Unsolvable block transitive automorphism groups of $2-(v, k, 1)$ ($k = 6, 7, 8, 9$) designs

Guangguo Han

Institute of Mathematics, Hangzhou Dianzi University, Hangzhou, Zhejiang, 310018, China

Received 6 October 2004; received in revised form 14 October 2007; accepted 18 October 2007
Available online 20 February 2008

Abstract

In this paper, we study block transitive automorphism groups of $2-(v, k, 1)$ block designs. Let $D$ be a $2-(v, k, 1)$ ($k = 6, 7, 8, 9$) design admitting a block transitive, point primitive but not flag transitive group $G$ of automorphisms. We prove that if $G$ is unsolvable, then $G$ does not admit an exceptional simple group of Lie type as its socle. Moreover, for a $2-(v, 9, 1)$ design, we also prove that there does not exist any block transitive, point imprimitive, unsolvable group $G$ of automorphisms.

Keywords: Block design; Block transitive; Automorphism group; Unsolvable group

1. Introduction

A $2-(v, k, 1)$ design $D = (\mathcal{P}, \mathcal{B})$ is a pair consisting of a finite set $\mathcal{P}$ of $v$ points and a collection $\mathcal{B}$ of $k$-subsets of $\mathcal{P}$, called blocks, such that any $2$-subset of $\mathcal{P}$ is contained in exactly one block. We shall always assume that $2 < k < v$.

Recall that an automorphism of a $2-(v, k, 1)$ design $D$ is a permutation of the set $\mathcal{P}$ of points which maps blocks to blocks. The set of all automorphisms is called the automorphism group $\text{Aut}(D)$ of $D$, a subgroup of $\text{Sym}(\mathcal{P})$. Let $G \leq \text{Aut}(D)$, then $G$ is said to be block transitive on $D$ if $G$ is transitive on $\mathcal{B}$, and is said to be point transitive (point primitive) on $D$ if $G$ is transitive (primitive) on $\mathcal{P}$. A flag of $D$ is a pair consisting of a point and a block through that point. Then $G$ is flag transitive on $D$ if $G$ is transitive on the set of flags.

The classification of block transitive $2-(v, 3, 1)$ designs was completed about thirty years ago (see [10]). Camina and Siemons [6] classified $2-(v, 4, 1)$ designs with a block transitive, solvable group of automorphisms. In [14] Li classified $2-(v, 4, 1)$ designs admitting a block transitive, unsolvable group of automorphisms. Liu classified $2-(v, k, 1)$ ($k = 6, 7, 8, 9$) designs with a block transitive, solvable group of automorphisms (see [18–21]). So for block transitive $2-(v, k, 1)$ ($k = 6, 7, 8, 9$) designs, we need to study the case in which the given block transitive group of automorphisms is unsolvable. In this paper we consider this case and prove the following theorems:

Theorem 1. Let $D$ be a $2-(v, k, 1)$ ($k = 6, 7, 8, 9$) design, $G \leq \text{Aut}(D)$ be block transitive, point primitive but not flag transitive. If $G$ is unsolvable, then $\text{Soc}(G)$, the socle of $G$, is not an exceptional simple group of Lie type.

Supported by the National Natural Science Foundation of China (Grant No. 60672064) and the Scientific Research Startup Foundation of Hangzhou Dianzi University.

E-mail address: hangg@hdu.edu.cn.

0012-365X/$ - see front matter © 2007 Elsevier B.V. All rights reserved.
doi:10.1016/j.disc.2007.10.051
Remark 1. In [3], Camina and Mischke classified block transitive, point imprimitive 2-(v, k, 1) designs with k < 9. For block transitive 2-(v, 9, 1) designs, Liu [18,19] considered the case in which the given block transitive group of automorphisms is solvable. In Section 4, we deal with 2-(v, 9, 1) designs admitting a block transitive, point imprimitive, unsolvable group of automorphisms. We have

**Theorem 2.** Let \( D \) be a 2-(v, 9, 1) design, \( G \leq \text{Aut}(D) \) be block transitive. If \( G \) is unsolvable, then \( G \) is point primitive.

**Remark 2.** There have been a number of contributions to the study of the block transitive automorphism groups of 2-(v, k, 1) designs (see [3–8,10–12,14,16–22]). By Camina and Spiezia [7], the socle of \( G \) is not a sporadic simple group. Also, by the result of Camina, Neumann, and Praeger [4], the socle of \( G \) is not an alternating group. Recently, Camina and Zalesski [8] classified 2-(v, k, 1) designs admitting a block transitive group of automorphisms whose socle is isomorphic to simple large-rank classical groups. Since then, the efforts have been to classify the block transitive examples whose socle is isomorphic to simple small-rank classical groups. This is still an open problem.

This paper is organized as follows: In Section 2, we collect some results and we use them to prove the theorems in Sections 3 and 4.

### 2. Preliminary results

Let \( D \) be a 2-(v, k, 1) design defined on the point set \( P \), and suppose that \( G \) is an automorphism group of \( D \) that acts transitively on blocks. For a 2-(v, k, 1) design, as usual, \( b \) denotes the number of blocks and \( r \) denotes the number of blocks through a given point. If \( B \) is a block, \( G_B \) denotes the setwise stabilizer of \( B \) in \( G \) and \( G_{(B)} \) is the pointwise stabilizer of \( B \) in \( G \). Also, \( G_B \) denotes the permutation group induced by the action of \( G_B \) on the points of \( B \), and so \( G_B \cong G_B / G_{(B)} \).

For the basic notions and results of design theory and finite permutation groups, the reader is referred to [1,23]. We will follow the notations of [9] for simple groups of the Lie type. Let \( W \) be the Weyl group associated with the simple group \( T \) of Lie type, \( N \) the monomial subgroup of \( T \), and \( H \) the diagonal subgroup of \( T \). From [9, Theorem 7.2.2], it is well known that there exists a homomorphism \( \phi : N \to W \) such that \( N/H \cong W \). Let \( \Phi \) be the root system corresponding to \( T \) with the fundamental system \( \Pi \), also let \( \Phi^+ (\Phi^-) \) be the set of positive (negative) roots in \( \Phi \). If \( J \) is a subset of the set \( II \) of fundamental roots and \( V_J \) is the subspace of \( V \) spanned by \( J \), then \( \Phi_J \) denotes the set of roots of \( \Phi \) lying in the subspace \( V_J \). We use the standard labelling for Dynkin diagrams with fundamental roots \( \alpha_i \) as in [2, pp. 250–275].

By an exceptional simple group of Lie type, we mean a finite non-Abelian simple group associated with one of the families \( G_2, F_4, E_6, E_7, E_8, 2B_2, 2G_2, 2F_4, 3D_4 \) and \( 2E_6 \), excluding \( 2G_2(2)' \) and \( 2G_2(3)' \) in view of the isomorphisms \( 2G_2(2)' \cong U_3(3) \) and \( 2G_2(3)' \cong L_2(8) \). Also, for twisted groups our notation for \( q \) is such that \( 2B_2(q), 2G_2(q), 2F_4(q), 3D_4(q), \) and \( 2E_6(q) \) are the twisted groups contained in \( B_2(q), G_2(q), F_4(q), D_4(q^3), \) and \( E_6(q^2) \), respectively.

