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Painlevé III and a singular linear statistics in Hermitian random matrix ensembles, I

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Abstract

In this paper, we study a certain linear statistics of the unitary Laguerre ensembles, motivated in part by an integrable quantum field theory at finite temperature. It transpires that this is equivalent to the characterization of a sequence of polynomials orthogonal with respect to the weight

$$w(x) = w(x,s) := x^{\alpha}e^{-x}e^{-s/x}, \quad 0 \le x < \infty, \alpha > 0, s > 0,$$

namely, the determination of the associated Hankel determinant and recurrence coefficients. Here $w(x,s)$ is the Laguerre weight $x^{\alpha}e^{-x}$ perturbed by a multiplicative factor $e^{-s/x}$, which induces an infinitely strong zero at the origin.

For polynomials orthogonal on the unit circle, a particular example where there are explicit formulas, the weight of which has infinitely strong zeros, was investigated by Pollaczek and Szegö many years ago. Such weights are said to be singular or irregular due to the violation of the Szegö condition.

In our problem, the linear statistics is a sum of the reciprocal of positive random variables $\{x_j : j = 1,\dots,n\}$; $\sum_{j=1}^{n} 1/x_j$.

We show that the moment generating function, or the Laplace transform of the probability density function of this linear statistics, can be expressed as the ratio of Hankel determinants and as an integral involving a particular third Painlevé function.

Keywords: Orthogonal polynomials; Painlevé functions and integrable systems

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1. Introduction

It is a well known fact that the joint probability density of the eigenvalues, $\{x_j : j = 1, \dots, n\}$ of any Hermitian matrix ensemble is [34]

$$p(x_1, \dots, x_n)\mathrm{d}x_1 \dots \mathrm{d}x_n = \frac{1}{D_n[w]} \prod_{1 \le j < k \le n} (x_j - x_k)^2 \prod_{l=1}^{n} w(x_l)\mathrm{d}x_l, \tag{1.1}$$

where D_n, the normalization constant, reads

$$D_n[w] = \frac{1}{n!} \int_{\mathbb{R}_+^n} \prod_{1 \le j < k \le n} (x_j - x_k)^2 \prod_{l=1}^{n} w(x_l)\mathrm{d}x_l. \tag{1.2}$$

For the sake of concreteness the domain of integration is $\mathbb{R}_+^n := [0, \infty)^n$.

Here w (≥ 0) is a weight supported on $\mathbb{R}_+$. Furthermore we suppose that w has moments of all orders, that is,

$$\mu_j := \int_{\mathbb{R}_+} x^j w(x)\mathrm{d}x, \quad j \in \{0, 1, \dots\}, \tag{1.3}$$

exist.

We shall see that D_n will play a fundamental role in this paper.

It is also well known that the normalization constant defined above has two more alternative representations; the first as the determinant of the Hankel or moment matrix and the second as a product,

$$\begin{aligned} D_n[w] &= \det \left(\mu_{j+k}\right)_{j,k=0}^{n-1} \\ &= \prod_{j=0}^{n-1} h_j. \end{aligned} \tag{1.4}$$

The quantity h_j in the second equality of (1.4) is the square of the weighted L^2 norm of the monic polynomials of degree j, orthogonal with respect to w;

$$\int_{\mathbb{R}_+} P_j(x) P_k(x) w(x)\mathrm{d}x = h_j \delta_{j,k}. \tag{1.5}$$

The random variable known as the linear statistics is a sum of a function of the random variables $\{x_j : j = 1, \dots, n\}$;

$$\sum_{j=1}^{n} f(x_j),$$

whose probability density function is determined by the standard formula,

$$\mathbb{P}_f(Q) = \frac{1}{n!} \int_{\mathbb{R}_+^n} p(x_1, \dots, x_n)\delta\left(Q - \sum_{j=1}^{n} f(x_j)\right) \mathrm{d}x_1 \dots \mathrm{d}x_n. \tag{1.6}$$

The moment generating function denoted as $M_f(s)$ (assuming $f(x) > 0$ for all $x \in \mathbb{R}_+$) is the Laplace transform of $\mathbb{P}_f(Q)$ with respect to s and has the following form:

$$M_f(s) = \int_0^\infty \mathbb{P}_f(Q)\mathrm{e}^{-sQ}\mathrm{d}Q \tag{1.7}$$

$$= \frac{\det\left(\mu_{j+k}(s)\right)_{j,k=0}^{n-1}}{\det\left(\mu_{j+k}(0)\right)_{j,k=0}^{n-1}} = \frac{D_n[w(.,s)]}{D_n[w(.,0)]} = \frac{\prod_{j=0}^{n-1} h_j(s)}{\prod_{j=0}^{n-1} h_j(0)}, \tag{1.8}$$

where

$$\mu_j(s) := \int_0^\infty x^j w(x)\mathrm{e}^{-sf(x)}\mathrm{d}x, \quad j \in \{0, 1, \ldots\} \tag{1.9}$$

and

$$h_j(s) = \int_0^\infty P_j^2(x)w(x)\mathrm{e}^{-sf(x)}\mathrm{d}x, \tag{1.10}$$

are the moments and the square of the L^2 norm of the polynomials P_j orthogonal with respect to $w\exp(-sf)$, respectively. Therefore we see that the moment generating function is the ratio of the Hankel determinant generated by the perturbed weight $w\exp(-sf)$ to the corresponding quantity generated by the original weight w.

Because the moments depend on s, the coefficients of the polynomials $P_j(z)$ also depend on s; however, we shall not display this dependence most of the time.

Our monic polynomials are normalized so that

$$P_n(z,s) = z^n + \mathsf{p}_1(n,s)\, z^{n-1} + \cdots + P_n(0,s), \tag{1.11}$$

with $P_0(z,s) := 1$ and $\mathsf{p}_1(0,s) := 0$.

We note here Heine's multiple-integral representation of $P_n(z)$,

$$P_n(z) = \frac{1}{n!}\int_{\mathbb{R}_+^n} \prod_{j=1}^n (z - x_j)p(x_1,\ldots,x_n)\mathrm{d}x_1\cdots\mathrm{d}x_n. \tag{1.12}$$

In Section 2, a description is given for a pair of ladder operators for smooth weights. These will lead to a linear second-order ordinary differential equation satisfied by $P_n(z)$ and two fundamental compatibility conditions valid for all $z \in \mathbb{C}\cup\{\infty\}$.

We denote the compatibility conditions as (S_1) and (S_2).

The compatibility conditions are essentially a consequence of the recurrence relations:

$$zP_n(z) = P_{n+1}(z) + \alpha_n(s)P_n(z) + \beta_n(s)P_{n-1}(z), \tag{1.13}$$

together with the initial conditions: $P_0(z) = 1$, and $\beta_0 P_{-1}(z) = 0$, and the Christoffel–Darboux formula (also a consequence of (1.13)).

In (1.13),

$$\alpha_n(s) \in \mathbb{R}, \quad n = 0, 1, \ldots$$

and

$$\beta_n(s) = \frac{h_n}{h_{n-1}} = \frac{D_{n+1}D_{n-1}}{D_n^2} > 0, \quad n = 1, 2, \ldots$$

are the recurrence coefficients.

An easy consequence of (1.11) and (1.13) is that

$$\mathsf{p}_1(n,s) - \mathsf{p}_1(n+1,s) = \alpha_n(s). \tag{1.14}$$

Taking a telescopic sum of (1.14), together with $\mathsf{p}_1(0,s) = 0$, implies

$$\sum_{j=0}^{n-1} \alpha_j(s) = -\mathsf{p}_1(n,s). \tag{1.15}$$

We refer the readers to [40] for more on the basic facts about orthogonal polynomials.

In our problem,

$$f(x) := \frac{1}{x}, \quad 0 \le x < \infty, \tag{1.16}$$

and the unperturbed weight is given by the equation

$$w_0(x) = x^{\alpha} \mathrm{e}^{-x}, \quad \alpha > 0.$$

The compatibility conditions (S_1) and (S_2), and a combination of these, $(S_2{}')$, produce a pair of non-linear difference equations, satisfied by the auxiliary quantities a_n and b_n. See (2.16) and (2.17).

The recurrence coefficients α_n and β_n are ultimately expressed in terms of a_n and b_n. See (2.9) and (2.14).

The linear statistics (1.16) leads to the weight

$$w(x,s) = w_0(x)\mathrm{e}^{-s/x} := x^{\alpha}\mathrm{e}^{-x}\mathrm{e}^{-s/x}, \quad \alpha > 0, s \ge 0.$$

Such weights arise from a certain problem in mathematical physics: An integrable quantum field theory at finite temperature [31].

In the theory of orthogonal polynomials, the effect of infinitely strong zeros on the Hankel determinants, recurrence coefficients and polynomials themselves is of considerable interest.

For orthogonal polynomials with weight w supported on $[-1,1]$, the classical Szegö theory gives a comprehensive account of the large n behavior of the recurrence coefficients and the polynomials (both outside $[-1,1]$ and on $(-1,1)$), if w is absolutely continuous and satisfies the Szegö condition,

$$\int_{-1}^{1} \frac{|\ln w(x)|}{\sqrt{1-x^2}} \mathrm{d}x < \infty.$$

See [pp. 296–312, [40]], [24,35] regarding Szegö's theory.

However, there is a class of orthogonal polynomials discovered by Pollaczek and extended by Szegö which is in some sense irregular. See [pp. 393–400, [40]] and [41] on this class of orthogonal polynomials. The Pollaczek–Szegö weight behaves like

$$\exp\left(-\frac{c}{\sqrt{1-x^2}}\right), \quad c > 0,$$

near ± 1, and consequently just violates the Szegö condition.

We reproduce here some of the results of [41] to illustrate the irregularity.

