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Stability Properties of Volterra Integrodifferential Equations

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1. INTRODUCTION

In this paper we are concerned with the asymptotic behavior of solutions of Volterra integrodifferential systems of the form

$$x'(t) = Ax(t) + \int_0^t B(t, s)\, x(s)\, ds + f(t), \qquad (' = d/dt), \tag{1}$$

where $0 \leqslant t < \infty$, A and B are $n \times n$ matrices, and x and f are n-vector valued functions.

To obtain our results for integrodifferential equations we first examine a general functional differential equation with unbounded delay and determine conditions of the Liapunov–Razumikhin type which guarantee that all solutions are bounded and tend to zero as t tends to infinity. The results for functional differential equations assume the existence of a Liapunov function defined on $R \times R^n$ rather than a Liapunov functional, and the techniques involved are similar to those employed by Seifert [11, 12] and R. Driver [3][1] in obtaining similar results for functional differential equations with unbounded delay. These results are then applied to Eq. (1) to obtain corresponding results for the solutions of (1). In particular, we are able to obtain results for the resolvent operator associated with Eq. (1) which when used in conjunction with the variation of constants formulas obtained by Grossman and Miller [5] yield immediate perturbation results for the nonlinear system

$$x'(t) = Ax(t) + \int_0^t B(t, s)\{x(s) + (g_1x)(s)\}\, ds + (g_2x)(t) + f(t), \tag{2}$$

where g_i, $i = 1, 2$, is a nonlinear functional of higher order.

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[1] The results in [11] on stability were in fact already obtained by Driver in [3].

Our results are also valid for the convolution case, $B(t, s) = B(t - s)$, if $B(t)$ is integrable on the half line $[0, \infty)$. Although necessary and sufficient conditions have been given for the resolvent of (1) to be integrable on $[0, \infty)$, these conditions are often difficult to verify and our results will provide an alternate test which may often be easier to use than the known criteria. In general, we shall assume that A is a stable matrix. However, in the convolution case we show that Eq (1) may be transformed into another equation of the same form where the matrix A in the new equation is stable and that the resolvent associated with Eq. (1) is in $L^1(0, \infty)$ if and only if the resolvent of the new equation is $L^1(0, \infty)$.

As Eq. (1) is not assumed to be of convolution type our results can be used to obtain global results for nonlinear equations of the form

$$x'(t) = Ax(t) + \int_0^t B(t, s)\, g(x(s))\, ds + f(t),$$

when g is uniformly Lipschitzian. This is accomplished by examining the difference of two solutions and applying the results and techniques previously obtained for the linear equation.

2. Functional Differential Equations

Consider the equation

$$x'(t) = f(t, x_t, \mu), \qquad t > 0, \tag{3μ}$$

where $\mu \in \mathcal{O}\!\ell$, an arbitrary set of parameters and for each fixed μ, $f: R^+ \times CB \to R^n$ is continuous. Here R^n is n-dimensional Euclidean space with the usual inner product and norm, R^+ is the set of nonnegative reals, and CB is the vector space of bounded continuous functions $\varphi: (-\infty, 0] \to R^n$ with the usual sup norm: $\| \ \|$. Given a function $x(t)$ defined for $t < T$, x_t is defined by $x_t(s) = x(t + s)$ for $s \leqslant 0$. By a solution of (3μ) is meant a function $x(t)$ defined on $(-\infty, T)$, $T > 0$, which is continuously differentiable on $(0, T)$ with $x_t \in CB$ for $t < T$, and which satisfies (3μ) on $(0, T)$. Throughout our discussion of Eq. (3μ), it is assumed that if $x(t)$ is a solution of (3μ) on $(-\infty, T)$, $0 < T < \infty$, then either $x(t)$ can be continued past T or $\| x_t \|$ is not bounded as $t \to T-$. A solution to the initial value problem is defined in the usual manner and the reader is referred to Driver [3] on the subjects of existence, uniqueness, and continuability of solutions of (3μ).

