ELSEVIER Journal of Pure and Applied Algebra 103 (1995) 287–302

JOURNAL OF PURE AND APPLIED ALGEBRA

Closed model structures for algebraic models of n-types☆

Julia G. Cabello*, Antonio R. Garzón

Department of Algebra, University of Granada, Granada 18071, Spain

Communicated by F.W. Lawvere; received 20 October 1993; revised 20 May 1994

Abstract

In this paper we give a general method to obtain a closed model structure, in the sense of Quillen, on a category related to the category of simplicial groups by a suitable adjoint situation. Applying this method, categories of algebraic models of connected types such as those of crossed modules of groups (2-types), 2-crossed modules of groups (3-types) or, in general, n-hypercrossed complexes of groups (($n + 1$)-types), as well as that of n-simplicial groups (all types), inherit such a closed model structure.

0. Introduction

The problem of giving algebraic models for the homotopy theory of spaces has been studied in the last years by several authors [3, 5, 15, 19, 20]. Classical references about it are the results by Eilenberg and Mac Lane [9] giving the well known equivalence between the homotopy category of pointed connected CW-complexes, with only one non-trivial homotopy group in dimension n, and the "homotopy" category of groups (abelian if $n \geq 2$), and the results given by Mac Lane and Whitehead [16, 24], proving a similar equivalence between the pointed connected CW-complexes such that $\Pi_i = 0$ for $i \geq 3$ (homotopy 2-types) and crossed modules.

The category of 2-crossed modules in the sense of Conduché [7] generalizes Mac Lane and Whitehead's results since it is an appropriate "algebraic category" to model arbitrary homotopy 3-types [3, 7] and in [5] Carrasco and Cegarra extended these partial results by giving algebraic models for all connected n-types. For this, they considered a category, n-HXC(Gp), consisting of certain complexes of non-abelian groups, called by them "n-hypercrossed complexes of groups", and showed that a certain localization of it, in the sense of Gabriel and Zissman [10] is equivalent to the homotopy category of connected CW-complexes ($n + 1$)-coconnected (i.e., $\Pi_i = 0$, for all $i \geq n + 2$).

☆ This paper has been supported financially by DGICYT: PS90-0226.

0022-4049/95/$09.50
SSDI 0022-4049(94)00105-7

The category n-HXC(Gp) is just, for $n = 1$, the category XM(Gp) of crossed modules of groups (2-types) and, for $n = 2$, the category 2-XM(Gp) of 2-crossed modules of groups in the sense of Conduché (3-types) and, on the other hand, it is equivalent to the full subcategory of the category of simplicial groups consisting of those simplicial groups with trivial Moore complex in dimensions $>n$, i.e., the category of n-hypergroupoids of groups in the sense of Duskin–Glenn, so that this category of n-hypergrupoids of groups, n-Hypdg(Gp), provides also algebraic models for $(n + 1)$-types.

The theory developed by Carrasco and Cegarra was based on the classical theory of Kan [14] showing that the category of simplicial groups, Simp(Gp), models all connected types. This category is a main example of what is a closed model category in the sense of Quillen [21], which means that in Simp(Gp) it is possible to do homotopy theory as well as for categories of topological spaces, i.e., to have analogues of most of constructions and results which are inherent to the homotopy theory of spaces as loops and suspensions of objects, (co)-fibration sequences, etc.

The main object of this paper is to get a closed model structure for the category n-Hypgd(Gp), and so for n-HXC(Gp), where the weak equivalences are precisely those morphisms which are inverted to do the localization which determines that category as a category of algebraic models of $(n + 1)$-types.

The technique used to define the structure and to check the axioms of closed model category, lies strongly on the adjoint situation connecting the category n-Hypgd(Gp) to Simp(Gp), and "lifting" then, to that category (which is in fact a reflexive full subcategory of Simp(Gp)), the well-known closed model structure of Simp(Gp) [21] following an analogous process, suggested by Kan and used by Thomason in [23], to define in Cat a closed model structure.

The same method is then also used to get a closed model structure for the category of n-simplicial groups, $\text{Simp}^n(\text{Gp})$, a category which also provides algebraic models for all types [4].

We want to note here that in 1983 Loday gave in [15] the foundation of a theory of another category of algebraic models for $(n + 1)$-types of spaces, called firstly "n-cat-groups" and renamed later more appropriately as "Cat^n-groups" [2]. Proofs making more clear the original one of Loday have been given by Porter [20] and Bullejos et al. [4]. The possibility of giving a closed model structure to this other category of n-types is also discussed and we conjecture that it can be achieved using the same method as for n-Hypgd(Gp).

The plan of this paper is briefly as follows. In Section 1 we recall the set of axioms most frequently used to define a closed model category and also some characterizations of (trivial) fibrations in the category of simplicial groups which will be useful along the paper. In Section 2 we formulate the general problem of "lifting" the closed model structure of simplicial groups to a category $\underline{C}$ related to it by a pair of adjoint functors, showing conditions to assert that $\underline{C}$ inherits this kind of structure (Theorem 2.5). As a direct application of these results we see in Section 3 that the category n-Hypgd(Gp), and so that of n-hypercrossed complexes of groups, n-HXC(Gp), is a closed model category; in particular we describe the structure for $n = 1$ (crossed modules) and for $n = 2$ (2-crossed modules). In a similar way we also see that $\mathrm{Simp}^n(\mathrm{Gp})$ is a closed model category and finally, in Section 3.3., we analyse in some detail the possibility of getting a closed model structure for the category $\mathrm{Cat}^n(\mathrm{Gp})$ using the same method as in Section 2.