Associated with the weight

$$w(x;a,b) := \frac{\mathrm{e}^{(2\theta-\pi)\phi(\theta)}}{\cosh[\pi\phi(\theta)]}$$

where $x := \cos\theta, 0 < \theta < \pi$ and

$$\phi(\theta) := \frac{a\cos\theta + b}{2\sin\theta}, \quad a, b \in \mathbb{R}, a \geq |b|,$$

are the normalized Pollaczek polynomials $\{p_n(x; a, b)\}$,

$$\int_{-1}^{1} [p_n(x; a, b)]^2 w(x; a, b) \mathrm{d}x = 1.$$

If $x \to 1$, then

$$w(x; a, b) \simeq 2\mathrm{e}^{(a+b)(1-\pi/\theta)}, \quad \text{as } \theta \to 0,$$

which shows that the weight vanishes exponentially at $x = 1$. An easy computation demonstrates the same behavior at $x = -1$.

The large n behaviors of $p_n(x; a, b)$ are as follows:

(a) $p_n(1; a, b) \sim n^{1/4}\mathrm{e}^{2\sqrt{a+b}\sqrt{n}}$,

(b) $p_n(x; a, b) \sim n^K [x + (x^2 - 1)]^n, \qquad K = K(x) = \dfrac{ax + b}{2\sqrt{x^2 - 1}}, \quad x \notin [-1, 1]$,

(c) $p_n(\cos\theta; a, b) = A(\theta)\cos[n\theta - \phi(\theta)\ln n + B(\theta)] + \epsilon_n(\theta)$,
$\lim_{n\to\infty} \epsilon_n(\theta) = 0, 0 < \theta < \pi$,

where the n independent functions $A(\theta)$ (> 0), $B(\theta)$ are analytic in $(0, \pi)$.

This is to be contrasted with the large n behavior of the normalized Jacobi polynomials $\{p_n(x)\}$ associated with the weight

$(1 - x)^\alpha (1 + x)^\beta, \quad \alpha > -1, \beta > -1, x \in [-1, 1]$,

(a$'$) $p_n(1) \sim n^{\alpha+1/2}$,

(b$'$) $p_n(x) \sim \left[x + \sqrt{x^2 - 1}\right]^n, \quad x \notin [-1, 1]$

(c$'$) $p_n(\cos\theta) = A_1(\theta)\cos[n\theta + B_1(\theta)] + \epsilon_n(\theta)$,

where $A_1(\theta)$ and $B_1(\theta)$ are functions of the same kind.

The symbol $\sim$ indicates that the ratio of the given quantities approaches a non-zero limit, while $\simeq$ indicates that the limit is 1. This is the convention adopted by [41] and it will not be used later.

Our paper is a first step in the study of the Pollaczek–Szegö type orthogonal polynomials supported on infinite intervals.

With reference to the Heine formula, we see that

$$(-1)^n P_n(0, s) = \frac{D_n[w(., s, \alpha + 1)]}{D_n[w(., s, \alpha)]}. \tag{1.17}$$

For the unperturbed or Laguerre weight, we have the explicit evaluation

$$(-1)^n P_n(0, 0) = \frac{\Gamma(n + \alpha + 1)}{\Gamma(\alpha + 1)}, \tag{1.18}$$

since

$$D_n[w(.,0,\alpha)] = \frac{G(n+1)G(n+\alpha+1)}{G(\alpha+1)}, \tag{1.19}$$

where $G(z)$ is the Barnes G-function that satisfies the functional relation $G(z+1) = \Gamma(z)G(z)$, together with $G(1) = 1$.

In Section 3, by taking the derivative with respect to s on the orthogonality relations we obtain a pair of differential–difference equations or the Toda equations. Combining the Toda equations and the non-linear difference equations obtained in Section 2, we produce a particular Painlevé III equation satisfied by $\alpha_n(s)$, up to linear shift in n.

The τ-function for this $P_{\rm III}$ turns out to be intimately related to the Hankel determinant

$$D_n(s) = \det\left(\int_0^\infty x^{j+k}\, x^\alpha \mathrm{e}^{-x-s/x}\right)_{j,k=0}^{n-1}. \tag{1.20}$$

We also express the recurrence coefficients α_n and β_n in terms of the logarithmic derivative of the Hankel determinant

$$H_n := s\frac{\mathrm{d}}{\mathrm{d}s}\ln D_n(s),$$

and obtain a functional equation involving H_n, H_n' and H_n''. The resulting second-order non-linear ordinary differential equation satisfied by H_n is recognized to be the Jimbo–Miwa–Okamoto σ-form of our $P_{\rm III}$.

In Section 4, we show, with the aid of the non-linear difference equations derived in Section 2, another functional equation involving H_n, H_{n+1} and H_{n-1}. We call the resulting non-linear second-order difference equation satisfied by H_n the discrete σ-form of $P_{\rm III}$.

In Section 5, the Riemann–Hilbert approach to orthogonal polynomials and the isomonodromy deformation theory of Jimbo and Miwa are used to re-derive the $P_{\rm III}$ and thereby identify the auxiliary quantities, a_n and b_n, introduced in Section 2, with the objects of the Jimbo–Miwa isomonodromy theory of Painlevé equations.

In Section 6, we show that the Hankel determinant is the isomonodromy τ-function in the sense of Jimbo and Miwa, and put into the context of the general theory of integrable systems the identities derived in Sections 2 and 3.

As we are studying an example of orthogonal polynomials where the otherwise classical weight

$$w_0(x) = x^\alpha \mathrm{e}^{-x}, \quad x \in \mathbb{R}_+, \alpha > 0$$

is perturbed by an infinitely strong zero,

$$w(x,s) := w_0(x)\mathrm{e}^{-s/x}, \quad s \geq 0,$$

the natural questions of interest are about the large n behavior of the Hankel determinant, recurrence coefficients and the orthogonal polynomials. Such investigations will therefore provide valuable insights into the asymptotics of the associated Painlevé transcendent. These results will be published in a forthcoming paper [14].

We want to emphasize that in this paper and in its follow-up we do not claim the introduction of new concepts. Our main aim is to investigate a concrete important example of the linear statistics which leads to a strong zero at $x = 0$ using the known techniques. In addition, taking this example as a "case study" we show how the types of apparatus which are used by the two communities — the orthogonal polynomial community and the integrable system community — match with each other.

2. Ladder operators and non-linear difference equations

The pair of ladder operators has been known to various authors over many years. Here we provide a brief guide to the literature on this subject. See, for example, [3,4,10,7,9,8,12,19,20,32,33,38]. In fact, Magnus in [33] noted that such operators were known to Laguerre. However, we find that (2.1)–(2.5) suit our purpose well.

Because the associated fundamental compatibility conditions and their use in the derivation of the Painlevé transcendent [16,11,19,21,33] are perhaps less well known we summarize these findings in (2.1)–(2.5), (S_1), (S_2) and (S_2') in a form which we find particularly easy to use. We note here that (S_1), (S_2) and (S_2') were also known to Magnus [33] and (S_2) also appeared in [26]. See also [21].

For polynomials orthogonal on the unit circle the analogous ladder operators can be found in [27]. See [22] for the circular case applicable to bi-orthogonal polynomials. The compatibility condition in the circular case can be found in [1] where it was used to obtain in explicit form the Toeplitz determinant generated from the pure Fisher–Hartwig symbol and the discriminant of the associated orthogonal polynomials.

The compatibility conditions can also be adapted to the situation where the weight has discontinuities. See [16,2].

The ladder operators are

$$\left(\frac{\mathrm{d}}{\mathrm{d}z} + B_n(z)\right) P_n(z) = \beta_n A_n(z) P_{n-1}(z), \tag{2.1}$$

$$\left(\frac{\mathrm{d}}{\mathrm{d}z} - B_n(z) - \mathsf{v}'(z)\right) P_{n-1}(z) = -A_{n-1}(z) P_n(z), \tag{2.2}$$

where

$$A_n(z) = \frac{1}{h_n}\int_0^\infty \frac{\mathsf{v}'(z)-\mathsf{v}'(y)}{z-y} P_n^2(y) w(y)\mathrm{d}y, \tag{2.3}$$

$$B_n(z) = \frac{1}{h_{n-1}}\int_0^\infty \frac{\mathsf{v}'(z)-\mathsf{v}'(y)}{z-y} P_n(y) P_{n-1}(y) w(y)\mathrm{d}y, \tag{2.4}$$

$$\mathsf{v}(z) := -\ln w(z), \tag{2.5}$$

and the associated fundamental compatibility conditions are

$$B_{n+1}(z) + B_n(z) = (z-\alpha_n) A_n(z) - \mathsf{v}'(z), \tag{S$_1$}$$

$$1 + (z-\alpha_n)(B_{n+1}(z) - B_n(z)) = \beta_{n+1} A_{n+1}(z) - \beta_n A_{n-1}(z). \tag{S$_2$}$$

In the case of rational $\mathsf{v}'(z)$, these compatibility conditions are valid for all $z \in \mathbb{C}\cup\{\infty\}$. See [13] for a recent derivation of the compatibility conditions. To arrive at Eqs. (2.3) and (2.4) we have assumed that $w(0) = w(\infty) = 0$. This is certainly the situation for our problem since we have assume that $\alpha > 0$ and $s \geq 0$.

Combining suitably (S_1) and (S_2) gives an expression involving $\sum_{j=0}^{n-1} A_j(z)$, $B_n(z)$ and $\mathsf{v}'(z)$ from which further insight into recurrence coefficients may be gained.

The equation (S_2') may be thought of as the first integral of (S_1) and (S_2).

Although (S_2') first appeared in [33] in a slightly different form, we present here a derivation of a version which we find useful in practice.

