Journal of Mathematical Analysis and Applications **247,** 183–197 (2000)
doi:10.1006/jmaa.2000.6847, available online at http://www.idealibrary.com on IDEAL®

Blow-up for the Porous Media Equation with Source Term and Positive Initial Energy[1]

Enzo Vitillaro

Dipartimento di Matematica e Informatica, Università di Perugia, Via Vanvitelli, 1, 06123 Perugia, Italy
E-mail: enzo@unipg.it

Submitted by William F. Ames

Received March 3, 2000

DEDICATED TO THE MEMORY OF MY FATHER

We study the Cauchy–Dirichlet problem for the porous media equation with nonlinear source term in a bounded subset of $\mathbb{R}^n$. The problem describes the propagation of thermal perturbations in a medium with a nonlinear heat-conduction coefficient and a heat source depending on the temperature. The aim of the paper is to extend the unstable set to a part of the positive energy region, a phenomenon which was known only for linear conduction. © 2000 Academic Press

1. INTRODUCTION AND MAIN RESULT

We consider the Cauchy–Dirichlet problem for the porous media equation with source term

$$\begin{cases} u_t = \Delta(|u|^{m-1}u) + |u|^{p-2}u, & \text{in } [0,\infty) \times \Omega, \\ u = 0 & \text{in } [0,\infty) \times \partial\Omega, \\ u(0) = u_0, \end{cases} \tag{1}$$

where Ω is a bounded and smooth subset of $\mathbb{R}^n$, $n \geq 1$, $m > 0$, and $p \geq 2$. Problem (1) (see [9, 21]) describes the propagation of thermal perturbations in a medium with a nonlinear heat-conduction coefficient and a heat

[1] Work done in the framework of the M.U.R.S.T. project "Metodi variazionali ed equazioni differenziali nonlineari."

0022-247X/00 $35.00

source depending on the temperature when $u_0 \geq 0$. Local existence for the solutions of (1) has been proved when $m > 1$ (the so-called *slow diffusion case*) in [9, 14, 16] (see also the recent book [21]) and, when $0 < m < 1$ (the *fast diffusion case*) in [6]. More precisely, in the slow diffusion case, if $|u_0|^{m-1}u_0 \in H_0^1(\Omega)$ and

$$p < 1 + m + \frac{2(m+1)}{n}, \tag{2}$$

local existence of a solution u such that

$$|u|^{m-1}u \in L^\infty\big(0,T; H_0^1(\Omega)\big), \qquad |u|^{(m-1)/2}u \in L^\infty\big(0,T; L^2(\Omega)\big),$$
$$\big(|u|^{(m-1)/2}u\big)_t \in L^2\big((0,T)\times\Omega\big), \tag{3}$$

has been proved in [9, 16]. If $u_0 \in L^\infty(\Omega)$ (see [14]), local existence of a solution u such that

$$\begin{aligned} u \in L^\infty\big((0,T)\times\Omega\big), \qquad |u|^{m-1}u \in L^2\big(0,T; H_0^1(\Omega)\big),\\ \big(|u|^{(m-1)/2}u\big)_t \in L^2\big((0,T)\times\Omega\big), \end{aligned} \tag{4}$$

is known, without adding restrictions from above on p. In the fast diffusion case (see [6]) local existence of a weak solution, when solely $u_0 \in L^\infty(\Omega)$, and of a strong solution, when also $|u_0|^{m-1}u_0 \in H_0^1(\Omega)$, is proved in the class of functions u such that

$$|u|^{m-1}u \in C\big([0,T]; L^2(\Omega)\big) \cap L^\infty\big(0,T; H_0^1(\Omega)\big) \cap L^\infty\big((0,T)\times\Omega\big),$$
$$|u|^{(m-1)/2}u \in H^1\big(0,T; L^2(\Omega)\big). \tag{5}$$

Global existence has been proved in the quoted papers when $p \leq \max\{2, m+1\}$ (see [21] for a more precise statement in the delicate case $p = m+1$), while the blow-up of the solutions is proved (in the fast diffusion case blow-up is proved only for strong solutions), when

$$p > \max\{2, m+1\}, \tag{6}$$

$|u_0|^{m-1}u_0 \in H_0^1(\Omega)$, and the initial energy

$$E(0) = \frac{1}{2}\big\|\nabla\big(|u_0|^{m-1}u_0\big)\big\|_2^2 - \frac{m}{m+p-1}\|u_0\|_{m+p-1}^{m+p-1}$$

is negative.

The same type of results holds for the heat equation with source, when $m = 1$. See, for example, [2, 7, 8, 11, 24]. However, other results are known for the heat equation when $2 < p \leq 2n/(n-2)$ (the last condition being necessary only when $n \geq 3$) and $u_0 \in H_0^1(\Omega)$. To describe them let us note that

$$E(0) \geq \frac{1}{2} B_1^{-2} \|u_0\|_p^2 - \frac{1}{p} \|u_0\|_p^p := g(\|u\|_p),$$

where B_1 is the optimal constant of Sobolev embedding $H_0^1(\Omega) \hookrightarrow L^p(\Omega)$. Then the behavior of the solution u is known when $(\|u_0\|_p, E(0)$, lies in the regions of A, B, and C of the plane characterized by (see Fig. 1)

