skew-product semiflows and gave examples of applications in the infinite-dimensional setting, including to parabolic partial differential equations and functional differential equations. They also considered the case of dichotomies for discrete time. We emphasize that all these works consider only the case of *uniform* exponential dichotomies, either with continuous or discrete time, and either in the finite-dimensional or infinite-dimensional setting.

2. Robustness of nonuniform exponential contractions

We consider in this section the problem of robustness of nonuniform exponential contractions. Roughly speaking, a nonuniform exponential contraction is *robust* if any sufficiently small of its linear perturbations is still a nonuniform exponential contraction.

Let X be a Banach space. We consider the linear equation (1), where $A: I \to \mathcal{B}(X)$ is a continuous function from an interval I to the space $\mathcal{B}(X)$ of bounded linear operators in X. We always assume that each solution of (1) is defined on the whole interval I. We denote by $T(t,s)$ the associated evolution operator, i.e., the operator such that $T(t,s)x(s) = x(t)$ for every $t, s \in I$, where $x(t)$ is any solution of (1). Clearly,

$$T(t,\tau)T(\tau,s) = T(t,s), \quad t,\tau,s \in I. \tag{4}$$

We say that (1) admits a *nonuniform exponential contraction in I* if for some constants a, $D > 0$ and $\varepsilon \geqslant 0$,

$$\|T(t,s)\| \leqslant De^{-a(t-s)+\varepsilon|s|}, \quad t \geqslant s \text{ in } I. \tag{5}$$

When $\varepsilon = 0$ we obtain the notion of *uniform exponential contraction*.

We also consider the perturbed equation (3), where $B : I \to \mathcal{B}(X)$ is a continuous function. When Eq. (1) has a nonuniform exponential contraction, we shall say that the contraction is *robust* in a given class of (sufficiently small) perturbations provided that for B in this class Eq. (3) still has a nonuniform exponential contraction.

The following statement provides a class of perturbations for which a given nonuniform exponential contraction is robust. Furthermore, we show that the constant a in (5) varies continuously with the perturbation. We shall consider contractions in an arbitrary interval $I \subset \mathbb{R}$.

Theorem 1. *Let $A, B : I \to \mathcal{B}(X)$ be continuous functions such that Eq.* (1) *admits a nonuniform exponential contraction in I, and $\|B(t)\| \leqslant \delta e^{-\varepsilon|t|}$, $t \in I$, with $\delta < a/D$. Then Eq.* (3) *admits a nonuniform exponential contraction in I, with the constant a replaced by $a - \delta D$, i.e.,*

$$\|U(t,s)\| \leqslant De^{-(a-\delta D)(t-s)+\varepsilon|s|}, \quad t \geqslant s \text{ in } I, \tag{6}$$

where $U(t,s)$ is the evolution operator associated with Eq. (3).

Proof. Set

$$J = \{(t,s) \in I \times I\colon t \geqslant s\}, \tag{7}$$

and consider the space

$$\mathcal{C} = \{U : J \to \mathcal{B}(X)\colon U \text{ is continuous and } \|U\| < \infty\} \tag{8}$$

with the norm

$$\|U\| = \sup\{\|U(t,s)\|e^{-\varepsilon|s|}\colon (t,s) \in J\}. \tag{9}$$

Clearly, $\mathcal{C}$ is a Banach space. We set

$$(LU)(t,s) = T(t,s) + \int_s^t T(t,\tau)B(\tau)U(\tau,s)\,d\tau$$

for each $U \in \mathcal{C}$. Note that

$$\begin{aligned}\|(LU)(t,s)\| &\leqslant \|T(t,s)\| + \int_s^t \|T(t,\tau)\| \cdot \|B(\tau)\| \cdot \|U(\tau,s)\|\,d\tau \\ &\leqslant De^{-a(t-s)+\varepsilon|s|} + D\delta e^{\varepsilon|s|}\|U\| \int_s^t e^{-a(t-\tau)}\,d\tau. \end{aligned} \tag{10}$$

Since $a > 0$ this implies that

$$\|LU\| \leqslant D + \delta \frac{D}{a} \|U\| < \infty.$$

Therefore, we have a well-defined operator $L : \mathcal{C} \to \mathcal{C}$. Proceeding in a similar manner to that in (10), we find that for any $U_1, U_2 \in \mathcal{C}$,

$$\|LU_1 - LU_2\| \leqslant \delta \frac{D}{a} \|U_1 - U_2\|.$$

Since $\delta < a/D$, the operator L is a contraction. Hence, there exists a unique $U \in \mathcal{C}$ such that $LU = U$, which thus satisfies the identity

$$U(t,s) = T(t,s) + \int_s^t T(t,\tau)B(\tau)U(\tau,s)\,d\tau. \tag{11}$$

Taking derivatives in (11) (or using the variation of constants formula), we find that for each $\xi \in X$ the function $t \mapsto U(t,s)\xi$, $t \geqslant s$ in I, is the solution of Eq. (3) with initial condition $U(s,s)\xi = T(s,s)\xi = \xi$. We will use the following statement to estimate the norm of the operator $U(t,s)$.

Lemma 1. *Let $x : I \cap [s, +\infty) \to [0, +\infty)$ be a bounded continuous function satisfying*

$$x(t) \leqslant De^{-a(t-s)+\varepsilon|s|} + \delta D \int_s^t e^{-a(t-\tau)} x(\tau)\,d\tau, \quad t \geqslant s \text{ in } I. \tag{12}$$

If $\delta < a/D$ then

$$x(t) \leqslant De^{-(a-\delta D)(t-s)+\varepsilon|s|}, \quad t \geqslant s \text{ in } I.$$

Proof. Consider the continuous function $\Phi(t)$ satisfying the integral equation

$$\Phi(t) = De^{-a(t-s)+\varepsilon|s|} + \delta D \int_s^t e^{-a(t-\tau)} \Phi(\tau)\,d\tau, \quad t \geqslant s. \tag{13}$$

One can easily verify that $\Phi' = (\delta D - a)\Phi$. Furthermore, by (13) we have $\Phi(s) = De^{\varepsilon|s|}$. Thus,

$$\Phi(t) = De^{\varepsilon|s|} e^{(\delta D - a)(t-s)}$$

(in particular this is the unique function satisfying the integral equation in (13)). Note that if $z(t) = x(t) - \Phi(t)$, then

$$z(t) \leqslant \delta D \int_s^t e^{-a(t-\tau)} z(\tau)\,d\tau, \quad t \geqslant s. \tag{14}$$

Set $z = \sup_{t \geqslant s} z(t)$. Since the functions x and Φ are bounded, the number z is finite. Taking the supremum in (14) we obtain

$$z \leqslant \delta D \sup_{t \geqslant s} \int_s^t e^{-a(t-\tau)} z(\tau)\, d\tau \leqslant \delta D z \sup_{t \geqslant s} \int_s^t e^{-a(t-\tau)}\, d\tau.$$

Hence, $z \leqslant (\delta D/a) z$. Since $\delta D/a < 1$, we have $z \leqslant 0$. Thus, $z(t) \leqslant 0$, i.e., $x(t) \leqslant \Phi(t)$ for $t \geqslant s$. □

Set now $x(t) = \|U(t,s)\|$. Proceeding in a similar manner to that in (10) we find that (12) holds. It follows from Lemma 1 that (6) holds. This completes the proof of the theorem. □

Under the hypotheses of Theorem 1, it follows from (11) that for $\delta > 0$,

$$\begin{aligned} \big\|U(t,s) - T(t,s)\big\| &\leqslant \delta D^2 \int_s^t e^{-a(t-\tau)} e^{-(a-\delta D)(\tau - s) + \varepsilon |s|}\, d\tau \\ &= \delta D^2 e^{-a(t-s)+\varepsilon|s|} \frac{e^{\delta D(t-s)} - 1}{\delta D} \\ &\leqslant D e^{-(a-\delta D)(t-s)+\varepsilon|s|}. \end{aligned}$$

In particular, whenever $[s, +\infty) \subset I$, for each fixed $\delta > 0$ we have

$$\lim_{t \to +\infty} \frac{1}{t} \log \big\|U(t,s) - T(t,s)\big\| = 0.$$

This shows that for each perturbation any two solutions $T(t,s)\xi$ and $U(t,s)\xi$ with the same initial condition $\xi \in X$ are forward (exponentially) asymptotic. Furthermore, for each $T > 0$ we have $(e^{\delta D(t-s)} - 1)/(\delta D) \leqslant 2T$ for any sufficiently small $\delta > 0$ and any $t \in [s, T]$. Hence,

$$\lim_{\delta \to 0} \sup_{t \in [s,T]} \big\|U(t,s) - T(t,s)\big\| = 0.$$

That is, as $\delta \to 0$ the solution $U(t,s)$ of the perturbed equation (3) approaches uniformly the solution $T(t,s)$ of the unperturbed equation (1) on each compact interval.

We note that a slightly weaker statement than the one in Theorem 1 (with the constants a and D replaced by larger constants) can be obtained from Theorem 2 below for nonuniform exponential dichotomies. Indeed note that a nonuniform exponential contraction is a nonuniform exponential dichotomy with projections $P(t) = \mathrm{Id}$ for every $t \in I$.

3. Robustness of nonuniform exponential dichotomies

We discuss in this section the robustness of nonuniform exponential dichotomies, starting with the case of the interval $I = [0, +\infty)$. We refer to Section 5 for the cases of the intervals $(-\infty, 0]$ and $\mathbb{R}$.

3.1. Exponential dichotomies and main results

We continue to consider the equation in (1), and we assume that each of its solution is defined on the whole interval I. Also, let $T(t,s)$ be the associated evolution operator. We say that (1) admits a *nonuniform exponential dichotomy in I* if there exist projections $P(t): X \to X$ for each $t \in I$ such that

$$T(t,s)P(s) = P(t)T(t,s), \quad t \geqslant s, \tag{15}$$

and for some constants a, $D > 0$ and $\varepsilon \geqslant 0$,

$$\left\|T(t,s)P(s)\right\| \leqslant De^{-a(t-s)+\varepsilon|s|}, \quad t \geqslant s, \tag{16}$$

and

$$\left\|T(t,s)Q(s)\right\| \leqslant De^{-a(s-t)+\varepsilon|s|}, \quad s \geqslant t, \tag{17}$$

where $Q(t) = \mathrm{Id} - P(t)$ is the complementary projection of $P(t)$. When $\varepsilon = 0$ we obtain the classical notion of *uniform exponential dichotomy*. When Eq. (1) has a nonuniform exponential contraction, we shall say that the contraction is *robust* in a given class of (sufficiently small) perturbations provided that for B in this class Eq. (3) still has a nonuniform exponential dichotomy.

The following is our main robustness result. Set

$$\tilde{a} = a\sqrt{1 - 2\delta D/a} \quad \text{and} \quad \tilde{D} = \frac{D}{1 - \delta D/(\tilde{a} + a)}. \tag{18}$$

We consider dichotomies in an interval $I = [\varrho, +\infty)$ with $\varrho \leqslant 0$.

Theorem 2. *Let $A, B : I \to \mathcal{B}(X)$ be continuous functions such that Eq.* (1) *admits a nonuniform exponential dichotomy in I with $\varepsilon < a$, and assume that $\|B(t)\| \leqslant \delta e^{-2\varepsilon|t|}$ for every $t \in I$. If δ is sufficiently small, then Eq.* (3) *admits a nonuniform exponential dichotomy in I, with the constants a, D, and ε in* (16)–(17) *replaced respectively by $\tilde{a}$, $4D\tilde{D}$, and 2ε.*

The proof of Theorem 2 is given at the end of Section 3.2. Note that setting $\theta = 2\delta D/a$ the constants in (18) satisfy

$$\tilde{a} = a\left(1 - \frac{1}{2}\theta - \frac{1}{8}\theta^2 - \cdots\right) = a - \delta D - \frac{\delta^2 D^2}{2a} - \cdots$$

and

$$\tilde{D} = D\left(1 + \frac{1}{4}\theta + \frac{1}{8}\theta^2 + \cdots\right) = D + \frac{\delta D^2}{2a} + \frac{\delta^2 D^3}{2a^2} + \cdots.$$

We also establish a weaker statement with slightly weaker hypotheses. Namely, in the following theorem we obtain norm bounds along the stable and unstable directions for the perturbed equation (3). However, we give no information about the norms of the projections for the perturbed equation. This requires slightly stronger hypotheses and is included in the statement of Theorem 2. We shall denote by $\hat{T}(t,s)$ the evolution operator associated to Eq. (3).

Theorem 3. *Let $A, B : I \to \mathcal{B}(X)$ be continuous functions such that Eq.* (1) *admits a nonuniform exponential dichotomy in I with $\varepsilon < a$, and assume that $\|B(t)\| \leqslant \delta e^{-\varepsilon|t|}$ for every $t \in I$. If*

$$\theta = 2\delta D/a < 1, \tag{19}$$

then there exist projections $\hat{P}(t) : X \to X$ for each $t \in I$ such that

$$\hat{T}(t,s)\hat{P}(s) = \hat{P}(t)\hat{T}(t,s), \quad t \geqslant s, \tag{20}$$

and

$$\left\|\hat{T}(t,s)|\operatorname{Im}\hat{P}(s)\right\| \leqslant \tilde{D}e^{-\tilde{a}(t-s)+\varepsilon|s|}, \quad t \geqslant s, \tag{21}$$

$$\left\|\hat{T}(t,s)|\operatorname{Im}\hat{Q}(s)\right\| \leqslant \tilde{D}e^{-\tilde{a}(s-t)+\varepsilon|s|}, \quad s \geqslant t, \tag{22}$$

where $\hat{Q}(t) = \mathrm{Id} - \hat{P}(t)$ is the complementary projection of $\hat{P}(t)$.

We will start by proving this weaker statement in the following section. We note that to obtain Theorem 2 from Theorem 3 it remains to obtain sharp bounds for the norms of the projections $\hat{P}(t)$ and $\hat{Q}(t)$.