Multiplying (S_2) by $A_n(z)$ we see that the r.h.s. of the resulting equation is a first-order difference, while the l.h.s., with $(z-\alpha_n)A_n(z)$ replaced by $B_{n+1}(z)+B_n(z)+\mathsf{v}'(z)$ is a first-order difference plus $A_n(z)$. Taking a telescopic sum, together with the initial conditions $B_0(z)=A_{-1}(z)=0$, produces the lemma:

Lemma 1.

$$B_n^2(z)+\mathsf{v}'(z)B_n(z)+\sum_{j=0}^{n-1}A_j(z)=\beta_nA_n(z)A_{n-1}(z). \tag{S_2'}$$

If $\mathsf{v}'(z)$ is rational then so are $A_n(z)$ and $B_n(z)$. See (2.3) and (2.4). Furthermore, eliminating $P_{n-1}(z)$ from (2.1) and (2.2) it is easy to show that $y(z):=P_n(z)$ satisfies the second-order linear ordinary differential equation

$$y''(z)-\left(\mathsf{v}'(z)+\frac{A_n'(z)}{A_n(z)}\right)y'(z)+\left(B_n'(z)-B_n(z)\frac{A_n'(z)}{A_n(z)}+\sum_{j=0}^{n-1}A_j(z)\right)y(z)=0. \tag{2.6}$$

Note that (S_2') has been used to simplify the coefficient of $y(z)$ in (2.6).

Eq. (2.6) can also be found in [38], albeit in a different form.

For the problem at hand, $f(x)=1/x$, $x\ge 0$, the weight and associated quantities are

$$w(x,s)=x^{\alpha}\mathrm{e}^{-x-s/x},$$
$$\mathsf{v}(z)=z+s/z-\alpha\ln z,\qquad \mathsf{v}'(z)=1-s/z^2-\alpha/z,$$
$$\frac{\mathsf{v}'(z)-\mathsf{v}'(y)}{z-y}=\frac{1}{z}\left(\frac{\alpha}{y}+\frac{s}{y^2}\right)+\frac{s}{z^2y}.$$

Using these we have the next lemma.

Lemma 2. *The coefficients $A_n(z)$ and $B_n(z)$ appearing in the ladder operators are*

$$A_n(z)=\frac{1}{z}+\frac{a_n}{z^2}, \tag{2.7}$$
$$B_n(z)=-\frac{n}{z}+\frac{b_n}{z^2}, \tag{2.8}$$
$$a_n:=\frac{s}{h_n}\int_0^\infty\frac{P_n^2}{y}w\mathrm{d}y,\qquad a_n(0)=0,$$
$$b_n:=\frac{s}{h_{n-1}}\int_0^\infty\frac{P_nP_{n-1}}{y}w\mathrm{d}y,\qquad b_n(0)=0.$$

Proof. From the definitions of $A_n(z)$ and $B_n(z)$ and with the identities

$$1=\frac{1}{h_n}\int_0^\infty\left(\frac{\alpha}{y}+\frac{s}{y^2}\right)P_n^2w\mathrm{d}y,$$
$$-n=\frac{1}{h_{n-1}}\int_0^\infty\left(\frac{\alpha}{y}+\frac{s}{y^2}\right)P_nP_{n-1}w\mathrm{d}y,$$

obtained by integration by parts, we find (2.7) and (2.8). □

We see that at this stage there are four unknowns, α_n, β_n, a_n and b_n. In what follows we will show how (S$_1$) and (S$_2{}'$) can be applied to obtain amongst other things the pair of non-linear difference equations involving a_n and b_n mentioned earlier.

Equating the residues on both sides of (S$_1$), we find

$$\alpha_n = 2n + 1 + \alpha + a_n, \tag{2.9}$$

$$b_{n+1} + b_n = s - \alpha_n a_n. \tag{2.10}$$

Carrying out a similar calculation with (S$_2{}'$) gives

$$\beta_n = n(n+\alpha) + b_n + \sum_{j=0}^{n-1} a_j, \tag{2.11}$$

$$\beta_n(a_n + a_{n-1}) = ns - (2n+\alpha)b_n, \tag{2.12}$$

$$b_n^2 - sb_n = \beta_n a_n a_{n-1}. \tag{2.13}$$

The upshot of these equations is that α_n and β_n are entirely determined by a_n and b_n, where α_n is simply a_n plus $2n + 1 + \alpha$.

Eliminating a_{n-1} from (2.12) and (2.13) we have the next lemma.

Lemma 3.

$$\beta_n a_n^2 = [ns - (2n+\alpha)b_n]a_n - (b_n^2 - sb_n). \tag{2.14}$$

Therefore (2.14) expresses β_n in terms of a_n and b_n, and importantly bypasses the finite sum in (2.11). Eliminating β_n from (2.11) and (2.14), an expression can be found for $\sum_{j=0}^{n-1} a_j$, in terms of a_n and b_n.

We state this in the next lemma.

Lemma 4.

$$\sum_{j=0}^{n-1} a_j = -n(n+\alpha) - b_n + \frac{ns - (2n+\alpha)b_n}{a_n} - \frac{b_n^2 - sb_n}{a_n^2}. \tag{2.15}$$

Note that because $a_n(0) = b_n(0) = 0$, (2.9) and (2.11) reduce to $\alpha_n(0) = 2n + 1 + \alpha$ and $\beta_n(0) = n(n+\alpha)$, respectively, which we recognize as the recurrence coefficients of the monic Laguerre polynomials.

In summary, with reference to (2.13) and (2.14), we obtain two non-linear difference equations, satisfied by a_n and b_n,

$$b_{n+1} + b_n = s - (2n + 1 + \alpha + a_n)a_n, \tag{2.16}$$

$$(b_n^2 - sb_n)(a_n + a_{n-1}) = [ns - (2n+\alpha)b_n]a_n a_{n-1}, \tag{2.17}$$

to be iterated in n with the initial conditions

$$a_0(s) = \sqrt{s}\,\frac{K_\alpha(2\sqrt{s})}{K_{\alpha+1}(2\sqrt{s})}, \tag{2.18}$$

$$b_0(s) = 0, \tag{2.19}$$

where $K_\alpha(z)$ is the MacDonald function of the second kind.

We call (2.16) and (2.17) together with the initial conditions (2.18) and (2.19) the MacDonald hierarchy. See also [4] for a general treatment of a class of semi-classical weights.

Solutions for a_n and b_n that are rational functions of $2\sqrt{s}$ are found for $\alpha = p + 1/2,\ \ p \in \mathbb{Z}$, since

$$K_{p+1/2}(z) = K_{-p-1/2}(z) = \sqrt{\frac{\pi}{2z}}\mathrm{e}^{-z}\sum_{k=0}^{p}\frac{(p+k)!}{k!(p-k)!(2z)^k}.$$

3. Toda evolution and Painlevé III

Taking derivatives with respect to s on the orthogonality relation will give rise to Toda type equations, which we discuss below.

Because

$$h_n = \int_0^\infty P_n^2 w \mathrm{d}y,$$

we have

$$s\frac{\mathrm{d}}{\mathrm{d}s}h_n = -s\int_0^\infty \frac{P_n^2}{y} w \mathrm{d}y,$$

and hence

$$s\frac{\mathrm{d}}{\mathrm{d}s}\ln h_n = -a_n \tag{3.1}$$

$$s\frac{\mathrm{d}}{\mathrm{d}s}\ln \beta_n = a_{n-1} - a_n. \tag{3.2}$$

We also have

$$0 = \frac{\mathrm{d}}{\mathrm{d}s}\int_0^\infty P_n P_{n-1} w \mathrm{d}y \tag{3.3}$$

$$= h_{n-1}\frac{\mathrm{d}}{\mathrm{d}s}\mathsf{p}_1(n) - \int_0^\infty \frac{P_n P_{n-1}}{y} w \mathrm{d}y \tag{3.4}$$

$$s\frac{\mathrm{d}\mathsf{p}_1(n)}{\mathrm{d}s} = b_n \tag{3.5}$$

$$\begin{aligned} s\frac{\mathrm{d}\alpha_n}{\mathrm{d}s} &= s\frac{\mathrm{d}a_n}{\mathrm{d}s} \\ &= b_n - b_{n+1} \\ &= \beta_n - \beta_{n+1} + \alpha_n. \end{aligned} \tag{3.6}$$

Eq. (3.6) follows from (2.9), (1.14) and (2.11).

There is another identity involving $\sum_{j=0}^{n-1} a_j$:

$$s\frac{\mathrm{d}}{\mathrm{d}s}\sum_{j=0}^{n-1} a_j = -b_n, \tag{3.7}$$

which is an immediate consequence of a telescopic sum of the second equality of (3.6).

We now show that the Hankel determinant, D_n, is up to scaling transformation the τ-function of the Toda equations. Let

$$\widetilde{D}_n(s) := s^{-n(n+\alpha)} D_n(s).$$

We find, by summing (3.1),

$$s\frac{\mathrm{d}}{\mathrm{d}s}\ln D_n = -\sum_{j=0}^{n-1} a_j, \tag{3.8}$$

since

$$\sum_{j=0}^{n-1} \ln h_j = \ln D_n.$$

Applying $s\frac{\mathrm{d}}{\mathrm{d}s}$ to (3.8) and keeping in mind (3.7), (2.11) and (3.8) gives

$$\begin{aligned} s\frac{\mathrm{d}}{\mathrm{d}s}\left(s\frac{\mathrm{d}}{\mathrm{d}s}\ln D_n\right) &= b_n \\ &= \beta_n - n(n+\alpha) - \sum_{j=0}^{n-1} a_j \\ &= \beta_n - n(n+\alpha) + s\frac{\mathrm{d}}{\mathrm{d}s}\ln D_n. \end{aligned}$$

The last equation simplifies to

$$s^2\frac{\mathrm{d}^2}{\mathrm{d}s^2}\ln D_n(s) = \frac{D_{n+1}D_{n-1}}{D_n^2} - n(n+\alpha),$$

since

$$\beta_n = \frac{D_{n+1}D_{n-1}}{D_n^2}.$$

In terms of $\widetilde{D}_n(s)$ we have

$$\frac{\mathrm{d}^2}{\mathrm{d}s^2}\ln \widetilde{D}_n(s) = \frac{\widetilde{D}_{n+1}\widetilde{D}_{n-1}}{\widetilde{D}_n^2}. \tag{3.9}$$

Eq. (3.9) is the Toda molecule equation [39] and shows that $\widetilde{D}_n(s)$ is the corresponding τ-function of the Toda equations (3.2) and (3.6).