The main result regarding the maximal subgroups of the finite exceptional simple group of Lie type we are going to use is the following

**Lemma 2.1 (Liebeck and Saxl [15]).** Let \( T = T(q) \) be an exceptional simple group of Lie type over \( GF(q) \), where \( q = p^f \) for \( p \) a prime integer and \( f \) a positive integer, and let \( G \) be a group with \( T \leq G \leq \text{Aut}(T) \). Suppose that \( M \) is a maximal subgroup of \( G \) not containing \( T \), then one of the following holds:

1. \( |M| < q^{k(T)} |G : T| \), where \( q^{k(T)} \) is defined as in Table 1,
2. \( T \cap M \) is a parabolic subgroup of \( T \),
3. \( T \cap M \) is as in Table 1.

**Lemma 2.2 (Higman and Maclachlin [13]).** If \( G \leq \text{Aut}(D) \) is flag transitive, then \( G \) is point primitive.

Reference [6] is an interesting paper. In it, Camina and Siemons established a useful lemma in which the so-called Witt conditions (Section 9 in [23]) are involved. Thanks to this lemma, they were able to use an induction argument in the study of block transitive designs. By using this lemma, we often obtain a smaller block transitive design and a
Table 1

<table>
<thead>
<tr>
<th>$T$</th>
<th>$q^k(T)$</th>
<th>$T \cap M$</th>
<th>Condition</th>
</tr>
</thead>
<tbody>
<tr>
<td>$2B_2(q)$</td>
<td>$q^2$</td>
<td>None</td>
<td>$q = 2^{2m+1}$</td>
</tr>
<tr>
<td>$G_2(q)$</td>
<td>$q^6$</td>
<td>$SL_2(q).2$ $SU_3(q).2$ $G_2(q) \frac{1}{2}$ $G_2(q^2) G_2(2), L_2(13), 2^3 L_3(2)$</td>
<td>$q = 3^{2m+1}$ $q$ square $q = 3$ $q = 4$</td>
</tr>
<tr>
<td>$2G_2(q)$</td>
<td>$q^3$</td>
<td>None</td>
<td>$q = 3^{2m+1} \geq 27$</td>
</tr>
<tr>
<td>$3D_4(q)$</td>
<td>$q^{12}$</td>
<td>$G_2(q) \frac{1}{2} 3D_4(q) \frac{1}{2}$</td>
<td>$q$ square</td>
</tr>
<tr>
<td>$2F_4(q)$</td>
<td>$q^{12}$</td>
<td>$L_3(3).2, L_2(25)$</td>
<td>$q = 2$</td>
</tr>
<tr>
<td>$F_4(q)$</td>
<td>$q^{24}$</td>
<td>$B_4(q)$ $D_4(q).S_3$ $3D_4(q).3$</td>
<td>$B_4(q)$ universal $D_4(q)$ universal</td>
</tr>
<tr>
<td>$2E_6(q)$</td>
<td>$q^{37}$</td>
<td>$F_4(q) \frac{1}{2}$ $2F_4(q)$</td>
<td>$D_5(q)$ universal $G$ contains a graph automorphism</td>
</tr>
<tr>
<td>$E_6(q)$</td>
<td>$q^{37}$</td>
<td>$F_4(q) \frac{1}{2}$ $2E_6(q) \frac{1}{2}$</td>
<td>$D_5(q)$ universal, $G$ contains a graph automorphism</td>
</tr>
<tr>
<td>$E_7(q)$</td>
<td>$q^{64}$</td>
<td>$E_6(q) \frac{1}{2}$ $2E_7(q) \frac{1}{2}$ $E_7(q) \frac{1}{2}$</td>
<td>$E_6(q)$ universal $E_7(q)$ universal $E_8(q) \frac{1}{2}$</td>
</tr>
<tr>
<td>$E_8(q)$</td>
<td>$q^{110}$</td>
<td>$E_7(q) \frac{1}{2}$ $D_8(q) \frac{1}{2}$ $E_8(q) \frac{1}{2}$</td>
<td>$E_7(q) \frac{1}{2}$ $E_8(q) \frac{1}{2}$</td>
</tr>
</tbody>
</table>

group associated with it such that the structure of this group is easier to handle. We state this lemma below and use it in our proof of Theorem 2.

**Lemma 2.3** ([Camina and Siemons [6]]). Let $G$ be a block transitive automorphism group of a $2-(v, k, 1)$ design. Let $B$ be a block and $H$ a subgroup of $G_B$. Assume that $H$ satisfies the following two conditions:

(i) $|\text{Fix } H \cap B| \geq 2$ and

(ii) if $K \leq G_B$ and $|\text{Fix } K \cap B| \geq 2$ and $K$ is conjugate to $H$ in $G$ then $H$ is conjugate to $K$ in $G_B$.

Then either

(a) $\text{Fix } H \subseteq B$ or

(b) the induced structure on $\text{Fix } H$ is a $2-(v_0, k_0, 1)$ design where $v_0 = |\text{Fix } H|$, $k_0 = |\text{Fix } H \cap B|$. Further, $N_G(H)$ acts as a block transitive group on this design.

**Lemma 2.4** ([Camina and Siemons [6]]). Let $G \leq \text{Aut}(D)$, and let $H \neq 1$ be a subgroup of $G$. Then $|\text{Fix } H| \leq r + k - 3$. 
3. Proof of Theorem 1

We assume throughout this section that \( q = p^f \) for \( p \) a prime integer and \( f \) a positive integer. If \( n \) is a positive integer, then \( |n|_p \) denotes the \( p \)-part of \( n \) and \( |n|_{p'} \) denotes the \( p' \)-part of \( n \). In other words, \( |n|_p = p^f \) where \( p^f | n \) but \( p^{f+1} \nmid n \), and \( |n|_{p'} = n/|n|_p \).

