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# Does the Diagonal Bound Property Imply that $X$ Is a Hilbert Space?

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## INTRODUCTION

Recently the authors came upon a paper of J. L. Abreu and A. Alonso [1], where they show that if $x_1, ..., x_n$ are vectors in a Hilbert space, then $\inf_{\varepsilon(n)} \|\sum_{i=1}^n \varepsilon_i x_i\|^2 \leqslant \sum_{i=1}^n \|x_i\|^2 \leqslant \sup_{\varepsilon(n)} \|\sum_{i=1}^n \varepsilon_i x_i\|^2$ where the inf and the sup are taken over all the possible sequences $\varepsilon(n) = \{\varepsilon_1, ..., \varepsilon_n\}$ such that $\varepsilon_i = \pm 1$ for $i = 1, ..., n$.

They ask the following question: Is the converse of this true? That is, if there exists $K > 0$ such that for all $x_1, ..., x_n$ in a Banach space $X$,

$$\frac{1}{K} \inf_{\varepsilon(n)} \left\| \sum_{i=1}^n \varepsilon_i x_i \right\|^2 \leqslant \sum_{i=1}^n \|x_i\|^2 \leqslant K \sup_{\varepsilon(n)} \left\| \sum_{i=1}^n \varepsilon_i x_i \right\|^2,$$

is $X$ isomorphic to a Hilbert space?

We show here that if $K = 1$ then $X$ is isometric to a Hilbert space, if $1 \leqslant K < 2$ then the space is reflexive, and in general that $X$ is of type $2 - \varepsilon$ and of cotype $2 + \varepsilon$ for every $\varepsilon > 0$. Finally, if $X$ is a Banach lattice then $X$ is isomorphic to an inner product space independently of the value of $K$.

We want to remark that all the terminology and results concerning superproperties, finite representability, type and cotype can be found in [2, 6], those concerning Banach lattices in [5].

We wish to thank A. Alonso for his helpful comments.

First of all we need the following definition:

DEFINITION. We will say that a Banach space $X$ has the diagonal bound property (DBP) for the constant $K$, if for all $x_1, ..., x_n \in X$

$$\frac{1}{K} \inf_{\varepsilon(n)} \left\| \sum_{i=1}^n \varepsilon_i x_i \right\|^2 \leqslant \sum_{i=1}^n \|x_i\|^2 \leqslant K \sup_{\varepsilon(n)} \left\| \sum_{i=1}^n \varepsilon_i x_i \right\|^2,$$



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where the inf and the sup are taken over all the sequences $\varepsilon(n) = \{\varepsilon_1, ..., \varepsilon_n\}$ where $\varepsilon_i = \pm 1$, $i = 1, ..., n$.

## 1. The Cases $K = 1$ and $1 < K < 2$

THEOREM 1. *A real or complex Banach space $X$ is isometrically isomorphic to an inner product space if and only if* DBP *holds for* $K = 1$.

*Proof.* The proof that DBP holds for a Hilbert space for $K = 1$ can be found in [1]. Suppose conversely that for every $x_1, ..., x_n$ in $X$,

$$\inf_{\varepsilon(n)} \left\| \sum_{i=1}^{n} \varepsilon_i x_i \right\|^2 \leqslant \sum_{i=1}^{n} \|x_i\|^2 \leqslant \sup_{\varepsilon(n)} \left\| \sum_{i=1}^{n} \varepsilon_i x_i \right\|^2.$$

Then, for any two elements $x, y \in X$ with $\|x\| = \|y\| = 1$,

$$\begin{aligned} 4 &= \min\{\|(x+y)+(x-y)\|^2, \|(x+y)-(x-y)\|^2\} \\ &\leqslant \|x+y\|^2 + \|x-y\|^2 \\ &\leqslant \max\{\|(x+y)+(x-y)\|^2, \|(x+y)-(x-y)\|^2\} = 4 \end{aligned}$$

and this, by a result of M. Day [3] implies that $X$ is an inner product space.