1. Notation and preliminaries

We will denote through this paper by Simp(Gp) the category of simplicial groups, i.e., the category of functors $\mathrm{Gp}^{\Delta^{\mathrm{op}}}$ where Δ is the category whose objects are the ordered sets $[0] = \{0\}, [1] = \{0,1\}, [2] = \{0,1,2\}, \ldots,$ and whose morphisms are the order-preserving functions between them.

Recall that the nth-simplicial kernel of a $(n-1)$-truncated simplicial group $G_{\cdot_{tr}}$, $\Delta^n(G_{\cdot_{tr}})$, is the subgroup of $(G_{n-1})^{n+1}$ whose elements are those $(x_0, \ldots, x_n)$ such that $d_i x_j = d_{j-1} x_i$, for $i < j$. If $d_i : \Delta^n(G_{\cdot_{tr}}) \to G_{n-1}$ denotes the restriction of the canonical projection, there are unique homomorphisms $s_j : G_{n-1} \to \Delta^n(G_{\cdot_{tr}})$, $0 \le j \le n-1$, such that

$$\Delta^n(G_{\cdot_{tr}}) \;\underset{\longleftarrow}{\overset{\longrightarrow}{\vdots}}\; G_{n-1} \;\overset{\longrightarrow}{\vdots}\; \cdots \;\rightrightarrows\; G_1 \;\rightleftarrows\; G_0$$

is a n-truncated simplicial group. By iterating this simplicial kernel construction, one has a functor cosk^{n-1} from the category of $(n-1)$-truncated simplicial groups to the category Simp(Gp).

Given a simplicial group

$$G_\cdot : \; \ldots G_n \;\underset{\longleftarrow}{\overset{\longrightarrow}{\vdots}}\; G_{n-1} \;\overset{\longrightarrow}{\vdots}\; \cdots \;\rightrightarrows\; G_1 \;\rightleftarrows\; G_0$$

we denote by $N(G_\cdot)$ its Moore complex [21], i.e.,

$$N(G_\cdot) = \;\cdots N_q(G_\cdot) \xrightarrow{\delta_q} N_{q-1}(G_\cdot) \longrightarrow \cdots \longrightarrow N_1(G_\cdot) \xrightarrow{\delta_1} N_0(G_\cdot)$$

where $N_0(G_\cdot) = G_0$,

$$N_q(G_\cdot) = \bigcap_{i=0}^{q-1} \mathrm{Ker}\, d_i \subseteq G_q$$

and δ_q is the restriction of $d_q : G_q \to G_{q-1}$ to $N_q(G_\cdot)$.

Recall also that the homotopy groups of the underlying simplicial set of a simplicial group $G_\cdot$ (pointed by the identity element) are given by the homology groups of the

Moore complex of $G.$:

$$\Pi_n(G.) = \frac{\bigcap_{i=0}^{n} \mathrm{Ker}(d_i : G_n \longrightarrow G_{n-1})}{d_{n+1}\left(\bigcap_{i=0}^{n} \mathrm{Ker}(d_i : G_{n+1} \longrightarrow G_n)\right)}, \quad n \geq 0.$$

A closed model category in the sense of Quillen [22] is a category $\underline{C}$ with three distinguished classes of morphisms called fibrations, cofibrations and weak equivalences, satisfying the following axioms:

CM1. The category $\underline{C}$ has finite limits and colimits.

CM2. For any composable pair f, g of morphisms in $\underline{C}$, if any two of f, g, gf are weak equivalences, so is the third.

CM3. If f is a retract of g and g is a fibration, cofibration or weak equivalence then f is also such.

Recall a morphism $f: X \to Y$ is a retract of $g: W \to Z$ if there are morphisms $i: X \to W$, $r: W \to X$, $j: Y \to Z$ and $s: Z \to Y$ such that $ri = \mathrm{Id}_X$, $sj = \mathrm{Id}_Y$ and also $gi = jf$, $fr = sg$.

CM4 (*Lifting axiom*). Given a solid arrow diagram

$$\begin{array}{ccc} A & \longrightarrow & X \\ {\scriptstyle i}\downarrow & \nearrow & \downarrow{\scriptstyle p} \\ B & \longrightarrow & Y \end{array} \qquad (*)$$

where i is a cofibration, p is a fibration and either i or p is a weak equivalence, then there exists the dotted arrow making the diagram commutative.

CM5 (*Factorization axiom*). Any morphism f in $\underline{C}$ may be factored both as $f = pi$ and $f = qj$, where p and q are fibrations, i and j are cofibrations, and p and j are weak equivalences.

Let us note that this set of axioms is equivalent to the original one given by Quillen in [21].

We will say that a morphism in $\underline{C}$ $i: A \to B$ has the *left lifting property* (LLP) with respect to another morphism $p: X \to Y$ and p is said to have the *right lifting property* (RLP) with respect to i if the dotted arrow exists in any diagram of the form (*).