As the Hankel determinant is now identified with the τ-function (see Section 6 for more on this issue), we may expect the emergence of a Painlevé equation. In fact, $a_n(s)$ satisfies a particular P_{III}. To see this, we first investigate the evolution of a_n and b_n as functions of s.

Lemma 5. *For a fixed n, the auxiliary quantities a_n and b_n satisfy the following coupled Riccati equations:*

$$s\frac{\mathrm{d}a_n}{\mathrm{d}s} = 2b_n + (2n+1+\alpha+a_n)a_n - s \tag{3.10}$$

$$s\frac{\mathrm{d}b_n}{\mathrm{d}s} = \frac{2}{a_n}(b_n^2 - sb_n) + (2n+\alpha+1)b_n - ns. \tag{3.11}$$

Proof. Eq. (3.10) follows from applying $s\frac{\mathrm{d}}{\mathrm{d}s}$ to (2.9) together with the first equality of (3.6) and with (2.10) to replace b_{n+1} by $s - \alpha_n a_n - b_n$.

A little bit more work is required to prove (3.11). Apply $s\frac{\mathrm{d}}{\mathrm{d}s}$ to (2.11) to find

$$\begin{aligned} s\frac{\mathrm{d}\beta_n}{\mathrm{d}s} &= s\frac{\mathrm{d}b_n}{\mathrm{d}s} + s\frac{\mathrm{d}}{\mathrm{d}s}\sum_{j=0}^{n-1} a_j \\ &= s\frac{\mathrm{d}b_n}{\mathrm{d}s} - b_n \\ &= \beta_n a_{n-1} - \beta_n a_n \\ &= \frac{b_n^2 - sb_n}{a_n} - \left[ns - (2n+\alpha)b_n - \frac{b_n^2 - sb_n}{a_n}\right], \end{aligned}$$

where the last three equalities follow from (3.7), (3.2), (2.13) and (2.14). Now the second and last equalities imply (3.11). □

The next theorem identifies a_n as a particular third Painlevé function.

Theorem 1. *For a fixed* $n \in \{0, 1, 2, \ldots\}$ *the auxiliary quantity* a_n *satisfies*

$$a_n'' = \frac{(a_n')^2}{a_n} - \frac{a_n'}{s} + (2n+1+\alpha)\frac{a_n^2}{s^2} + \frac{a_n^3}{s^2} + \frac{\alpha}{s} - \frac{1}{a_n}, \tag{3.12}$$

with the initial conditions

$$a_n(0) = 0, \qquad a_n'(0) = \frac{1}{\alpha}, \quad \alpha > 0. \tag{3.13}$$

If $a_n(s) := -\mathsf{q}(s)$, *then* $\mathsf{q}(s)$ *is* $P_{\mathrm{III}'}(-4(2n+1+\alpha), -4\alpha, 4, -4)$, *following the convention of* [36].

Proof. Eliminating b_n from (3.10) and (3.11) gives (3.12). The initial conditions follow from a straightforward computation. □

Remark I. If $n = 0$, then $b_0 = 0$ and (3.10) is solved by

$$\sqrt{s}\frac{K_\alpha(2\sqrt{s})}{K_{\alpha+1}(2\sqrt{s})}$$

which is (2.18). We observe that the above also solves (3.12) for $n = 0$.

Remark II. An alternative form, obtained from

$$a_n(s) =: \frac{s}{X_n(s)},$$

reads

$$X_n'' = \frac{(X_n')^2}{X_n} - \frac{X_n'}{s} - \frac{\alpha X_n^2}{s^2} - \frac{2n+1+\alpha}{s} + \frac{X_n^3}{s^2} - \frac{1}{X_n}, \tag{3.14}$$

which is a $P_{\mathrm{III}'}(-4\alpha, -4(2n+1+\alpha), 4, -4)$. If the derivatives in (3.14) are neglected, then X_n solves the quartic

$$X^4 - \alpha X^3 - (2n+1+\alpha)sX - s^2 = 0. \tag{3.15}$$

We may interpret an appropriate solution of (3.15) as the geometric mean of the end points of the support of a single-interval equilibrium density. This appears in a potential theoretic minimization problem, the detail of which is in a forthcoming paper [14]. We note that another P_{III} associated with the Toeplitz determinant

$$\det[I_{j-k+\nu}(\sqrt{t})]_{0\le j,k\le n-1},$$

was found in [22] (see also [23]). Here $I_r(z)$ is the modified Bessel's function of the first kind. We note that this Toeplitz determinant can be thought of as a Toeplitz analog of the Hankel determinant of this paper. The above Toeplitz determinant with $\nu = 0$ appeared in connection with a certain ensemble of $n\times n$ unitary matrices and Ulam's problem in combinatorics. See [42].

In the next theorem, we display two alternative integral representations of D_n, in terms of a_n and X_n.

Theorem 2.

$$\ln\frac{D_n(s)}{D_n(0)} = \int_0^s\left[\frac{t}{2}-\frac{1}{4}\left(\frac{t}{a_n}-\alpha\right)^2 - a_n\left(n+\frac{\alpha}{2}\right)-\frac{a_n^2}{4}+\frac{1}{4}\left(1-\frac{ta_n'}{a_n}\right)^2\right]\frac{\mathrm{d}t}{t} \tag{3.16}$$

$$= \int_0^s\left[\frac{t}{2}-\frac{1}{4}(X_n-\alpha)^2-\left(n+\frac{\alpha}{2}\right)\frac{t}{X_n}-\frac{t^2}{4X_n^2}+\frac{t^2X_n'^2}{4X_n^2}\right]\frac{\mathrm{d}t}{t}. \tag{3.17}$$

Proof. From (3.8) and (2.15) we see that the logarithmic derivative of $D_n(s)$ is expressed in terms of a_n and b_n. If we use (3.10) to eliminate b_n in favor of a_n and a_n' from the resulting equation, then (3.16) follows after some simplification. Eq. (3.17) follows from the substitution $a_n(s) =: s/X_n(s)$. Note that $D_n(0)$ is given by (1.19). □

Put

$$H_n := s\frac{\mathrm{d}}{\mathrm{d}s}\ln D_n. \tag{3.18}$$

In Section 6 we will show that the Hankel determinant $D_n(s)$ can be identified with the Jimbo–Miwa τ-function corresponding to the solution $a_n(s)$ of the Painlevé III′. See (3.12). Namely, we will show that

$$D_n(s) = \text{const}\,\tau(s)\mathrm{e}^{\frac{s}{2}}s^{\frac{n(n+\alpha)}{2}}. \tag{3.19}$$

In turn, this relation yields the following formula for the quantity H_n:

$$H_n = \sigma(s)+\frac{s}{2}+\frac{n(n+\alpha)}{2}, \tag{3.20}$$

where $\sigma(s) \equiv s\frac{\mathrm{d}}{\mathrm{d}s}\ln\tau$ is the Jimbo–Miwa–Okamoto σ-function corresponding to the equation $P_{\text{III}'}$. Therefore, H_n, H_n' and H_n'' should satisfy a functional equation

$$f(H_n, H_n', H_n'', n, s,) = 0,$$

known as the Jimbo–Miwa–Okamoto σ-form of our $P_{\text{III}'}$.

With H_n defined above, it is easy to see from (3.7), (3.8) and (2.11) that

$$b_n = sH_n' \tag{3.21}$$

$$\beta_n = n(n+\alpha)+sH_n'-H_n. \tag{3.22}$$

In the next theorem we state the non-linear second-order ordinary differential equation satisfied by H_n.

Theorem 3. *If*

$$H_n := s\frac{\mathrm{d}}{\mathrm{d}s}\ln D_n(s), \tag{3.23}$$

then

$$(sH_n'')^2 = [n-(2n+\alpha)H_n']^2 - 4[n(n+\alpha)+sH_n'-H_n]H_n'(H_n'-1). \tag{3.24}$$

Proof. First we rewrite (2.14) and (3.11) as

$$\beta_n a_n + \frac{b_n^2 - sb_n}{a_n} = ns-(2n+\alpha)b_n \tag{3.25}$$

$$\frac{2}{a_n}(b_n^2-sb_n) = sb_n' - b_n + ns - (2n+\alpha)b_n \tag{3.26}$$

respectively. Eliminating a_n from (3.25) and (3.26) produces

$$\left(sb_n'-b_n\right)^2 = [ns-(2n+\alpha)b_n]^2 - 4\beta_n(b_n^2-sb_n). \tag{3.27}$$

Eq. (3.24) follows by substituting b_n and β_n from (3.21) and (3.22) into (3.27). □

Hence the recurrence coefficients α_n and β_n of the orthogonal polynomials associated with our weight,

$$w(x,s) = x^{\alpha}\mathrm{e}^{-x-s/x}, \quad 0\le x<\infty, \alpha>0, s>0,$$

are expressed in terms of H_n, H_n' and H_n'', as follows:

$$\alpha_n = 2n+1+\alpha+\frac{2s(H_n')^2 - sH_n'}{sH_n''+n-(2n+\alpha)H_n'} \tag{3.28}$$

$$\beta_n = n(n+\alpha)+sH_n'-H_n, \tag{3.29}$$

and H_n itself satisfies (3.24).

Remark III. With the identification (3.20) of the function H_n as the σ-function (up to a linear shift), Eq. (3.24) coincides, up to a change of the independent variable s to the variable $t=\sqrt{s}$, with equation (C.29) of [29][1].