3.2. Proofs

Proof of Theorem 3. The proof is much more delicate than that of Theorem 1 and we shall divide it into several steps. We first prove some auxiliary results.

Step 1. Construction of bounded solutions. Set J as in (7), and consider the Banach space $\mathcal{C}$ in (8).

Lemma 2. *The equation $Z' = (A(t) + B(t))Z$ has a unique solution $U \in \mathcal{C}$ such that for each $(t,s) \in J$,*

$$U(t,s) = T(t,s)P(s) + \int_s^t T(t,\tau)P(\tau)B(\tau)U(\tau,s)\,d\tau$$
$$- \int_t^\infty T(t,\tau)Q(\tau)B(\tau)U(\tau,s)\,d\tau. \tag{23}$$

Proof. Assume that some function $U \in \mathcal{C}$ satisfies (23). Then $t \mapsto U(t,s)$ is differentiable (since $t \mapsto T(t,s)$ is differentiable), and taking derivatives with respect to t in (23) a simple computation shows that $t \mapsto U(t,s)\xi$, $t \geqslant s$, is a solution of Eq. (3) for each $\xi \in X$. Thus, we must show that the operator L defined by

$$(LU)(t,s) = T(t,s)P(s) + \int_s^t T(t,\tau)P(\tau)B(\tau)U(\tau,s)\,d\tau - \int_t^\infty T(t,\tau)Q(\tau)B(\tau)U(\tau,s)\,d\tau \tag{24}$$

has a unique fixed point in the space $\mathcal{C}$. We have

$$\begin{aligned}\|(LU)(t,s)\| &\leqslant \|T(t,s)P(s)\| \\ &\quad + \int_s^t \|T(t,\tau)P(\tau)\| \cdot \|B(\tau)\| \cdot \|U(\tau,s)\|\,d\tau \\ &\quad + \int_t^\infty \|T(t,\tau)Q(\tau)\| \cdot \|B(\tau)\| \cdot \|U(\tau,s)\|\,d\tau \\ &\leqslant De^{-a(t-s)+\varepsilon|s|} + D\delta e^{\varepsilon|s|}\|U\| \int_s^t e^{-a(t-\tau)}\,d\tau \\ &\quad + D\delta e^{\varepsilon|s|}\|U\| \int_t^\infty e^{-a(\tau-t)}\,d\tau. \end{aligned} \tag{25}$$

Since $a > 0$, in view of (19) this implies that

$$\|LU\| \leqslant D + \theta\|U\| < \infty.$$

Therefore, we have a well-defined operator $L : \mathcal{C} \to \mathcal{C}$. Using the identity (24) for $U_1, U_2 \in \mathcal{C}$, and proceeding in a similar manner to that in (25) we obtain

$$\|LU_1 - LU_2\| \leqslant \theta\|U_1 - U_2\|.$$

It follows from (19) that L is a contraction, and thus there exists a unique $U \in \mathcal{C}$ such that $LU = U$. This completes the proof of the lemma. □

Lemma 3. *For any $t \geqslant \tau \geqslant s$ in I we have*

$$U(t,\tau)U(\tau,s) = U(t,s).$$

Proof. Write

$$X(t,s) = T(t,s)P(s)B(s) \quad \text{and} \quad Y(t,s) = T(t,s)Q(s)B(s).$$

Since $P(t)$ and $Q(t)$ are complementary projections, it follows from (4) and (15) together with (23) that $U(t,\tau)U(\tau,s)$ is equal to

$$
\begin{aligned}
&T(t,\tau)P(\tau)T(\tau,s)P(s)\\
&\quad+T(t,\tau)P(\tau)\left(\int_s^\tau X(\tau,u)U(u,s)\,du-\int_\tau^\infty Y(\tau,u)U(u,s)\,du\right)\\
&\quad+\left(\int_\tau^t X(t,u)U(u,\tau)\,du-\int_t^\infty Y(t,u)U(u,\tau)\,du\right)U(\tau,s),
\end{aligned}
$$

and thus,

$$
\begin{aligned}
U(t,\tau)U(\tau,s)=T(t,s)P(s)+\int_s^\tau X(t,u)U(u,s)\,du+\int_\tau^t X(t,u)U(u,\tau)U(\tau,s)\,du\\
-\int_t^\infty Y(t,u)U(u,\tau)U(\tau,s)\,du.
\end{aligned}
$$

Using again (23) this yields

$$
\begin{aligned}
&U(t,\tau)U(\tau,s)-U(t,s)\\
&\quad=\int_s^\tau X(t,u)U(u,s)\,du+\int_\tau^t X(t,u)U(u,\tau)U(\tau,s)\,du\\
&\qquad-\int_t^\infty Y(t,u)U(u,\tau)U(\tau,s)\,du-\int_s^t X(t,u)U(u,s)\,du+\int_t^\infty Y(t,u)U(u,s)\,du\\
&\quad=\int_\tau^t X(t,u)\big[U(u,\tau)U(\tau,s)-U(u,s)\big]\,du-\int_t^\infty Y(t,u)\big[U(u,\tau)U(\tau,s)-U(u,s)\big]\,du.
\end{aligned}
$$

Setting

$$
Z(u)=U(u,\tau)U(\tau,s)-U(u,s), \tag{26}
$$

we can rewrite the above identity in the form

$$
Z(t)=\int_\tau^t X(t,u)Z(u)\,du-\int_t^\infty Y(t,u)Z(u)\,du. \tag{27}
$$

For each fixed $\tau \geqslant s$ in I, we consider the operator N defined by

$$
(NW)(t)=\int_\tau^t X(t,u)W(u)\,du-\int_t^\infty Y(t,u)W(u)\,du, \tag{28}
$$

in the Banach space

$$\mathcal{E} = \big\{ W : [\tau, +\infty) \to \mathcal{B}(X)\colon\ W \text{ is continuous and } \|W\| < \infty \big\} \tag{29}$$

with the supremum norm $\|W\| = \sup\{\|W(u)\|\colon\ u \in [\tau, +\infty)\}$. By (28),

$$\big\|(NW)(t)\big\| \leqslant D \int_\tau^t e^{-a(t-u)+\varepsilon|u|} \big\|B(u)\big\| \cdot \big\|W(u)\big\| \, du$$
$$+ D \int_t^\infty e^{-a(u-t)+\varepsilon|u|} \big\|B(u)\big\| \cdot \big\|W(u)\big\| \, du \leqslant \theta \|W\|,$$

and thus $N(\mathcal{E}) \subset \mathcal{E}$. Furthermore, proceeding in a similar manner we find that for $W_1, W_2 \in \mathcal{E}$,

$$\|NW_1 - NW_2\| \leqslant \theta \|W_1 - W_2\|.$$

By hypothesis (19), N is a contraction, and hence there is a unique function $W \in \mathcal{E}$ satisfying (27). On the other hand, $0 \in \mathcal{E}$ also satisfies (27) and thus we must have $W = 0$. By Lemma 2, the function Z in (26) is in $\mathcal{E}$, and since it also satisfies (27), we conclude that for any $t \geqslant \tau \geqslant s$ in I,

$$Z(t) = U(t, \tau)U(\tau, s) - U(t, s) = 0.$$

Therefore, $U(t, \tau)U(\tau, s) = U(t, s)$. □

Step 2. Projections and invariance of the evolution operator. Recall that $\hat{T}(t, s)$ denotes the evolution operator associated to Eq. (3). For each $t \in I$ we define the linear operators

$$\hat{P}(t) = \hat{T}(t, 0)U(0, 0)\hat{T}(0, t) \quad \text{and} \quad \hat{Q}(t) = \mathrm{Id} - \hat{P}(t). \tag{30}$$

We want to show that the evolution operator admits a nonuniform exponential dichotomy with projections $\hat{P}(t)$. We start by showing that the linear operators $\hat{P}(t)$ are indeed projections, leaving invariant $\hat{T}(t, s)$.

Lemma 4. *The operator $\hat{P}(t)$ is a projection for each $t \in I$, and* (20) *holds.*

Proof. Set $R = U(0, 0)$. By Lemma 3, we have $R^2 = R$. Since $\hat{T}(t, t) = \mathrm{Id}$,

$$\hat{P}(t)\hat{P}(t) = \hat{T}(t, 0)R\hat{T}(0, t)\hat{T}(t, 0)R\hat{T}(0, t) = \hat{T}(t, 0)R^2\hat{T}(0, t) = \hat{P}(t),$$

and $\hat{P}(t)$ is a projection. Moreover, for $t \geqslant s$,

$$\hat{P}(t)\hat{T}(t, s) = \hat{T}(t, 0)R\hat{T}(0, t)\hat{T}(t, s) = \hat{T}(t, s)\hat{T}(s, 0)R\hat{T}(0, s) = \hat{T}(t, s)\hat{P}(s).$$

This completes the proof. □

Step 3. Characterization of bounded solutions.

Lemma 5. *Given $s \in I$, if $y : [s, +\infty) \to X$ is a bounded solution of Eq.* (3) *with $y(s) = \xi$, then*

$$y(t) = T(t,s)P(s)\xi + \int_s^t T(t,\tau)P(\tau)B(\tau)y(\tau)\,d\tau - \int_t^\infty T(t,\tau)Q(\tau)B(\tau)y(\tau)\,d\tau.$$

Proof. By the variation of constants formula, for $t \geqslant s$ in I,

$$P(t)y(t) = T(t,s)P(s)\xi + \int_s^t T(t,\tau)P(\tau)B(\tau)y(\tau)\,d\tau \tag{31}$$

and

$$Q(t)y(t) = T(t,s)Q(s)\xi + \int_s^t T(t,\tau)Q(\tau)B(\tau)y(\tau)\,d\tau. \tag{32}$$

Equivalently, the last formula can be written in the form

$$Q(s)\xi = T(s,t)Q(t)y(t) - \int_s^t T(s,\tau)Q(\tau)B(\tau)y(\tau)\,d\tau. \tag{33}$$

Since $y(t)$ is bounded, we have

$$\left\|T(s,t)Q(t)y(t)\right\| \leqslant CDe^{-a(t-s)+\varepsilon|t|},$$

where $C = \sup\{\|y(t)\|: t \geqslant s \text{ in } I\} < \infty$. Furthermore,

$$\int_s^\infty \left\|T(s,\tau)Q(\tau)\right\| \cdot \left\|B(\tau)\right\| \cdot \left\|y(\tau)\right\| d\tau \leqslant D\delta C \int_s^\infty e^{-a(\tau-s)}\,d\tau = \frac{D\delta C}{a}.$$

Taking limits in (33) when $t \to +\infty$, since $a > \varepsilon$ we obtain

$$Q(s)\xi = -\int_s^\infty T(s,\tau)Q(\tau)B(\tau)y(\tau)\,d\tau.$$

It follows from (32) that

$$\begin{aligned} Q(t)y(t) &= -\int_s^\infty T(t,\tau)Q(\tau)B(\tau)y(\tau)\,d\tau + \int_s^t T(t,\tau)Q(\tau)B(\tau)y(\tau)\,d\tau \\ &= -\int_t^\infty T(t,\tau)Q(\tau)B(\tau)y(\tau)\,d\tau. \end{aligned}$$

The desired statement follows readily from adding this identity to (31). □

Lemma 6. *The function $[s,+\infty)\cap I \ni t \mapsto \hat P(t)\hat T(t,s)$ is bounded, and for any $t \geqslant s$ in I we have*

$$\hat P(t)\hat T(t,s) = T(t,s)P(s)\hat P(s) + \int_s^t T(t,\tau)P(\tau)B(\tau)\hat P(\tau)\hat T(\tau,s)\,d\tau - \int_t^\infty T(t,\tau)Q(\tau)B(\tau)\hat P(\tau)\hat T(\tau,s)\,d\tau.$$

Proof. By Lemma 2, the function $t \mapsto U(t,0)\xi$, $t \geqslant 0$, is a solution of Eq. (3) with initial condition at time zero equal to $U(0,0)\xi$. Therefore, $U(t,0) = \hat T(t,0)U(0,0)$. By Lemma 4 (see (20)),

$$\begin{aligned} \hat P(t)\hat T(t,s) &= \hat T(t,s)\hat P(s) \\ &= \hat T(t,s)\hat T(s,0)U(0,0)\hat T(0,s) \\ &= \hat T(t,0)U(0,0)\hat T(0,s) = U(t,0)\hat T(0,s). \end{aligned} \tag{34}$$

Again by Lemma 2, for each $\xi \in X$ the function

$$y(t) = \hat P(t)\hat T(t,s)\xi = U(t,0)\hat T(0,s)\xi$$

is a solution of (3). Furthermore, by the definition of the space $\mathcal{C}$ in (8)–(9) this solution is bounded for $t \geqslant s$, and by (34),

$$y(s) = U(s,0)\hat T(0,s)\xi = \hat P(s)\hat T(s,s)\xi = \hat P(s)\xi.$$

The desired identity follows now readily from Lemma 5. □

Step 4. Auxiliary bounds.