The reader will notice that there is no requirement that $f(t, \varphi, \mu)$ depend on μ in a continuous fashion. The reason for this is that we are not concerned with continuity of solutions with respect to the parameter but rather we wish to make clear the rather obvious observation that if one has a Liapunov function which does not depend on μ but which "works" equally well for each $\mu \in \mathcal{U}$ then the solutions of (3μ) behave in a uniform fashion independent of μ. Although this is basically an artificial problem in this form, it will arise in a rather natural fashion as one attempts to analyze Eq. (1) and its nonlinear generalization when the function $f(t)$ is asymptotically periodic or asymptotically almost periodic.

In our discussion of boundedness and asymptotic behavior of solutions of (3μ), V will be a continuous function defined on $R \times R^n$ into R and $V'(t, x(t))$, $t \geqslant 0$, will denote the upper right-hand derivative of $V(t, x(t))$, where $x(t)$ is a solution of (3μ). Also, h will be a function defined and continuous on R^+ into R^+, satisfying $h(0) = 0$ and $h(s) > s$ for $s > 0$.

Given $\varphi \in CB$ and $t_0 \geqslant 0$, $x(t, t_0, \varphi, \mu)$ will denote a solution of (3μ) which satisfies $x(t_0 + s) = \varphi(s)$ for $s \leqslant 0$. If $f(t, 0, \mu)$ is identically zero, we say that the zero solution of (3μ) is

(i) uniformly stable if for each $\epsilon > 0$ there exists $\delta = \delta(\epsilon) > 0$ so that $t_0 \geqslant 0$, $\mu \in \mathcal{U}$, and $\| \varphi \| < \delta$ imply $| x(t, t_0, \varphi, \mu)| < \epsilon$ for $t \geqslant t_0$,

(ii) uniformly asymptotically stable if it is uniformly stable and there exists a constant $\delta_0 > 0$ so that for any $\eta > 0$ there exists $T(\eta) > 0$ such that $\mu \in \mathcal{U}$, $t_0 \geqslant 0$, and $\| \varphi \| < \delta_0$ imply $| x(t, t_0, \varphi, \mu)| < \eta$ for $t \geqslant t_0 + T(\eta)$.

We say the the solutions of (3μ) are

(iii) uniform bounded if for any constant $M > 0$ there exists $N = N(M) > 0$ so that $\| \varphi \| < M$, $\mu \in \mathcal{U}$ and $t_0 \geqslant 0$ imply $| x(t, t_0, \varphi, \mu)| < N$ for $t \geqslant t_0$.

THEOREM 1. *Suppose $V: R \times R^n \to R$ is continuous and satisfies:*

(i) *$a(| x |) \leqslant V(t, x) \leqslant b(| x |)$ for all (t, x) where $a, b: R^+ \to R^+$ are continuous and increasing with $a(r) \to \infty$ as $r \to \infty$,*

(ii) *there exists $M \geqslant 0$ so that if $x(t)$ is a solution of (3μ) with $| x(t)| \geqslant M$ for some $t \geqslant 0$ and $V(s, x(s)) < h(V(t, x(t)))$ for $s \leqslant t$, then $V'(t, x(t)) \leqslant 0$. Then the solutions of (3μ) are uniform bounded.*

This theorem is analogous to Theorem 2 in [11], and its proof is similar, and therefore is omitted. We remark that the hypothesis $a(r) \to \infty$ as $r \to \infty$ of our theorem should also have been included in Theorem 2 in [11].

We also note that if $f(t, 0, \mu) \equiv 0$, $b(0) = 0$, and $M = 0$, the hypotheses of Theorem 1 imply uniform stability of the zero solution; we omit the details.