Recall now that Simp(Gp) is a closed model category [21] where the firbrations are the Kan fibrations, the weak equivalences are those morphisms which induce isomorphisms on the homotopy groups and the cofibrations are defined by the LLP with respect to the trivial fibrations. By using the simplicial sets $\Delta[n,k]$, $\Delta[n]$ and $\dot{\Delta}[n]$ [17] and the free group functor F: Simp(Sets) $\to$ Simp(Gp), (trivial) fibrations are characterized as follows [21, Section 2, Propositions 1 and 2]:

Proposition 1.1. *Let $f.: X. \to Y.$ be a morphism of simplicial groups.*

(i) *$f.$ is a fibration iff $f.$ has the RLP with respect to the family of morphisms $F\Delta[n,k] \to F\Delta[n]$, $0 \le k \le n$, $n > 0$, induced by the inclusions $\Delta[n,k] \hookrightarrow \Delta[n]$, $0 \le k \le n$, $n > 0$.*

(ii) *$f.$ is a trivial fibration iff $f.$ has the RLP with respect to the family of morphisms $F\dot{\Delta}[n] \to F\Delta[n]$, for all $n \ge 0$, induced by the inclusions $\dot{\Delta}[n] \hookrightarrow \Delta[n]$, for all $n \ge 0$.*

2. Lifting closed model structures from Simp(Gp)

Let us consider Simp(Gp) with its Quillen's closed model structure and suppose through all this section that $\underline{C}$ is a category which has finite limits and colimits, related to Simp(Gp) by an adjunction

$$\underline{C} \underset{R}{\overset{L}{\rightleftarrows}} \mathrm{Simp(Gp)} \tag{A}$$

with L the left adjoint functor to R, and for which

$$\varphi : \mathrm{Hom}_{\underline{C}}(LG., X) \longrightarrow \mathrm{Hom}_{\mathrm{Simp(Gp)}}(G., RX)$$

will denote the corresponding natural bijective map.

The aim for this section is to prove that, under suitable conditions for this adjoint situation, the category $\underline{C}$ acquires a closed model structure in the Quillen's sense, which is the "lifted" one from that of Simp(Gp) in the following sense (see [23]):

Definition 2.1. A morphism f in $\underline{C}$ is said to be fibration (weak equivalence) if Rf is a fibration (weak equivalence) in Simp(Gp). A morphism f in $\underline{C}$ is a cofibration if it has the LLP with respect to the trivial fibrations.

We will use now the characterizations of (trivial) fibrations in Simp(Gp) (see Proposition 1.1) and the adjunction (A) to prove:

Proposition 2.2. (i) *The functor L: Simp(Gp) $\to \underline{C}$ preserves cofibrations.*

(ii) *A morphism f in $\underline{C}$ is a fibration iff it has the RLP with respect to the family of morphisms $LF\Delta[n,k] \to LF\Delta[n]$, $0 \le k \le n$, $n > 0$, induced by the inclusions $\Delta[n,k] \hookrightarrow \Delta[n]$, $0 \le k \le n$, $n > 0$.*

(iii) *A morphism f in $\underline{C}$ is a trivial fibration iff it has the RLP with respect to the family of morphisms $LF\dot{\Delta}[n] \to LF\Delta[n]$, $n \ge 0$, induced by the inclusions $\Delta[n] \hookrightarrow \Delta[n]$, $n \ge 0$.*

(iv) *If $A \to B$ is a cofibration in $\underline{C}$ and $A \to C$ is any morphism, the induced morphism into the pushout*

$$C \longrightarrow B \coprod_A C$$

is a cofibration.

(v) *If* $C_0 \to C_1 \to C_2 \to \cdots$ *is a sequence of cofibrations in* $\underline{C}$, *the canonical morphism*

$$C_0 \longrightarrow C_\infty = \varinjlim C_n$$

is a cofibration.

Proof. (i) is a consequence of the natural bijective correspondence, induced by φ, between diagrams in $\underline{C}$

$$\begin{array}{ccc} LG. & \longrightarrow & X \\ {\scriptstyle Li.}\big\downarrow & \nearrow & \big\downarrow{\scriptstyle p} \\ LH. & \longrightarrow & Y \end{array}$$

and diagrams in Simp(Gp)

$$\begin{array}{ccc} G. & \longrightarrow & RX \\ {\scriptstyle i.}\big\downarrow & \nearrow & \big\downarrow{\scriptstyle Rp} \\ H. & \longrightarrow & RY \end{array}$$

(ii) and (iii) follow from Proposition 1.1. and the above correspondence between diagrams in $\underline{C}$ and Simp(Gp).

As for (iv) and (v), note that the morphisms

$$C \longrightarrow B \coprod_A C$$

and $C_0 \to C_\infty$ are cofibrations since they have the LLP with respect to the trivial fibrations, as can be easily deduced from the universal property of pushouts and directed limits. □

An object A in a category $\underline{C}$ is said to be "small" if

$$\mathrm{Hom}_{\underline{C}}\left(A, \varinjlim B_m\right) \cong \varinjlim \mathrm{Hom}_{\underline{C}}(A, B_m)$$

for any directed system $\{B_m\}$ in $\underline{C}$. A family of objects $\{A_n\}$ in $\underline{C}$ is said to be "sequentally small" if the objects A_n are small.

We can then prove:

Proposition 2.3. *If the functor* $R: \underline{C} \to \mathrm{Simp(Gp)}$ *preserves directed limits, then:*

(i) *The objects* $LF\Delta[n,k]$ *and* $LF\dot{\Delta}[n]$, $0 \le k \le n$, $n \ge 0$, *are sequentially small.*

(ii) *For any sequence of weak equivalences in* $\underline{C}$, $C_0 \to C_1 \to C_2 \to \cdots$, *the canonical morphism*

$$C_0 \longrightarrow C_\infty = \varinjlim C_n$$

is a weak equivalence.