Lemma 7. *Given $s \in \mathbb{R}$ and $\varsigma \in (s,+\infty]$, let $x:[s,\varsigma) \to [0,+\infty)$ be a continuous function satisfying*

$$x(t) \leqslant De^{-a(t-s)+\varepsilon|s|}\gamma + \delta D\int_s^t e^{-a(t-\tau)}x(\tau)\,d\tau + \delta D\int_t^\varsigma e^{-a(\tau-t)}x(\tau)\,d\tau \tag{35}$$

for every $t \in [s,\varsigma)$, and assumed to be bounded when $\varsigma = +\infty$. Then

$$x(t) \leqslant \tilde D\gamma e^{-\tilde a(t-s)+\varepsilon|s|}, \quad t \in [s,\varsigma).$$

Proof. The proof is an elaboration of the one of Lemma 1. We will show that $x(t) \leqslant \Phi(t)$, where $\Phi(t)$ is any bounded continuous function satisfying the integral equation

$$\Phi(t) = De^{-a(t-s)+\varepsilon|s|}\gamma + \delta D\int_s^t e^{-a(t-\tau)}\Phi(\tau)\,d\tau + \delta D\int_t^\varsigma e^{-a(\tau-t)}\Phi(\tau)\,d\tau \tag{36}$$

for $t \geqslant s$. Clearly, $\Phi(t)$ satisfies the differential equation

$$z'' - a^2(1-\theta)z = 0. \tag{37}$$

Notice that $-\tilde{a} = -a\sqrt{1-\theta}$ is the negative root of the corresponding characteristic equation. In order that Φ is a bounded function when $\varsigma = +\infty$, we must have $\Phi(t) = \Phi(s)e^{-\tilde{a}(t-s)}$ (when $\varsigma < +\infty$ we simply take $\Phi(t)$ to be of this form). Furthermore, by (36), substituting $\Phi(t)$ and setting $t = s$,

$$\Phi(s) = De^{\varepsilon|s|}\gamma + \delta D\Phi(s)\int_s^\varsigma e^{-(a+\tilde{a})(\tau-s)}\,d\tau \leqslant De^{\varepsilon|s|}\gamma + \Phi(s)\frac{\delta D}{a+\tilde{a}}.$$

Since $a + \tilde{a} > 0$, this yields

$$\Phi(s) \leqslant \frac{D}{1-\delta D/(\tilde{a}+a)}e^{\varepsilon|s|}\gamma,$$

and thus $\Phi(t) \leqslant \tilde{D}\gamma e^{-\tilde{a}(t-s)+\varepsilon|s|}$. We now set $z(t) = x(t) - \Phi(t)$ for $t \geqslant s$. It follows from (35) and (36) that

$$z(t) \leqslant \delta D\int_s^t e^{-a(t-\tau)}z(\tau)\,d\tau + \delta D\int_t^\varsigma e^{-a(\tau-t)}z(\tau)\,d\tau.$$

Set also $z = \sup_{t\geqslant s} z(t)$. Since the functions x and Φ are bounded, z is finite, and taking the supremum in the above inequality we obtain

$$z \leqslant \delta Dz\sup_{t\geqslant s}\int_s^t e^{-a(t-\tau)}\,d\tau + \delta Dz\sup_{t\geqslant s}\int_t^\varsigma e^{-a(\tau-t)}\,d\tau.$$

Hence, $z \leqslant \theta z$. It follows from (19) that $z \leqslant 0$, and thus $z(t) \leqslant 0$, i.e., $x(t) \leqslant \Phi(t)$ for $t \geqslant s$. □

Lemma 8. *Given $s \in \mathbb{R}$ and $\varrho \in [-\infty, s)$, let $y : (\varrho, s] \to [0, +\infty)$ be a continuous function satisfying*

$$y(t) \leqslant De^{-a(s-t)+\varepsilon|s|}\gamma + \delta D\int_\varrho^t e^{-a(t-\tau)}y(\tau)\,d\tau + \delta D\int_t^s e^{-a(\tau-t)}y(\tau)\,d\tau \tag{38}$$

for $t \in (\varrho, s]$, and assumed to be bounded when $\varrho = -\infty$. Then

$$y(t) \leqslant \tilde{D}\gamma e^{-\tilde{a}(s-t)+\varepsilon|s|}, \quad t \in (\varrho, s].$$

Proof. Proceeding in a similar manner to that in the proof of Lemma 7 we can show that $y(t) \leqslant \Psi(t)$, where $\Psi(t)$ is any bounded continuous function satisfying

$$\Psi(t) = De^{-a(s-t)+\varepsilon|s|}\gamma + \delta D\int_\varrho^t e^{-a(t-\tau)}\Psi(\tau)\,d\tau + \delta D\int_t^s e^{-a(\tau-t)}\Psi(\tau)\,d\tau$$

for $t \leqslant s$. Note first that $\Psi(t)$ also satisfies the differential equation (37). Substituting $\Psi(t) = \Psi(s)e^{-\tilde{a}(s-t)}$ in the above identity and setting $t = s$ we obtain

$$\Psi(s) = De^{\varepsilon|s|}\gamma + \delta D\Psi(s)\int_{\varrho}^{s} e^{-(a+\tilde{a})(s-\tau)}\,d\tau \leqslant De^{\varepsilon|s|}\gamma + \Psi(s)\frac{\delta D}{a+\tilde{a}}.$$

Hence,

$$\Psi(s) \leqslant \frac{D}{1-\delta D/(a+\tilde{a})}e^{\varepsilon|s|}\gamma,$$

and $\Psi(t) \leqslant \Psi(s)e^{-\tilde{a}(s-t)}$. Proceeding in a similar manner to that in Lemma 7 we find that

$$y(t) \leqslant \Psi(t) \leqslant \tilde{D}\gamma e^{-\tilde{a}(s-t)+\varepsilon|s|}.$$

This completes the proof of the lemma. □

Step 5. Norm bounds for the evolution operator. We now establish norm bounds for $\hat{T}(t,s)|\operatorname{Im}\hat{P}(s)$ when $t \geqslant s$ and $\hat{T}(t,s)|\operatorname{Im}\hat{Q}(s)$ when $t \leqslant s$. We recall that the constants $\tilde{a}$ and $\tilde{D}$ are given by (18).

Lemma 9. *The inequality* (21) *holds for every $t \geqslant s$ in I.*

Proof. Let $\xi \in X$. Setting $x(t) = \|\hat{P}(t)\hat{T}(t,s)\xi\|$ for $t \geqslant s$, and $\gamma = \|\hat{P}(s)\xi\|$ it follows from Lemma 6 and (16)–(17) that the function x is bounded, and satisfies the inequality in (35) with $\varsigma = +\infty$. Therefore, by Lemma 7,

$$\|\hat{P}(t)\hat{T}(t,s)\xi\| \leqslant \tilde{D}e^{-\tilde{a}(t-s)+\varepsilon|s|}\|\hat{P}(s)\xi\|, \quad t \geqslant s.$$

By Lemma 4 we have

$$\hat{P}(t)\hat{T}(t,s) = \hat{T}(t,s)\hat{P}(s) = \hat{T}(t,s)\hat{P}(s)\hat{P}(s),$$

and hence, setting $\eta = \hat{P}(s)\xi$,

$$\|\hat{T}(t,s)\hat{P}(s)\eta\| \leqslant \tilde{D}e^{-\tilde{a}(t-s)+\varepsilon|s|}\|\eta\|, \quad t \geqslant s.$$

This establishes the desired inequality. □

Lemma 10. *The inequality* (22) *holds for every $t \leqslant s$ in I.*

Proof. We first derive an equation for $\hat{Q}(t)\hat{T}(t,s)$. By the variation of constants formula, i.e.,

$$\hat{T}(t,s) = T(t,s) + \int_{s}^{t} T(t,\tau)B(\tau)\hat{T}(\tau,s)\,d\tau,$$

the function $y(t) = \hat{T}(t,0)\hat{Q}(0)$ satisfies

$$y(t) = T(t,0)\hat{Q}(0) + \int_0^t T(t,\tau)B(\tau)y(\tau)\,d\tau. \tag{39}$$

On the other hand, using (23) with $t = s = 0$,

$$\hat{P}(0) = U(0,0) = P(0) - \int_0^\infty Q(0)T(0,\tau)B(\tau)U(\tau,0)\,d\tau. \tag{40}$$

Since $P(0)$ and $Q(0)$ are complementary projections, by (40) we have

$$P(0)\hat{P}(0) = P(0). \tag{41}$$

Therefore,

$$Q(0)\hat{Q}(0) = \big(\mathrm{Id} - P(0)\big)\big(\mathrm{Id} - \hat{P}(0)\big) = \mathrm{Id} - \hat{P}(0) = \hat{Q}(0). \tag{42}$$

It follows from (39) that

$$\begin{aligned} y(s) &= T(s,0)\hat{Q}(0) + \int_0^s T(s,\tau)B(\tau)y(\tau)\,d\tau \\ &= T(s,0)Q(0)\hat{Q}(0) + \int_0^s T(s,\tau)B(\tau)y(\tau)\,d\tau. \end{aligned} \tag{43}$$

Multiplying (43) on the left by $T(t,s)Q(s)$ and using again (42), we obtain

$$\begin{aligned} T(t,s)Q(s)y(s) &= T(t,0)Q(0)\hat{Q}(0) + \int_0^s T(t,\tau)Q(\tau)B(\tau)y(\tau)\,d\tau \\ &= T(t,0)\hat{Q}(0) + \int_0^s T(t,\tau)Q(\tau)B(\tau)y(\tau)\,d\tau. \end{aligned} \tag{44}$$

Combining (39) and (44) yields

$$\begin{aligned} y(t) &= T(t,s)Q(s)y(s) - \int_0^s T(t,\tau)Q(\tau)B(\tau)y(\tau)\,d\tau + \int_0^t T(t,\tau)B(\tau)y(\tau)\,d\tau \\ &= T(t,s)Q(s)y(s) + \int_0^t T(t,\tau)P(\tau)B(\tau)y(\tau)\,d\tau - \int_t^s T(t,\tau)Q(\tau)B(\tau)y(\tau)\,d\tau. \end{aligned} \tag{45}$$

On the other hand, it follows readily from Lemma 4 that

$$\hat{Q}(t)\hat{T}(t,s) = \hat{T}(t,s)\hat{Q}(s). \tag{46}$$

Since $y(\tau) = \hat{T}(\tau,0)\hat{Q}(0)$, we obtain $y(\tau)\hat{T}(0,s) = \hat{Q}(\tau)\hat{T}(\tau,s)$. Thus, multiplying (45) on the right by $\hat{T}(0,s)$ we find that for $t \leqslant s$,

$$\hat{Q}(t)\hat{T}(t,s) = T(t,s)Q(s)\hat{Q}(s) + \int_0^t T(t,\tau)P(\tau)B(\tau)\hat{Q}(\tau)\hat{T}(\tau,s)\,d\tau$$
$$- \int_t^s T(t,\tau)Q(\tau)B(\tau)\hat{Q}(\tau)\hat{T}(\tau,s)\,d\tau. \tag{47}$$

Let now $\xi \in X$, and set $y(t) = \|\hat{T}(t,s)\hat{Q}(s)\xi\|$ for $t \leqslant s$ in I, and $\gamma = \|\hat{Q}(s)\xi\|$. It follows readily from (47) and (46) that the function y satisfies the inequality (38). Using Lemma 8 and proceeding in a similar manner to that in the proof of Lemma 9 we readily obtain the desired inequality. □

We proceed with the proof of the theorem. We have shown that there exist projections $\hat{P}(t)$ (see (30)) leaving invariant the evolution operator $\hat{T}(t,s)$ (see Lemma 4). The corresponding norms bounds for $\hat{T}(t,s)|\operatorname{Im}\hat{P}(t)$ and $\hat{T}(t,s)|\operatorname{Im}\hat{Q}(t)$ are given respectively by Lemmas 9 and 10. This completes the proof of the theorem. □

We can now establish our main robustness result.

Proof of Theorem 2. Applying Theorem 3 we obtain projections $\hat{P}(t)$ satisfying (20) as well as the norm bounds in (21) and (22). We also obtain norm bounds for the projections. We note that this is the only place in the proof where the assumption $\|B(t)\| \leqslant \delta e^{-\varepsilon|t|}$ in Theorem 3 must be replaced by the new assumption $\|B(t)\| \leqslant \delta e^{-2\varepsilon|t|}$.

Lemma 11. *Provided that δ is sufficiently small, for any $t \in I$ we have*

$$\|\hat{P}(t)\| \leqslant 4De^{\varepsilon|t|} \quad \text{and} \quad \|\hat{Q}(t)\| \leqslant 4De^{\varepsilon|t|}. \tag{48}$$

Proof. By Lemma 6 with $t = s$, since $P(t)$ and $Q(t)$ are complementary projections,

$$Q(t)\hat{P}(t) = -\int_t^\infty T(t,\tau)Q(\tau)B(\tau)\hat{P}(\tau)\hat{T}(\tau,t)\,d\tau. \tag{49}$$

By Lemmas 9 and 4 (see (20)) we have that for $\tau \geqslant t$ in I,

$$\|\hat{P}(\tau)\hat{T}(\tau,t)\| \leqslant \tilde{D}e^{-\tilde{a}(\tau-t)+\varepsilon|t|}\|\hat{P}(t)\|. \tag{50}$$

By (49), using (17) we obtain

$$\begin{aligned}\|Q(t)\hat{P}(t)\| &\leqslant \delta D\tilde{D}\|\hat{P}(t)\| \int_t^\infty e^{-a(\tau-t)+\varepsilon|\tau|} e^{-2\varepsilon|\tau|} e^{-\tilde{a}(\tau-t)+\varepsilon|t|}\, d\tau \\ &\leqslant \delta D\tilde{D}\|\hat{P}(t)\| \int_t^\infty e^{-(a+\tilde{a}-\varepsilon)(\tau-t)}\, d\tau = \frac{D\tilde{D}\delta}{a+\tilde{a}-\varepsilon}\|\hat{P}(t)\|, \end{aligned} \tag{51}$$

since $a > \varepsilon$. Similarly, it follows from (47) with $t = s$ that

$$P(t)\hat{Q}(t) = \int_0^t T(t,\tau)P(\tau)B(\tau)\hat{Q}(\tau)\hat{T}(\tau,t)\, d\tau. \tag{52}$$

By Lemma 10 and (46) we have that for $\tau \leqslant t$ in I,

$$\|\hat{Q}(\tau)\hat{T}(\tau,t)\| \leqslant \tilde{D}e^{-\tilde{a}(t-\tau)+\varepsilon|t|}\|\hat{Q}(t)\|. \tag{53}$$