THEOREM 2. *Suppose all solutions of (3μ) are bounded. Let $V: R \times R^n \to R$ be continuous and satisfy:*

(i) *$a(|x|) \leqslant V(t, x) \leqslant b(|x|)$ for all (t, x), where $a, b\colon R^+ \to R^+$ are continuous and increasing.*

(ii) *Given $M > 0$, there exist monotone sequences of positive reals $\{r_j\}$ and $\{u_j\}$ with $r_j \to \infty$ and $u_j \to 0$ as $j \to \infty$, and continuous functions $w_j(s)$ which are positive on the set $u_j/2 \leqslant s \leqslant M$ so that given a solution $x(s) = x(s, t_0, \varphi, \mu)$ of (3μ) with $|x(s)| \leqslant M$ for all s, if for some $t \geqslant r_j + t_0$ one has $u_j \leqslant |x(t)|$ and $V(s, x(s)) < h(V(t, x(t)))$ for $t - r_j \leqslant s \leqslant t$, then $V'(t, x(t)) \leqslant -w_j(|x(t)|)$.*
Then every solution of (3μ) tends to zero as $t \to \infty$. In particular, given $M > 0$ and $\eta > 0$ there exists $T(\eta, M) > 0$ so that if $x(t) = x(t, t_0, \varphi, \mu)$ is a solution of (3μ) with $|x(t)| \leqslant M$ for all t, then $|x(t)| < \eta$ for all $t \geqslant T(\eta, M) + t_0$.

Proof. Let $M > 0$ be given and $x(t) = x(t, t_0, \varphi, \mu)$ be a solution of (3μ) with the property that $|x(t)| \leqslant M$ for all t. Let $\eta > 0$ be given, $\eta < M$.

As h is continuous and $h(s) > s$ for $s > 0$, there exists $\epsilon_0 > 0$ so that $h(s) - s \geqslant \epsilon_0$ for $a(\eta)/2 \leqslant s \leqslant b(M)$. Now, choose a positive integer N so that $a(\eta) \geqslant b(M) - N\epsilon_0 \geqslant a(\eta)/2$. (This choice is always possible if ϵ_0 is chosen sufficiently small.) Let $\epsilon_j = b(M) - j\epsilon_0$ and choose $J > 0$ so that $u_J < b^{-1}(\epsilon_N)$.

If $V(t, x(t)) > \epsilon_1$ for $t \geqslant r_J + t_0$ then

$$h(V(t, x(t))) \geqslant V(t, x(t)) + \epsilon_0 > \epsilon_1 + \epsilon_0 = b(M) \geqslant V(s, x(s))$$

for $t - r_J \leqslant s \leqslant t$. Also, $b(|x(t)|) \geqslant V(t, x(t)) \geqslant \epsilon_1 > \epsilon_N$ for $t \geqslant r_J + t_0$, so $|x(t)| > b^{-1}(\epsilon_N) > u_J$ for $t > r_J + t_0$. Hence,

$$V'(t, x(t)) \leqslant -w_J(|x(t)|) \leqslant -\gamma < 0$$

for $t \geqslant r_J + t_0$, where $w_J(s) \geqslant \gamma > 0$ for $u_J \leqslant s \leqslant M$. Thus,

$$V(t, x(t)) - b(M) \leqslant V(t, x(t)) - V(r_J + t_0, x(r_J + t_0)) \leqslant -\gamma(t - r_J - t_0)$$

for $t \geqslant r_J + t_0$ which contradicts $V(t, x) \geqslant 0$, and so there must be a value of t, say t_1, in the interval $[r_J + t_0, r_J + t_0 + \epsilon_0/\gamma]$ so that $V(t_1, x(t_1)) \leqslant \epsilon_1$. If there exists $t_2 > t_1$ for which $V(t_2, x(t_2)) > \epsilon_1$, there must exist t_3, $t_1 \leqslant t_3 < t_2$, for which $V(t_3, x(t_3)) = \epsilon_1$ and $V(t, x(t)) \geqslant \epsilon_1$ for $t_3 \leqslant t \leqslant t_2$. If $t_3 \leqslant t \leqslant t_2$, we see that $|x(t)| > b^{-1}(\epsilon_N) > u_J$ and