4. Discrete σ-form

We may anticipate due to the recurrence relations and non-linear difference equations, (2.16) and (2.17), that, for a fixed t, H_n, and $H_{n\pm 1}$ would satisfy a discrete analog of (3.24). It turns out

[1] When comparing Eq. (3.24) with equation (C.29) of [29], one also has to take into account that, if we denote the σ-function of [29], for which equation (C.29) is written, as σ_{JM} then the precise relation with our σ-function is given by the equation, $\sigma_{JM}(t) = 2\sigma(t^2)$.

that in this instance a_n has a simpler expression in terms of $H_{n\pm1}$. We shall obtain an expression for b_n in terms of H_n and $H_{n\pm1}$. From

$$H_n = -\sum_{j=0}^{n-1} a_j,$$

we have

$$a_n = H_n - H_{n+1}, \tag{4.1}$$

and

$$a_n + a_{n-1} = H_{n-1} - H_{n+1} =: \delta^2 H_n.$$

Multiplying the above by β_n we find

$$\beta_n(a_n + a_{n-1}) = \beta_n \delta^2 H_n \tag{4.2}$$

$$= ns - (2n+\alpha)b_n, \tag{4.3}$$

where the last equation follows from (2.12). Now (2.11) becomes

$$\beta_n = n(n+\alpha) + b_n - H_n.$$

Substituting the above into (4.2) produces a linear equation in b_n whose solution is

$$b_n = \frac{ns + \delta^2 H_n[H_n - n(n+\alpha)]}{2n+\alpha+\delta^2 H_n}. \tag{4.4}$$

Hence the auxiliary quantities a_n and b_n, and the β_n, are now expressed in terms of H_n, and $H_{n\pm1}$. Substituting these into (2.13) give rise to the discrete σ-form stated in the next theorem.

Theorem 4. *If*

$$H_n := s\frac{\mathrm{d}}{\mathrm{d}s} \ln D_n,$$

then

$$\{[H_n - n(n+\alpha)]\delta^2 H_n + ns\}\{[H_n - n(n+\alpha) - s]\delta^2 H_n - (n+\alpha)s\}$$
$$= (2n+\alpha+\delta^2 H_n)\{ns + (2n+\alpha)[n(n+\alpha) - H_n]\}(H_n - H_{n+1})(H_{n-1} - H_n). \tag{4.5}$$

We also have the discrete analog of (3.28) and (3.29),

$$\alpha_n = 2n + 1 + \alpha + H_n - H_{n+1} \tag{4.6}$$

$$\beta_n = n(n+\alpha) + \frac{ns - n(n+\alpha)\delta^2 H_n - (2n+\alpha)H_n}{2n+\alpha+\delta^2 H_n}. \tag{4.7}$$

Now the obvious equalities, (4.6) = (3.28) and (4.7) = (3.29), imply two further differential–difference equations which H_n must satisfy:

$$H_n - H_{n+1} = \frac{2s(H_n')^2 - sH_n'}{sH_n'' + n - (2n+\alpha)H_n'} \tag{4.8}$$

$$\frac{ns - n(n+\alpha)\delta^2 H_n - (2n+\alpha)H_n}{2n+\alpha+\delta^2 H_n} = sH_n' - H_n. \tag{4.9}$$

Remark IV. From the point of view of general Jimbo–Miwa isomonodromy theory of Painlevé equations, which we will be discussing in Sections 5 and 6, Eq. (4.5) should be related to the Bäcklund–Schlesinger transformations of the τ-function. However, we failed to identify Eq. (4.5) with any of the difference equations for the τ-function discussed in [29] and to describe the possible Schlesinger transformations. Since it is written for the logarithmic derivative of the τ-function, and not for the τ-function itself as in [29], Eq. (4.5) might be in fact of a different nature than the ones considered in [29]. We would also like to mention that Eq. (4.5) is an integrable discrete equation — its Lax pair is formed by the first and the third equations of the triple (see (5.17)) of Section 5, and the equation itself, as has already been noticed, represents a Bäcklund–Schlesinger transformation of the third Painlevé equation. Therefore, we expect this equation to be equivalent to one of the known discrete Painlevé equations, which we have not yet identified. Apparently this identification is not quite straightforward. One of the referees has suggested that (4.5) could be a composition of the basic Schlesinger transformations T_1 and T_2 found in [23] (see Proposition (4.6) of [23]). We should also mention here that Okamoto's paper [37], the key reference for the symmetries and the transformation theory of the third Painlevé equation, might provide further insight into (4.5).

5. An alternative derivation of the Painlevé III equation

In this section we present an alternative derivation of the third Painlevé equation (3.12) for the quantity $a_n(s)$. This derivation is based on the Riemann–Hilbert point of view [19,20] on orthogonal polynomials and makes use of the general Jimbo–Miwa–Ueno theory of isomonodromy deformations. This in turn allows us to place some of the key identities of the preceding sections into the general framework of integrable systems.

The Riemann–Hilbert problem for the orthogonal polynomials at hand is the following:

- $Y : \mathbb{C} \setminus \mathbb{R}_+ \to \mathbb{C}^{2\times 2}$ is analytic.
- $Y_+(x) = Y_-(x) \begin{pmatrix} 1 & x^\alpha e^{-x-s/x} \\ 0 & 1 \end{pmatrix}$ for $x \in \mathbb{R}_+ \setminus \{0\}$, with $\mathbb{R}_+$ oriented from left to right.
- $Y(z) = (I + O(1/z)) \begin{pmatrix} z^n & 0 \\ 0 & z^{-n} \end{pmatrix}$ as $z \to \infty$.
- $Y(z) = O(1)$ as $z \to 0$.

Here $Y_\pm(z)$ denote the non-tangential limiting values of $Y(z)$ on $\mathbb{R}_+$ taken (in the usual pointwise sense) from the $\pm$ side. The Riemann–Hilbert problem has the unique solution expressed in terms of the orthogonal polynomials $P_n(z)$

$$Y(z) = \begin{pmatrix} P_n(z) & \dfrac{1}{2\pi i} \displaystyle\int_{\mathbb{R}_+} \dfrac{P_n(x) x^\alpha e^{-x-s/x}}{x-z} dx \\ -\dfrac{2\pi i}{h_{n-1}} P_{n-1}(z) & -\dfrac{1}{h_{n-1}} \displaystyle\int_{\mathbb{R}_+} \dfrac{P_{n-1}(x) x^\alpha e^{-x-s/x}}{x-z} dx \end{pmatrix}. \tag{5.1}$$

We also note that an immediate consequence of the unimodularity of the jump matrix of the above Riemann–Hilbert problem is the identity[2]

$$\det Y(z) \equiv 1. \tag{5.2}$$

[2] A more conventional derivation of this identity is based on the use of the basic three-term recurrence Eq. (1.13) (see e.g. [15]).

Eq. (5.1) implies, in particular, that the asymptotic behavior of the function $Y(z)$ at $z = \infty$ and $z = 0$ can be specified as the following full asymptotic series:

$$Y(z) \sim \left(I + \sum_{k=1}^{\infty} \frac{Y_{-k}}{z^k}\right) z^{n\sigma_3}, \quad z \to \infty, \tag{5.3}$$

and

$$Y(z) \sim Q\left(I + \sum_{k=1}^{\infty} Y_k z^k\right), \quad z \to 0, \tag{5.4}$$

where σ_3 denotes, as usual, the third Pauli matrix

$$\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Moreover, the matrix coefficients $Y_{\pm k}$ and Q of these series are the smooth functions of n and s (and of course of α), and they all can be easily expressed in terms of the fundamental objects associated with the orthogonal polynomials $P_n(z)$, i.e. in terms of the functions $\mathsf{p}_1(n, s)$, $h_n(s)$, and the negative moments of the polynomials $P_n(z)$. Indeed, by a straightforward calculation we have from (5.1) the following expressions for the coefficient Y_{-1} and for the matrix multiplier $Q \equiv Y(0)$:

$$Y_{-1} = \begin{pmatrix} \mathsf{p}_1(n) & -\dfrac{h_n}{2\pi \mathrm{i}} \\ -\dfrac{2\pi \mathrm{i}}{h_{n-1}} & -\mathsf{p}_1(n) \end{pmatrix}, \tag{5.5}$$

$$Q = \begin{pmatrix} 1 & p_n \\ -q_n & 1 - p_n q_n \end{pmatrix} P_n^{\sigma_3}(0), \tag{5.6}$$

where

$$p_n = \frac{P_n(0)}{2\pi \mathrm{i}} \int_{\mathbb{R}_+} \frac{P_n(x) x^{\alpha} \mathrm{e}^{-x-s/x}}{x} \mathrm{d}x, \qquad q_n = \frac{2\pi \mathrm{i}}{h_{n-1}} \frac{P_{n-1}(0)}{P_n(0)}, \tag{5.7}$$

and we have taken into account the determinant identity (5.2).

We are now going to write down a triple of differential and difference equations for the function $Y_n(z, s)$ following the standard procedure of the theory of integrable systems (see [17,30,29]; see also [19,28,15,18]).

Put

$$\Psi(z) \equiv \Psi(z, n, s, \alpha) := Y(z) \mathrm{e}^{-\frac{1}{2}\left(z + \frac{s}{z}\right)\sigma_3} z^{\frac{\alpha}{2}\sigma_3}, \tag{5.8}$$

where the branch of the function $z^{\frac{\alpha}{2}}$ is defined by the condition, $-\pi < \arg z < \pi$. The Riemann–Hilbert relations in terms of the function $\Psi(z)$ are as follows:

(i) $\Psi : \mathbb{C} \setminus \mathbb{R} \to \mathbb{C}^{2\times 2}$ is analytic.

(ii) $\Psi_+(x) = \Psi_-(x) \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ for $x \in \mathbb{R}$ and $x > 0$.

(iii) $\Psi_+(x) = \Psi_-(x) \mathrm{e}^{\pi \mathrm{i}\alpha\sigma_3}$ for $x \in \mathbb{R}$ and $x < 0$.

(iv) $\Psi(z) \sim \left(I + \sum_{k=1}^{\infty} \frac{\Psi_{-k}}{z^k}\right) z^{(n+\frac{\alpha}{2})\sigma_3} \mathrm{e}^{-\frac{z}{2}\sigma_3}$ as $z \to \infty$.