By (52), using (16) we obtain

$$\begin{aligned}\|P(t)\hat{Q}(t)\| &\leqslant \delta D\tilde{D}\|\hat{Q}(t)\| \int_0^t e^{-a(t-\tau)+\varepsilon|\tau|} e^{-2\varepsilon|\tau|} e^{-\tilde{a}(t-\tau)+\varepsilon|t|}\, d\tau \\ &\leqslant \delta D\tilde{D}\|\hat{Q}(t)\| \int_0^t e^{-(a+\tilde{a}-\varepsilon)(t-\tau)}\, d\tau = \frac{D\tilde{D}\delta}{a+\tilde{a}-\varepsilon}\|\hat{Q}(t)\|. \end{aligned} \tag{54}$$

Observe now that

$$\begin{aligned}\hat{P}(t) - P(t) &= \hat{P}(t) - P(t)\hat{P}(t) - P(t) + P(t)\hat{P}(t) \\ &= \big(\mathrm{Id} - P(t)\big)\hat{P}(t) - P(t)\big(\mathrm{Id} - \hat{P}(t)\big) \\ &= Q(t)\hat{P}(t) - P(t)\hat{Q}(t). \end{aligned} \tag{55}$$

It follows from (51) and (54) that

$$\|\hat{P}(t) - P(t)\| \leqslant \frac{\delta D\tilde{D}}{a+\tilde{a}-\varepsilon}\big(\|\hat{P}(t)\| + \|\hat{Q}(t)\|\big). \tag{56}$$

On the other hand, by (16)–(17) with $t = s$, we have

$$\|P(t)\| \leqslant De^{\varepsilon|t|} \quad \text{and} \quad \|Q(t)\| \leqslant De^{\varepsilon|t|}.$$

It follows from (56) that

$$\|\hat{P}(t)\| \leqslant \|\hat{P}(t) - P(t)\| + \|P(t)\| \leqslant \frac{\delta D\tilde{D}}{a+\tilde{a}-\varepsilon}\big(\|\hat{P}(t)\| + \|\hat{Q}(t)\|\big) + De^{\varepsilon|t|},$$

and since $\|\hat{Q}(t)-Q(t)\|=\|\hat{P}(t)-P(t)\|$ we also have

$$\|\hat{Q}(t)\| \leqslant \|\hat{P}(t)-P(t)\|+\|Q(t)\| \leqslant \frac{\delta D\tilde{D}}{a+\tilde{a}-\varepsilon}\big(\|\hat{P}(t)\|+\|\hat{Q}(t)\|\big)+De^{\varepsilon|t|}.$$

Therefore,

$$\|\hat{P}(t)\|+\|\hat{Q}(t)\| \leqslant \frac{2\delta D\tilde{D}}{a+\tilde{a}-\varepsilon}\big(\|\hat{P}(t)\|+\|\hat{Q}(t)\|\big)+2De^{\varepsilon|t|},$$

and

$$\left(1-\frac{2\delta D\tilde{D}}{a+\tilde{a}-\varepsilon}\right)\big(\|\hat{P}(t)\|+\|\hat{Q}(t)\|\big) \leqslant 2De^{\varepsilon|t|}.$$

Taking δ sufficiently small so that $2\delta D\tilde{D}/(a+\tilde{a}-\varepsilon) \leqslant 1/2$ we obtain

$$\|\hat{P}(t)\|+\|\hat{Q}(t)\| \leqslant 4De^{\varepsilon|t|}.$$

This yields the desired inequalities. □

Combining (50) with (48) we find that for $\tau \geqslant t$ in I,

$$\|\hat{P}(\tau)\hat{T}(\tau,t)\| \leqslant \tilde{D}e^{-\tilde{a}(\tau-t)+\varepsilon|t|}\|\hat{P}(t)\| \leqslant 4D\tilde{D}e^{-\tilde{a}(\tau-t)+2\varepsilon|t|}.$$

Similarly, combining (53) with (48) we find that for $\tau \leqslant t$ in I,

$$\|\hat{Q}(\tau)\hat{T}(\tau,t)\| \leqslant \tilde{D}e^{-\tilde{a}(t-\tau)+\varepsilon|t|}\|\hat{Q}(t)\| \leqslant 4D\tilde{D}e^{-\tilde{a}(t-\tau)+2\varepsilon|t|}.$$

This completes the proof of the theorem. □

4. Stable and unstable subspaces

4.1. Dimension of the stable and unstable subspaces

Consider the linear subspaces

$$E(t)=P(t)X \quad \text{and} \quad F(t)=Q(t)X$$

for each $t \geqslant 0$. We call $E(t)$ and $F(t)$ respectively the *stable* and *unstable subspaces* at time t associated to the exponential dichotomy of Eq. (1). Clearly, $X=E(t)\oplus F(t)$ for every $t\in I$, and the dimensions $\dim E(t)$ and $\dim F(t)$ are independent of t. Similarly, under the hypotheses of Theorem 2 we consider the linear subspaces

$$\hat{E}(t)=\hat{P}(t)X \quad \text{and} \quad \hat{F}(t)=\hat{Q}(t)X \tag{57}$$

for each $t\in I$ (see (30) for the definition of the projections $\hat{P}(t)$ and $\hat{Q}(t)$). These are respectively the *stable* and *unstable subspaces* at time t associated to the exponential dichotomy of Eq. (3). The dimensions $\dim\hat{E}(t)$ and $\dim\hat{F}(t)$ are also independent of t.

Theorem 4. *Under the hypotheses of Theorem* 2, $\dim \hat{E}(t) = \dim E(t)$ *and* $\dim \hat{F}(t) = \dim F(t)$ *for each* $t \in I$.

Proof. In view of the above discussion, it is sufficient to consider $t = 0$. Fix $\tau \in I$ and set

$$Z(t) = U(t,\tau)\bigl(P(\tau) - \mathrm{Id}\bigr), \quad t \geqslant \tau.$$

Using (23) it is straightforward to verify that Z satisfies (27). By Lemma 2, we have $Z \in \mathcal{E}$ (see (29)) and proceeding as in the proof of Lemma 3 we find that $Z = 0$, i.e.,

$$Z(t) = U(t,\tau)\bigl(P(\tau) - \mathrm{Id}\bigr) = U(t,\tau)P(\tau) - U(t,\tau) = 0.$$

In particular, setting $t = \tau = 0$,

$$\hat{P}(0)P(0) = \hat{P}(0). \tag{58}$$

We now consider the linear operators

$$S = \mathrm{Id} - P(0) + \hat{P}(0) \quad \text{and} \quad T = \mathrm{Id} + P(0) - \hat{P}(0).$$

It follows easily from (41) and (58) that $ST = \mathrm{Id}$. Therefore, S is invertible and $S^{-1} = T$. Furthermore, a simple computation shows that

$$SP(0)S^{-1} = SP(0)T = \hat{P}(0),$$

and $P(0)$ and $\hat{P}(0)$ are similar. The same happens with $Q(0)$ and $\hat{Q}(0)$. In particular, $\dim E(0) = \dim \hat{E}(0)$ and $\dim F(0) = \dim \hat{F}(0)$. This implies the desired statement. □

4.2. Angles between the stable and unstable subspaces

We discuss briefly in this section how the spaces $\hat{E}(t)$ and $\hat{F}(t)$ in (57) may vary with the perturbation $B(t)$ or more precisely with the parameter δ in Theorem 2.

Theorem 5. *Under the hypotheses of Theorem* 2, *for any* $t \geqslant s$ *in* I *we have*

$$\bigl\| P(t) - \hat{P}(t) \bigr\| \leqslant \delta \frac{8D^2\tilde{D}}{a + \tilde{a}} e^{\varepsilon|t|}. \tag{59}$$

In particular, for each fixed $t \in I$ *we have* $\hat{P}(t) \to P(t)$ *as* $\delta \to 0$.

Proof. The inequality (59) follows immediately from (56) and (48). □

We now define

$$\alpha(t) = \inf\bigl\{\|x - y\|\colon\ x \in E(t),\ y \in F(t),\ \|x\| = \|y\| = 1\bigr\},$$
$$\hat{\alpha}(t) = \inf\bigl\{\|x - y\|\colon\ x \in \hat{E}(t),\ y \in \hat{F}(t),\ \|x\| = \|y\| = 1\bigr\}.$$

When X is a Hilbert space we have

$$\alpha(t) = 2\sin\bigl(\theta(t)/2\bigr) \quad \text{and} \quad \hat{\alpha}(t) = 2\sin\bigl(\hat{\theta}(t)/2\bigr),$$

where $\theta(t)$ and $\hat{\theta}(t)$ are respectively the angle between $E(t)$ and $F(t)$, and the angle between $\hat{E}(t)$ and $\hat{F}(t)$. We have (see for example [15])

$$\frac{1}{\|P(t)\|} \leqslant \alpha(t) \leqslant \frac{2}{\|P(t)\|} \quad \text{and} \quad \frac{1}{\|\hat{P}(t)\|} \leqslant \hat{\alpha}(t) \leqslant \frac{2}{\|\hat{P}(t)\|}. \tag{60}$$

It follows from Theorem 5 that

$$\bigl|\|P(t)\| - \|\hat{P}(t)\|\bigr| \leqslant \|P(t) - \hat{P}(t)\| \leqslant \delta \frac{8D^2\tilde{D}}{a+\tilde{a}} e^{\varepsilon|t|}.$$

Hence, by (60), for each fixed $t \in I$,

$$\lim_{\delta \to 0} \bigl|\alpha(t) - \hat{\alpha}(t)\bigr| \leqslant \frac{1}{\|P(t)\|} \leqslant \alpha(t).$$

5. Robustness of dichotomies in the interval $\mathbb{R}$

We first consider the case of the interval $I = (-\infty, \varsigma]$. We continue to consider the constants $\tilde{a}$ and $\tilde{D}$ in (18).

Theorem 6. *The statement in Theorem* 2 *holds for* $I = (-\infty, \varsigma]$ *with* $\varsigma \geqslant 0$.

Proof. The proof is analogous to the proof of Theorem 2, and hence we will only indicate the main differences. Set

$$K = \bigl\{(t,s) \in I \times I \colon t \leqslant s\bigr\},$$

and consider the Banach space

$$\mathcal{D} = \bigl\{V : K \to \mathcal{B}(X) \colon V \text{ is continuous and } \|V\| < \infty\bigr\} \tag{61}$$

with the norm

$$\|V\| = \sup\bigl\{\bigl\|V(t,s)\bigr\| e^{-\varepsilon|s|} \colon (t,s) \in K\bigr\}.$$

Similar arguments to those in the proofs of Lemmas 2 and 3 establish the following statement.

Lemma 12. *The equation* $Z' = (A(t) + B(t))Z$ *has a unique solution* $V \in \mathcal{D}$ *such that for each* $(t,s) \in K$,

$$V(t,s) = T(t,s)Q(s) - \int_t^s T(t,\tau)Q(\tau)B(\tau)V(\tau,s)\,d\tau$$
$$+ \int_{-\infty}^t T(t,\tau)P(\tau)B(\tau)V(\tau,s)\,d\tau. \tag{62}$$

Furthermore, $V(s,\tau)V(\tau,t) = V(s,t)$ *for any* $t \geqslant \tau \geqslant s$ *in* K.

Let now $\hat{T}(t,s)$ be the evolution operator associated to Eq. (3). For each $t \in I$ we consider the linear operators

$$\hat{Q}(t) = \hat{T}(t,0)V(0,0)\hat{T}(0,t) \quad \text{and} \quad \hat{P}(t) = \mathrm{Id} - \hat{Q}(t).$$

Lemma 13. *The operator* $\hat{P}(t)$ *is a projection, and*

$$\hat{P}(t)\hat{T}(t,s) = \hat{T}(t,s)\hat{P}(s), \quad t \geqslant s.$$

The proof of the lemma is analogous to the one of Lemma 4. To obtain the norm bounds for $\hat{T}(t,s)\hat{P}(s)$ when $t \geqslant s$ and $\hat{T}(t,s)\hat{Q}(t)$ when $t \leqslant s$ we start with the following statement.

Lemma 14. *Given* $s \in K$, *if* $y:(-\infty,s] \to X$ *is a bounded solution of Eq.* (3) *with* $y(s) = \xi$, *then*

$$y(t) = T(t,s)Q(s)\xi - \int_s^t T(t,\tau)Q(\tau)B(\tau)y(\tau)\,d\tau + \int_{-\infty}^t T(t,\tau)P(\tau)B(\tau)y(\tau)\,d\tau.$$

Proof. By the variation of constants formula, for $t \leqslant s$ in K,

$$P(s)\xi = T(s,t)P(t)y(t) + \int_t^s T(s,\tau)P(\tau)B(\tau)y(\tau)\,d\tau \tag{63}$$

and

$$Q(s)\xi = T(s,t)Q(t)y(t) + \int_t^s T(s,\tau)Q(\tau)B(\tau)y(\tau)\,d\tau. \tag{64}$$

Since $y(t)$ is bounded, we have

$$\|T(s,t)P(t)y(t)\| \leqslant CDe^{-a(s-t)+\varepsilon|t|},$$

where $C = \sup\{\|y(t)\|\colon t \leqslant s \text{ in } K\} < \infty$. Furthermore,

$$\int_{-\infty}^s \|T(s,\tau)P(\tau)\| \cdot \|B(\tau)\| \cdot \|y(\tau)\|\,d\tau \leqslant D\delta C \int_{-\infty}^s e^{-a(s-\tau)}\,d\tau = \frac{D\delta C}{a}.$$

Taking limits in (63) when $t \to -\infty$, since $a > \varepsilon$ we obtain

$$P(s)\xi = \int_{-\infty}^{s} T(s,\tau)P(\tau)B(\tau)y(\tau)\,d\tau.$$

By (64) we conclude that

$$\begin{aligned}P(t)y(t) &= \int_{-\infty}^{s} T(t,\tau)P(\tau)B(\tau)y(\tau)\,d\tau - \int_{t}^{s} T(t,\tau)P(\tau)B(\tau)y(\tau)\,d\tau\\ &= \int_{-\infty}^{t} T(t,\tau)Q(\tau)B(\tau)y(\tau)\,d\tau.\end{aligned}$$

The desired statement follows from this identity and (64). □

Proceeding as in the proof of Lemma 6 we obtain the following.