$$h(V(t, x(t))) \geqslant V(t, x(t)) + \epsilon_0 \geqslant \epsilon_1 + \epsilon_0 = b(M) \geqslant V(s, x(s))$$

for $t - r_J \leqslant s \leqslant t$, and, hence, on (t_3, t_2),

$$V'(t, x(t)) \leqslant -w_J(|x(t)|) \leqslant -\gamma < 0,$$

and we must have $V(t_2, x(t_2)) \leqslant V(t_3, x(t_3)) = \epsilon_1$ which is a contradiction. It now follows from this contradiction that $V(t, x(t)) \leqslant \epsilon_1$ for $t \geqslant r_J + \epsilon_0/\gamma + t_0$. If we now consider $x(t)$ on the interval $[2r_J + \epsilon_0/\gamma + t_0, \infty)$, by a slight modification of the above argument it can be shown that $V(t, x(t)) \leqslant \epsilon_2$ for $t \geqslant 2r_J + 2\epsilon_0/\gamma + t_0$. Continuing this process, $V(t, x(t)) \leqslant \epsilon_N = b(M) - N\epsilon_0 \leqslant a(\eta)$ for $t \geqslant N(r_J + \epsilon_0/\gamma) + t_0$. Defining $T(\eta, M)$ by $T(\eta, M) = N(r_J + \epsilon_0/\gamma)$ we obtain $|x(t)| \leqslant \eta$ for $t \geqslant T(\eta, M) + t_0$. This completes the proof.

Just as Theorem 1 may be slightly altered to provide a result on uniform stability for (3μ), Theorem 2 may be modified to give sufficient conditions for uniform asymptotic stability. We state this result in a form easily applied to integrodifferential equations.

THEOREM 3. *Suppose that* $f(t, 0, \mu) \equiv 0$ *and that* $x = 0$ *is uniformly stable. Let* $V: R \times R^n \to R$ *be continuous and satisfy*:

(i) $a(|x|) \leqslant V(t, x) \leqslant b(|x|)$ *for all* (t, x) *where* $a, b: R^+ \to R^+$ *are continuous and increasing*,

(ii) *there exists* $M > 0$ *and monotone sequences of positive reals* $\{r_j\}$ *and* $\{u_j\}$ *with* $r_j \to \infty$ *and* $u_j \to 0$ *as* $j \to \infty$, *and continuous functions* $w_j(s)$ *which are positive on* $u_j/2 \leqslant s \leqslant M$ *so that given a solution* $x(s) = x(s, t_0, \varphi, \mu)$ *of* (3μ) *with* $|x(s)| \leqslant M$ *for all* s, *if for some* $t \geqslant r_j + t_0$ *one has* $u_j \leqslant |x(t)|$ *and* $V(s, x(s)) < h(V(t, x(t)))$ *for* $t - r_j \leqslant s \leqslant t$, *then* $V'(t, x(t)) \leqslant -w_j(|x(t)|)$. *Then* $x = 0$ *is uniformly asymptotically stable.*

The proof of this result is almost identical to the proof of Theorem 2 and is similar to Theorem 1 of [12]. This result differs from the result of Driver [3, Theorem 7] in that Driver assumed that for μ fixed, $f(t, x_t, \mu)$ depended only on t and $x(s)$ for $g(t) \leqslant s \leqslant t$ where $g(t) \to \infty$ as $t \to \infty$ to obtain asymptotic stability and to obtain uniform asymptotic stability it was further assumed that $g(t) \geqslant t - r$ for some constant r.

We note that an example exists showing that the uniform asymptotic stability of the zero solution does not follow by adding to the conditions sufficient for the uniform stability of the zero solution the requirement that for a solution $x(t)$ such that $h(V(t, x(t))) > V(s, x(s))$ for all $s \leqslant t$ we have $V'(t, x(t))$ bounded above by a negative definite function of $|x(t)|$; cf. [11]. For the case where the delay interval is bounded, say by $r > 0$, for solutions with $h(V(t, x(t))) > V(s, x(s))$ for $t - r \leqslant s \leqslant t$, such an additional condition on $V'(t, x(t))$ does imply the uniform asymptotic stability of the zero solution; cf. [3], for example.