(v) $\Psi(z) \sim Q\left(I + \sum_{k=1}^{\infty} \Psi_k z^k\right) z^{\frac{\alpha}{2}\sigma_3} \mathrm{e}^{-\frac{s}{2z}\sigma_3}$ as $z \to 0$, where the real line $\mathbb{R}$ is oriented as usual from left to right. The coefficients $\Psi_{\pm k}$ of the asymptotic series are easily evaluated via combinations of the coefficients $Y_{\pm k}$ and the coefficients of the expansions of the exponential function $e^{-\frac{s}{2z}}$ and $e^{-\frac{z}{2}}$ near $z = \infty$ and $z = 0$, respectively. In particular, we have that

$$\Psi_{-1} = \begin{pmatrix} \mathsf{p}_1(n) - \frac{s}{2} & -\frac{h_n}{2\pi i} \\ -\frac{2\pi i}{h_{n-1}} & -\mathsf{p}_1(n) + \frac{s}{2} \end{pmatrix}. \tag{5.9}$$

The important feature of the Ψ-RH problem is that the jump matrices of the jump relations (ii) and (iii) do not depend on z, s, and n. Therefore, by standard arguments based on the Liouville theorem (cf. [17,19]), we conclude that the logarithmic derivatives,

$$\begin{aligned} &A(z) := \frac{\partial \Psi(z)}{\partial z} \Psi^{-1}(z), \qquad B(z) := \frac{\partial \Psi(z)}{\partial s} \Psi^{-1}(z), \quad \text{and} \\ &U(z) := \Psi(z, n+1)\, \Psi^{-1}(z, n), \end{aligned} \tag{5.10}$$

are rational functions of z. Using the asymptotic expansions (iv) and (v), we can evaluate the respective principal parts at the poles at the points $z = 0$ and $z = \infty$ and arrive, taking into account (5.9) and (5.6), at the following explicit formulae for the function $A(z)$, $B(z)$, and $U(z)$:

$$A(z) = -\frac{1}{2}\sigma_3 + \frac{A_1}{z} + \frac{A_2}{z^2}, \tag{5.11}$$

$$B(z) = -\frac{A_2}{sz}, \tag{5.12}$$

and

$$U(z) = z \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + U_0, \tag{5.13}$$

where[3]

$$\begin{aligned} A_1 &= \frac{1}{2}[\sigma_3, \Psi_{-1}] + \left(n + \frac{\alpha}{2}\right)\sigma_3 \\ &= \begin{pmatrix} n + \frac{\alpha}{2} & -\frac{h_n}{2\pi i} \\ \frac{2\pi i}{h_{n-1}} & -n - \frac{\alpha}{2} \end{pmatrix}, \end{aligned} \tag{5.14}$$

$$A_2 = \frac{s}{2} Q \sigma_3 Q^{-1} = \frac{s}{2} \begin{pmatrix} 1 - 2p_n q_n & -2p_n \\ 2q_n(p_n q_n - 1) & 2p_n q_n - 1 \end{pmatrix}, \tag{5.15}$$

and

$$\begin{aligned} U_0 &= \Psi_{-1}(n+1) \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \Psi_{-1}(n) \\ &= \begin{pmatrix} \mathsf{p}_1(n+1) - \mathsf{p}_1(n) & \frac{h_n}{2\pi i} \\ -\frac{2\pi i}{h_n} & 0 \end{pmatrix} \equiv \begin{pmatrix} -\alpha_n & \frac{h_n}{2\pi i} \\ -\frac{2\pi i}{h_n} & 0 \end{pmatrix}. \end{aligned} \tag{5.16}$$

[3] The notation $[M_1, M_2]$ means the usual commutator of two matrices, $[M_1, M_2] = M_1 M_2 - M_2 M_1$.

According to the standard methodology (cf. [30,29,19]), relations (5.10) should be now re-interpreted as a system of linear differential–difference equations:

$$\begin{cases} \frac{\partial \Psi(z)}{\partial z} = A(z)\Psi(z) \\ \frac{\partial \Psi(z)}{\partial s} = B(z)\Psi(z) \\ \Psi(z, n+1) = U(z)\Psi(z, n), \end{cases} \tag{5.17}$$

which we call the Lax triple.

The compatibility conditions of this system, i.e. the equations

$$\frac{\partial A(z)}{\partial s} - \frac{\partial B(z)}{\partial z} = [B(z), A(z)] \quad \left(\Psi_{zs} = \Psi_{sz}\right) \tag{5.18}$$

$$\frac{\partial U(z)}{\partial s} = B(z, n+1)U(z) - U(z)B(z, n), \quad \Big((\Psi(z, n+1))_s = \Psi_s(z, n+1)\Big) \tag{5.19}$$

and

$$\frac{\partial U(z)}{\partial z} = A(z, n+1)U(z) - U(z)A(z, n), \quad \Big((\Psi(z, n+1))_z = \Psi_z(z, n+1)\Big), \tag{5.20}$$

yield the Painlevé type (Eq. (5.18)), the Toda type (Eq. (5.19)) and the discrete Painlevé or Freud type (Eq. (5.20)) equations, respectively, for a proper combination of functions h_n, p_n and q_n. Moreover, using the Jimbo–Miwa list of the Lax pairs for Painlevé equations [29] and noticing that the "master equation", i.e. the first equation of system (5.17), is a 2×2 system with two irregular singular points of Poincaré rank 1, one concludes that the relevant Painlevé equation is, in fact, the third Painlevé equation. In order to make a precise statement, i.e. to point out the exact combination of functions h_n, p_n and q_n which make up the solution of Painlevé III equation, we only need to perform a simple scaling transformation of the system (5.17) which would bring it to the normal form of [29]. To this end, we introduce the new independent variables

$$\lambda := s^{-1/2}z, \quad \text{and} \quad t := \sqrt{s}, \tag{5.21}$$

and pass from the function $\Psi(z, s)$ to the function $\Phi(\lambda, t)$ defined by the equation

$$\Phi(\lambda, t) := t^{-(n+\frac{\alpha}{2})\sigma_3}\,\Psi(t\lambda, t^2). \tag{5.22}$$

We notice that in terms of the function $\Phi(\lambda, t)$ the asymptotic relations (iv) and (v) transform into the relations

$$\Phi(\lambda) \sim \left(I + \sum_{k=1}^{\infty} \frac{\Phi_{-k}}{\lambda^k}\right) \lambda^{(n+\frac{\alpha}{2})\sigma_3} e^{-\frac{t\lambda}{2}\sigma_3}, \quad \lambda \to \infty, \tag{5.23}$$

and

$$\Phi(\lambda) \sim R\left(I + \sum_{k=1}^{\infty} \Phi_k \lambda^k\right) \lambda^{\frac{\alpha}{2}\sigma_3} e^{-\frac{t}{2\lambda}\sigma_3}, \quad \lambda \to 0, \tag{5.24}$$

with the new coefficients connected to the old ones by the equations

$$R = t^{-(n+\frac{\alpha}{2})\sigma_3} Q t^{\frac{\alpha}{2}\sigma_3}, \quad \Phi_{-k} = t^{-k} t^{-(n+\frac{\alpha}{2})\sigma_3}\,\Psi_{-k} t^{(n+\frac{\alpha}{2})\sigma_3}, \quad \Phi_k = t^k t^{-\frac{\alpha}{2}\sigma_3}\,\Psi_k t^{\frac{\alpha}{2}\sigma_3}.$$

Simultaneously, the first two equations of system (5.17) transform into the *Jimbo–Miwa Lax pair* for the third Painlevé equation,

$$\begin{cases} \dfrac{\partial \Phi(\lambda)}{\partial \lambda} = \left(-\dfrac{t}{2}\sigma_3 + \dfrac{A_{-1}}{\lambda} + \dfrac{A_{-2}}{\lambda^2}\right)\Phi(\lambda) \equiv A_{JM}(\lambda)\Phi(\lambda) \\ \dfrac{\partial \Phi(\lambda)}{\partial t} = \left(-\dfrac{\lambda}{2}\sigma_3 + B_0 + \dfrac{B_{-1}}{\lambda}\right)\Phi(\lambda) \equiv B_{JM}(\lambda)\Phi(\lambda). \end{cases} \tag{5.25}$$

Here, the matrix coefficients A_{-1}, A_{-2}, B_0, and B_{-1} are given by the equations

$$A_{-1} = t^{-(n+\frac{\alpha}{2})\sigma_3} A_1 t^{(n+\frac{\alpha}{2})\sigma_3} = \begin{pmatrix} -\dfrac{\theta_\infty}{2} & u \\ v & \dfrac{\theta_\infty}{2} \end{pmatrix}, \tag{5.26}$$

$$A_{-2} = \frac{1}{t} t^{-(n+\frac{\alpha}{2})\sigma_3} A_2 t^{(n+\frac{\alpha}{2})\sigma_3} = \begin{pmatrix} \zeta + \dfrac{t}{2} & -w\zeta \\ \dfrac{\zeta + t}{w} & -\zeta - \dfrac{t}{2} \end{pmatrix}, \tag{5.27}$$

$$B_0 = \frac{1}{t} A_{-1} - \frac{1}{t}\left(n + \frac{\alpha}{2}\right)\sigma_3 = \frac{1}{t}\begin{pmatrix} 0 & u \\ v & 0 \end{pmatrix}, \tag{5.28}$$

$$B_{-1} = -\frac{1}{t} A_{-2} = -\frac{1}{t}\begin{pmatrix} \zeta + \dfrac{t}{2} & -w\zeta \\ \dfrac{\zeta + t}{w} & -\zeta - \dfrac{t}{2} \end{pmatrix}, \tag{5.29}$$

where

$$\theta_\infty = -\alpha - 2n, \tag{5.30}$$

and the new scalar functional parameters u, v, ζ, and w are defined in terms of the original functions h_n, h_{n-1}, p_n, and q_n via the formulae

$$u = -\frac{h_n}{2\pi \mathrm{i}} t^{-2n-\alpha}, \qquad v = \frac{2\pi \mathrm{i}}{h_{n-1}} t^{2n+\alpha}, \tag{5.31}$$

$$\zeta = -t p_n q_n, \qquad w = -\frac{1}{q_n} t^{-2n-\alpha}. \tag{5.32}$$

With the u, v, ζ, w notation, the Lax pair (5.25) matches, up to the replacement $t \to -t$ and the use of the letter z instead of the letter ζ, the Lax pair presented on page 439 of [29], and hence we can use the general results of [29].