Lemma 15. *The function $(-\infty,s]\cap I \ni t \mapsto \hat{Q}(t)\hat{T}(t,s)$ is bounded, and for any $t \leqslant s$ in K we have*

$$\begin{aligned}\hat{Q}(t)\hat{T}(t,s) = T(t,s)Q(s)\hat{Q}(s) &- \int_{t}^{s} T(t,\tau)Q(\tau)B(\tau)\hat{T}(\tau,s)\hat{Q}(s)\,d\tau\\ &+ \int_{-\infty}^{t} T(t,\tau)P(\tau)B(\tau)\hat{T}(\tau,s)\hat{Q}(s)\,d\tau.\end{aligned}$$

Lemma 16. *We have*

$$\begin{aligned}\big\|\hat{T}(t,s)|\operatorname{Im}\hat{P}(s)\big\| &\leqslant \tilde{D}e^{-\tilde{a}(t-s)+\varepsilon|s|}, \quad t \geqslant s \text{ in } K,\\ \big\|\hat{T}(t,s)|\operatorname{Im}\hat{Q}(s)\big\| &\leqslant \tilde{D}e^{-\tilde{a}(s-t)+\varepsilon|s|}, \quad t \leqslant s \text{ in } K.\end{aligned}$$

Proof. The proof of the second statement is analogous to the proof of Lemma 9. Namely, it follows from Lemma 15 that for each $\xi \in X$ the function $y:(-\infty,s] \to [0,+\infty)$ given by $y(t) = \hat{Q}(t)\hat{T}(t,s)\xi$ is a bounded solution of (3). Thus the desired inequality follows readily from Lemma 8 with $\varrho = -\infty$.

The proof of the first statement is analogous to the proof of Lemma 10. Namely, using similar arguments we can show that for $t \geqslant s$,

$$\begin{aligned}\hat{P}(t)\hat{T}(t,s) = T(t,s)P(s)\hat{P}(s) &+ \int_{0}^{t} T(t,\tau)Q(\tau)B(\tau)\hat{P}(\tau)\hat{T}(\tau,s)\,d\tau\\ &+ \int_{s}^{t} T(t,\tau)P(\tau)B(\tau)\hat{P}(\tau)\hat{T}(\tau,s)\,d\tau.\end{aligned}$$

The desired statement follows now easily from Lemma 7. □

Finally, for any sufficiently small δ, proceeding as in the proof of Lemma 11 we obtain the norm bounds

$$\|\hat{P}(t)\| \leqslant 4De^{\varepsilon|t|} \quad \text{and} \quad \|\hat{Q}(t)\| \leqslant 4De^{\varepsilon|t|}$$

for any $t \in K$. Combined with Lemma 16 we find that

$$\|\hat{T}(t,s)\hat{P}(s)\| \leqslant \tilde{D}e^{-\tilde{a}(t-s)+\varepsilon|s|}\|\hat{P}(s)\| \leqslant 4D\tilde{D}e^{-\tilde{a}(t-s)+2\varepsilon|s|}$$

for $t \geqslant s$ in K, and that

$$\|\hat{T}(t,s)\hat{Q}(s)\| \leqslant \tilde{D}e^{-\tilde{a}(s-t)+\varepsilon|s|}\|\hat{Q}(s)\| \leqslant 4D\tilde{D}e^{-\tilde{a}(s-t)+2\varepsilon|s|}$$

for $t \leqslant s$ in K. This completes the proof of the theorem. □

We now consider the case of dichotomies in the interval $\mathbb{R}$.

Theorem 7. *The statement in Theorem* 2 *holds for* $I = \mathbb{R}$.

Proof. Consider the sets

$$J = \{(t,s) \in \mathbb{R} \times \mathbb{R}: t \geqslant s\} \quad \text{and} \quad K = \{(t,s) \in \mathbb{R} \times \mathbb{R}: t \leqslant s\}.$$

We also consider the spaces $\mathcal{C}$ and $\mathcal{D}$ in (8) and (61) associated respectively with J and K. We note that by repeating arguments in the proofs of Theorems 3 and 6 using these sets we find that the operators

$$\hat{P}_+(t) := \hat{T}(t,0)U(0,0)\hat{T}(0,t), \qquad \hat{Q}_-(t) := \hat{T}(t,0)V(0,0)\hat{T}(0,t)$$

are projections for each $t \in \mathbb{R}$, such that for every $t, s \in \mathbb{R}$,

$$\hat{P}_+(t)\hat{T}(t,s) = \hat{T}(t,s)\hat{P}_+(s), \qquad \hat{Q}_-(t)\hat{T}(t,s) = \hat{T}(t,s)\hat{Q}_-(s),$$

and

$$\|\hat{T}(t,s)|\operatorname{Im}\hat{P}_+(s)\| \leqslant \tilde{D}e^{-\tilde{a}(t-s)+\varepsilon|s|}, \quad t \geqslant s, \tag{65}$$

$$\|\hat{T}(t,s)|\operatorname{Im}\hat{Q}_-(s)\| \leqslant \tilde{D}e^{-\tilde{a}(s-t)+\varepsilon|s|}, \quad t \leqslant s. \tag{66}$$

Indeed, notice that Lemmas 7 and 8 hold respectively for functions of the form $x : [s, +\infty) \to [0, +\infty)$ and $y : (-\infty, s] \to [0, +\infty)$, for any $s \in \mathbb{R}$. This allows us to establish respectively (65) (see the proof of Lemma 9) and (66) (see the proof of Lemma 16). Using the identities

$$P(0)\hat{P}_+(0) = P(0), \qquad \hat{P}_+(0)P(0) = \hat{P}_+(0) \tag{67}$$

(see (41) and (58)), and the corresponding

$$Q(0)\hat{Q}_-(0) = Q(0), \qquad \hat{Q}_-(0)Q(0) = \hat{Q}_-(0), \tag{68}$$

we can establish the following statement.

Lemma 17. *If δ is sufficiently small, then the operator $S = \hat{P}_+(0) + \hat{Q}_-(0)$ is invertible.*

Proof. Setting $\hat{P}_-(0) = \mathrm{Id} - \hat{Q}_-(0)$, it follows readily from (68) that $P(0)\hat{P}_-(0) = \hat{P}_-(0)$. Using also (67) we obtain

$$\begin{aligned}\hat{P}_+(0) + \hat{Q}_-(0) - \mathrm{Id} &= \hat{P}_+(0) - P(0) + P(0) - \hat{P}_-(0)\\ &= \hat{P}_+(0) - P(0)\hat{P}_+(0) + P(0) - P(0)\hat{P}_-(0)\\ &= Q(0)\hat{P}_+(0) + P(0)\hat{Q}_-(0).\end{aligned}$$

By Lemma 12,

$$P(0)\hat{Q}_-(0) = P(0)V(0,0) = -\int_{-\infty}^{0} T(0,\tau)P(\tau)B(\tau)V(\tau,0)\,d\tau,$$

and by Lemma 2,

$$Q(0)\hat{P}_+(0) = Q(0)U(0,0) = -\int_{0}^{\infty} T(0,\tau)Q(\tau)B(\tau)U(\tau,0)\,d\tau.$$

To estimate the two integrals, we need to obtain the bounds for $U(t,0)$ when $t \geqslant 0$ and for $V(t,0)$ when $t \leqslant 0$. Using (23) and (16)–(17) we obtain

$$\begin{aligned}\|U(t,0)\| &\leqslant \|T(t,0)P(0)\| + \int_0^t \|T(t,\tau)P(\tau)\| \cdot \|B(\tau)\| \cdot \|U(\tau,0)\|\,d\tau\\ &\quad + \int_t^\infty \|T(t,\tau)Q(\tau)\| \cdot \|B(\tau)\| \cdot \|U(\tau,0)\|\,d\tau\\ &\leqslant De^{-at} + D\delta\int_0^t e^{-a(t-\tau)}\|U(\tau,0)\|\,d\tau + D\delta\int_t^\infty e^{-a(\tau-t)}\|U(\tau,0)\|\,d\tau.\end{aligned}$$

Setting $x(t) = \|U(t,0)\|$ and $\gamma = 1$, it follows from Lemma 7 that

$$\|U(t,0)\| \leqslant \tilde{D}e^{-\tilde{a}t}, \quad t \geqslant 0. \tag{69}$$

To estimate $V(t,0)$ for $t \geqslant 0$, we note that using (62) and again (16)–(17),

$$\begin{aligned}\|V(t,0)\| &\leqslant \|T(t,0)Q(0)\| + \int_t^0 \|T(t,\tau)Q(\tau)\| \cdot \|B(\tau)\| \cdot \|V(\tau,0)\|\,d\tau\\ &\quad + \int_{-\infty}^t \|T(t,\tau)P(\tau)\| \cdot \|B(\tau)\| \cdot \|V(\tau,0)\|\,d\tau\\ &\leqslant De^{at} + D\delta\int_t^0 e^{-a(\tau-t)}\|V(\tau,0)\|\,d\tau + D\delta\int_{-\infty}^t e^{-a(t-\tau)}\|V(\tau,0)\|\,d\tau.\end{aligned}$$

Setting $x(t) = \|V(t,0)\|$ and $\gamma = 1$, it follows from Lemma 8 (for functions in the interval $(-\infty, s]$) that

$$\left\| V(t,0) \right\| \leqslant \tilde{D} e^{\tilde{a} t}, \quad t \leqslant 0. \tag{70}$$

It follows from (69) and (70) that

$$\begin{aligned}
\left\| \hat{P}_+(0) + \hat{Q}_-(0) - \mathrm{Id} \right\| &\leqslant \int_0^\infty \left\| T(0,\tau) Q(\tau) \right\| \cdot \left\| B(\tau) \right\| \cdot \left\| U(\tau,0) \right\| d\tau \\
&\quad + \int_{-\infty}^0 \left\| T(0,\tau) P(\tau) \right\| \cdot \left\| B(\tau) \right\| \cdot \left\| V(\tau,0) \right\| d\tau \\
&\leqslant \delta D \tilde{D} \int_0^\infty e^{-(a+\tilde{a})\tau} \, d\tau + \delta D \tilde{D} \int_{-\infty}^0 e^{(a+\tilde{a})\tau} \, d\tau \\
&\leqslant \frac{2\delta D \tilde{D}}{a + \tilde{a}}.
\end{aligned}$$

Hence, taking δ sufficiently small, we can make $\| \hat{P}_+(0) + \hat{Q}_-(0) - \mathrm{Id} \|$ as small as desired, and thus $S = \hat{P}_+(0) + \hat{Q}_-(0)$ becomes invertible. □

For each $t \in \mathbb{R}$ we set

$$\tilde{P}(t) = \hat{T}(t,0) S P(0) S^{-1} \hat{T}(0,t).$$

We have

$$\tilde{P}(t)^2 = \hat{T}(t,0) \tilde{P}(0)^2 \hat{T}(0,t) = \hat{T}(t,0) S P(0)^2 S^{-1} \hat{T}(0,t) = \tilde{P}(t),$$

and $\tilde{P}(t)$ is a projection for each t. Furthermore,

$$\hat{T}(t,s) \tilde{P}(s) = \hat{T}(t,0) S P(0) S^{-1} \hat{T}(0,s) = \tilde{P}(t) \hat{T}(t,s). \tag{71}$$

Thus, to show that Eq. (3) admits a nonuniform exponential dichotomy in $\mathbb{R}$ with projections $\tilde{P}(t)$, it remains to obtain norm bounds for $\hat{T}(t,s)\tilde{P}(s)$ when $t \geqslant s$, and $\hat{T}(t,s)\tilde{Q}(s)$ when $t \leqslant s$. These will be a consequence of (65)–(66). Observe first that by (67)–(68),

$$\begin{aligned}
S P(0) &= \hat{P}_+(0) P(0) + \hat{Q}_-(0) P(0) = \hat{P}_+(0), \\
S Q(0) &= \hat{P}_+(0) Q(0) + \hat{Q}_-(0) Q(0) = \hat{Q}_-(0).
\end{aligned}$$

Therefore, setting

$$S(t) = \hat{T}(t,0) S \hat{T}(0,t) = \hat{P}_+(t) + \hat{Q}_-(t),$$

we obtain

$$\tilde{P}(t)S(t) = \hat{T}(t,0)SP(0)S^{-1}S\hat{T}(0,t) = \hat{T}(0,t)SP(0)\hat{T}(0,t) = \hat{P}_+(t),$$

and thus also $\tilde{Q}(t)S(t) = \hat{Q}_-(t)$, where $\tilde{Q}(t) = \mathrm{Id} - \tilde{P}(t)$. Therefore,

$$\operatorname{Im}\tilde{P}(t) \supset \operatorname{Im}\hat{P}_+(t) \quad \text{and} \quad \operatorname{Im}\tilde{Q}(t) \supset \operatorname{Im}\hat{Q}_-(t).$$

Since $S(t)$ is invertible it follows that indeed

$$\operatorname{Im}\tilde{P}(t) = \operatorname{Im}\hat{P}_+(t) \quad \text{and} \quad \operatorname{Im}\tilde{Q}(t) = \operatorname{Im}\hat{Q}_-(t).$$

By (65) we obtain that for $t \geqslant s$,

$$\begin{aligned} \|\hat{T}(t,s)\tilde{P}(s)\| &\leqslant \|\hat{T}(t,s)|\operatorname{Im}\tilde{P}(s)\| \cdot \|\tilde{P}(s)\| \\ &= \|\hat{T}(t,s)|\operatorname{Im}\hat{P}_+(s)\| \cdot \|\tilde{P}(s)\| \\ &\leqslant \tilde{D}e^{-\tilde{a}(t-s)+\varepsilon|s|}\|\tilde{P}(s)\|. \end{aligned} \tag{72}$$

Similarly, it follows from (66) that for $t \leqslant s$,

$$\|\hat{T}(t,s)\tilde{Q}(s)\| \leqslant \|\hat{T}(t,s)|\operatorname{Im}\hat{Q}_-(s)\| \cdot \|\tilde{P}(s)\| \leqslant \tilde{D}e^{-\tilde{a}(s-t)+\varepsilon|s|}\|\tilde{Q}(s)\|. \tag{73}$$