$$A = \{(\lambda, E) \in [0, \infty) \times \mathbb{R} : g(\lambda) \leq E < E_1, \lambda < \lambda_1\},$$
$$B = \{(\lambda, E) \in [0, \infty) \times \mathbb{R} : \max\{g(\lambda), 0\} \leq E < E_1, \lambda > \lambda_1\},$$
$$C = \{(\lambda, E) \in [0, \infty) \times \mathbb{R} : g(\lambda) \leq E < 0\},$$

where λ_1 is the absolute maximum point of g and $E_1 = g(\lambda_1) > 0$. In particular, if $(\|u_0\|_p, E(0)) \in A$ the solution is global (see [24]), while if $(\|u_0\|_p, E(0)) \in B \cup C$ blow-up in finite time occurs (see [10] and the more recent paper [18] when $(\|u_0\|_p, E(0)) \in B$). Actually E_1 in the literature quoted has the variational characterization

$$E_1 = \inf_{u \in H_0^1(\Omega), u \neq 0} \sup_{\lambda > 0} J(\lambda u),$$

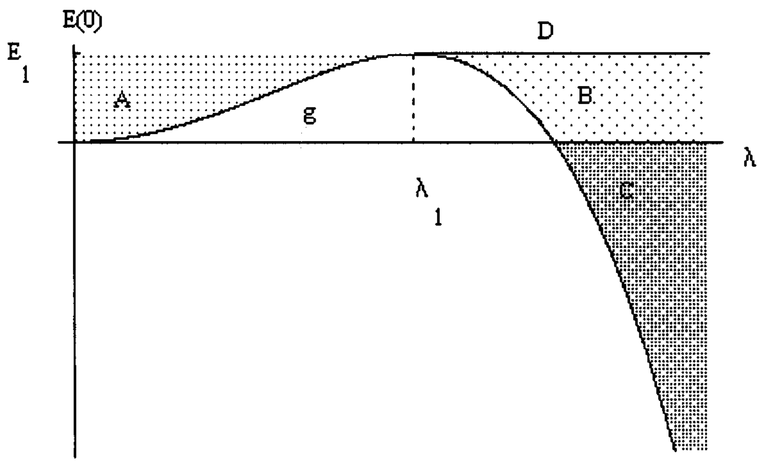

FIG. 1. The four regions A, B, C, and D in the plane $(\lambda, E(0))$, where $\lambda = \|u_0\|_p$.

where

$$J(u) = \frac{1}{2}\|\nabla u\|_2^2 - \frac{1}{p}\|u\|_p^p.$$

Simple calculations show that the two characterizations of E_1 agree and that E_1 is also the Mountain Pass level of the functional J on $H_0^1(\Omega)$ (see Section 4 below).

The global existence result for initial data in the region A has been extended in [9] to the slow diffusion case, while there are not extensions, in author's knowledge, of the blow-up theorem for initial data in the region B. The aim of this paper is to solve this problem. The method used in the proof is inspired by the arguments of [18], where the classical convexity method is adapted to handle with positive initial energy for abstract evolution equations of hyperbolic type. The main idea in [12], in which (1) is treated by the change of variable $v = |u|^{m-1}u$, cannot be extended here to the important slow diffusion case, due to the singularity that appears in the transformed equation. However, we adapt the method of [18] using the change of variable in a somewhat implicit way.

Moreover, with a simple argument, we extend the result to the region

$$D := \{(\lambda, E) \in [0,\infty) \times \mathbb{R}^2 : \lambda > \lambda_1, E = E_1\}.$$

In order to have an unified proof for the slow and fast diffusion cases, and to minimize the assumptions on p in the first one, when also $u_0 \in L^\infty(\Omega)$, we consider distributional solutions of (1) satisfying a regularity condition weaker than (3), (4), or (5), but strong enough to use the convexity method. Then we prove a global nonexistence result (see Theorem 3.1 below) for such type of solutions of (1). In this way we include in our study the solution founded in [9, 14, 16], and the strong solutions founded in [6], while weak solutions obtained in [6] are "too weak" to prove global nonexistence (also for the quoted author!).

This global nonexistence result can be conveniently applied to the local solution founded in [9, 14, 16], and the strong solutions founded in [6], to obtain the following blow-up result.

THEOREM 1.1. *Let u be a solution of* (1) *whose existence is proved in one of the papers quoted above. Assume that* (6) *holds, that*

$$p < 1 + m\frac{n+2}{n-2} \qquad (\text{when } n \geq 3), \tag{7}$$

and that $|u_0|^{m-1}u_0 \in H_0^1(\Omega), (\|u_0\|_p, E(0)) \in B \cup C \cup D$. *Then*

(i) *if* (2) *holds and* $m \geq 1$, *there is* $T_0 > 0$ *such that* $\|u(t)\|_{m+1} \to \infty$ *as* $t \to T_0^-$;

(ii) *if* $u_0 \in L^\infty(\Omega)$, *then there is* $T_1 > 0$ *such that* $\|u(t)\|_\infty \to \infty$ *as* $t \to T_1^-$.

The two different cases (i) and (ii) arise from local existence results available in the literature and quoted above.

The proof of the nonexistence result is given for a type of equations more general than (1); see (8) below, which generalizes (1) in several directions. First of all it contains the doubly nonlinear operator $\Delta_s(|u|^{m-1}u) = \operatorname{div}(|\nabla(|u|^{m-1}u)^{s-2}\nabla(|u|^{m-1}u)|)$, which appears in important equations modeling turbulent filtration (see [4] and the references therein; see also [17, 20]). Moreover, the evolution operator could be also time-dependent.