Theorem 5 (*[29]*). *Consider the overdetermined linear system* (5.25) *with the matrix coefficients defined by the equations*

$$A_{-1} = \begin{pmatrix} -\dfrac{\theta_\infty}{2} & u \\ v & \dfrac{\theta_\infty}{2} \end{pmatrix}, \qquad A_{-2} = \begin{pmatrix} \zeta + \dfrac{t}{2} & -w\zeta \\ \dfrac{\zeta + t}{w} & -\zeta - \dfrac{t}{2} \end{pmatrix}, \tag{5.33}$$

$$B_0 = \frac{1}{t}\begin{pmatrix} 0 & u \\ v & 0 \end{pmatrix}, \qquad B_{-1} = -\frac{1}{t}\begin{pmatrix} \zeta + \dfrac{t}{2} & -w\zeta \\ \dfrac{\zeta + t}{w} & -\zeta - \dfrac{t}{2} \end{pmatrix}, \tag{5.34}$$

i.e., by the right hand sides of the last equalities in the formulae (5.26)–(5.29) not necessarily assuming any specific, "orthogonal polynomial" choice of the parameters θ_∞, u, v, ζ and w. Then the following is true.

1. *The compatibility condition of system (5.25), i.e. the matrix equation*

$$\frac{\partial A_{JM}}{\partial t} - \frac{\partial B_{JM}}{\partial \lambda} = [B_{JM}, A_{JM}], \tag{5.35}$$

is equivalent to the following set of scalar equations:

$$t\frac{du}{dt} = \theta_\infty u + 2t\zeta w, \tag{5.36}$$

$$t\frac{dv}{dt} = -\theta_\infty v + \frac{2t}{w}(\zeta + t), \tag{5.37}$$

$$t\frac{d\zeta}{dt} = 2\zeta wv + \zeta + \frac{2u(\zeta + t)}{w}, \tag{5.38}$$

$$t\frac{d\ln w}{dt} = \frac{2u}{w} - 2wv - \theta_\infty, \tag{5.39}$$

with the quantity

$$\theta_0 := -\frac{\theta_\infty}{t}(2\zeta + t) + \frac{2u(\zeta + t)}{tw} - \frac{2\zeta}{t}wv, \tag{5.40}$$

being the first integral of system (5.36)–(5.39).

2. *The quantity θ_0 is very similar to the parameter θ_∞. Indeed, they describe the formal monodromy at the relevant irregular points: θ_∞– at $\lambda = \infty$ and θ_0– at $\lambda = 0$. This means that they appear as the branching exponents in the following formal matrix solutions*[4] *of the Lax pair (5.25) at the points $\lambda = \infty$ and $\lambda = 0$:*

$$\Phi^{(\infty)}_{formal}(\lambda) = \left(I + \sum_{k=1}^{\infty} \frac{\Phi^{(\infty)}_k}{\lambda^k}\right)\lambda^{-\frac{\theta_\infty}{2}\sigma_3} e^{-\frac{t\lambda}{2}\sigma_3}, \tag{5.41}$$

and

$$\Phi^{(0)}_{formal}(\lambda) = R^{(0)}\left(I + \sum_{k=1}^{\infty} \Phi^{(0)}_k \lambda^k\right)\lambda^{\frac{\theta_0}{2}\sigma_3} e^{-\frac{t}{2\lambda}\sigma_3}. \tag{5.42}$$

3. *The function*

$$y := -\frac{u}{\zeta w}, \tag{5.43}$$

satisfies the third Painlevé equation,

$$\frac{d^2y}{dt^2} = \frac{1}{y}\left(\frac{dy}{dt}\right)^2 - \frac{1}{t}\frac{dy}{dt} - \frac{1}{t}\Big(4\theta_0 y^2 + 4(1-\theta_\infty)\Big) + 4y^3 - \frac{4}{y}. \tag{5.44}$$

In the standard notation [25], *this is the Painlevé* III *equation,* $P_{\mathrm{III}}(-4\theta_0, -4(1-\theta_\infty), 4, -4)$.

[4] All the coefficients $\Phi^{(\infty)}_k$ of the series (5.41) are uniquely defined as rational functions of u, v, ζ, w, and t via simple recurrence relations [30]. The matrix factor $R^{(0)}$ is defined, up to a right diagonal multiplier, $R^{(0)} \to R^{(0)}\Lambda$, by the equations, $\frac{t}{2}R^{(0)}\sigma_3\left(R^{(0)}\right)^{-1} = A_{-2}$, $\det R^{(0)} = 1$. The coefficients $\Phi^{(0)}_k$ of the series (5.42) are defined as rational functions of u, v, ζ, w, and t up to the conjugation $\Phi^{(0)}_k \to \Lambda^{-1}\Phi^{(0)}_k\Lambda$. We will discuss this issue in more detail in the next section.

Comparing the general formal expansions (5.41) and (5.42) with the asymptotic series (5.23) and (5.24), we conclude that in our case the parameters θ_∞ and θ_0 assume the values (see also (5.30))

$$\theta_\infty = -\alpha - 2n, \quad \text{and} \quad \theta_0 = \alpha. \tag{5.45}$$

Simultaneously, from (5.31) and (5.32) we see that in our case

$$y = \frac{h_n}{2\pi \mathrm{i} t p_n}. \tag{5.46}$$

In other words, the relation (5.46) provides us with the combination of the functions h_n, p_n, q_n which satisfies the third Painlevé equations and which we have been looking for, while (5.45) specifies the parameters of the Painlevé equation. We conclude then that the function y defined in (5.46) satisfies the following Painlevé III equation:

$$\frac{\mathrm{d}^2 y}{\mathrm{d}t^2} = \frac{1}{y}\left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2 - \frac{1}{t}\frac{\mathrm{d}y}{\mathrm{d}t} - \frac{1}{t}\Big(4\alpha y^2 + 4(2n+1+\alpha)\Big) + 4y^3 - \frac{4}{y}. \tag{5.47}$$

What is still left for us is to establish the connection between the function $y(t)$ defined in (5.46) and the function $a_n(s)$ defined in (2.7). This is easy. Indeed, we have that

$$\begin{aligned}\int_0^\infty \frac{P_n^2}{x} w\mathrm{d}x &= \int_0^\infty P_n(x)\left(x^{n-1} + \mathsf{p}_1(n)x^{n-2} + \cdots + \frac{P_n(0)}{x}\right) w\mathrm{d}x \\ &= P_n(0)\int_0^\infty \frac{P_n}{x} w\mathrm{d}x,\end{aligned}$$

and hence (see (5.7)),

$$p_n = \frac{h_n}{2\pi \mathrm{i}}\frac{a_n}{s}, \tag{5.48}$$

which in turn implies that

$$y = \frac{t}{a_n} \equiv \frac{X_n}{t}. \tag{5.49}$$

It is an elementary exercise to check that the substitution

$$y = \frac{t}{a_n}, \qquad t = \sqrt{s},$$

transforms Eq. (5.47) into Eq. (3.12) for the quantity $a_n(s)$.

We conclude this section by revealing the connections of some of the key identities established in the previous sections with the general constructions of the isomonodromy theory of Painlevé equations discussed above.

We first note that the following equation, similar to Eq. (5.48), holds:

$$q_n = \frac{2\pi \mathrm{i}}{h_n}\frac{b_n}{a_n}. \tag{5.50}$$

Therefore all the matrix coefficients of the Lax triple (5.17) can be expressed in terms of the functions α_n, h_n, a_n, and b_n. Actually, we need only to rewrite the coefficient A_2 from (5.15),

$$A_2 = \frac{s}{2}\begin{pmatrix} 1 - \dfrac{2b_n}{s} & -\dfrac{h_n}{\pi \mathrm{i}}\dfrac{a_n}{s} \\ \dfrac{4\pi \mathrm{i}}{h_n}\dfrac{b_n}{a_n}\left(\dfrac{b_n}{s} - 1\right) & \dfrac{2b_n}{s} - 1 \end{pmatrix}. \tag{5.51}$$

Substituting this representation together with the similar formulae for A_1 and U_0 (see (5.14) and (5.16)) into the compatibility Eq. (5.20) we obtain the set of the scalar difference equations (2.9)–(2.13) of Section 2. Similar operation with the compatibility equation (5.19) results in the differential–difference equations (3.1)–(3.6) of Section 3. The scalar form of compatibility equation (5.18) we have already discussed in detail in this section. As we have seen, this equation is equivalent to the set of scalar equations (5.36)–(5.39) which, in terms of the functions $h_n(s), h_{n-1}(s), a_n(s), b_n(s)$, transforms into the system

$$s\frac{\mathrm{d}h_n}{\mathrm{d}s} = -h_n a_n, \tag{5.52}$$

$$s\frac{\mathrm{d}h_{n-1}}{\mathrm{d}s} = \frac{h_{n-1}^2}{h_n}\frac{b_n}{a_n}(s - b_n), \tag{5.53}$$

$$s\frac{\mathrm{d}b_n}{\mathrm{d}s} = b_n - \frac{b_n}{a_n}(s - b_n) - \frac{h_n}{h_{n-1}}a_n \tag{5.54}$$

$$s\frac{\mathrm{d}a_n}{\mathrm{d}s} = 2b_n + (2n + 1 + \alpha + a_n)a_n - s, \tag{5.55}$$

which is equivalent to the system of equations derived in Lemma 5 of Section 3.