Following Fang–Li (see [11]), we shall use the following parameters of \( 2-(v, k, 1) \) designs:

\[ k_v = (k, \alpha), \quad k_r = (k, \beta) = (k, v - 1), \quad b_v = (b, \alpha), \quad b_r = (b, \beta) = (b, v - 1). \]

It is easy to check that

\[ k = k_vk_r, \quad b = b_vb_r, \quad v = k_vk_v \quad \text{and} \quad r = k_rb_r. \]

**Proposition 3.1.** Let \( \mathcal{D} \) and \( G \) satisfy the conditions of Theorem 1, \( T = \text{Soc}(G) \) and \( T_\alpha = T \cap G_\alpha \), where \( \alpha \in \mathcal{P} \). Then we have the following properties:

1. \( v = 1 + k_r(k - 1)b_r; \)
2. \( \frac{v}{x} < (k_rk - k_r + 1)|G : T| \) or \( \frac{v - 1}{x} \leq (k_rk - k_r)|G : T| \), where \( x \) is the size of \( T_\alpha \)-orbit in \( \mathcal{P} \setminus \{ \alpha \}; \)
3. \( \frac{|T|}{|T_\alpha|} < \frac{k_x - k_\alpha + 1}{2} |G : T| \), where \( \frac{k_x - k_\alpha + 1}{2} \) is the ceiling of \( \frac{k_x - k_\alpha + 1}{2} \);
4. \( x(v - 1, q) = 1 \), then there exists in \( \mathcal{P} \setminus \{ \alpha \} \) a \( T_\alpha \)-orbit with size \( y \) such that \( y | |G_\alpha| \).
5. \( b_r | |G_\alpha| \).

**Proof.** (P1) Since \( kb = vr \) and \( r = \frac{v - 1}{k - 1} \), we have

\[ k(k - 1)b = v(v - 1), \]

which is the same as

\[ k_r(k - 1)b_r = v - 1. \]

Hence

\[ v = 1 + k_r(k - 1)b_r. \]

(P2) Let \( \Delta \) be any \( T_\alpha \)-orbit in \( \mathcal{P} \setminus \{ \alpha \} \) with size \( x \), and \( \Gamma \) be a nontrivial suborbit of \( G_\alpha \) such that \( \Delta \subseteq \Gamma \). Since

\[ \frac{|G|}{|G_\alpha|} = \frac{|T|}{|T_\alpha|}, \]

we have

\[ |G : T| = |G_\alpha : T_\alpha|, \]

and

\[ \frac{|\Gamma|}{|G_\alpha|} \leq \frac{|G_\alpha|}{|T_\alpha|} = \frac{|T_\alpha|}{|T_\alpha|} = x |G : T|, \]

where \( \beta \in \Delta \). Since \( v = 1 + k_r(k - 1)b_r \), then

\[ \frac{v}{b_r} < 1 + k_r(k - 1). \]

By Lemma 2.1 of [14], we have \( b_r ||\Gamma||, \) and \( b_r \leq |\Gamma| \). Thus

\[ \frac{v}{x |G : T|} \leq \frac{v}{|\Gamma|} \leq \frac{v}{b_r} < 1 + k_r(k - 1), \]

and we have the first inequality. The proof of the other inequality is similar.

(P3) Since \( T \) is not a Frobenius group (because a Frobenius group has a regular nilpotent normal subgroup), there exist \( \alpha, \beta \in \mathcal{P} \) such that \( |T_{\alpha\beta}| \neq 1 \). Then

\[ \frac{v}{x} = \frac{|T|}{|T_\alpha|^2} |T_{\alpha\beta}| \geq 2 \frac{|T|}{|T_\alpha|^2}. \]
Combining with (P₂), we get
\[
\frac{|T|}{|T_α|^2} < \left\lfloor \frac{k_r k_r + 1}{2} \right\rfloor |G : T|.
\]

(P₄) Let \( t \) be the size of any \( T_α \)-orbit in \( \mathcal{P} \setminus \{α\} \). Suppose to the contrary that \( t \nmid |T_α|p'. \) Since \( t || T_α \), we have \( p | t \). Furthermore, since \( \mathcal{P} \setminus \{α\} \) is a union of \( T_α \)-orbits, \( p | v - 1 \). Thus \( p | (v - 1, q) \), which contradicts \( (v - 1, q) = 1 \).

(P₅) This is a straightforward consequence of [14, Lemma 2.1]. □

**Proof of Theorem 1.** We first explain the steps we use to prove Theorem 1 when \( k = 9 \). By Lemma 2.1, the maximal subgroup of \( G \) is one of three possible cases. We show that each of these three cases is impossible. Each case involves rather lengthy and repetitious numerical calculations, and we are going to give just some sample calculations in detail.

When \( k = 6, 7, 8 \), the proof is completely analogous to the case of \( k = 9 \). We have checked all the cases carefully and we shall omit the proof here.

Now we give the proof when \( k = 9 \). Assume by way of contradiction that \( T = \text{Soc}(G) \) is an exceptional simple group of Lie type.

Since \( G \) is not flag transitive, then \( 9 \nmid v \). We have \( k_r = 9 \), since \( (k_r, k_r) = 1 \). So \( k_r(k_r - 1) = 72 \) and \( \left\lfloor \frac{k_r k_r + 1}{2} \right\rfloor = 37 \) from Proposition 3.1.

Because of the point primitivity of \( G \), the subgroup \( G_α \), the stabilizer of a point \( α \in \mathcal{P} \), is maximal in \( G \). By Lemma 2.1, the maximal subgroup \( M = G_α \) of \( G \) is one of three possible cases. We exclude these cases one by one.

Case 1 \(|M| < q^{k(T)}|G : T|\), where \( q^{k(T)} \) is defined as in Table 1.

By (P₃) of Proposition 3.1, we have an upper bound of \(|T|\),
\[
|T| < 37|T_α|^2 |G : T| < 37q^{2k(T)}|G : T|.
\] (1)

If \( q^N \geq q^{k(T)} \), where \( N \) is the number of positive roots in the root system of the Weyl group \( W \) associated with \( T \), then \( p | v = \frac{|T|}{|T_α|} \). Thus by (P₁), if \( q^N \geq q^{k(T)} \), we only have to consider the case with \( q \) odd. The proof of this case consists of two parts. Firstly, by Eq. (1), we eliminate almost all exceptional simple groups with \( q \) big enough. Then by (P₁), we eliminate the others by calculating \( v \) directly. As an example, we only give the proof in connection with the simple group \( T = E_6(q) \) in detail.

By Theorems 9.3.4 and 14.3.1 in [9], we have
\[
|T| = \frac{1}{d} q^N(q^{d_1} - ε_1)(q^{d_2} - ε_2) \cdots (q^{d_l} - ε_l),
\]
where \( d_1 > d_2 > \cdots > d_l, ε_i = ±1, i = 1, 2, \ldots, l \). Also we have
\[
d_1 + d_2 + \cdots + d_l = N + l.
\]

So
\[
|T| = \frac{1}{d} q^{36}(q^{12} - 1)(q^{9} + 1)(q^{6} - 1)(q^{5} + 1)(q^{2} - 1)
\geq \frac{1}{d} q^{36}(q^{42} - q^{40} + q^{37} - q^{36} - \cdots)
\geq \frac{1}{d} q^{36}(q^{42} - q^{40} - (2^6 - 6)q^{34}).
\]

Then
\[
\frac{|T|}{q^{2k(T)}} > \frac{q^{36}(q^{42} - q^{40} - (2^6 - 6)q^{34})}{dq^{74}}
\geq \frac{q^4 - q^2 - 58}{d}
\geq 37|G : T|, \quad \text{if } q \neq 2,
\]
contradicting property (P₃).
If \( q = 2 \), since \( |T_a| < 2^{37} \), then \( |T_a| \) satisfies either \( 2^{36} \nmid |T_a| \) or \( 2^{36} || T_a| \). If \( 2^{36} \nmid |T_a| \), then \( v = \frac{|T|}{|T_a|} \) is even, contradicting property (P1). If \( 2^{36} || T_a| \), then the possible order of \( T_a \) is only \( 2^{36} \). Calculating \( v \) directly, we have \( 9|v \), contradicting property (P1).