Proof. (i) It is well known that $F\Delta[n,k]$ and $F\dot{\Delta}[n]$ are small. Thus, using the adjunction, one has

$$\begin{aligned}
\mathrm{Hom}_{\underline{C}}\left(LF\Delta[n,k], \varinjlim B_m\right) &\cong \mathrm{Hom}_{\mathrm{Simp(Gp)}}\left(F\Delta[n,k], R \varinjlim B_m\right) \\
&\cong \mathrm{Hom}_{\mathrm{Simp(Gp)}}\left(F\Delta[n,k], \varinjlim RB_m\right) \cong \varinjlim \mathrm{Hom}_{\mathrm{Simp(Gp)}}(F\Delta[n,k], RB_m) \\
&\cong \varinjlim \mathrm{Hom}_{\underline{C}}(LF\Delta[n,k], B_m),
\end{aligned}$$

and similarly for $LF\dot{\Delta}[n]$.

(ii) Since R preserves directed limits we have $RC_\infty \cong \varinjlim RC_i$. As every $C_i \to C_{i+1}$ is a weak equivalence, using the fact that in Simp(Gp) the homotopy groups of a directed limit are the directed limits of the homotopy groups, one sees that, as each $RC_i \to RC_{i+1}$ is a weak equivalence, so is $RC_0 \to \varinjlim RC_i \cong RC_\infty$. Thus $C_0 \to C_\infty$ is a weak equivalence. □

Conditions for (A) to have a good behaviour with respect to taking pushouts in $\underline{C}$ of trivial cofibrations, are given in the following:

Proposition 2.4. *Suppose that the functor* $L:\mathrm{Simp(Gp)} \to \underline{C}$ *preserves weak equivalences and the counit of the adjunction* (A), $\varepsilon_C: LRC \to C$, *is an isomorphism for all* $C \in \underline{C}$. *Given a pushout diagram in* $\underline{C}$

$$\begin{array}{ccc}
LG_. & \xrightarrow{h} & B \\
{\scriptstyle Lf_.}\downarrow & & \downarrow{\scriptstyle g} \\
LH_. & \longrightarrow & Q
\end{array}$$

if $f_.$ *is a trivial cofibration in* Simp(Gp), *then* g *is so in* $\underline{C}$.

Proof. It remains only to prove, according to Proposition 2.2., that g is a weak equivalence. For this, taking the pushout diagram in Simp(Gp)

$$\begin{array}{ccc}
G_. & \xrightarrow{\varphi(h)} & RB \\
{\scriptstyle f_.}\downarrow & & \downarrow \\
H_. & \longrightarrow & P_.
\end{array}$$

we have, since $f_.$ is a trivial cofibration, that $RB \to P_.$ is a weak equivalence and, as L preserves them, $LRB \to LP_.$ is a weak equivalence in $\underline{C}$.

Now, considering the commutative diagram:

$$\begin{array}{ccccc} & & h & & \\ LG. & \xrightarrow{L\varphi(h)} & LRB & \xrightarrow{\varepsilon_B} & B \\ {\scriptstyle Lf.}\downarrow & & \downarrow & & \downarrow{\scriptstyle g} \\ LH & \longrightarrow & LP. & \longrightarrow & Q \end{array}$$

where the square on the left is a pushout, we have $LP. \cong Q$ since ε_B is an isomorphism, and so g is a weak equivalence. □

We can show now the main theorem of this section.

Theorem 2.5. *For the general adjunction* (A), *suppose the functor L preserves weak equivalences, R preserves directed limits and the counit of the adjunction,* ε_C, *is an isomorphism for each object* $C \in \underline{C}$. *Then, the category* $\underline{C}$ *is a closed model category under the structure proposed by Definition* 2.1.

Proof. Axiom CM1 is true by hypothesis and CM2 is immediate.

Axiom CM3 for weak equivalences and fibrations holds in $\underline{C}$ clearly. Let us prove it for cofibrations.

Let $f: X \to Y$ be a retract of a cofibration $g: W \to Z$ with morphisms i, r, j and s as in the formulation of CM3 (see Section 1), and take any commutative square in $\underline{C}$

$$\begin{array}{ccc} X & \xrightarrow{a} & E \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle p} \\ Y & \xrightarrow{b} & B \end{array} \qquad (*)$$

where p is a trivial fibration. With this square we construct the following commutative one

$$\begin{array}{ccc} W & \xrightarrow{ar} & E \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle p} \\ Z & \xrightarrow{bs} & B \end{array}$$

for which there exists a lifting $D: Z \to E$ because g is a cofibration; thus, a lifting for $(*)$ is a given by Dj.

CM5 (*Factorization axiom*). We start showing the factorization as a trivial cofibration followed by a fibration of any morphism $f: X \to Y$ in $\underline{C}$. For this consider the induced morphisms $LF\Delta[n,k] \to LF\Delta[n]$ and all the commutative diagrams

$$\begin{array}{ccc} LF\Delta[n,k] & \longrightarrow & X \\ \downarrow & \lambda & \downarrow f \\ LF\Delta[n] & \longrightarrow & Y \end{array} \qquad \text{for } 0 \le k \le n \text{ and } n > 0. \tag{$**$}$$

and let

$$\coprod_{\lambda} LF\Delta[n,k] \longrightarrow \coprod_{\lambda} LF\Delta[n]$$

be the induced morphism in the coproducts of $LF\Delta[n,k]$ and $LF\Delta[n]$ indexed by the set of all diagrams ($**$). The morphism

$$\coprod_{\lambda} F\Delta[n,k] \longrightarrow \coprod_{\lambda} F\Delta[n]$$

is a trivial cofibration as it is a coproduct of trivial cofibrations and then, considering the pushout

$$\begin{array}{ccc} \coprod_{\lambda} LF\Delta[n,k] & \longrightarrow & X \\ \downarrow & & \downarrow i_0 \\ \coprod_{\lambda} LF\Delta[n] & \xrightarrow[\alpha_0]{} & X_0 \end{array}$$

we have that the morphism i_0 is a trivial cofibration by Proposition 2.4. which has, in addition, the LLP with respect to all fibrations, and induced by i_0 and α_0, there exists a morphism $p_0 : X_0 \to Y$ such that $p_0 i_0 = f$. Moreover, by construction any morphism $LF\Delta[n] \to Y$ in any diagram ($**$) lifts through p_0 extending $LF\Delta[n,k] \to X \to X_0$.