THEOREM 2. *Suppose that $X$ has the* DBP *for* $1 \leqslant K < 2$. *Then $X$ is reflexive.*

*Proof.* Using the terminology and results in [2], we will prove that $X$ is uniformly nonsquare and hence reflexive. That is, we have to show that in this case there exists $\delta > 0$ such that for any $x, y \in X$ with

$$\|x\| = \|y\| = 1, \qquad \frac{\|x-y\|}{2} > 1 - \delta \text{ implies } \frac{\|x+y\|}{2} \leqslant 1 - \delta.$$

Let $\delta = 1 - \sqrt{K/2} > 0$ and suppose $\|x\| = \|y\| = 1$ and $\|x - y\|/2 > 1 - \delta$. Then

$$\begin{aligned} 4(1-\delta)^2 + \|x+y\|^2 &\leqslant \|x-y\|^2 + \|x+y\|^2 \\ &\leqslant K \max\{\|(x-y)+(x+y)\|^2, \|(x-y)+(x+y)\|^2\} = 4K. \end{aligned}$$

Hence $\|x + y\|^2 \leqslant 4(K - (1-\delta)^2) = 4(1-\delta)^2$ i.e.

$$\left\| \frac{x+y}{2} \right\| \leqslant 1 - \delta.$$

## 2. Consequences of the DBP

(i) It is easy to see that the diagonal bound property is inherited by subspaces.

(ii) The DBP is stable under isomorphisms; in fact, if $T: X \to Y$ is a Banach space isomorphism and $X$ has the DBP for $K$, then $Y$ has the DBP for $\|T\| \|T^{-1}\| K$.

(iii) The DBP is a superproperty and since $X^{**}$ is finitely representable in $X$, this implies that $X^{**}$ has DBP for the same $K$ as $X$.

(iv) Neither $l_\infty$ nor $c_0$ have the DBP for any $K$. This is obvious, for if $e_i = (0, ..., 1, 0, ...)$, then $\|\sum_{i=1}^n \varepsilon_i e_i\|^2 = 1$ for every $\varepsilon_i = \pm 1$ but $\sum_{i=1}^n \|e_i\|^2 = n$.

(v) From (i) and (iv) we get that neither $l_\infty$ nor $c_0$ can be isomorphic to a subspace of $X$ if $X$ has the DBP.

(vi) From (iii) and (iv) we get that neither $l_\infty$ nor $c_0$ are finitely representable in $X$, if $X$ has the DBP.

## 3. What Happens if $X$ Is a Banach Lattice

Theorem 3. *If $X$ is a Banach lattice and $X$ has the* DBP *for some $K \geq 1$ then any sequence $\{x_i\}_{i=1}^\infty$ of mutually disjoint elements of $X$ with $\|x_i\| = 1$, $i = 1, 2, ...$, is equivalent to the unit vector basis of $l_2$.*

*Proof.* We observe first, that if $x_1, ..., x_n$ are mutually disjoint elements in $X$, then $|\sum_{i=1}^n \varepsilon_i x_i| = \sum_{i=1}^n |x_i|$ for any sequence $\varepsilon(n) = \{\varepsilon_1, ..., \varepsilon_n\}$ with $\varepsilon_i = \pm 1$ $i = 1, ..., n$, and consequently $\|\sum_{i=1}^n \varepsilon_i x_i\| = \|\sum_{i=1}^n x_i\| = \|\sup_{1 \leq i \leq n} |x_i|\,\|$. Therefore, if $\{a_i\}_{i=1}^\infty$ is any sequence of scalars, for any $n$ we get

$$\begin{aligned} K^{-1} \left\| \sum_{i=1}^n a_i x_i \right\|^2 &= \inf_{\varepsilon(n)} K^{-1} \left\| \sum_{i=1}^n \varepsilon_i a_i x_i \right\|^2 \\ &\leq \sum_{i=1}^n \|a_i x_i\|^2 = \sum_{i=1}^n |a_i|^2 \\ &\leq K \sup_{\varepsilon(n)} \left\| \sum_{i=1}^n \varepsilon_i a_i x_i \right\|^2 = K \left\| \sum_{i=1}^n a_i x_i \right\|^2, \end{aligned}$$

where the inf and sup are taken over all sequences $\varepsilon(n) = \{\varepsilon_1, ..., \varepsilon_n\}$ with $\varepsilon_i = \pm 1$, $i = 1, ..., n$.

Theorem 4. *If $X$ is a Banach lattice and $X$ has the* DBP *for some $K \geq 1$ then $X$ is order continuous.*

*Proof.* Since by (2.v) $X$ does not contain a sequence of mutually disjoint elements equivalent to the unit vector basis of $c_0$, and $X$ does not contain a subspace isomorphic to $l_\infty$, we get that $X$ is $\sigma$-complete and $\sigma$-order continuous. Hence $X$ is order continuous.