Case 2 \( T \cap M \) is a parabolic subgroup of \( T \).

In this case, we continue to make use of properties (P1) and (P2). As an example, we give the proof in connection with the simple group \( T = E_6(q) \) in detail. Let \( \Pi = \{ \alpha_1, \alpha_2, \ldots, \alpha_6 \} \) be the fundamental root system of \( E_6(q) \), \( J_1 = \Pi - \{ \alpha_i \} \) and let \( P_{J_1} \) be the parabolic subgroup of \( E_6(q) \) determined by \( J_1 \).

(i) \( T_a = P_{J_1} \).

By [9], we have

\[
|P_{J_1}| = \frac{1}{d} q^{36} (q - 1)(q^2 - 1)(q^4 - 1)(q^5 - 1)(q^6 - 1)(q^8 - 1)
\]

and

\[
v = \frac{(q^{12} - 1)(q^6 - 1)}{(q - 1)(q^4 - 1)}.
\]

From [9], there exists a homomorphism \( \phi : N \to W \) inducing \( N/H \cong W \). Let \( \phi(n_1) = w_{\alpha_1} \), where \( n_1 \in N \), \( w_{\alpha_1} \) is the corresponding reflection of \( \alpha_1 \) in the Weyl group \( W \). Now we consider \( P_{J_1} \cap P_{J_1}^{\alpha_1} \). Since

\[
P_{J_1} = \langle X_r, H | r \in \Phi^+ \cup \Phi_{J_1} \rangle,
\]

then

\[
P_{J_1}^{\alpha_1} = \langle X_r, H | r \in (\Phi^+)^{\alpha_1} \cup (\Phi_{J_1})^{\alpha_1} \rangle
\]

\[= \langle X_r, H | r \in (\Phi^+ - \{ \alpha_1 \}) \cup \{ -\alpha_1 \} \cup \Phi_{w_{\alpha_1}(J_1)} \rangle.
\]

We have

\[
\langle X_r, H | r \in (\Phi^+ - \{ \alpha_1 \}) \cup \Phi_{J'} \rangle \leq P_{J_1} \cap P_{J_1}^{\alpha_1},
\]

where \( J' = \{ \alpha_2, \alpha_4, \alpha_5, \alpha_6 \} \). Let

\[
\tilde{P} = \langle X_r, H | r \in (\Phi^+ - \{ \alpha_1 \}) \cup \Phi_{J'} \rangle
\]

and

\[
\tilde{U} = \prod_{r \in (\Phi^+ - \{ \alpha_1 \}) \cap \Phi_{J'}} X_r \leq U_{J'}.
\]

We claim that \( \tilde{U} \trianglelefteq \tilde{P} \). We show that the subgroups generating \( \tilde{P} \) all normalize \( \tilde{U} \). It is clear that \( H \) normalizes \( \tilde{U} \). Let \( r \) be a positive root. If \( s \in (\Phi^+ - \{ \alpha_1 \}) \cap \Phi_{J'} \), all roots of the form \( ir + js \) where \( i \) and \( j \) are positive integers, are also in \( (\Phi^+ - \{ \alpha_1 \}) \cap \Phi_{J'} \). Thus the commutator formula (see [9]) shows that \( X_r \) normalizes \( \tilde{U} \). Now suppose \( r \in \Phi^+ \cap \Phi_{J'} \). Then \( -r \) is not in \( (\Phi^+ - \{ \alpha_1 \}) \cap \Phi_{J'} \), and if \( s \) is any root in \( (\Phi^+ - \{ \alpha_1 \}) \cap \Phi_{J'} \), all roots of the form \( ir + js \), where \( i \) and \( j \) are positive integers, are in \( (\Phi^+ - \{ \alpha_1 \}) \cap \Phi_{J'} \). For \( ir + js \) involves some fundamental root not in \( J' \) with a positive coefficient. Hence \( X_r \) normalizes \( \tilde{U} \) in this case also. Thus \( \tilde{U} \trianglelefteq \tilde{P} \).

Now we define \( L_{J'} \) to be the subgroup of \( G \) generated by \( H \) and the root subgroups \( X_r \) for all \( r \in \Phi_{J'} \). Then we have

\[
\tilde{P} = \tilde{U} L_{J'}, \quad |\tilde{P}| = \frac{1}{d} q^{35} (q - 1)^2(q^2 - 1)(q^3 - 1)(q^4 - 1)(q^5 - 1).
\]

Thus \( T_a \) has an orbit of size

\[
x = \frac{|P_{J_1}|}{|P_{J_1} \cap P_{J_1}^{\alpha_1}|} \leq \frac{|P_{J_1}|}{|P|} = \frac{q(q^6 - 1)(q^8 - 1)}{(q - 1)(q^3 - 1)}.
\]
Therefore
\[
\frac{v}{x} > q^5 - q^2 > 73|G : T| \quad \text{if } q \neq 2, 4,
\]
contradicting property (P2).

If \( q = 2 \) or 4, then \( v \) is correspondingly 16,863 or 574,905,133. This contradicts property (P1).

(ii) \( T_{\alpha} = P_{J_2} \).

By [9], we have
\[
|P_{J_2}| = \frac{1}{d} q^{36} \prod_{i=1}^{6} (q^i - 1)
\]
and
\[
v = \frac{(q^8 - 1)(q^9 - 1)(q^{12} - 1)}{(q - 1)(q^2 - 1)(q^4 - 1)}.
\]

Let \( n_2 \) be an inverse image of \( w_{\alpha_2} \) under the homomorphism \( \phi : N \to W \). Now we consider \( P_{J_2} \cap P_{J_2}^{n_2} \). Similarly we have
\[
P_{J_2} = \langle X_r, H | r \in \Phi^+ \cup \Phi_{J_2} \rangle
\]
and
\[
P_{J_2}^{n_2} = \langle X_r, H | r \in (\Phi^+)^{n_2} \cup (\Phi_{J_2})^{n_2} \rangle
\]
\[
= \langle X_r, H | r \in (\Phi^+ - \{\alpha_2\}) \cup \{-\alpha_2\} \cup \Phi_{w_{\alpha_2}(J_2)} \rangle.
\]

Then
\[
(X_r, H | r \in (\Phi^+ - \{\alpha_2\}) \cup \Phi_{J'}) \leq P_{J_2} \cap P_{J_2}^{n_2},
\]
where \( J' = \{\alpha_1, \alpha_3, \alpha_5, \alpha_6\} \). Hence
\[
|P_{J_2} \cap P_{J_2}^{n_2}| > \frac{1}{d} q^{35}(q - 1)^2(q^2 - 1)^2(q^3 - 1)^2.
\]

Thus \( T_{\alpha} \) has an orbit of size
\[
x = \frac{|P_{J_2}|}{|P_{J_2} \cap P_{J_2}^{n_2}|} \leq \frac{q(q^4 - 1)(q^5 - 1)(q^6 - 1)}{(q - 1)(q^2 - 1)(q^3 - 1)}.
\]

It follows that
\[
\frac{v}{x} > q^{10} > 73|G : T|,
\]
contradicting property (P2).