2. PRELIMINARIES

We consider the problem

$$\begin{cases} \rho(t)u_t = \Delta_s(|u|^{m-1}u) + f(x,u), & \text{in } [0,\infty) \times \Omega, \\ u = 0 & \text{in } [0,\infty) \times \partial\Omega, \\ u(0) = u_0, & \end{cases} \tag{8}$$

where Ω is a bounded and smooth subset of $\mathbb{R}^n$ as before, $m > 0$, $s > 1$, $\rho \in W^{1,\infty}_{\text{loc}}[0,\infty)$ is nonnegative and decreasing, and $|u_0|^{m-1}u_0 \in W^{1,s}_0(\Omega)$. We assume f to be a Caratheodory real function on $\Omega \times \mathbb{R}$ such that

(F1) there is $p > 1$ and $c_1 > 0$ such that for all $(x,u) \in \Omega \times \mathbb{R}$

$$|f(x,u)| \le c_1|u|^{p-1};$$

(F2) for all $(x,u) \in \Omega \times \mathbb{R}$

$$F_u(x,u)u \ge (m+p-1)F(x,u),$$

where

$$F(x,u) := \int_0^u m|\tau|^{m-1} f(x,\tau)\,d\tau.$$

Moreover, denoting by r the Sobolev critical exponent of the embedding $W^{1,s}_0(\Omega) \hookrightarrow L^q(\Omega)$, i.e., $r = ns/(n-s)$ if $n > s$, r arbitrarily large if $n = s$, $r = \infty$ if $n < s$, we shall suppose that

$$\max\{2, 1 + (s-1)m\} < p < 1 + m(r-1). \tag{9}$$

We say that u, defined on a suitable cylinder $Q_T := (0,T)\times\Omega$, is a solution of (8) if

$$|u|^{m-1}u \in L^s\big(0,T;W_0^{1,s}(\Omega)\big)\cap L^\infty(0,T;L^r(\Omega)), \tag{10}$$

$$|u|^{(m-1)/2}u \in H^1\big(0,T;L^2(\Omega)\big), \tag{11}$$

and u satisfies (8) in the following sense:

(a) u is a distributional solution, i.e.,

$$\int_\Omega \rho(\tau)u(\tau)\phi(\tau,x)\Big|_{\tau=0}^{\tau=t} = \int_0^t\int_\Omega \rho' u\phi + \rho u\phi_t + f(x,u)\phi - \Big|\nabla\big(|u|^{m-1}u\big)\Big|^{s-2}\nabla\big(|u|^{m-1}u\big)\nabla\phi \tag{12}$$

for a.a. $t\in[0,T]$ and $\phi\in C_c^\infty([0,\infty)\times\Omega)$, and u satisfies the initial condition

$$u(0)=u_0; \tag{13}$$

(b) u verifies the energy identity in the weak form of the inequality

$$E(t)\le E(0) - \frac{4m}{(m+1)^2}\int_0^t\int_\Omega \rho\Big|\big(|u|^{(m-1)/2}u\big)_t\Big|^2 \tag{14}$$

for a.a. $t\in[0,T]$, where the energy function E naturally associated to u is given by

$$E(t) = \frac{1}{s}\Big\|\nabla\big(|u(t)|^{m-1}u(t)\big)\Big\|_s^s - \int_\Omega F(x,u(t)). \tag{15}$$

Remark 2.1. It is easy to see, under the regularity assumptions on ρ and u, that all the terms in (12)–(15) make sense. Indeed, by (11), it follows immediately that

$$u\in C\big([0,T];L^{m+1}(\Omega)\big). \tag{16}$$

Then, using the regularity of ρ, the left hand side and the first two terms in the right hand side of (12) make sense, and by (F1), (9), and (10) also the remaining terms are well defined. By (16), u has a pointwise meaning on time, so also (13) can be written. The energy function E has an a.e. meaning because of (F1), (9), (10), and the Sobolev embedding (see also Lemma 2.1 below). Furthermore $E(0)$ is well definite because of (16) and the regularity of u_0. Finally we can write the integral in (14) because of (11).

Remark 2.2. It is worth observing that, in the case $\rho \equiv 1$, $s = 2$ and $f(x, u) = |u|^{p-2}u$; namely for Eq. (1), conditions (10)–(11) are a direct consequence of each one among (3), (4), or (5). Moreover, the solutions founded in all the papers quoted in the Introduction, with the mentioned exception of the weak solution introduced in [6], satisfy the energy relation (14) as an equality. However, the inequality (14) is enough for our purposes.

We start with a lemma, which is an easy consequence of the Sobolev embedding, Hölder inequality, and (9), stated in this form (for $s = 2$) in [9]

LEMMA 2.1. *There are positive constants B_0 and B_1 such that*

$$\|v\|_{m+p-1}^{m} \le B_0 \| \, |v|^{m-1} v\|_r \le B_1 \left\| \nabla\left(|v|^{m-1} v\right) \right\|_s$$

for $v \in W_0^{1,s}(\Omega)$, where the first inequality holds also for $v \in L^r(\Omega)$.