Applying this entire construction to $p_0 : X_0 \to Y$, one produces a new factorization

$$X_0 \xrightarrow{i_1} X_1 \xrightarrow{p_1} Y$$

and iterating it countably many times, one obtains a sequence of objects of $\underline{C}$, $\{X_m\}$ and morphisms $\{p_m : X_m \to Y\}$ such that the following diagram is commutative

$$\begin{array}{l} X \xrightarrow{i_0} X_0 \xrightarrow{i_1} X_1 \xrightarrow{i_2} \cdots \xrightarrow{i_m} X_m \longrightarrow \cdots \\ f \downarrow \quad \swarrow p_0 \quad \swarrow p_1 \quad \swarrow p_m \\ Y \end{array}$$

where each i_m is a trivial cofibration.

Call

$$X_\infty = \varinjlim X_m$$

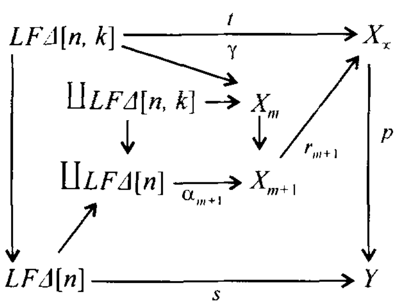
and let $i: X \to X_\infty$ be the canonical morphism. Then, by Propositions 2.2. and 2.3., i is a trivial cofibration which has, in addition, the LLP with respect to all fibrations since each i_m has. On the other hand, the morphisms p_m induce another one $p: X_\infty \to Y$ such that $pi = f$ and we will prove now that p is a fibration in $\underline{C}$. To do this, we see, according to Proposition 2.2., that it has the RLP with respect to the family of morphisms $LF\Delta[n,k] \to LF\Delta[n]$, $0 \le k \le n$, $n > 0$. Take then any commutative diagram

$$\begin{array}{ccc} LF\Delta[n,k] & \xrightarrow{t} & X_\infty \\ \downarrow & & \downarrow p \\ LF\Delta[n] & \xrightarrow{s} & Y \end{array} \qquad (1)$$

and note that, since the objects $LF\Delta[n,k]$ are small by Proposition 2.3, the morphism t factors through some X_m, that is, there is a commutative diagram

$$\begin{array}{ccc} LF\Delta[n,k] & \xrightarrow{t} & X_\infty \\ & {\scriptstyle\gamma}\searrow \quad \nearrow{\scriptstyle r_m} & \\ & X_m & \end{array}$$

Thus, using the following commutative diagram obtained from the above ones:



(2)

it is straightforward to see that the composition

$$LF\Delta[n] \longrightarrow \coprod LF\Delta[n] \xrightarrow{\alpha_{m+1}} X_{m+1} \xrightarrow{r_{m+1}} X_\infty$$

is the required lifting for the diagram (1).

As for the factorization of $f: X \to Y$ into a cofibration followed by a trivial fibration, we repeat the same process as above, starting now from all commutative diagrams of the form

$$\begin{array}{ccc} LF\dot{\Delta}[n] & \longrightarrow & X \\ \downarrow & & \downarrow f \\ LF\Delta[n] & \longrightarrow & Y \end{array} \quad n \ge 0$$

having then a factorization of f, $X \xrightarrow{j} X_\infty \xrightarrow{q} Y$, where j is a cofibration by Proposition 2.2, and, again by this Proposition, $q: X \to Y$ is a trivial fibration since the objects $LF\dot{\Delta}[n]$ are small by Proposition 2.3. and so, a similar diagram to (2) gives the required RLP for f.

CM4 (*Lifting axiom*). The only non-trivial part of this axiom consists of showing the existence of lifting in commutative diagrams of the form

$$\begin{array}{ccc} A & \xrightarrow{a} & X \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle q} \\ B & \xrightarrow[b]{} & Y \end{array} \tag{3}$$

where f is a trivial cofibration and q is a fibration. For this, factor the morphism $f: A \to B$ as in the first half of CM5, that is, $f = pi$ with $p: A_\infty \to B$ a fibration and $i: A \to A_\infty$ a trivial cofibration which has the LLP with respect to all fibrations; axiom CM2 gives then that p is also a weak equivalence and, in the diagram

$$\begin{array}{ccc} A & \xrightarrow{i} & A_\infty \\ {\scriptstyle f}\downarrow & \overset{s}{\nearrow} & \downarrow{\scriptstyle p} \\ B & \xrightarrow[\mathrm{Id}_B]{} & B \end{array}$$

the dotted arrow exists since p is then a trivial fibration and f is a cofibration. On the other hand, since i has the LLP with respect to the fibrations there exists a morphism $t: A_\infty \to X$ such that $ti = a$ and $qt = bp$, and thus, finally, the lifting for the diagram (3) is given by the composition $ts: B \to X$. □

Cofibrations in $\underline{C}$ can be now characterized as follows:

Proposition 2.6. *Let $f: A \to B$ be a morphism in $\underline{C}$. Then, f is a cofibration iff f is a strong retract of the morphism $j: A \to A_\infty$ obtained from the factorization $f = qj$, given in the verification of* CM5, *into a cofibration followed by a trivial fibration.*