(iii) \( T_{\alpha} = P_{J_3} \).

By [9], we have
\[
|P_{J_3}| = \frac{1}{d} q^{36}(q - 1)(q^2 - 1)^2(q^3 - 1)(q^4 - 1)(q^5 - 1)
\]
and
\[
v = \frac{(q^3 + 1)(q^4 + 1)(q^9 - 1)(q^{12} - 1)}{(q - 1)(q^2 - 1)}.
\]

Let \( n_3 \) be an inverse image of \( w_{\alpha_3} \) under the homomorphism \( \phi : N \to W \). Now we consider \( P_{J_3} \cap P_{J_3}^{n_3} \). Similarly to the above we have
\[
P_{J_3} = \langle X_r, H | r \in \Phi^+ \cup \Phi_{J_3} \rangle
\]
and
\[ P_{J_3}^{n_3} = \langle X_r, H | r \in (\phi^+)^{n_3} \cup (\phi_{J_3})^{n_3} \rangle = \langle X_r, H | r \in (\phi^+ - \{\alpha_3\}) \cup \{-\alpha_3\} \cup \phi_{w_{J_3}(J_3)} \rangle. \]

Then
\[ \langle X_r, H | r \in (\phi^+ - \{\alpha_3\}) \cup \phi_{J'} \rangle \leq P_{J_3} \cap P_{J_3}^{n_3}, \]
where \( J' = \{\alpha_2, \alpha_5, \alpha_6\} \). Hence
\[ |P_{J_3} \cap P_{J_3}^{n_3}| > \frac{1}{d} q^{35}(q - 1)^3(q^2 - 1)^2(q^3 - 1). \]

So \( T_\alpha \) has an orbit of size
\[ x = \frac{|P_{J_3}|}{|P_{J_3} \cap P_{J_3}^{n_3}|} \leq \frac{q(q^4 - 1)(q^5 - 1)}{(q - 1)^2}. \]
It follows that
\[ \frac{v}{x} \geq \frac{(q - 1)(q^3 + 1)(q^4 + 1)(q^9 - 1)(q^{12} - 1)}{q(q^2 - 1)(q^4 - 1)(q^5 - 1)} > q^{16} > 73|G : T|, \]
contradicting (P2).

(iv) \( T_\alpha = P_{J_1} \). The proof is similar to (i).
(v) \( T_\alpha = P_{J_2} \). There exists an element \( g \in \text{Aut}(T) \) such that \( P_{J_2} = P_{J_3}^g \), a contradiction.
(vi) \( T_\alpha = P_{J_4} \). There exists an element \( g_1 \in \text{Aut}(T) \) such that \( P_{J_4} = P_{J_4}^{g_1} \), a contradiction.

Case 3 \( T \cap M \) as in Table 1.

For all simple groups of Lie type in Table 1, we have \( q^N \nmid |T \cap M| \). Then \( p|v = \frac{|T|}{|T \cap M|} \). By (P1), \( q \) is odd in this case. In Table 2, we give the arguments dealing with these groups in Table 1. For instance, if the argument in the last column of Table 2 is (P4), it means that a contradiction can be deduced by using (P4). As an example, we give the proof in detail for the case \( T = 2E_6(q), T_\alpha = (2D_5(q) \circ (q + 1)/e_{-1}).f_{-1} \). Then we have
\[ |T_\alpha| = \frac{1}{(3, q + 1)} q^{20}(q + 1)(q^5 + 1)(q^2 - 1)(q^4 - 1)(q^6 - 1)(q^8 - 1) \]
and
\[ v = \frac{q^{16}(q^{12} - 1)(q^9 + 1)}{(q^4 - 1)(q + 1)}. \]
Since \( (v - 1, q) = 1 \), (P3), the group \( T_\alpha \) has an orbit of size \( x \) such that
\[ x \leq |T_\alpha|_{p'} \leq \frac{1}{(3, q + 1)}(q + 1)(q^5 + 1)(q^2 - 1)(q^4 - 1)(q^6 - 1)(q^8 - 1). \]
It follows that
\[ \frac{v}{x} \geq \frac{(3, q + 1)q^{16}(q^{12} - 1)(q^9 + 1)}{(q + 1)^2(q^5 + 1)(q^2 - 1)(q^4 - 1)^2(q^6 - 1)(q^8 - 1)} > (3, q + 1)q^5 > 73|G : T|, \]
a contradiction.

Now we have proved that the maximal subgroup of \( G \) does not satisfy any of the 3 possibilities in Lemma 2.1. Hence \( T \) is not an exceptional simple group of Lie type. This completes the proof of Theorem 1 when \( k = 9 \). □
Table 2