In what follows we set $q = (m + p - 1)/m$ and choose B_1 as the optimal constant of the Sobolev embedding $W_0^{1,s}(\Omega) \hookrightarrow L^q(\Omega)$. We shall use the well known interpolation inequality (see [3]),

$$\|v\|_\beta \le \|v\|_\alpha^\theta \|v\|_\gamma^{1-\theta}, \qquad v \in L^\alpha \cap L^\beta, \tag{17}$$

where $1 \le \alpha \le \beta \le \gamma$ and

$$\frac{1}{\beta} = \frac{\theta}{\alpha} + \frac{1-\theta}{\gamma}.$$

Note, for future reference, that any solution u of (8) satisfies also the further regularity

$$u \in C\left([0, T]; L^{m+p-1}(\Omega)\right). \tag{18}$$

Indeed, by (10), $u \in L^\infty(0, T; L^{mr}(\Omega))$. Moreover, as $m + 1 < m + p - 1 < mr$, using (17), for $\tau, t \in [0, T]$,

$$\|u(\tau) - u(t)\|_{m+p-1} \le \|u(\tau) - u(t)\|_{m+1}^\theta \|u(\tau) - u(t)\|_{mr}^{1-\theta},$$

where

$$\frac{1}{m+p-1} = \frac{\theta}{m+1} + \frac{1-\theta}{mr}.$$

Then (18) follows by (16).

The aim of the next lemma is to point out an approximation property which allows us to take $|u|^{m-1}u$ as a test function in (12).

LEMMA 2.2. *Let $k, l > 1$, and $T > 0$. Then $C_c^\infty([0,T] \times \Omega)$ is dense in $L^l(Q_T) \cap L^k(0,T; W_0^{1,k}(\Omega))$ and in $W^{1,l}(0,T; L^l(\Omega)) \cap L^k(0,T; W_0^{1,k}(\Omega))$, endowed with the natural norms.*

Proof. The proof is standard, based on convolution arguments (see [5, Theorem 1.22]), after reduction to compact support functions in Ω. This is done by applying the arguments of [23] to the evolution case (see also [25, Proof of Theorem 7, Step 1]. ∎

The next lemma points out an identity which is obvious for classical solutions of (8), but needs some care due to the distributional nature of the solutions we consider.

LEMMA 2.3. *Let u be a solution of* (8). *Then*

$$\frac{2}{m+1}\int_0^t\int_\Omega \rho(\tau)\left(|u|^{(m-1)/2}u\right)_t |u|^{(m-1)/2}u$$
$$= -\int_0^t\int_\Omega \left|\nabla\left(|u|^{m-1}u\right)\right|^s - f(x,u)|u|^{m-1}u \qquad \text{in } [0,T).$$

Proof. We distinguish two cases: $m \geq 1$ and $0 < m < 1$. In the first using the characterization of Sobolev functions and the chain rule in Sobolev spaces due to J. Serrin (see [15, Lemmas 1.5 and 2.1]), it is easy to see that

$$\left(|u|^{m-1}u\right)_t = \frac{2m}{m+1}|u|^{(m-1)/2}\left(|u|^{(m-1)/2}u\right)_t. \tag{19}$$

Moreover, by (11), (16), and the Hölder inequality, $(|u|^{m-1}u)_t$ is in $L^{1+1/m}(Q_T)$. Then $|u|^{m-1}u \in W^{1,1+1/m}(0,T; L^{1+1/m}(\Omega))$ again by (16). By (10) we can apply Lemma 2.2 with $k = s$ and $l = 1 + 1/m$, so there is a sequence $(\phi_\varepsilon)_\varepsilon$ in $C_c^\infty([0,T] \times \Omega)$ such that

$$\phi_\varepsilon \to |u|^{m-1}u \qquad \text{in } W^{1,1+1/m}\left(0,T; L^{1+1/m}(\Omega)\right) \cap L^s\left(0,T; W_0^{1,s}(\Omega)\right).$$

Passing to the limit in (12) with $\phi = \phi_\varepsilon$ and using the regularity properties of u and ρ in addition to (F1) and (9), we obtain

$$\int_\Omega \rho(\tau)|u(\tau)|^{m+1}\Big|_{\tau=0}^{\tau=t} = \int_0^t\int_\Omega \rho'|u|^{m+1} + \rho u\left(|u|^{m-1}u\right)_t$$
$$-\left|\nabla\left(|u|^{m-1}u\right)\right|^s + f(x,u)|u|^{m-1}u \tag{20}$$

and then, by (11) and (19), we conclude the proof when $m \geq 1$.

In the second case $0 < m < 1$ we cannot directly put $\phi = |u|^{m-1}u$ in (12), because of the singular term appearing in (19). On the other hand, by the same arguments used before,

$$u_t = \frac{2}{m+1}|u|^{(1-m)/2}\left(|u^{(m-1)/2}u\right)_t. \tag{21}$$

Again by (11), (16), and the Hölder inequality, u_t is in $L^{m+1}(Q_T)$. Then again by (16),

$$u \in W^{1,m+1}\big(0,T; L^{m+1}(\Omega)\big). \tag{22}$$

Moreover, by Leibnitz' formula in Sobolev spaces,

$$\frac{d}{dt}\int_\Omega \rho u\phi = \int_\Omega \rho' u\phi + \rho u_t \phi + \rho u \phi_t$$

for all $\phi \in C_c^\infty([0,T] \times \Omega)$; hence (12) can be equivalently written as

$$\int_0^t\int_\Omega - \rho u_t \phi - \left|\nabla\left(|u|^{m-1}u\right)\right|^{s-2}\nabla\left(|u|^{m-1}u\right)\nabla\phi + f(x,u)\phi = 0. \tag{23}$$