Proof. Suppose f is a cofibration in $\underline{C}$ and factor it by CM5 as $f = qj$. Thus in the diagram

$$\begin{array}{ccc} A & \xrightarrow{j} & A_\infty \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle q} \\ B & \xrightarrow[\mathrm{Id}_B]{} & B \end{array}$$

there exists a lifting $s: B \to A_\infty$ which shows, by the following diagram, that f is a strong retract of j

$$\begin{array}{ccc} A & \xrightarrow{\mathrm{Id}_A} & A \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle j} \\ B & \underset{q}{\overset{s}{\rightleftarrows}} & A_\infty \end{array}$$

Conversely, if f is a strong retract of the morphism $j: A \to A_\infty$ obtained from the factorization of f, as in the second half of CM5, f is a retract of a cofibration, and so, by CM3, f is. □

Corollary 2.7. *Given $X \in \underset{\sim}{C}$, X is a cofibrant object (i.e., the unique morphism $\phi \to X$ is a cofibration) iff X is a retract of the objects ϕ_∞ obtained from the factorization of $\phi \to X$ into a cofibration followed by a trivial fibration as in CM5.*

3. Closed model structures for algebraic models of $(n + 1)$-types

We will use the general method of Section 2 to discuss closed model structures on categories of algebraic models of connected types.

3.1. n-HXC(Gp) as a closed model category

The non-abelian version of the classical Dold–Kan's theorem given in [5] provided, by a canonical process of truncation, a new category of algebraic models for $(n + 1)$-types. This category, n-HXC(Gp), consists of certain complexes of non-abelian groups, called n-hypercrossed complexes of groups; n-HXC(Gp) is equivalent to the full subcategory of Simp(Gp) formed by those simplicial groups with trivial Moore complex in dimensions $> n$, which we will denote by n-Hypgd(Gp) since it is just the category of n-hypergrupoids of groups in the sense of Duskin–Glenn [11].

n-Hypgd(Gp) is a reflexive subcategory of Simp(Gp), where the reflection functor P: Simp(Gp) $\to$ n-Hypgd(Gp), left adjoint to the inclusion functor J, is explicitely given by

$$P(G.) = \mathrm{cosk}^{n+1}\left(\frac{G_{n+1}}{H_{n+1}} \;\vdots\; \frac{G_n}{d_{n+1}(N_{n+1}G.)} \;\vdots\; G_{n-1} \rightrightarrows \cdots \rightrightarrows G_1 \rightrightarrows G_0 \right)$$

where H_{n+1} is the normal subgroup of G_{n+1} formed by those $x \in G_{n+1}$ such that $d_i x \in d_{n+1}(N_{n+1}G.)$, $0 \le i \le n + 1$ (compare with that given in [11]).

For this adjoint situation, it is clear that $PJ = \mathrm{Id}$ and P preserves weak equivalences. Also, note that J preserves directed limits since the Moore functor is given by means of finite limits and, for groups, these commute with directed limits.

Recalling Definition 2.1., we say that a morphism f of n-Hypgd(Gp) is a fibration (weak equivalence) if Jf is a fibration (weak equivalence) in Simp(Gp) and f is a cofibration if it has the LLP with respect to the trivial fibrations.

With these definitions we have then, as a direct consequence of Theorem 2.5., the following:

Theorem 3.1.1. *n-Hypgd(Gp) is a closed model category under the structure proposed above.*

Using now the equivalence of categories between n-Hypgd(Gp) and n-HXC(Gp) [5] we have:

Theorem 3.1.2. *The category of algebraic models of $(n+1)$-types, n-HXC(Gp), is a closed model category.*

Particularly, let us note that 1-HXC(Gp) is just the category of crossed modules of groups and 2-HXC(Gp) is that of 2-crossed modules of groups in the sense of Conduché [7] so that we have:

Corollary 3.1.3. *The category XM(Gp) of crossed modules of groups (2-types) is a closed model category where the fibrations are those morphisms $\Gamma = (f_1, f_0)$: $(G \xrightarrow{\rho} H) \to (G' \xrightarrow{\rho'} H')$ such that f_1 is surjective and the weak equivalences are those morphisms Γ inducing isomorphisms $\operatorname{Ker}\rho \cong \operatorname{Ker}\rho'$ and $\operatorname{Coker}\rho \cong \operatorname{Coker}\rho'$.*

Corollary 3.1.4. *The category 2-XM(Gp) of 2-crossed modules of groups (3-types) is a closed model category where the fibrations are those morphisms $\Gamma = (f_2, f_1, f_0): (L \xrightarrow{\varphi} M \xrightarrow{\rho} N) \to (L' \xrightarrow{\varphi'} M' \xrightarrow{\rho'} N')$ such that f_2 and f_1 are surjective and the weak equivalences are those morphisms Γ inducing isomorphisms $\operatorname{Ker}\varphi \cong \operatorname{Ker}\varphi'$, $\operatorname{Ker}\rho/\operatorname{Im}\varphi \cong \operatorname{Ker}\rho'/\operatorname{Im}\varphi'$ and $\operatorname{Coker}\rho \cong \operatorname{Coker}\rho'$.*

3.2. Simpl^n(Gp) as a closed model category

Simp^n(Gp) denotes the category of n-simplicial groups, that is, the category of functors $\mathrm{Gp}^{\Delta^{op} \times \overset{n)}{\cdots} \times \Delta^{op}}$, so that an n-simplical group has n independent simplicial structures (one for each of the "coordinate" directions).