<table>
<thead>
<tr>
<th>T</th>
<th>$q^{k(T)}$</th>
<th>$T \cap M$</th>
<th>Argument</th>
</tr>
</thead>
<tbody>
<tr>
<td>$^2B_2(q)$</td>
<td>$q^2$</td>
<td>None</td>
<td>(P_3), (P_1)</td>
</tr>
<tr>
<td>$G_2(q)$</td>
<td>$q^6$</td>
<td>$SL_3(q).2$</td>
<td>(P_4), (P_1)</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$SU_3(q).2$</td>
<td>(P_4), (P_1)</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$^2G_2(q)$</td>
<td>(P_1)</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$G_2(q^{1/2})$</td>
<td>(P_1)</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$G_2(2), L_2(13), 2^3.L_3(2)$</td>
<td>(P_1)</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$J_2$</td>
<td>(P_1)</td>
</tr>
<tr>
<td>$^2G_2(q)$</td>
<td>$q^3$</td>
<td>None</td>
<td>(P_4), (P_1)</td>
</tr>
<tr>
<td>$^3D_4(q)$</td>
<td>$q^{12}$</td>
<td>$G_2(q)$</td>
<td>(P_4), (P_1)</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$^3D_4(q^{1/2})$</td>
<td>(P_1)</td>
</tr>
<tr>
<td>$^2F_4(q)$</td>
<td>$q^{12}$</td>
<td>$L_3(3).2, L_2(25)$</td>
<td>(P_1)</td>
</tr>
<tr>
<td>$F_4(q)$</td>
<td>$q^{24}$</td>
<td>$B_3(q)$</td>
<td>(P_1)</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$D_4(q).S_3$</td>
<td>(P_1)</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$^3D_4(q^{1/2})$</td>
<td>(P_1)</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$F_4(q^{7})$</td>
<td>(P_1)</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$^2F_4(q)$</td>
<td>(P_1)</td>
</tr>
<tr>
<td>$^2E_6(q)$</td>
<td>$q^{37}$</td>
<td>$F_4(q)$</td>
<td>(P_1)</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$(^2D_5(q) \circ (q + 1)/e_{-1}).f_{-1}$</td>
<td>(P_4), (P_1)</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$(SL_2(q) \circ A_5(q)).d$</td>
<td>(P_4), (P_1)</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$Fi_{22}$</td>
<td>(P_1)</td>
</tr>
<tr>
<td>$E_6(q)$</td>
<td>$q^{37}$</td>
<td>$F_4(q)$</td>
<td>(P_1)</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$(SL_2(q) \circ A_5(q)).d$</td>
<td>(P_1)</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$E_6(q^{1/2})$</td>
<td>(P_1)</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$^2E_6(q^{1/2})$</td>
<td>(P_1)</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$(D_5(q) \circ (q - 1)/e_{+1}).f_{+1}$</td>
<td>(P_4), (P_1)</td>
</tr>
<tr>
<td>$E_7(q)$</td>
<td>$q^{64}$</td>
<td>$(E_6(q) \circ (q - 1)/d).e_{+1.2}$</td>
<td>(P_1)</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$(^2E_6(q) \circ (q + 1)/d).e_{-1.2}$</td>
<td>(P_1)</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$(SL_2(q) \circ D_6(q)).d$</td>
<td>(P_1)</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$E_7(q^{1/2}).d$</td>
<td>(P_1)</td>
</tr>
<tr>
<td>$E_8(q)$</td>
<td>$q^{110}$</td>
<td>$(SL_2(q) \circ E_7(q)).d$</td>
<td>(P_1)</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$D_8(q).d$</td>
<td>(P_4), (P_1)</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$E_8(q^{1/2})$</td>
<td>(P_1)</td>
</tr>
</tbody>
</table>

Table 3

<table>
<thead>
<tr>
<th>x</th>
<th>c</th>
<th>d</th>
<th>v</th>
<th>y</th>
<th>r</th>
<th>b</th>
</tr>
</thead>
<tbody>
<tr>
<td>(I)</td>
<td>1</td>
<td>71</td>
<td>71</td>
<td>5041</td>
<td>$\frac{1}{2}$</td>
<td>630</td>
</tr>
<tr>
<td>(II)</td>
<td>2</td>
<td>29</td>
<td>5</td>
<td>145</td>
<td>1</td>
<td>18</td>
</tr>
<tr>
<td>(III)</td>
<td>14</td>
<td>5</td>
<td>29</td>
<td>145</td>
<td>7</td>
<td>18</td>
</tr>
<tr>
<td>(IV)</td>
<td>4</td>
<td>17</td>
<td>17</td>
<td>289</td>
<td>2</td>
<td>36</td>
</tr>
<tr>
<td>(V)</td>
<td>10</td>
<td>7</td>
<td>31</td>
<td>217</td>
<td>5</td>
<td>27</td>
</tr>
<tr>
<td>(VI)</td>
<td>2</td>
<td>31</td>
<td>7</td>
<td>217</td>
<td>1</td>
<td>27</td>
</tr>
</tbody>
</table>

4. Proof of Theorem 2

**Lemma 4.1.** Let $\mathcal{D}$ be a 2-$(v, 9, 1)$ design, $G \leq \text{Aut}(\mathcal{D})$ be block transitive and point primitive. Let $y$ and $d$ be the number of inner pairs in a given block and the common length of imprimitivity blocks, respectively. Then the possible values of the parameters of $G$ and $\mathcal{D}$ are as in Table 3.
Proof. Since $G$ is imprimitive on $\mathcal{P}$, the length of any imprimitivity block is of the form $1 + x \frac{v-1}{72}$ for some integer $x$. So there is an integer $c$ such that

$$
\left(1 + x \frac{v-1}{72}\right) c = v.
$$

That is,

$$(72 - x)c = (72 - xc)v.$$

Because $c$ is a divisor of $v$, we have $72 - xc | 72 - x$. Obviously, $36 > x \geq 1$ and $c \geq 2$. By direct calculation, we have the following possibilities for the pair $(x, c)$:

$$(1, 71); (2, 29); (2, 31); (3, 23); (4, 17); (6, 11); (8, 8); (9, 7); (10, 7); (12, 5); (14, 5); (16, 4); (18, 3).$$

Calculating $v$ directly, we can eliminate the cases when $v$ is less than 72 or $72 \nmid v - 1$, and we get the following possible values of the triad $(x, c, v)$:

$$(1, 71, 5041); (2, 31, 217); (2, 29, 145); (4, 17, 289); (10, 7, 217); (14, 5, 145).$$

Then it is not difficult to have Table 3. □

Proof of Theorem 2. By Lemma 2.2, if $G$ is flag transitive, then $G$ is point primitive.

Now let $G$ be block transitive but not flag transitive. Assume by way of contradiction that $G$ is imprimitive on $\mathcal{P}$. Then by Lemma 4.1, we have six cases as in Table 3. We will rule out these cases one by one.

Case (I) in Table 3 cannot occur.

In Table 3, there is $\frac{1}{2}$ inner pair in a given block, which is impossible.

Case (II) in Table 3 cannot occur.

In this case, $G$ has 29 imprimitivity blocks, denoted by $\Delta_0, \Delta_1, \ldots, \Delta_{28}$, and each imprimitivity block has size 5. Also there is only one inner pair in a block.

Let $B$ be a block of design $\mathcal{D}$. Without loss of generality, let $B = \{\alpha, \beta, \gamma_1, \ldots, \gamma_7\}$ with $\alpha, \beta \in \Delta_0$ and $\gamma_i \in \Delta_i$ $(i = 1, 2, \ldots, 7)$.

Let $C$ be the set of imprimitivity blocks. That is, $C = \{\Delta_0, \ldots, \Delta_{28}\}$. Now we consider the action of $G$ on $C$. Then $G^C$ is a primitive group of degree 29. By the classification of primitive groups, $G^C$ contains $A_{29}$ or is a subgroup of $AGL(1, 29)$.

Now we shall prove that $G^C$ does not contain $A_{29}$. It is sufficient to prove that $G$ does not have an element of order 13. Otherwise, let $T$ be a Sylow 13-subgroup of $G$. Since 13 cannot divide $b$, $T$ fixes a block of $\mathcal{D}$. By the block transitivity of $G$, we can assume $T \leq G_B$.