By (10), (16), and applying Lemma 2.2, for every $\varepsilon > 0$ there is $\phi_\varepsilon \in C_c^\infty([0,T] \times \Omega)$ such that $\phi_\varepsilon \to |u|^{m-1}u$ in $L^{1+1/m}(Q_T) \cap L^s(0,T; W_0^{1,s}(\Omega))$. Passing to the limit in (23), with ϕ replaced by ϕ_ε, we conclude that

$$\int_0^t\int_\Omega - \rho|u|^{m-1}uu_t - \left|\nabla\left(|u|^{m-1}u\right)\right|^s + f(\cdot,u)|u|^{m-1}u = 0,$$

which, by (21), concludes the proof. ∎

3. THE GLOBAL NONEXISTENCE RESULT

We set

$$\lambda_1 = \left(c_1 B_1^s\right)^{-1/(q-s)}, \qquad E_1 = \left(\frac{1}{s} - \frac{1}{q}\right)\left(c_1 B_1^q\right)^{-s/(q-s)}.$$

Our main global nonexistence result is the following

THEOREM 3.1. *If* $\|u_0\|_{m+p-1}^m > \lambda_1$ *and* $E(0) \le E_1$, *then no global solutions of* (8) *can exist in the whole* $[0,\infty)$.

We now set

$$\lambda_0 = \|u_0\|_{m+p-1}^m, \qquad E_0 = E(0), \qquad \Gamma = \left\{(\lambda, E) \in \mathbb{R}^2 : \lambda > \lambda_1, E < E_1\right\}.$$

As the proof of Theorem 3.1 will be done by contradiction, in what follows we can suppose that T can be taken arbitrarily large. We start with

LEMMA 3.1. *Let* $(\lambda_0, E_0) \in \Gamma$. *Then*

(i) $E(t) \le E_0$ *for all* $t \ge 0$;

(ii) *there is* $\lambda_2 > \lambda_1$ *such that* $\|u(t)\|_{m+p-1}^m \ge \lambda_2$ *for all* $t \ge 0$;

(iii) *there is* $\lambda_3 > B_1^{-1}\lambda_1$ *such that* $\|\nabla(|u|^{m-1}u)\|_s \ge \lambda_3$.

Proof. It is sufficient to prove only (ii), as (i) immediately follows by (14) and (iii) is a consequence of (ii) and Lemma 2.1. By Lemma 2.1 and (F1)

$$E(t) \ge \frac{1}{sB_1^s}\|u(t)\|_{m+p-1}^{ms} - \frac{c_1}{q}\|u(t)\|_{m+p-1}^{m+p-1} = g\big(\|u(t)\|_{m+p-1}^m\big), \quad (24)$$

where $g(\lambda) := 1/(sB_1^s)\lambda^s - (c_1/q)\lambda^q$, $\lambda \ge 0$, and $q = (m+p-1)/m$. It is easy to see that g takes its maximum for $\lambda = \lambda_1$, with $g(\lambda_1) = E_1$, that g is strictly decreasing for $\lambda \ge \lambda_1$, and that $g(\lambda) \to -\infty$ as $\lambda \to \infty$. Consequently, as $E_0 < E_1$, there is $\lambda_2 > \lambda_1$ such that $g(\lambda_2) = E_0$. Since $g(\lambda_0) \le E_0 = g(\lambda_2)$ by (24), it follows that $\lambda_2 \le \lambda_0$. Now suppose by contradiction that $\|u(t_0)\|_{m+p-1}^m < \lambda_2$ for some $t_0 \ge 0$. By (18) we can suppose that $\lambda_1 < \|u(t_0)\|_{m+p-1}^m$. Hence $E(t_0) \ge g(\|u(t_0)\|_{m+p-1}^m) > g(\lambda_2) = E_0$ by (24), which contradicts (i). ∎

We are now ready to give the

Proof. (Proof of Theorem 3.1). When $E_0 < E_1$ we can apply Lemmas 2.1–3.1 and adapt the arguments of [18] to our situation. More precisely, for any solution of (8) in $[0,\infty)$ we define

$$\mathscr{I}(t) = \int_0^t\int_\Omega \rho|u|^{m+1} + \int_0^t\int_\Omega (\tau - t)\rho'|u|^{m+1} + (T_0 - t)\int_\Omega \rho(0)|u_0|^{m+1} + \beta(t+t_0)^2, \quad (25)$$

where t_0, T_0, and β are positive constants, which will be fixed later (see [11, 13]). Then, using (11),

$$\mathscr{I}'(t) = \int_\Omega \rho(t)|u(t)|^{m+1} - \int_0^t\int_\Omega \rho'(\tau)|u(\tau)|^{m+1} - \int_\Omega \rho(0)|u_0|^{m+1} + 2\beta(t+t_0)$$
$$= 2\int_0^t\int_\Omega \rho(\tau)\big(|u|^{(m-1)/2}u\big)_t(\tau)|u(\tau)|^{(m-1)/2}u(\tau) + 2\beta(t+t_0). \quad (26)$$