Lemma 18. *Provided that δ is sufficiently small, for any $t \in \mathbb{R}$ we have*

$$\|\tilde{P}(t)\| \leqslant 4De^{\varepsilon|t|} \quad \text{and} \quad \|\tilde{Q}(t)\| \leqslant 4De^{\varepsilon|t|}.$$

Proof. It follows from (72) that for each $\xi \in X$ the function $y(t) = \hat{T}(t,s)\tilde{P}(s)\xi$, $t \geqslant s$, is bounded. Since $y(s) = \tilde{P}(s)\xi$, it follows from Lemma 5 that for $t \geqslant s$,

$$\tilde{P}(t)\hat{T}(t,s) = T(t,s)P(s)\tilde{P}(s) + \int_s^t T(t,\tau)P(\tau)B(\tau)\tilde{P}(\tau)\hat{T}(\tau,s)\,d\tau$$
$$- \int_t^\infty T(t,\tau)Q(\tau)B(\tau)\tilde{P}(\tau)\hat{T}(\tau,s)\,d\tau.$$

Setting $t = s$, since $P(t)$ and $Q(t)$ are complementary projections we obtain

$$Q(t)\tilde{P}(t) = -\int_t^\infty T(t,\tau)Q(\tau)B(\tau)\tilde{P}(\tau)\hat{T}(\tau,t)\,d\tau.$$

By (72) and (71) we have that for $\tau \geqslant t$,

$$\|\tilde{P}(\tau)\hat{T}(\tau,t)\| \leqslant \tilde{D}e^{-\tilde{a}(\tau-t)+\varepsilon|t|}\|\tilde{P}(t)\|.$$

Proceeding as in (51) we obtain

$$\|Q(t)\tilde{P}(t)\| \leqslant \frac{D\tilde{D}\delta}{a+\tilde{a}-\varepsilon}\|\tilde{P}(t)\|.$$

Similarly, by (73), for each $\xi \in X$ the function $y(t) = \hat{T}(t,s)\tilde{Q}(s)\xi$, $t \leqslant s$, is bounded. Since $y(s) = \tilde{Q}(s)\xi$, it follows from Lemma 14 that for $t \leqslant s$,

$$\tilde{Q}(t)\hat{T}(t,s) = T(t,s)Q(s)\tilde{Q}(s) - \int_s^t T(t,\tau)Q(\tau)B(\tau)\tilde{Q}(\tau)\hat{T}(\tau,s)\,d\tau$$
$$+ \int_{-\infty}^t T(t,\tau)P(\tau)B(\tau)\tilde{Q}(\tau)\hat{T}(\tau,s)\,d\tau.$$

Setting $t = s$ we obtain

$$P(t)\tilde{Q}(t) = \int_{-\infty}^t T(t,\tau)P(\tau)B(\tau)\tilde{Q}(\tau)\hat{T}(\tau,t)\,d\tau. \tag{74}$$

By (73) and (71) we have that for $\tau \leqslant t$,

$$\|\tilde{Q}(\tau)\hat{T}(\tau,t)\| \leqslant \tilde{D}e^{-\tilde{a}(t-\tau)+\varepsilon|t|}\|\tilde{Q}(t)\|.$$

Therefore, in view of (74),

$$\|P(t)\tilde{Q}(t)\| \leqslant \frac{D\tilde{D}\delta}{a+\tilde{a}-\varepsilon}\|\tilde{Q}(t)\|.$$

Observe now that replacing $\hat{P}(t)$ by $\tilde{P}(t)$ in (55) we obtain

$$\tilde{P}(t) - P(t) = Q(t)\tilde{P}(t) - P(t)\tilde{Q}(t).$$

The desired statement can be obtained by repeating arguments in the proof of Lemma 11, replacing $\hat{P}(t)$ by $\tilde{P}(t)$ and $\hat{Q}(t)$ by $\tilde{Q}(t)$. □

The theorem follows readily from (72), (73), and Lemma 18. □

6. Discrete time case

Let again X be a Banach space. We consider a sequence $A_m \in \mathcal{B}(X)$, $m \in \mathbb{N}$, where $\mathcal{B}(X)$ is the space of bounded linear operators in X. We assume that each A_m is invertible and has bounded inverse. Set

$$\mathcal{A}(m,n) = \begin{cases} A_{m-1}\cdots A_n & \text{if } m > n,\\ \mathrm{Id} & \text{if } m = n,\\ A_m^{-1}\cdots A_{n-1}^{-1} & \text{if } m < n. \end{cases}$$

Clearly,

$$\mathcal{A}(m,k)\mathcal{A}(k,n)=\mathcal{A}(m,n), \quad m,k,n\in\mathbb{N}.$$

We say that the sequence $(A_m)_{m\in\mathbb{N}}$ admits a *nonuniform exponential contraction* if there exist constants a, $D>0$ and $\varepsilon\geqslant 0$ such that

$$\left\|\mathcal{A}(m,n)\right\|\leqslant De^{-a(m-n)+\varepsilon|n|}, \quad m\geqslant n.$$

We say that the sequence $(A_m)_{m\in\mathbb{N}}$ admits a *nonuniform exponential dichotomy* if there exist projections $P_m: X\to X$ for each $m\in\mathbb{N}$ such that

$$\mathcal{A}(m,n)P_n=P_m\mathcal{A}(m,n), \quad m\geqslant n,$$

and there exist constants a, $D>0$ and $\varepsilon\geqslant 0$ such that

$$\begin{aligned}\left\|\mathcal{A}(m,n)P_n\right\|&\leqslant De^{-a(m-n)+\varepsilon|n|}, \quad m\geqslant n,\\ \left\|\mathcal{A}(m,n)Q_n\right\|&\leqslant De^{-a(n-m)+\varepsilon|n|}, \quad n\geqslant m,\end{aligned} \tag{75}$$

where $Q_m=\mathrm{Id}-P_m$ is the complementary projection.

6.1. Robustness of nonuniform exponential contractions

The following is a version of Theorem 1 in the case of discrete time.

Theorem 8. *Let $A_m, B_m\in\mathcal{B}(X)$, $m\in\mathbb{N}$. If the sequence $(A_m)_{m\in\mathbb{N}}$ admits a nonuniform exponential contraction, and $\|B_m\|\leqslant\delta e^{-\varepsilon|m+1|}$, $m\in\mathbb{N}$, with $\delta D<1-e^{-a}$, then the sequence $(A_m+B_m)_{m\in\mathbb{N}}$ admits a nonuniform exponential contraction, with the constant a replaced by $a-\log(1+\delta De^{a})$.*

Proof. One can easily verify that

$$\hat{\mathcal{A}}(m,n)=\mathcal{A}(m,n)+\sum_{l=n}^{m-1}\mathcal{A}(m,l+1)B_l\hat{\mathcal{A}}(l,n), \tag{76}$$

where

$$\hat{\mathcal{A}}(m,n)=\begin{cases}(A_{m-1}+B_{m-1})\cdots(A_n+B_n) & \text{for } m>n,\\ \mathrm{Id} & \text{for } m=n.\end{cases} \tag{77}$$

Setting $x_m=\|\hat{\mathcal{A}}(m,n)\|$, we find that

$$\begin{aligned}x_m&\leqslant\left\|\mathcal{A}(m,n)\right\|+\sum_{l=n}^{m-1}\left\|\mathcal{A}(m,l+1)\right\|\cdot\|B_l\|x_l\\ &\leqslant De^{-a(m-n)+\varepsilon|n|}+\delta D\sum_{l=n}^{m-1}e^{-a(m-l-1)}x_l.\end{aligned} \tag{78}$$

Lemma 19. *Let $(x_m)_{m\geqslant n}\subset[0,+\infty)$ be a bounded sequence satisfying*

$$x_m \leqslant De^{-a(m-n)+\varepsilon|n|} + \delta D\sum_{l=n}^{m-1} e^{-a(m-l-1)}x_l.$$

If $\delta D < 1-e^{-a}$ then

$$x_m \leqslant De^{-(a-\log(1+\delta De^a))(m-n)+\varepsilon|n|}, \quad m\geqslant n.$$

Proof. Consider the sequence Φ_m defined recursively by

$$\Phi_m = De^{-a(m-n)+\varepsilon|n|} + \delta D\sum_{l=n}^{m-1} e^{-a(m-l-1)}\Phi_l. \tag{79}$$

Setting $\Gamma_m = e^{a(m-n)}\Phi_m$, we can rewrite (79) in the form

$$\Gamma_m = De^{\varepsilon|n|} + \delta De^a\sum_{l=n}^{m-1}\Gamma_l.$$

One can easily verify that $\Gamma_{m+1}-\Gamma_m = \delta De^a\Gamma_m$, i.e.,

$$\Gamma_{m+1} = \left(1+\delta De^a\right)\Gamma_m.$$

Furthermore, again by (79), $\Gamma_n = \Phi_n = De^{\varepsilon|n|}$ and for any $m\geqslant n$,

$$\Gamma_m = \left(1+\delta De^a\right)^{m-n}\Gamma_n = De^{\varepsilon|n|}\left(1+\delta De^a\right)^{m-n} = De^{\log(1+\delta De^a)(m-n)+\varepsilon|n|}.$$

Set $z_m = x_m - \Phi_m$. Then

$$z_m \leqslant \delta D\sum_{l=n}^{m-1} e^{-a(m-l-1)}z_l, \quad m\geqslant n. \tag{80}$$

Set now $z = \sup_{m\geqslant n} z_m$. Since the sequences $(x_m)_{m\geqslant n}$ and $(\Phi_m)_{m\geqslant n}$ are bounded, the number z is finite. Taking the supremum in (80) we obtain

$$z \leqslant \delta Dz\sup_{m\geqslant n}\sum_{l=n}^{m-1} e^{-a(m-l-1)}.$$

Hence, $z\leqslant (\delta D/(1-e^{-a}))z$. Since $\delta D<1-e^{-a}$, we obtain that $z\leqslant 0$. Thus, $z_m\leqslant 0$, i.e., $x_m\leqslant\Phi_m$ for $m\geqslant n$. Therefore,

$$x_m \leqslant \Phi_m = e^{-a(m-n)}\Gamma_m \leqslant De^{-(a-\log(1+\delta De^a))(m-n)+\varepsilon|n|},$$

and we obtain the desired inequality. □

It follows from (78) and Lemma 19 that

$$\|\hat{\mathcal{A}}(m,n)\| \leqslant De^{-(a-\log(1+\delta De^{a}))(m-n)+\varepsilon|n|}, \quad m \geqslant n.$$

This completes the proof of the theorem. □

We note that in a similar manner to that in Theorem 1 we can consider nonuniform exponential contractions in any interval $I \subset \mathbb{N}$. The statement in Theorem 8 remains valid for all these contractions.

6.2. Robustness of nonuniform exponential dichotomies

We consider in this section the problem of robustness of nonuniform exponential dichotomies in "the interval" $\mathbb{N}$. As in the continuous time setting, the case of other intervals can be treated in a similar manner and thus we omit this material. Set

$$\tilde{a} = -\log\bigl(\cosh a - \sqrt{\cosh^2 a - 2(1+\delta D \sinh a)}\bigr),$$
$$\tilde{D} = \frac{D}{1-\delta De^{\tilde{a}}/(e^{a+\tilde{a}}-1)}.$$

The following is our robustness result.

Theorem 9. *Let $A_m, B_m \in \mathcal{B}(X)$, $m \in \mathbb{N}$. If the sequence $(A_m)_{m\in\mathbb{N}}$ admits a nonuniform exponential dichotomy with $\varepsilon < a$, and $\|B_m\| \leqslant \delta e^{-2\varepsilon|m+1|}$, $m \in \mathbb{N}$, with δ sufficiently small, then the sequence $(A_m + B_m)_{m\in\mathbb{N}}$ admits a nonuniform exponential dichotomy, with constants a, D, and ε replaced respectively by $\tilde{a}$, $4D\tilde{D}$, and 2ε.*

Proof. Consider the Banach space

$$\mathcal{C} = \bigl\{U = \bigl(U(m,n)\bigr)_{m\geqslant n} \in \mathcal{B}(X)\colon \|U\| < \infty\bigr\}$$

with the norm

$$\|U\| = \sup\bigl\{\|U(m,n)\|e^{-\varepsilon|n|}\colon m \geqslant n\bigr\}.$$

Set

$$C_{nl} = \mathcal{A}(n,l+1)P_{l+1}B_l \quad \text{and} \quad D_{nl} = \mathcal{A}(n,l+1)Q_{l+1}B_l.$$

We will always assume that

$$\theta := \delta D\frac{1+e^{-a}}{1-e^{-a}} < 1. \tag{81}$$

Lemma 20. *The difference equation* $Z_{m+1} = (A_m + B_m)Z_m$ *has a unique solution* $U \in \mathcal{C}$ *satisfying*

$$U(m,n) = \mathcal{A}(m,n)P_n + \sum_{l=n}^{m-1} C_{ml} U(l,n) - \sum_{l=m}^{\infty} D_{ml} U(l,n). \tag{82}$$

Proof. Assume that $U \in \mathcal{C}$ satisfies (82). We have

$$\begin{aligned} U(m+1,n) - A_m U(m,n) &= C_{m+1,m} U(m,n) + D_{m+1,m} U(m,n) \\ &= B_m U(m,n), \end{aligned}$$

and hence $U(m,n)$ solves the difference equation. Thus, we must show that the operator

$$(LU)(m,n) = \mathcal{A}(m,n)P_n + \sum_{l=n}^{m-1} C_{ml} U(l,n) - \sum_{l=m}^{\infty} D_{ml} U(l,n)$$

has a unique fixed point in $\mathcal{C}$. Clearly,

$$\begin{aligned} \big\|(LU)(m,n)\big\| &\leqslant \big\|\mathcal{A}(m,n)P_n\big\| + \sum_{l=n}^{m-1} \big\|\mathcal{A}(m,l+1)P_{l+1}\big\| \cdot \|B_l\| \cdot \big\|U(l,n)\big\| \\ &\quad + \sum_{l=m}^{\infty} \big\|\mathcal{A}(m,l+1)Q_{l+1}\big\| \cdot \|B_l\| \cdot \big\|U(l,n)\big\| \\ &\leqslant De^{-a(m-n)+\varepsilon|n|} + D\delta e^{\varepsilon|n|}\|U\| \sum_{l=n}^{m-1} e^{-a(m-l-1)} \\ &\quad + D\delta e^{\varepsilon|n|}\|U\| \sum_{l=m}^{\infty} e^{-a(m-l-1)}. \end{aligned}$$

Since $a > 0$, this implies that $\|LU\| \leqslant D + \theta\|U\| < \infty$, and hence, $L : \mathcal{C} \to \mathcal{C}$. For $U_1, U_2 \in \mathcal{C}$, proceeding in a similar manner to the one above we obtain

$$\|LU_1 - LU_2\| \leqslant \theta\|U_1 - U_2\|,$$

and L is a contraction. Thus, there is a unique $U \in \mathcal{C}$ such that $LU = U$. □

Lemma 21. *For any* $m \geqslant k \geqslant n$ *we have* $U(m,k)U(k,n) = U(m,n)$.