By Lemma 2.3, we have $Fix T \nsubseteq B$ since 13 $| (145 - 9)$ and the induced structure on $Fix T$ is a 2-$(v_0, 9, 1)$ design, where $v_0 = |Fix T|$. By Lemma 2.4 and Fisher’s inequality, we have $r + k - 3 = 24 \geq v_0 \geq 9^2 - 9 + 1 = 73$. This is a contradiction. Hence $G^C$ does not contain $A_{29}$.

Next we shall prove that $G$ does not have an element of order 3. Otherwise, let $T$ be a Sylow 3-subgroup of $G$. Since 3 cannot divide $b$, we have $T \leq G_B$. By the point distribution of $B$, $T$ then fixes at least 3 points of $B$, say $\alpha_1, \alpha_2, \gamma_1$. Let $g \in T$ and $\tilde{g}$ be the induced element of $g$ in $G^C$. Since $\alpha_i^g = \alpha_i$ $(i = 1, 2)$ and $\gamma_i^g = \gamma_i$, we have $\Delta_0^g = \Delta_0$ and $\Delta_i^g = \Delta_i$. Then $\tilde{g}$ fixes $\Delta_0$ and $\Delta_1$. Since $G^C$ is a subgroup of $AGL(1, 29)$, we have $\tilde{g} = 1$. Hence $\Delta_0^g = \Delta_0$ $(i = 0, 1, \ldots, 28)$. It is evident that $|Fix T \cap \Delta_i| = 2$ or 5 $(i = 0, 1, \ldots, 28)$. Then $|Fix T| \geq 58$. By Lemma 2.4, we also have $|Fix T| \leq r + k - 3 = 24$. This is a contradiction.

Let $K$ be the kernel of the action of $G$ on $C$. Then $K$ consists of the elements of $G$ which fix every imprimitivity block $\Delta_i$ $(i = 0, 1, 20)$. By the classification of primitive groups of degree 5, $K^\Delta_i$ contains $A_5$ or is a subgroup of $AGL(1, 5)$. Since $3 \nmid |G|$ and $3 \nmid |K^\Delta_i|$, we have $K^\Delta_i$ is a subgroup of $AGL(1, 5)$. Hence $K^\Delta_i$ is a solvable group. It follows easily that $K$ is solvable, and hence $G$ is solvable. This is a contradiction.

Case (III) in Table 3 cannot occur.

In Table 3, $G$ has 5 imprimitivity blocks $\Delta_0, \Delta_1, \ldots, \Delta_4$ with size 29.

By the same argument as in case (II), we can prove that $G$ does not have an element of order 13. Since $13 \nmid |G|$, $K^\Delta_i \leq AGL(1, 29)$. 
Let $C$ be the set of imprimitivity blocks and $K$ be the kernel of the action of $G$ on $C$. Next we will prove that $K$ acts faithfully on every imprimitivity block. It is sufficient to prove that $K$ acts faithfully on $\Delta_0$. Let $L$ be the kernel of the action of $K$ on $\Delta_0$. If $L \neq 1$, then $\Delta_0 \subseteq \text{Fix } L$. So $|\text{Fix } L| \geq 29$. On the other hand, by Lemma 2.4 we have $|\text{Fix } L| \leq r + k - 3 = 24$. This is a contradiction. Hence $L = 1$ and $K$ acts faithfully on $\Delta_0$.

Let $H$ be the Sylow 29-subgroup of $K$, then $H \triangleleft K$. So $H$ is a normal Hall-subgroup of $G$. By the Schur–Zassenhaus Theorem, there exists a complement of $H$ in $G$, say $L$, such that $G = HL$.

Because $|G : G_\alpha| = 145 (\alpha \in \mathcal{P})$, we have $G_\alpha \leq \langle g \in G \rangle$. Then $|L : G_\alpha| = 5$ and $G$ has an imprimitivity block with size 5. This takes us back to case (II).

*Case (IV) in Table 3 cannot occur.*

In Table 3, $G$ has 17 imprimitivity blocks $\Delta_0, \Delta_1, \ldots, \Delta_{16}$ with size 17. Also there are 2 inner pairs in each block. Then, without loss of generality, we may assume

$$B = \{\alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_1, \gamma_2, \ldots, \gamma_5\},$$

where $\alpha_1, \alpha_2 \in \Delta_0$, $\beta_1, \beta_2 \in \Delta_1$ and $\gamma_i \in \Delta_{i+1} (i = 1, 2, \ldots, 5)$.

Consider the action of $G$ on $C$, the set of imprimitivity blocks. Then $G^C$ is a primitive group of degree 17. By the classification of primitive groups, $G^C$ then contains $A_{17}$ or $PSL(2, 16)$, or is a subgroup of $AGL(1, 17)$.

Next we shall prove that $G$ does not have an element of order 3. Otherwise, since $3 \nmid b$, we have $3 || G_B|$. If $3 || G_B|$, let $T$ be a Sylow 3-subgroup of $G_B$, then $|\text{Fix } T \cap B| = 6$. Because $3 \nmid 289$, $\text{Fix } T \nsubseteq B$. By Lemma 2.3, there exists a 2-$(v_0, 6, 1)$ design on $\text{Fix } T$, where $v_0 = |\text{Fix } T|$. Further, $N_G(T)$ acts as a block transitive group on this design. Since

$$6^2 - 6 + 1 \leq |\text{Fix } T| \leq r + k - 3 = 42,$$

and $6 - 1 = 5|v_0 - 1$, we have $v_0 \in \{31, 36, 41\}$. Also since $v_0 \equiv 289 \pmod{3}$, $v_0 \neq 36, 41$. Then $v_0 = 31$. Because of the transitivity of $N_G(T)$ on this design, $31 || |G|$. This is impossible.

Suppose $3 || G_B|$ and let $T$ be a Sylow 3-subgroup of $G_B$. Since $3 \nmid 289$, $|\text{Fix } T \cap B| = 6$. By Lemma 2.3, there exists a 2-$(v_0, 9, 1)$ design on $\text{Fix } T$, where $v_0 = |\text{Fix } T|$. Hence $v_0 \geq 9^2 - 9 + 1 = 73$. But by Lemma 2.4, $v_0 \leq r + k - 3 = 42$. We get a contradiction. Hence $3 \nmid |G|$ and $G^C$ is a subgroup of $AGL(1, 17)$.

Let $K$ be the kernel of the action of $G$ on $C$. Since $3 \nmid |G|$, we have $3 \nmid |K|$. And then the induced group of $K$ on every $\Delta_i$ is contained in $AGL(1, 17)$. Hence $G$ is solvable. This is a contradiction.