Using Lemma 2.3, we get

$$\frac{1}{2}\mathscr{I}'' = \frac{m+1}{2}\left[-\left\|\nabla(|u|^{m-1}u)\right\|_s^s + \int_\Omega f(x,u)|u|^{m-1}u\right] + \beta. \quad (27)$$

Next, using (15) and (F2),

$$\begin{aligned}\frac{1}{2}\mathscr{I}'' &= \frac{m+1}{2}\left[\left(\frac{q}{s}-1\right)\left\|\nabla(|u|^{m-1}u)\right\|_s^s + \int_\Omega f(x,u)|u|^{m-1}u \right.\\ &\qquad \left. -q\int_\Omega F(x,u) - qE\right] + \beta\\ &\geq \frac{m+1}{2}\left[\left(\frac{q}{s}-1\right)\left\|\nabla(|u|^{m-1}u)\right\|_s^s - qE\right] + \beta. \quad (28)\end{aligned}$$

By Lemma 3.1 and (14), using the explicit values of λ_1 and E_1,

$$\begin{aligned}\frac{1}{2}\mathscr{I}'' &\geq \frac{m+1}{2}\left[\left(\frac{q}{s}-1\right)(B_1^q c_1)^{-s/(q-s)} - qE_0\right.\\ &\qquad \left. + \frac{4mq}{(m+1)^2}\int_0^t\int_\Omega \rho\left|\left(|u|^{(m-1)/2}u\right)_t\right|^2\right] + \beta\\ &= \frac{(m+1)q}{2}\left[E_1 - E_0 + \frac{4m}{(m+1)^2}\int_0^t\int_\Omega \rho(\tau)\left|\left(|u|^{(m-1)/2}u\right)_t\right|^2\right] + \beta.\end{aligned}$$

Now choose $\beta = (m+1)^2(E_1 - E_0)/2m > 0$ since $E_1 > E_0$. Then

$$\mathscr{I}'' \geq 2\left(\frac{mq}{m+1} + 1\right)\beta + \frac{4mq}{m+1}\int_0^t\int_\Omega \rho(\tau)\left|\left(|u|^{(m-1)/2}u\right)_t\right|^2. \quad (29)$$

Clearly $\mathscr{I}'(0) = 2\beta t_0 > 0$, $\mathscr{I}(0) = T_0\int_\Omega \rho(0)|u_0|^{m+1} + \beta t_0^2 > 0$. Moreover, $\mathscr{I}'' > 0$ by (29), so $\mathscr{I}'$ and $\mathscr{I}$ are both positive. We claim that

$$\mathscr{I}\mathscr{I}'' - \alpha(\mathscr{I}')^2 \geq 0 \qquad \text{on } [0, T_0], \quad (30)$$

where $\alpha = [1 + mq/(m+1)]/2$. Indeed, let

$$\mathbb{A} = \int_0^t\int_\Omega \rho|u|^{m+1} + \beta(t+t_0)^2, \quad (31)$$

$$\mathbb{C} = \int_0^t\int_\Omega \rho\left|\left(|u|^{(m-1)/2}u\right)_t\right|^2 + \beta, \quad (32)$$

and $\mathbb{B} = \mathscr{I}'/2$. By (25), being $\rho' \leq 0$,

$$\mathbb{A} \leq \mathscr{I} \qquad \text{on } [0, T_0]. \tag{33}$$

Moreover, by (29), using the fact that $p > 2$,

$$\mathscr{I}'' \geq 2[mq/(m+1) + 1]\mathbb{C}. \tag{34}$$

Now for all $(\xi, \tau) \in \mathbb{R}^2$ and $t \geq 0$, using (26), (31), and (32), we get

$$\mathbb{A}\xi^2 + 2\mathbb{B}\xi\tau + \mathbb{C}\tau^2$$
$$= \int_0^t \int_\Omega \rho(\tau)\left[(|u|^{(m-1)/2}u)_t + |u|^{(m-1)/2}u\right]^2 + \beta\left[(t - t_0)\xi + \tau\right]^2 \geq 0;$$

hence $\mathbb{A}\mathbb{C} - \mathbb{B}^2 \geq 0$. Then the claim (30) follows by (33) and (34). Reproducing verbatim the proof of [18, Theorem 1(i)] we conclude the proof when $E_0 < E_1$.

Consider now the case $E_0 = E_1$. By Lemma 3.1(i), $E(t) \leq E_1$ on $[0, \infty)$. Then, since $\lambda_0 > \lambda_1$, by the continuity of $\|u(t)\|_{m+p-1}$, only two possibilities can occur:

(a) there is $t_0 \geq 0$ such that $E(t_0) < E_1$ and $\|u(t_0)\|_{m+p-1}^m > \lambda_1$;

(b) there is $\varepsilon_0 >$ such that $E(t) = E_1$ for all $t \in [0, \varepsilon_0)$.