Proof. Since P_m and Q_m are complementary projections, it follows from (82) that

$$\begin{aligned} U(m,k)U(k,n) &= \mathcal{A}(m,n)P_n + \sum_{l=n}^{k-1} C_{ml} U(l,n) + \sum_{l=k}^{m-1} C_{ml} U(l,k)U(k,n) \\ &\quad - \sum_{l=m}^{\infty} D_{ml} U(l,k)U(k,n). \end{aligned}$$

Subtracting the identity from (82), and setting $Z_l = U(l,k)U(k,n) - U(l,n)$ we obtain

$$Z_m = \sum_{l=k}^{m-1} C_{ml} Z_l - \sum_{l=m}^{\infty} D_{ml} Z_l. \tag{83}$$

For fixed k and n we consider the operator

$$(NW)_m = \sum_{l=k}^{m-1} C_{ml} W_l - \sum_{l=m}^{\infty} D_{ml} W_l$$

in the Banach space

$$\mathcal{E} = \left\{ W = (W_m)_{m \geqslant k} \subset \mathcal{B}(X) \colon \ \|W\| < \infty \right\}$$

with the norm $\|W\| = \sup\{\|W_l\|\colon\ l \geqslant k\}$. We have

$$\begin{aligned}\left\|(NW)_m\right\| &\leqslant D \sum_{l=k}^{m-1} e^{-a(m-l-1)+\varepsilon|l+1|} \|B_l\| \cdot \|W_l\| \\ &\quad + D \sum_{l=m}^{\infty} e^{-a(l+1-m)+\varepsilon|l+1|} \|B_l\| \cdot \|W_l\| \leqslant \theta \|W\|,\end{aligned}$$

and thus $N \colon \mathcal{E} \to \mathcal{E}$. We can also show that

$$\|NW_1 - NW_2\| \leqslant \theta \|W_1 - W_2\|.$$

Hence, the operator N is a contraction, and there exists a unique $W \in \mathcal{E}$ satisfying (83), which thus must be the above sequence $Z \in \mathcal{E}$. Since $0 \in \mathcal{E}$ also satisfies (83) we conclude that for any $m \geqslant k \geqslant n$,

$$Z_m = U(m,k)U(k,n) - U(m,n) = 0.$$

This completes the proof of the lemma. □

Let $\hat{\mathcal{A}}(m,n)$ be as in (77) for $m \geqslant n$, and let $\hat{\mathcal{A}}(m,n) = \hat{\mathcal{A}}(n,m)^{-1}$ for $m < n$. We set

$$\hat{P}_m = \hat{\mathcal{A}}(m,1)U(1,1)\hat{\mathcal{A}}(1,m) \quad \text{and} \quad \hat{Q}_m = \mathrm{Id} - \hat{P}_m.$$

Lemma 22. *The operator $\hat{P}_m$ is a projection, and*

$$\hat{P}_m \hat{\mathcal{A}}(m,n) = \hat{\mathcal{A}}(m,n)\hat{P}_n, \quad m \geqslant n.$$

The proof is analogous to that of Lemma 4 and thus it is omitted.

Lemma 23. *If $(y_m)_{m\geqslant n}$ is a bounded sequence satisfying the identities $y_{m+1}=(A_m+B_m)y_m$ for $m\geqslant n$, then*

$$y_m=\mathcal{A}(m,n)P_ny_n+\sum_{l=n}^{m-1}C_{ml}y_l-\sum_{l=m}^{\infty}D_{ml}y_l.$$

Proof. Clearly, for $m\geqslant n$,

$$P_my_m=\mathcal{A}(m,n)P_ny_n+\sum_{l=n}^{m-1}C_{ml}y_l \tag{84}$$

and

$$Q_my_m=\mathcal{A}(m,n)Q_ny_n+\sum_{l=n}^{m-1}D_{ml}y_l. \tag{85}$$

Since the sequence $(y_m)_{m\geqslant n}$ is bounded, we have

$$\sum_{l=n}^{\infty}\|D_{nl}y_l\|\leqslant D\delta\sum_{l=n}^{\infty}e^{-a(l+1-n)}\sup_{l\geqslant n}\|y_l\|<\infty.$$

Rewriting (85) in the form

$$Q_ny_n=\mathcal{A}(n,m)Q_my_m-\sum_{l=n}^{m-1}D_{nl}y_l,$$

and letting $m\to\infty$, since $a>\varepsilon$ we obtain $Q_ny_n=-\sum_{l=n}^{\infty}D_{nl}y_l$. It follows from (85) that

$$Q_my_m=-\sum_{l=n}^{\infty}D_{ml}y_l+\sum_{l=n}^{m-1}D_{ml}y_l=-\sum_{l=m}^{\infty}D_{ml}y_l. \tag{86}$$

The statement follows from adding (84) and (86). □

Lemma 24. *For any $m\geqslant n$,*

$$\hat{P}_m\hat{\mathcal{A}}(m,n)=\hat{\mathcal{A}}(m,n)P_n\hat{P}_n+\sum_{l=n}^{m-1}C_{ml}\hat{P}_l\hat{\mathcal{A}}(l,n)-\sum_{l=m}^{\infty}D_{ml}\hat{P}_l\hat{\mathcal{A}}(l,n).$$

The proof is analogous to that of Lemma 6 and thus it is omitted.

Lemma 25. *Let $(x_m)_{m\geqslant n}\subset[0,+\infty)$ be a bounded sequence such that*

$$x_m\leqslant De^{-a(m-n)+\varepsilon n}\gamma+\delta D\sum_{l=n}^{m-1}e^{-a(m-l-1)}x_l+\delta D\sum_{l=m}^{\infty}e^{-a(l+1-m)}x_l \tag{87}$$

for $m\geqslant n$. Then $x_m\leqslant\tilde{D}\gamma e^{-\tilde{a}(m-n)+\varepsilon n}$ for $m\geqslant n$.

Proof. Consider the sequence Φ_m such that for $m \geqslant n$,

$$\Phi_m = De^{-a(m-n)+\varepsilon|n|}\gamma + \delta D \sum_{l=n}^{m-1} e^{-a(m-l-1)}\Phi_l + \delta D \sum_{l=m}^{\infty} e^{-a(l+1-m)}\Phi_l. \tag{88}$$

One can show that it satisfies the recurrence

$$\Phi_{m+1} = \big(e^a + e^{-a}\big)\Phi_m - \big(1 + \delta D\big(e^a - e^{-a}\big)\big)\Phi_{m-1}. \tag{89}$$

Indeed,

$$\begin{aligned}
\Phi_{m+1} &= e^{-a} De^{-a(m-n)+\varepsilon|n|}\gamma + e^{-a}\delta D \sum_{l=n}^{m} e^{-a(m-l-1)}\Phi_l \\
&\quad + e^{a}\delta D \sum_{l=m+1}^{\infty} e^{-a(l+1-m)}\Phi_l \\
&= e^{-a} De^{-a(m-n)+\varepsilon|n|}\gamma + e^{-a}\delta D \sum_{l=n}^{m-1} e^{-a(m-l-1)}\Phi_l + e^{-a}\delta De^{a}\Phi_m \\
&\quad + e^{a}\delta D \sum_{l=m}^{\infty} e^{-a(l+1-m)}\Phi_l - e^{a}\delta De^{-a}\Phi_m \\
&= e^{-a} De^{-a(m-n)+\varepsilon|n|}\gamma + e^{-a}\delta D \sum_{l=n}^{m-1} e^{-a(m-l-1)}\Phi_l \\
&\quad + e^{a}\delta D \sum_{l=m}^{\infty} e^{-a(l+1-m)}\Phi_l \\
&= e^{-a}\Phi_m - e^{-a}\delta D \sum_{l=m}^{\infty} e^{-a(l+1-m)}\Phi_l + e^{a}\delta D \sum_{l=m}^{\infty} e^{-a(l+1-m)}\Phi_l.
\end{aligned}$$

Similarly,

$$\begin{aligned}
\Phi_{m-1} &= e^{a} De^{-a(m-n)+\varepsilon|n|}\gamma + e^{a}\delta D \sum_{l=n}^{m-2} e^{-a(m-l-1)}\Phi_l \\
&\quad + e^{-a}\delta D \sum_{l=m-1}^{\infty} e^{-a(l+1-m)}\Phi_l \\
&= e^{a} De^{-a(m-n)+\varepsilon|n|}\gamma + e^{a}\delta D \sum_{l=n}^{m-1} e^{-a(m-l-1)}\Phi_l - e^{a}\delta D\Phi_{m-1} \\
&\quad + e^{-a}\delta D \sum_{l=m}^{\infty} e^{-a(l+1-m)}\Phi_l + e^{-a}\delta D\Phi_{m-1}
\end{aligned}$$

$$= e^a \Phi_m - e^a \delta D \sum_{l=m}^{\infty} e^{-a(l+1-m)} \Phi_l + e^{-a} \delta D \sum_{l=m}^{\infty} e^{-a(l+1-m)} \Phi_l$$
$$- e^a \delta D \Phi_{m-1} + e^{-a} \delta D \Phi_{m-1}.$$

Summing the two identities we obtain (89). We can easily verify that in order that the solution Φ_m of the recurrence in (89) is bounded we must have $\Phi_m = \Phi_n e^{-\tilde{a}(m-n)}$ (notice that $e^{-\tilde{a}} < 1$). By (88), setting $m = n$,

$$\Phi_n = De^{\varepsilon|n|}\gamma + \delta De^{-a}\Phi_n \sum_{l=n}^{\infty} e^{-(a+\tilde{a})(l-n)} = De^{\varepsilon|n|}\gamma + \Phi_n \frac{\delta De^{-a}}{1 - e^{-(a+\tilde{a})}},$$

which yields

$$\Phi_n = \frac{De^{\varepsilon|n|}\gamma(e^{a+\tilde{a}} - 1)}{e^{a+\tilde{a}} - 1 - \delta De^{\tilde{a}}}.$$

This implies that $\Phi_m = \tilde{D}\gamma e^{-\tilde{a}(m-n)+\varepsilon|n|}$. Set now $z_m = x_m - \Phi_m$ for $m \geqslant n$. Then

$$z_m \leqslant \delta D \sum_{l=n}^{m-1} e^{-a(m-l-1)} z_l + \delta D \sum_{l=m}^{\infty} e^{-a(l+1-m)} z_l.$$

Setting $z = \sup_{m \geqslant n} z_m$, we obtain

$$z \leqslant \delta Dz \sup_{m \geqslant n} \sum_{l=n}^{m-1} e^{-a(m-l-1)} + \delta Dz \sup_{m \geqslant n} \sum_{l=m}^{\infty} e^{-a(l+1-m)}.$$

Hence, $z \leqslant \theta z$ and by (81), we obtain that $z \leqslant 0$. Thus, $z_m \leqslant 0$ and $x_m \leqslant \Phi_m$ for $m \geqslant n$. This yields the desired result. □

Lemma 26. *Let $(y_m)_{1 \leqslant m \leqslant n} \subset [0, +\infty)$ be a finite sequence such that*

$$y_m \leqslant De^{-\tilde{a}(n-m)+\varepsilon n}\gamma + \delta D \sum_{l=m}^{n-1} e^{-a(l+1-m)} y_l + \delta D \sum_{l=0}^{m-1} e^{-a(m-l-1)} y_l \tag{90}$$

for $m \leqslant n$. Then $y_m \leqslant \tilde{D}\gamma e^{-\tilde{a}(n-m)+\varepsilon n}$ for $m \leqslant n$.

The proof is analogous to that of Lemma 25.

Lemma 27. $\|\hat{\mathcal{A}}(m,n) | \operatorname{Im} \hat{P}_n\| \leqslant \tilde{D}e^{-\tilde{a}(m-n)+\varepsilon n}$ *for $m \geqslant n$.*

Proof. Let $\xi \in X$. Setting $x_m = \|\hat{P}_m \hat{\mathcal{A}}(m,n)\xi\|$ for $m \geqslant n$ and $\gamma = \|\hat{P}_n \xi\|$. It follows from Lemma 24 and (75) that x_m satisfies (87). By Lemmas 22 and 25 we obtain

$$\|\hat{\mathcal{A}}(m,n)\hat{P}_n \xi\| = \|\hat{P}_m \hat{\mathcal{A}}(m,n)\xi\| \leqslant \tilde{D}e^{-\tilde{a}(m-n)+\varepsilon|n|} \|\hat{P}_n \xi\|.$$

This establishes the desired inequality. □

Lemma 28. $\|\hat{\mathcal{A}}(m,n)|\operatorname{Im}\hat{Q}_n\| \leqslant \tilde{D}e^{-\tilde{a}(n-m)+\varepsilon n}$ *for* $m \leqslant n$.