We want to highlight the theoretical meaning of the important identity (2.14). It is, in fact, the formal monodromy identity (5.40) written in terms of the functions $h_n(s), h_{n-1}(s), a_n(s), b_n(s)$.

It is also worth mentioning that the ladder equations (2.1) and (2.2) are just the first column of the master, z-equation of the Lax triple (5.17). The first column of the second equation of the triple (5.17) yields the ladder operators in s,

$$\left(zs\frac{\mathrm{d}}{\mathrm{d}s} - b_n\right)P_n(z) = -\beta_n a_n P_{n-1}(z), \tag{5.56}$$

$$\left(zs\frac{\mathrm{d}}{\mathrm{d}s} - b_{n-1} - \alpha_{n-1}a_{n-1} + za_{n-1}\right)P_{n-1}(z) = a_{n-1}P_n(z). \tag{5.57}$$

When deriving these equations we have made use of the relations (2.10) and (2.13). Eqs. (5.56) and (5.57) of course can be derived using the orthogonality relations, bypassing the isomonodromy theory.

Finally, the third equation in (5.17) is equivalent to the basic recurrence relations (1.13) for the polynomials P_n. It should be pointed out that for the general solutions of the Painlevé III equations, only the first two equations, which are equivalent under scaling to the Lax pair (5.25), hold. From the point of view of the general isomonodromy theory of Painlevé equations, the third equation of the triple (5.17) describes the Bäcklund–Schlesinger transformation of the third Painlevé equation (see [30]; see also Chapter 6 of [18]).

6. The Hankel determinant as the isomonodromy τ-function

Let us remind the reader of the Jimbo–Miwa definition of the τ-function corresponding to the third Painlevé equation (5.44).

Consider the formal series (5.41) and (5.42) of Theorem 5. In the footnote to this theorem we have already mentioned that all the coefficients of these series can be evaluated as rational functions of the parameters u, v, ζ, w, and t. For example, the first coefficients in each of the series are given by the relations (see p. 440 of [29])

$$\Phi_1^{(\infty)} = \frac{1}{t}\begin{pmatrix} uv - t\zeta - \frac{t^2}{2} & u \\ -v & -uv + t\zeta + \frac{t^2}{2} \end{pmatrix}, \tag{6.1}$$

and

$$\Phi_1^{(0)} = \frac{1}{t}\begin{pmatrix} \widetilde{u}\widetilde{v} - t\zeta - \frac{t^2}{2} & -\widetilde{u} \\ \widetilde{v} & -\widetilde{u}\widetilde{v} + t\zeta + \frac{t^2}{2} \end{pmatrix}, \tag{6.2}$$

where the parameters $\widetilde{u}$ and $\widetilde{v}$ are related to the basic parameters u and v through the equation

$$\left(R^{(0)}\right)^{-1}\begin{pmatrix} -\frac{\theta_\infty}{2} & u \\ v & \frac{\theta_\infty}{2} \end{pmatrix} R^{(0)} = \begin{pmatrix} \frac{\theta_0}{2} & \widetilde{u} \\ \widetilde{v} & -\frac{\theta_0}{2} \end{pmatrix}, \tag{6.3}$$

which, in particular, means the identity

$$\widetilde{u}\widetilde{v} = uv + \frac{\theta_\infty^2 - \theta_0^2}{4}. \tag{6.4}$$

Denote as $\widehat{Y}_\infty(\lambda)$ and $\widehat{Y}_0(\lambda)$ the series in the brackets of formulae (5.41) and (5.42), i.e.

$$\widehat{Y}_\infty(\lambda) = \left(I + \sum_{k=1}^{\infty}\frac{\Phi_k^{(\infty)}}{\lambda^k}\right), \tag{6.5}$$

and

$$\widehat{Y}_0(\lambda) = \left(I + \sum_{k=1}^{\infty}\Phi_k^{(0)}\lambda^k\right). \tag{6.6}$$

The Jimbo–Miwa–Ueno isomonodromy τ-function [30] in the case of the Lax pair (5.25) is defined by the formula

$$\begin{aligned} \mathrm{d}\ln\tau &= -\mathrm{Trace}\,\mathrm{Res}_{\lambda=0}\widehat{Y}_0^{-1}(\lambda)\frac{\partial\widehat{Y}_0}{\partial\lambda}(\lambda)\mathrm{d}T_0(\lambda) \\ &\quad - \mathrm{Trace}\,\mathrm{Res}_{\lambda=\infty}\widehat{Y}_\infty^{-1}(\lambda)\frac{\partial\widehat{Y}_\infty}{\partial\lambda}(\lambda)\mathrm{d}T_\infty(\lambda), \end{aligned} \tag{6.7}$$

where

$$\mathrm{d}T_0(\lambda) = -\frac{1}{2\lambda}\sigma_3 \mathrm{d}t, \quad \text{and} \quad \mathrm{d}T_\infty(\lambda) = -\frac{\lambda}{2}\sigma_3 dt.$$

Substituting (6.5) and (6.6) into (6.7), we arrive at the equation

$$\mathrm{d}\ln\tau = \frac{1}{2}\mathrm{Trace}\left(\Phi_1^{(0)}\sigma_3 + \Phi_1^{(\infty)}\sigma_3\right)\mathrm{d}t,$$

which, taking into account (6.1), (6.2) and (6.4), implies that

$$\mathrm{d}\ln\tau(t) = H_{\mathrm{III}}(u(t), v(t), \zeta(t); t)\mathrm{d}t, \tag{6.8}$$

where

$$H_{\mathrm{III}}(u, v, \zeta; t) = \frac{1}{t}\left(2uv - 2t\zeta - t^2 - \frac{\theta_0^2 - \theta_\infty^2}{4}\right). \tag{6.9}$$

Let us now turn to the Hankel determinant $D_n(s)$ and consider the quantity

$$H_n := s\frac{\mathrm{d}}{\mathrm{d}s}\ln D_n,$$

which has played a central role in Sections 3 and 4. From (3.8) and (2.15) we have that

$$H_n = n(n+\alpha) + b_n - \frac{ns - (2n+\alpha)b_n}{a_n} + \frac{b_n^2 - sb_n}{a_n^2}. \tag{6.10}$$

At the same time, in our case (see (5.45)),

$$\theta_0 = \alpha, \qquad \theta_\infty = -\alpha - 2n, \qquad uv = -\beta_n, \quad \text{and} \quad \zeta = -\frac{b_n}{t},$$

where for the last two formulae we have used (5.31), (5.32), (5.48) and (5.50). Substituting these formulae into (6.9) we obtain that for our special solution of the system (5.36)–(5.39), the function H_{III} assumes the form

$$H_{\mathrm{III}} = \frac{1}{t}\Big(-2\beta_n + 2b_n - s + n(n+\alpha)\Big). \tag{6.11}$$

Recalling now identity (2.14) (which we remind the reader is the formal monodromy identity (5.40) in disguise) we arrive, after some simple algebra, at the relation

$$H_{\mathrm{III}} = \frac{2}{t}H_n - t - \frac{n(n+\alpha)}{t}. \tag{6.12}$$

This equation, in turn, implies that

$$\begin{aligned} s\frac{\mathrm{d}}{\mathrm{d}s}\ln\tau &= \frac{t}{2}\frac{\mathrm{d}}{\mathrm{d}t}\ln\tau = \frac{t}{2}H_{\mathrm{III}} = H_n - \frac{s}{2} - \frac{n(n+\alpha)}{2} \\ &= s\frac{\mathrm{d}}{\mathrm{d}s}\ln D_n - \frac{s}{2} - \frac{n(n+\alpha)}{2}, \end{aligned} \tag{6.13}$$

and hence we obtain the following relation between the Hankel determinant D_n and the τ-function of the third Painlevé equation (compare with [28] where similar formula is derived for a class of Toeplitz determinants),

$$D_n(s) = \text{const}\,\tau(s)\mathrm{e}^{\frac{s}{2}}s^{\frac{n(n+\alpha)}{2}}. \tag{6.14}$$

Remark V. The function $H_{\mathrm{III}}(u, v, \zeta; t)$ actually depends on ζ and the product uv. The latter can be expressed, with the help of the formal monodromy relation (5.40), in terms of ζ, t, and y. Indeed, we have that

$$2uv = (\theta_0 + \theta_\infty)ty + 2\zeta y\theta_\infty + 2y^2\zeta(\zeta + t),$$

and expression (6.9) can be transformed to the following equation defining H_{III} as a function of y, ζ and t:

$$H_{\mathrm{III}}(y, \zeta; t) = \frac{1}{t}\left(2y^2\zeta^2 + 2(ty^2 + \theta_\infty y - t)\zeta + (\theta_0 + \theta_\infty)ty - t^2 - \frac{\theta_0^2 - \theta_\infty^2}{4}\right). \tag{6.15}$$

This is the canonical (see again p. 440 of [29]) representation of the logarithmic derivative of the τ-function for Painlevé III equation (5.44). The remarkable fact of the general Jimbo–Miwa–Ueno theory is that the function $H_{\mathrm{III}}(y, \zeta; t)$ is the Hamiltonian of the third Painlevé equation.

Remark VI. In Sections 2 and 3, an important role has been played by Eq. (2.15) which transforms a *non-local* object—the sum of a_j from $j = 0$ to $j = n - 1$, to a *local* expression, which involves only a_n and b_n. We can see now an intrinsic reason for that. Indeed, on the one hand, the sum mentioned is, by its very nature, the logarithmic derivative of the Hankel determinant. On the other hand, the latter is a τ-function and hence its logarithmic derivative *must* admit a local representation in view of the general formula (6.7).

Remark VII. The isomonodromy context for orthogonal polynomials and, in particular, the interpretation of the Hankel determinants as isomonodromy τ-functions have been well understood for some time, since the works from the early nineties [19,33]. For the most general semi-classical weights this fact was established in the recent paper [5] (see also [6]).

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