In the first case, shifting the time origin, we can apply the previous case and conclude the proof. In the latter we claim that

$$-\left\|\nabla(|u|^{m-1}u)\right\|_s^s + \int_\Omega f(x,u)|u|^{m-1}u = 0 \tag{35}$$

a.e. on a sufficiently small interval $[0, \varepsilon_2)$. To prove (35) we consider two subcases: either $\rho(0) > 0$ or $\rho(0) = 0$. In the first ρ is positive on some small interval $[0, \varepsilon_1)$, then $(|u|^{(m-1)/2}u)_t = 0$ a.e. on $[0, \varepsilon_2)$ by (14), where $\varepsilon_2 = \min\{\varepsilon_0, \varepsilon_1\}$. Hence, by (20) (when $m \geq 1$) or (23) (when $0 < m < 1$), we obtain (35). If $\rho(0) = 0$ then, as ρ is decreasing and nonnegative, $\rho \equiv 0$, so (12) becomes

$$\int_0^t \int_\Omega -\left|\nabla(|u|^{m-1}u)\right|^{s-2} \nabla(|u|^{m-1}u)\nabla\phi + f(x,u)\phi = 0 \qquad \text{for a.a. } t \geq 0.$$

Now, taking $\phi(t,x) = \phi_0(x)\phi_1(t)$, with $\phi_0 \in C_c^\infty(\Omega)$, yields that, for a.a. $t \in [0, \infty)$,

$$\int_\Omega -\left|\nabla(|u|^{m-1}u)\right|^{s-2} \nabla(|u|^{m-1}u)\nabla\phi_0 + f(x,u)\phi_0 = 0.$$

Hence (35) follows. By the form of the energy function (15) and (F2),

$$E_1 \geq \frac{1}{s}\left\| \nabla\left(|u|^{m-1}u\right)\right\|_s^s - \frac{m}{m+p-1}\int_\Omega f(x,u)|u|^{m-1}u.$$

Therefore, by (35),

$$E_1 \geq \left(\frac{1}{s} - \frac{1}{q}\right)\left\| \nabla\left(|u|^{m-1}u\right)\right\|_s^s,$$

so, by Lemma 3.1(iii),

$$E_1 > \left(\frac{1}{s} - \frac{1}{q}\right) B_1^{-s}\lambda_1^s,$$

which contradicts the definition of E_1. ∎

4. PROOF OF THEOREM 1.1 AND FINAL REMARKS

Proof. (Proof of Theorem 1.1). Blow-up follows from Theorem 3.1, the local existence results of [6, 9, 14], together with a standard continuation procedure. Nevertheless, for the sake of clearness, we explain briefly how to complete the proof. Clearly, applying Theorem 3.1 with $\rho \equiv 1$, $s = 2$, and $f(x,u) = |u|^{p-2}u$, there are no global solutions of (1) on $[0,\infty)$; hence there are no global solutions of the types considered in [6, 9, 14]. Hence

$$T_{\max} = \sup\{T > 0 : \text{there is a solution of } (1) \text{ on } [0,T)\} < \infty.$$

Now, in the case (i), considering local solution of [9] it is easy to see that the arguments of the author show the existence of the solution on an interval $[0,T)$, where $T = T(\|u_0\|_{m+1})$ is a decreasing function $\|u_0\|_{m+1}$. If there is a sequence $t_n \to T_{\max}$ such that $\|u(t_n)\|_{m+1}$ is bounded, as the equation is autonomous, there is a corresponding sequence of intervals $[t_n, t_n + T_1)$, where $T_1 = \inf_n T(\|u(t_n)\|_{m+1}) \geq T(\sup_n \|u(t_n)\|_{m+1}) > 0$. This contradicts the definition of $T_{\max}$, so $\|u(t)\|_{m+1} \to \infty$ as $t \to T_{\max}^-$.

In the case (ii), we consider solutions of the type given in [6, 14], and we conclude the proof arguing as in [6, Proof of Theorem 3.1; 14, Corollary 4.1]. ∎

Final Remarks. In the case of Eq. (1) clearly

$$\lambda_1 = B_1^{-2/(q-2)}, \qquad E_1 = \left(\frac{1}{2} - \frac{1}{q}\right) B_1^{-2q/(q-2)}.$$

The energy level E_1 can be also characterized in a variational form, i.e., as

$$d_1 = \inf_{u \in H_0^1(\Omega)\setminus\{0\}} \sup_{\lambda > 0} J(\lambda u), \tag{36}$$

where

$$J(u) = \frac{1}{2}\|\nabla u\|_2^2 - \frac{1}{q}\|u\|_q^q. \tag{37}$$

Indeed, for $u \neq 0$, an easy calculation shows that

$$\sup_{\lambda > 0} J(\lambda u) = \left(\frac{1}{2} - \frac{1}{q}\right)\left(\frac{\|\nabla u\|_2}{\|u\|_q}\right)^{2q/(q-2)};$$

hence $d_1 = E_1$, since B_1 is the optimal constant in the Sobolev embedding.

Moreover, as stated in the Introduction, E_1 can be characterized as the Mountain Pass level of the functional J on $H_0^1(\Omega)$, i.e., as

$$d_2 = \inf_{\gamma \in \Gamma_1} \sup_{t \in [0,1]} J(\gamma(t)),$$

where $\Gamma_1 = \{\gamma \in C([0,1]; H_0^1(\Omega)) : \gamma(0) = 0, J(\gamma(1)) < 0\}$.

Indeed, due to the form of J (see Lemma 3.1) it is easy to see that for any $u \in H_0^1(\Omega)$, $u \neq 0$, there is a unique $\lambda(u) > 0$ which is a critical point of $\lambda \mapsto J(\lambda u)$, and that $J(\lambda(u)u) = \max_{\lambda > 0} J(\lambda u)$. Then, if we define $\gamma_u(t) = Rtu$, with R so large that $J(Ru) < 0$, then $\gamma_u \in \Gamma_1$ and $J(\lambda(u)u) = \max_{t \in [0,1]} J(\gamma_u(t)) \geq d_2$. This proves that $d_1 \geq d_2$. Conversely, if u is a critical point of J on $H_0^1(\Omega)$ such that $J(u) = d_2$ (it is well known that it exists; see, for example, [1, 19, 22]), clearly $J(u) = \max_{\lambda > 0} J(\lambda u) \geq d_1$, so $d_2 \geq d_1$, proving our claim.