Proof. In a similar manner to the proof of Lemma 10, we first derive an identity for $\hat{Q}_m\hat{\mathcal{A}}(m,n)$. Since

$$\hat{\mathcal{A}}(m,n) = \mathcal{A}(m,n) + \sum_{l=n}^{m-1} \mathcal{A}(m,l+1)B_l\hat{\mathcal{A}}(l,n)$$

(see (76)–(77)), the sequence $y_m = \hat{\mathcal{A}}(m,1)\hat{Q}_1$ satisfies

$$y_m = \mathcal{A}(m,1)\hat{Q}_1 + \sum_{l=1}^{m-1} \mathcal{A}(m,l+1)B_l y_l. \tag{91}$$

On the other hand, using (82) with $m = n = 1$,

$$\hat{P}_1 = U(1,1) = P_1 - \sum_{l=1}^{\infty} Q_1\mathcal{A}(1,l+1)B_l U(l,1). \tag{92}$$

It follows from (92) that $P_1\hat{P}_1 = P_1$. Therefore,

$$Q_1\hat{Q}_1 = (\mathrm{Id} - P_1)(\mathrm{Id} - \hat{P}_1) = \mathrm{Id} - \hat{P}_1 = \hat{Q}_1. \tag{93}$$

By (91), we obtain

$$y_n = \mathcal{A}(n,1)\hat{Q}_1 + \sum_{l=1}^{n-1} \mathcal{A}(n,l+1)B_l y_l = \mathcal{A}(n,1)Q_1\hat{Q}_1 + \sum_{l=1}^{n-1} \mathcal{A}(n,l+1)B_l y_l. \tag{94}$$

Multiplying (94) on the left by $\mathcal{A}(m,n)Q_n$ and using (93), we obtain

$$\mathcal{A}(m,n)Q_n y_n = \mathcal{A}(m,1)Q_1\hat{Q}_1 + \sum_{l=1}^{n-1} \mathcal{A}(m,l+1)Q_m B_l y_l = \mathcal{A}(m,1)\hat{Q}_1 + \sum_{l=1}^{n-1} D_{ml} y_l.$$

Combined with (91) this yields

$$\begin{aligned} y_m &= \mathcal{A}(m,n)Q_n y_n - \sum_{l=1}^{n-1} D_{ml} y_l + \sum_{l=1}^{m-1} \mathcal{A}(m,l+1)B_l y_l \\ &= \mathcal{A}(m,n)Q_n y_n + \sum_{l=1}^{m-1} C_{ml} y_l - \sum_{l=m}^{n-1} D_{ml} y_l. \end{aligned} \tag{95}$$

On the other hand, by Lemma 22,

$$\hat{Q}_m\hat{\mathcal{A}}(m,n) = \hat{\mathcal{A}}(m,n)\hat{Q}_n, \tag{96}$$

and thus, $y_l\hat{\mathcal{A}}(1,n) = \hat{Q}_l\hat{\mathcal{A}}(l,n)$. Multiplying (95) on the right by $\hat{\mathcal{A}}(1,n)$ we obtain

$$\hat{Q}_m \hat{\mathcal{A}}(m,n) = \mathcal{A}(m,n) Q_n \hat{Q}_n + \sum_{l=1}^{m-1} C_{ml} \hat{Q}_l \hat{\mathcal{A}}(l,n) - \sum_{l=m}^{n-1} D_{ml} \hat{Q}_l \hat{\mathcal{A}}(l,n). \tag{97}$$

Let now $\xi \in X$. Set $y_m = \|\hat{\mathcal{A}}(m,n)\hat{Q}_n\xi\|$ for $m \leqslant n$, and $\gamma = \|\hat{Q}_n\xi\|$. It follows from (97) and (96) that the sequence y_m satisfies the inequality in (90). Using Lemma 26 we readily obtain the desired inequality. □

Lemma 29. *Provided that δ is sufficiently small, for any $m \in \mathbb{N}$,*

$$\|\hat{P}_m\| \leqslant 4De^{\varepsilon m} \quad \textit{and} \quad \|\hat{Q}_m\| \leqslant 4De^{\varepsilon m}.$$

Proof. By Lemma 24 with $m = n$,

$$Q_m \hat{P}_m = -\sum_{l=m}^{\infty} D_{ml} \hat{P}_l \hat{\mathcal{A}}(l,m). \tag{98}$$

By Lemmas 27 and 22 we have that for $l \geqslant m$,

$$\left\| \hat{P}_l \hat{\mathcal{A}}(l,m) \right\| \leqslant \tilde{D} e^{-\tilde{a}(l-m)+\varepsilon m} \|\hat{P}_m\|.$$

By (98), using (75) we obtain

$$\begin{aligned}
\|\hat{Q}_m \hat{P}_m\| &\leqslant \delta D \tilde{D} \|\hat{P}_m\| \sum_{l=m}^{\infty} e^{-a(l+1-m)+\varepsilon(l+1)} e^{-2\varepsilon(l+1)} e^{-\tilde{a}(l-m)+\varepsilon m} \\
&\leqslant \delta D \tilde{D} e^{-a-\varepsilon} \|\hat{P}_m\| \sum_{l=m}^{\infty} e^{-(a+\tilde{a}+\varepsilon)(l-m)} \\
&= \frac{\delta D \tilde{D} e^{-a-\varepsilon}}{1 - e^{-(a+\tilde{a}+\varepsilon)}} \|\hat{P}_m\|.
\end{aligned} \tag{99}$$

Similarly, it follows from (97) with $m = n$ that

$$P_m \hat{Q}_m = \sum_{l=1}^{m-1} C_{ml} \hat{Q}_l \hat{\mathcal{A}}(l,m). \tag{100}$$

By Lemma 28 and (96) we have that for $l \leqslant m$,

$$\left\| \hat{Q}_l \hat{\mathcal{A}}(l,m) \right\| \leqslant \tilde{D} e^{-\tilde{a}(m-l)+\varepsilon m} \|\hat{Q}_m\|.$$

By (100), using (75) we obtain

$$\begin{aligned}\|P_m \hat{Q}_m\| &\leqslant \delta D\tilde{D}\|\hat{Q}_m\| \sum_{l=1}^{m-1} e^{-a(m-l-1)+\varepsilon(l+1)} e^{-2\varepsilon(l+1)} e^{-\tilde{a}(m-l)+\varepsilon m} \\ &\leqslant \delta D\tilde{D}e^{a-\varepsilon}\|\hat{Q}_m\| \sum_{l=1}^{m-1} e^{-(a+\tilde{a}-\varepsilon)(m-l)} = \frac{\delta D\tilde{D}e^{a-\varepsilon}}{e^{a+\tilde{a}-\varepsilon}-1}\|\hat{Q}_m\|. \end{aligned} \tag{101}$$

Proceeding in a similar manner to that in (55) we find that

$$\hat{P}_m - P_m = Q_m \hat{P}_m - P_m \hat{Q}_m.$$

It follows from (99) and (101) that

$$\|\hat{P}_m - P_m\| \leqslant \frac{1}{4}\big(\|\hat{P}_m\| + \|\hat{Q}_m\|\big) \tag{102}$$

provided that δ is sufficiently small. On the other hand, by (75) with $m = n$ we have

$$\|P_m\| \leqslant De^{\varepsilon m} \quad \text{and} \quad \|Q_m\| \leqslant De^{\varepsilon m}.$$

It follows from (102) that

$$\|\hat{P}_m\| \leqslant \|\hat{P}_m - P_m\| + \|P_m\| \leqslant \frac{1}{4}\big(\|\hat{P}_m\| + \|\hat{Q}_m\|\big) + De^{\varepsilon m},$$

and since $\|\hat{Q}(t) - Q(t)\| = \|\hat{P}(t) - P(t)\|$ we also have

$$\|\hat{Q}_m\| \leqslant \|\hat{P}_m - P_m\| + \|Q_m\| \leqslant \frac{1}{4}\big(\|\hat{P}_m\| + \|\hat{Q}_m\|\big) + De^{\varepsilon m}.$$

Therefore,

$$\|\hat{P}_m\| + \|\hat{Q}_m\| \leqslant \frac{1}{2}\big(\|\hat{P}_m\| + \|\hat{Q}_m\|\big) + 2De^{2\varepsilon m},$$

and $\|\hat{P}_m\| + \|\hat{Q}_m\| \leqslant 4De^{\varepsilon m}$. □

This completes the proof of the theorem. □

References

[1] L. Barreira, Ya. Pesin, Lyapunov Exponents and Smooth Ergodic Theory, Univ. Lecture Ser., vol. 23, Amer. Math. Soc., 2002.

[2] L. Barreira, C. Valls, Smoothness of invariant manifolds for nonautonomous equations, Comm. Math. Phys. 259 (2005) 639–677.

[3] L. Barreira, C. Valls, A Grobman–Hartman theorem for nonuniformly hyperbolic dynamics, J. Differential Equations 228 (2006) 285–310.

[4] L. Barreira, C. Valls, Smooth invariant manifolds in Banach spaces with nonuniform exponential dichotomy, J. Funct. Anal. 238 (2006) 118–148.

[5] L. Barreira, C. Valls, Stable manifolds for nonautonomous equations without exponential dichotomy, J. Differential Equations 221 (2006) 58–90.

[6] L. Barreira, C. Valls, Nonuniform exponential dichotomies and Lyapunov regularity, J. Dynam. Differential Equations 19 (2007) 215–241.

[7] C. Blázquez, Transverse homoclinic orbits in periodically perturbed parabolic equations, Nonlinear Anal. 10 (1986) 1277–1291.

[8] C. Chicone, Yu. Latushkin, Evolution Semigroups in Dynamical Systems and Differential Equations, Math. Surveys Monogr., vol. 70, Amer. Math. Soc., 1999.

[9] S.-N. Chow, H. Leiva, Existence and roughness of the exponential dichotomy for skew-product semiflow in Banach spaces, J. Differential Equations 120 (1995) 429–477.

[10] W. Coppel, Dichotomies and reducibility, J. Differential Equations 3 (1967) 500–521.

[11] W. Coppel, Dichotomies in Stability Theory, Lecture Notes in Math., vol. 629, Springer, 1978.

[12] Ju. Dalec'kiĭ, M. Kreĭn, Stability of Solutions of Differential Equations in Banach Space, Transl. Math. Monogr., vol. 43, Amer. Math. Soc., 1974.

[13] J. Hale, Asymptotic Behavior of Dissipative Systems, Math. Surveys Monogr., vol. 25, Amer. Math. Soc., 1988.

[14] D. Henry, Geometric Theory of Semilinear Parabolic Equations, Lecture Notes in Math., vol. 840, Springer, 1981.

[15] D. Henry, Exponential dichotomies, the shadowing lemma and homoclinic orbits in Banach spaces, Resenhas 1 (1994) 381–401.

[16] N. Huy, Exponential dichotomy of evolution equations and admissibility of function spaces on a half-line, J. Funct. Anal. 235 (2006) 330–354.

[17] A. Katok, B. Hasselblatt, Introduction to the Modern Theory of Dynamical Systems, with a supplement by A. Katok and L. Mendoza, Encyclopedia Math. Appl., vol. 54, Cambridge Univ. Press, Cambridge, 1995.

[18] X.-B. Lin, Exponential dichotomies and homoclinic orbits in functional-differential equations, J. Differential Equations 63 (1986) 227–254.

[19] J. Massera, J. Schäffer, Linear differential equations and functional analysis, I, Ann. of Math. (2) 67 (1958) 517–573.

[20] J. Massera, J. Schäffer, Linear Differential Equations and Function Spaces, Pure Appl. Math., vol. 21, Academic Press, 1966.

[21] N. Minh, N. Huy, Characterizations of dichotomies of evolution equations on the half-line, J. Math. Anal. Appl. 261 (2001) 28–44.

[22] N. Minh, F. Räbiger, R. Schnaubelt, Exponential stability, exponential expansiveness, and exponential dichotomy of evolution equations on the half-line, Integral Equations Operator Theory 32 (1998) 332–353.

[23] R. Naulin, M. Pinto, Admissible perturbations of exponential dichotomy roughness, Nonlinear Anal. 31 (1998) 559–571.

[24] K. Palmer, Exponential dichotomies and transversal homoclinic points, J. Differential Equations 55 (1984) 225–256.

[25] O. Perron, Die Stabilitätsfrage bei Differentialgleichungen, Math. Z. 32 (1930) 703–728.

[26] V. Pliss, G. Sell, Robustness of exponential dichotomies in infinite-dimensional dynamical systems, J. Dynam. Differential Equations 11 (1999) 471–513.

[27] L. Popescu, Exponential dichotomy roughness on Banach spaces, J. Math. Anal. Appl. 314 (2006) 436–454.

[28] P. Preda, A. Pogan, C. Preda, (L^p, L^q)-admissibility and exponential dichotomy of evolutionary processes on the half-line, Integral Equations Operator Theory 49 (2004) 405–418.

[29] H. Rodrigues, J. Ruas-Filho, Evolution equations: Dichotomies and the Fredholm alternative for bounded solutions, J. Differential Equations 119 (1995) 263–283.

[30] H. Rodrigues, M. Silveira, Properties of bounded solutions of linear and nonlinear evolution equations: Homoclinics of a beam equation, J. Differential Equations 70 (1987) 403–440.

[31] R. Sacker, G. Sell, Dichotomies for linear evolutionary equations in Banach spaces, J. Differential Equations 113 (1994) 17–67.

[32] G. Sell, Y. You, Dynamics of Evolutionary Equations, Applied Math. Sci., vol. 143, Springer, 2002.

[33] W. Zeng, Transversality of homoclinic orbits and exponential dichotomies for parabolic equations, J. Math. Anal. Appl. 216 (1997) 466–480.

[34] W. Zhang, The Fredholm alternative and exponential dichotomies for parabolic equations, J. Math. Anal. Appl. 191 (1985) 180–201.