*Case (V) in Table 3 cannot occur.*

In Table 3, $G$ has 7 imprimitivity blocks $\Delta_0, \Delta_1, \ldots, \Delta_6$ with size 31. Also every block of design $D$ contains 5 inner pairs. Then we can suppose

$$B = \{\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \gamma_1, \gamma_2, \delta_1, \delta_2\},$$

where $\alpha_1, \alpha_2, \alpha_3 \in \Delta_0$, $\beta_1, \beta_2 \in \Delta_1$, $\gamma_1, \gamma_2 \in \Delta_2$ and $\delta_i \in \Delta_{i+2} (i = 1, 2)$.

Now we claim that there is no element of order 5 in $G$. Otherwise, let $T$ be a Sylow 5-subgroup of $G$. Since $5 \nmid b$, there is a block, say $B$, such that $T \leq G_B$. By Lemma 2.3, we have $|\text{Fix } T \cap B| = 9^2 - 9 + 1 = 73$. But by Lemma 2.4, $|\text{Fix } T| \leq r + k - 3 = 33$. Hence $G_B$ does not have an element of order 31.

Let $K$ be the subgroup of $G$, consisting of the elements of $G$ fixing every imprimitivity block, and let $\bar{G} = G/K$. Because $5 \nmid |G|$, $\bar{G}$ contains $PSL(2, 7)$, or is a subgroup of $AGL(1, 7)$. Moreover $G_{\Delta_0}$ is a primitive group of degree 31. Also since $5 \nmid |G|$, $G_{\Delta_0} \leq AGL(1, 31)$.

Next we shall prove that $K$ acts faithfully on every imprimitivity block. It is sufficient to prove that $K$ acts faithfully on $\Delta_0$. Let $L$ be the kernel of the action of $K$ on $\Delta_0$. If $L \neq 1$, then there exists an imprimitivity block $\Delta_j$ such that $L$ acts nontrivially on $\Delta_j$. Because $L \triangleleft K$, $L$ is transitive on $\Delta_j$. Moreover, the Sylow 31-subgroup of $L$ fixes every
point of $\Delta_0$. Then the Sylow 31-subgroup fixes some block. This contradicts the fact $31 \nmid |G_B|$. Thus we have $L = 1$ and $K$ acts faithfully on $\Delta_0$.

Let $g \in G$ such that $g$ fixes every point of $\Delta_0$. Because of the point distribution of the blocks, there exists one block, denoted by $D$, such that $D$ intersects $\Delta_0$ at exactly two points. Then $g$ fixes $D$. Suppose $D$ intersects $\Delta_j$ at three points, then $g$ fixes $\Delta_j$ too. Since $G_{\Delta_0} \leq AGL(1, 31)$, we have $g \in K$. On the other hand, since $K$ acts faithfully on $\Delta_0$, $g = 1$. This implies that $G_{\Delta_0}$ acts faithfully on $\Delta_0$. Hence $|G| = 7|G_{\Delta_0}|$ is a divisor of $7|AGL(1, 31)|$. In particular, $4 \nmid |G|$. Hence $G$ is solvable, a contradiction.

Case (VI) in Table 3 cannot occur.

In Table 3, $G$ has 31 imprimitivity blocks $\Delta_0, \Delta_1, \ldots, \Delta_{30}$ with size 7. Without loss of generality, we may assume

$$B = \{\alpha_1, \alpha_2, \gamma_1, \gamma_2, \ldots, \gamma_7\}$$

where $\alpha_1, \alpha_2 \in \Delta_0$ and $\gamma_i \in \Delta_i, i = 1, 2, \ldots, 7$. Let $C = \{\Delta_0, \Delta_1, \ldots, \Delta_{30}\}$. Then $G^C$ is a primitive group of degree 31. So the socle of $G^C$ is $A_{31}, PSL(3, 5)$ or $PSL(5, 2)$, or $G^C$ is a subgroup of $AGL(1, 31)$.

Next we shall prove that $G$ does not have an element of order 5. Otherwise, let $T$ be a Sylow 5-subgroup of $G$. Since $5 \nmid |B|$, there is a block, say $B$, such that $T \leq G_B$. If $5 \nmid |G^B|$, by Lemma 2.3, we have $Fix T \nsubseteq B$ since $5 \nmid (217 - 9)$ and the induced structure on $Fix T$ is a 2-(v0, 9, 1) design, where $v_0 = |Fix T|$. By Lemma 2.4, we have

$$r + k - 3 = 33 \geq v_0 \geq 9^2 - 9 + 1 = 73.$$ 

This is a contradiction. If $5||G^B|$, then $|Fix T \cap B| = 4$. Because $5 \nmid (217 - 4)$, Fix $T \nsubseteq B$. By Lemma 2.3, there exists a 2-$(v_0, 4, 1)$ design on $Fix T$, where $v_0 = |Fix T|$. Further, $N_G(T)$ acts as a block transitive group on this design. Since

$$4^2 - 4 + 1 = 13 \leq |Fix T| \leq r + k - 3 = 33,$$

and $4 = 1 = 3|v_0 - 1$, we have $v_0 = 3n + 1$ (n = 4, 5, ..., 10). But this contradicts the fact $v_0 \equiv 217$ (mod 5). Hence $G$ does not have an element of order 5 and the socle of $G^C$ is not $A_{31}, PSL(3, 5)$, or $PSL(5, 2)$. Consequently, we have $G^C \leq AGL(1, 31)$.

Now we shall prove that $G$ does not have an element of order 3. Otherwise, let $T$ be a Sylow 3-subgroup of $G$. Since 3 cannot divide $b$, we have $T \leq G_B$. By the point distribution of $B$, $T$ then fixes at least 3 points of $B$, say $\alpha_1, \alpha_2, \gamma_1$. Let $g \in T$ and $\bar{g}$ be the induced element of $g$ in $G^C$. Since $\alpha_i^g = \alpha_i$ (i = 1, 2) and $\gamma_1^g = \gamma_1$, we have $\Delta_0^g = \Delta_0$ and $\Delta_1^g = \Delta_1$. Then $\bar{g}$ fixes $\Delta_0$ and $\Delta_1$. Since $G^C$ is a subgroup of $AGL(1, 31)$, we have $\bar{g} = \bar{1}$. Hence $\Delta_0^g = \Delta_i$ (i = 0, 1, ..., 30). It is evident that $|Fix T \cap \Delta_i| \geq 1$ (i = 1, 2, ..., 30) and $|Fix T \cap \Delta_0| \geq 4$. Then $|Fix T| \geq 34$. But by Lemma 2.4, we also have $|Fix T| \leq r + k - 3 = 33$. This is a contradiction.

Let $K$ be the kernel of the action of $G$ on $C$. Since $3 \nmid |G|$, we have $K^{\Delta_i}$ is a subgroup of $AGL(1, 7)$. Hence $K^{\Delta_i}$ is a solvable group. It follows easily that $K$ is solvable, and thence $G$ is solvable. This is a contradiction. □

Acknowledgments

The authors would like to thank the referees for their valuable comments and suggestions on this paper. The first author thanks Professor Shiu-chun Wong and Professor Huiling Li for their guidance and help.

References