REFERENCES

1. A. Ambrosetti, "Critical Points and Nonlinear Variational Problems," Supplément au Bulletin de la Societé Mathématique de France, Mémoires, Vol. 120, 1992.
2. J. Ball, Remarks on blow-up and nonexistence theorems for nonlinear evolution equations, *Quart. J. Math. Oxford Ser. (2)* **28** (1977), 473–486.
3. H. Brezis, "Analyse Fonctionelle, Théorie et applications," Masson, Paris, 1983.
4. J. R. Esteban and J. L. Vazquez, Homogeneous diffusion in $\mathbb{R}$ with power-like nonlinear diffusivity, *Arch. Rational Mech. Anal.* **103** (1988), 39–88.
5. L. C. Evans and R. F. Gariepy, Measure theory and fine properties of functions, *in* "Studies in Advanced Mathematics," CRC Press, Boca Raton, FL, 1992.
6. J. Filo, On solutions of a perturbed fast diffusion equation, *Appl. Math.* **32**, No. 5 (1987), 364–380.
7. H. Fujita, On the blowing up of solutions of the Cauchy problem for $u_t = \Delta u + u^{1+\alpha}$, *J. Fac. Sci. Univ. Tokyo Sect. IA Math.* **13** (1966), 109–124.

8. H. Fujita, On some nonexistence and nonuniqueness theorems for nonlinear parabolic equations, *in* "Nonlinear Functional Analysis," Proc. Sympos. Pure Math., Vol. 18, pp. 105–124, Amer. Math. Soc., Providence, 1968.
9. V. A. Galaktionov, Boundary value problem for the nonlinear parabolic equation $u_t = \Delta u^{\sigma+1} + u^{\beta+1}$, *Differentsial'nye Uravneniya* **17**, No. 5 (1981), 836–842 [In Russian]; English translation, *Differential Equations* **17** (1981), 551–555.
10. H. Ishii, Asymptotic stability and blowing up of solutions of some nonlinear equations, *J. Differential Equations* **26** (1977), 291–319.
11. H. A. Levine, Some nonexistence and instability theorems for solutions of formally parabolic equations of the form $Pu_t = -Au + \mathscr{F}(u)$, *Arch. Rational Mech. Anal.* **51** (1973), 371–386.
12. H. A. Levine, S. R. Park, and J. Serrin, Global existence and nonexistence theorems for quasilinear evolution equations of formally parabolic type, *J. Differential Equations* **142** (1998), 212–229.
13. H. A. Levine, P. Pucci, and J. Serrin, Some remarks on global nonexistence for nonautonomous abstract evolution equations, *in* "Harmonic Analysis and Nonlinear Differential Equations," (M. L. Lapidus, L. H. Harper, and A. J. Rumbos, Eds.), Contemp. Math., Vol. 208, pp. 253–263, Amer. Math. Soc., Providence, 1997.
14. H. A. Levine and P. E. Sacks, Some existence and nonexistence theorems for solutions of degenerate parabolic equations, *J. Differential Equations* **52** (1984), 135–161.
15. M. Marcus and V. J. Mizel, Absolute continuity of tracks and mappings on Sobolev spaces, *Arch. Rational Mech. Anal.* **45** (1972), 294–320.
16. M. Nakao, Existence, nonexistence and some asymptotic behavior of global solutions of a nonlinear degenerate parabolic equation, *Math. Rep. Kyushu Univ.* **14** (1983), 1–21.
17. M. M. Porzio, L^∞_{loc}—estimates for a class of doubly nonlinear parabolic equation with source, *Rend. Mat. Appl. (7)*, **16** (1996), 433–456.
18. P. Pucci and J. Serrin, Global nonexistence for abstract evolution equations with positive initial energy, *J. Differential Equations* **150** (1998), 203–214.
19. P. H. Rabinowitz, Minimax methods in critical point theory with applications to differential equations, *in* CMBS Regional Conference Series in Mathematics, Vol. 65, Amer. Math. Soc., Providence, 1986.
20. P. A. Raviart, Sur la résolution de certain equations paraboliques non linéaires, *J. Funct. Anal.* **5** (1970), 299–328.
21. A. A. Samarskii, V. A. Galaktionov, S. P. Kurdyumov, and A. P. Mikhailov, "Blow-up in Quasilinear Parabolic Equations," de Gruyter Exp. Math., Vol. 19, de Gruyter, Berlin/New York, 1995.
22. M. Struwe, "Variational Methods," Springer-Verlag, New York, 1990.
23. Luc Tartar, "Partial Differential Equations, I," Carnegie Mellon University, Pittsburgh, 1997.
24. M. Tsutsumi, Existence and nonexistence of global solutions for nonlinear parabolic equations, *Publ. Res. Inst. Math. Sci.* **8** (1972), 211–229.
25. E. Vitillaro, Global nonexistence theorems for a class of evolution equations with dissipation and application, *Arch. Rational Mech. Anal.* **149** (1999), 155–182.