Simp^n(Gp) is related to Simp(Gp) by an adjoint situation

$$\mathrm{Simp}^n(\mathrm{Gp}) \underset{\overline{\mathbb{W}}}{\overset{\mathbb{T}}{\rightleftarrows}} \mathrm{Simp}(\mathrm{Gp})$$

where the functor $\mathbb{T}$ is an extension [4] of Illusie's Total Dec functor [13] and its right adjoint $\overline{\mathbb{W}}$, is a generalization of the Artin-Mazur's total complex [4, 17].

Following Definition 2.1. we say that a morphism f in $\mathrm{Simp}^n(\mathrm{Gp})$ is a fibration (weak equivalence) if $\overline{\mathbb{W}}f$ is a fibration (weak equivalence) of simplicial groups, and cofibrations in $\mathrm{Simp}^n(\mathrm{Gp})$ are defined by the LLP with respect to the trivial fibrations.

Note that in [18] Moerdijk defined, in a similar way, a closed model structure for $\mathrm{Simp}^2(\mathrm{Sets})$, the category of bisimplicial sets.

The unit of the above adjunction $G. \to \overline{\mathbb{W}}\mathbb{T}G.$ is a weak equivalence of simplicial groups [4] and so it is clear that the functor $\mathbb{T}$ preserves weak equivalences. Moreover, using the fact that in Simp(Gp), and hence in $\mathrm{Simp}^n(\mathrm{Gp})$, directed limits commute with finite limits and taking into account that the construction of $\overline{\mathbb{W}}$ is given by means of finite limits, this functor preserves directed limits. Although the counit of this adjunction is not an isomorphism, the proof of the Theorem 2.5 works by replacing the required Proposition 2.4 by the following:

Lemma 3.2.1. *In any pushout diagram in* $\mathrm{Simp}^n(\mathrm{Gp})$

$$\begin{array}{ccc} \mathbb{T}F\Delta[r,k] & \longrightarrow & X \\ \downarrow & & \downarrow{\scriptstyle f} \\ \mathbb{T}F\Delta[r] & \longrightarrow & P, \end{array} \quad 0 \le k \le r,\ r > 0$$

the morphism f is a trivial cofibration.

Proof. Since the morphisms $\mathbb{T}F\Delta[n,k] \to \mathbb{T}F\Delta[n]$ are trivial cofibrations it is clear that f is a cofibration. To prove that f is a weak equivalence, i.e., that $\overline{\mathbb{W}}f$ is of simplicial groups, we use the diagonal functor $\mathbb{D}:\mathrm{Simp}^n(\mathrm{Gp}) \to \mathrm{Simp}(\mathrm{Gp})$ which is weak equivalent to $\overline{\mathbb{W}}$, that is, there exists a natural transformation $\nu:\mathbb{D} \to \overline{\mathbb{W}}$ such that $\nu_X:\mathbb{D}X \to \overline{\mathbb{W}}X$ is a weak equivalence of simplicial groups [4]. Now, applying $\mathbb{D}$ to the given pushout diagram, we have a pushout in Simp(Gp)

$$\begin{array}{ccc} \mathbb{D}\mathbb{T}F\Delta[r,k] & \longrightarrow & \mathbb{D}X \\ \downarrow & & \downarrow{\scriptstyle \mathbb{D}f} \\ \mathbb{D}\mathbb{T}F\Delta[r] & \longrightarrow & \mathbb{D}P \end{array}$$

where $\mathbb{D}\mathbb{T}F\Delta[r,k] \to \mathbb{D}\mathbb{T}F\Delta[r]$ is a weak equivalence and also a cofibration because, in fact, it is clear that this morphism is a free map in the sense of Quillen [21]. Thus $\mathbb{D}f$ is a weak equivalence and so f is. $\square$

We can now assert the following:

Theorem 3.2.1. *The category* $\mathrm{Simp}^n(\mathrm{Gp})$ *with the above structure is a closed model category.*

3.3. A closed model structure for $Cat^n(Gp)$?

Let $\mathrm{Cat}^n(\mathrm{Gp})$ be the category of cat^n-groups [2, 15], that is the category of n-fold internal categories in Gp, or equivalently, that of n-fold internal groupoids in Gp. By using the n independent category structures of a cat^n-group there is a multinerve functor

$$\mathbb{N} : \mathrm{Cat}^n(\mathrm{Gp}) \to \mathrm{Simp}^n(\mathrm{Gp})$$

which embeds $\mathrm{Cat}^n(\mathrm{Gp})$ into $\mathrm{Simp}^n(\mathrm{Gp})$ as a reflexive subcategory whose image consists just (up to natural equivalence) of those n-simplicial groups which have trivial Moore complex in dimensions ≥ 2, for each of the n independent simplicial structures. Let us note that, for $n = 1$, $\mathrm{Cat}^n(\mathrm{Gp})$ is the category of internal groupoids in Gp and $\mathbb{N}$ is the usual nerve functor in the sense of Grothendieck, which gives an equivalence of categories between $\mathrm{Cat}^1(\mathrm{Gp})$ and 1-Hypgd(Gp). In this case the reflector functor $\mathbb{P} : \mathrm{Simp}(\mathrm{Gp}) \to \mathrm{Cat}^1(\mathrm{Gp})$ is given by the fundamental groupoid construction

$$\mathbb{P}(G.) = \frac{G_1}{d_2 N_2(G.)} \rightleftarrows G_0$$

where the source, target and identity morphisms are induced by d_0, d_1 and s_0 respectively.

For $n \geq 2$, the reflection functor

$$\mathbb{P} : \mathrm{Simp}^n(\mathrm{Gp}) \longrightarrow \mathrm{Cat}^n(\mathrm{Gp})$$

is then obtained by taking fundamental groupoid in each of the n independent directions.