THEOREM 5. *If $X$ is a Banach lattice and $X$ has the* DBP *for some $K>1$ then $X$ is isomorphic to $L_2(\mu)$ for some measure $\mu$. In particular $X$ is isomorphic to an inner product space.*

*Proof.* This follows directly from Theorems 4 and 5 and by a theorem found in [5].

The following result was proved in [1], but here we get it as a corollary.

COROLLARY 6. *If $X$ is a Banach space with an unconditional basis which has the* DBP *for some $K$, then $X$ is isomorphic to an inner product space.*

*Proof.* This follows readily from the fact that every Banach space with an unconditional basis can be viewed as a Banach lattice.

## 4. THE GENERAL CASE

DEFINITION 7. (a) Let $p$ be such that $1 \leqslant p \leqslant 2$. We say that a Banach space $X$ is of type $p$, if there exists a constant $C$ such that for any finite family $x_1, ..., x_n$ of points in $X$,

$$\left(2^{-n} \sum_{n=1}^{2^n} \left\| \sum_{\varepsilon(n)} \varepsilon_i x_i \right\|^p \right)^{1/p} \leqslant C \left( \sum_{i=1}^{n} \|x_i\|^p \right)^{1/p},$$

where $\varepsilon(n)$ ranges over all sequences

$$\varepsilon(n) = \{\varepsilon_1, ..., \varepsilon_n\} \qquad \text{with} \quad \varepsilon_i = \pm 1,\ i = 1, ..., n.$$

Similarly

(b) If $2 \leqslant q$ we say that a Banach space $X$ is of cotype $q$ if there is a constant $D$ such that for all $n \geqslant 1$ and all points $x_1, ..., x_n$ we have

$$\left( \sum_{i=1}^{n} \|x_i\|^q \right)^{1/q} \leqslant D \left(2^{-n} \sum_{n=1}^{2^n} \left\| \sum_{\varepsilon(n)} \varepsilon_i x_i \right\|^q \right)^{1/q}$$

where $\varepsilon(n)$ is as above.

It is not difficult to show that a Hilbert space is of type 2 and cotype 2. What is more surprising is the fact that a Banach space which is of type 2 and cotype 2 is isomorphic to a Hilbert space. This result is due to Kwapien [4].

THEOREM 8. *If $X$ is a Banach space with the* DBP *for some $K$, then $X$ is of type $2-\varepsilon$ and of cotype $2+\varepsilon$ for every $\varepsilon>0$.*

*Proof.* Pisier [6] showed that if $p(X)=\sup\{r: X \text{ is of type } r\}$ and $q(X)=\inf\{q: X \text{ is of cotype } q\}$, then $p(X)=\inf\{r: l_r \text{ is finitely representable in } X\}$ and $q(X)=\sup\{s: l_s \text{ is finitely representable in } X\}$. From Corollary 6 and the fact that for $1\leqslant r<\infty$, $l_r$ has an unconditional basis, it follows that $l_r$ does not have the DBP if $r\neq 2$. Also from (2.v) we know that $l_\infty$ does not have the DBP.

Since the DBP is a superproperty, (2.iii), this means that $l_r$ is not finitely representable in $X$ for $r\neq 2$. Hence $p(X)=q(X)=2$, which proves the theorem.

Observe that there are spaces $X$ such that $X$ has type $2-\varepsilon$ and cotype $2+\varepsilon$ for every $\varepsilon>0$ which are not isomorphic to a Hilbert space. In [5], for instance, it is shown that there exists a sequence of integers $\{k_n\}_{n=1}^{\infty}$ and a sequence of numbers $\{p_n\}$, with $p_n\to 2$ so that $X=(\sum_{n=1}^{\infty}\oplus\, l_{p_n}^{k_n})$ is not isomorphic to $l_2$ but $X$ is of type $2-\varepsilon$ and cotype $2+\varepsilon$ for every $\varepsilon>0$. Note, however, that this space does not have the DBP for any $K$.

As we mentioned in (2.iii), if $X$ has the DBP then $X^{**}$ also has the DBP, and hence $X^{**}$ also is of type $2-\varepsilon$ and cotype $2+\varepsilon$ for every $\varepsilon>0$.

It is also true that in this case $X^*$ is of type $2-\varepsilon$ and cotype $2+\varepsilon$ for every $\varepsilon>0$; Pisier [7].

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