Considering then the following adjoint situation

$$\mathrm{Cat}^n(\mathrm{Gp}) \underset{\mathbb{N}}{\overset{\mathbb{P}}{\rightleftarrows}} \mathrm{Simp}^n(\mathrm{Gp}) \underset{\overline{\mathbb{W}}}{\overset{\mathbb{T}}{\rightleftarrows}} \mathrm{Simp}(\mathrm{Gp})$$

we propose, as in Definition 2.1, that a morphism f in $\mathrm{Cat}^n(\mathrm{Gp})$ is a fibration (weak equivalence) if $\overline{\mathbb{W}}\mathbb{N}f$ is a fibration (weak equivalence) of simplicial groups and that f is a cofibration if it has the LLP with respect to the trivial fibrations.

For the above adjunction it is clear that the functor $\mathbb{N} : \mathrm{Cat}^n(\mathrm{Gp}) \to \mathrm{Simp}^n(\mathrm{Gp})$ preserves directed limits so the composition $\overline{\mathbb{W}}\mathbb{N}$ does; also the functor $\mathbb{P}\mathbb{T} : \mathrm{Simp}(\mathrm{Gp}) \to \mathrm{Cat}^n(\mathrm{Gp})$ preserves weak equivalences since the unit of the adjunction, $\eta_{G.} : G. \to \overline{\mathbb{W}}\mathbb{N}\mathbb{P}\mathbb{T}G.$, induces isomorphisms on the homotopy groups Π_i, $0 \leq i \leq n$, for each simplicial group $G.$ [4].

As in the case of $\mathrm{Simp}^n(\mathrm{Gp})$, the counit of this adjunction is not an isomorphism but we conjecture that it is possible to use the constructions given at the proof of the Theorem 2.5, to show that $\mathrm{Cat}^n(\mathrm{Gp})$, $n \geq 2$, is a closed model category with the above proposed structure.

References

[1] R. Brown and M. Golasinski, A model structure for the homotopy theory of crossed complexes, Cahiers Topologie Géom. Differentielle Catégoriques 30 (1989) 61–82.

[2] R. Brown and P.J. Loday, Van Kampen theorems for diagrams of spaces, Topology 26 (1987) 311–335.

[3] M. Bullejos and A.M. Cegarra, A 3-dimensional non-abelian cohomology of groups with applications to homotopy classification of continuous maps, Canad. J. Math. 43 (1991) 265–296.

[4] M. Bullejos, A.M. Cegarra and J. Duskin, On cat^n-groups and homotopy types, J. Pure Appl. Algebra 86 (1993) 135–154.

[5] P. Carrasco and A.M. Cegarra, Group-theoretic algebraic models for homotopy types, J. Pure Appl. Algebra 75 (1991) 195–235.

[6] A.M. Cegarra, M. Bullejos and A.R. Garzón, Higher dimensional obstruction theory in algebraic categories, J. Pure Appl. Algebra 49 (1987) 43–102.

[7] D. Conduché, Modules croisés généralisés de longueur 2, J. Pure Appl. Algebra 34 (1984) 155–178.

[8] E.B. Curtis, Simplicial Homotopy Theory, Adv. in Math. 6 (1971) 107–209.

[9] S. Eilenberg and S. Mac Lane, Relations between homology and homotopy groups, Proc. Nat. Acad. Sci. U.S.A. 29 (1943).

[10] P. Gabriel and M. Zisman, Calculus of Fractions and Homotopy Theory (Springer, Berlin, 1967).

[11] P. Glenn, Realization of cohomology classes in arbitrary exact categories, J. Pure Appl. Algebra 25 (1982) 33–107.

[12] L. Hernández and T. Porter, Categorical models of n-types for pro-crossed complexes and J_n-prospaces, Lecture Notes in Mathematics Vol. 1509 (1992) 148–185.

[13] L. Illusie, Complex cotangent et deformations II, Lecture Notes in Mathematics, Vol. 283 (Springer, Berlin, 1972).

[14] D.M. Kan, On homotopy theory and C.S.S. groups, Ann. of Math. 68(1) (1958) 38–53.

[15] J.L. Loday, Spaces with finitely many non-trivial homotopy groups, J. Pure Appl. Algebra 24 (1982) 179–202.

[16] S. Mac Lane and J.H.C. Whitehead, On 3-type of a complex, Proc. Nat. Acad. Sci. U.S.A. 30 (1956) 41–48.

[17] J.P. May, Simplicial Objects in Algebraic Topology (Nostrand, New York, 1976).

[18] I. Moerdijk, Bisimplicial sets and the group-completion theorem, in: J.F. Jardine and V.P. Snaith, eds., Algebraic K-theory: Connections with Geometry and Topology (Reidel, Dordrecht, 1989) 225–240.

[19] I. Moerdijk and J.A. Svensson, Algebraic classification of equivariant homotopy 2-types, I, J. Pure Appl. Algebra 89 (1993) 187–216.

[20] T. Porter, n-types of simplicial groups and crossed n-cubes, Topology 32(1) (1993) 5–24.

[21] D. Quillen, Homotopical algebra, Lecture Notes in Mathematics, Vol. 43 (Springer, Berlin, 1967).

[22] D. Quillen, Rational homotopy theory, Ann. of Math. 90 (1969) 205–295.

[23] R.W. Thomason, Cat as a closed model category, Cahiers Topologie Géométrie Différentielle XXI-3 (1980) 304–324.

[24] J.H.C. Whitehead, Combinatorial homotopy I, II, Bull. Amer. Math. Soc. 55 (1949) 213–245, 496–543.