3. Integrodifferential Equations

We now consider the integrodifferential equation

$$x'(t) = Ax(t) + \int_0^t B(t, s)\, x(s)\, ds + f(t), \tag{1}$$

where $t \geqslant t_0 \geqslant 0$ and where $x(t) = \theta(t)$ on $0 \leqslant t \leqslant t_0$. Here A and B are $n \times n$ matrices with B defined for $t, s \geqslant 0$ and locally integrable on $R^+ \times R^+$, and $f\colon R^+ \to R^n$ and $\theta\colon [0, t_0] \to R^n$ are continuous.

This type of initial value problem for (1) is not the usual one. The usual initial value problem for (1), and the one we are ultimately concerned with here, is when $t_0 = 0$ and θ is an initial vector $\theta(0)$. It can be very useful, however, to study the initial value problem where $t_0 > 0$ is also considered and the reader is referred to Miller [8] and Grossman and Miller [6] where this type of initial value problem has been examined to obtain significant results on the behavior of solutions to the usual initial value problem where $t_0 = 0$.

In what follows, we shall assume

$$\lim_{h \to 0} \int_0^t | B(t, s) - B(t + h, s)|\, ds = 0 \tag{H1}$$

and

$$\lim_{h \to 0} \int_t^{t+h} | B(t + h, s)|\, ds = 0$$

for each $t \geqslant 0$.

To see that the results obtained for Eq. (3μ) can be applied to (1), let $x^*(t)$ be a solution of (1). Then $x(t) = x^*(t)$ for $t \geqslant 0$, $x(t) = \theta(0)$ for $t < 0$ is clearly a solution of $x'(t) = F(t, x_t)$ which is of the form of (3μ); here

$$F(t, \varphi) = A\varphi(0) + \int_{-t}^0 B(t, t + s)\, \varphi(s)\, ds + f(t);$$

$\varphi \in CB$, and $x_t(s) = x(t + s)$, $s \leqslant 0$; as before. The fact that F is continuous on $R^+ \times CB$ follows from our hypotheses on B and f. When applying our results for Eq. (3μ) to (1) we shall assume $x(t) = \theta(0)$ for $t \leqslant 0$ and we denote this solution $x(t, t_0, \theta)$.

It is of interest to note that if $f(t)$ is identically zero that the definitions given by Miller [8] for the uniform stability and uniform asymptotic stability of the zero solution of (1) are equivalent to the definitions for uniform stability and uniform asymptotic stability of the zero solution of (1) when it is written as $x'(t) = F(t, x_t)$ where F is as above.

Associated with eq. (1) is the resolvent equation

$$\frac{\partial}{\partial s} R(t, s) = -R(t, s) A - \int_s^t R(t, u) B(u, s) \, du, \qquad R(t, t) = I \tag{4}$$

on the interval $0 \leqslant s \leqslant t$. Grossman and Miller, in [5, Lemma 1], have shown under weaker assumptions than we have imposed on (1) that $R(t, s)$ is defined and continuous on the set $0 \leqslant s \leqslant t$ and that $(\partial/\partial s) R(t, s)$ exists a.e. on the set $0 \leqslant s \leqslant t$ and is locally integrable on that set. Further, the solution of (1) with initial condition $t_0 = 0$, $x(0) = x_0$, can be written in the form

$$x(t) = R(t, 0) x_0 + \int_0^t R(t, s) f(s) \, ds. \tag{5}$$

In order to obtain results on the asymptotic behavior of the solutions of (2) it will be necessary to examine the operators $\rho, \dot{\rho}: C \to C$ defined by

$$\rho(\varphi)(t) = \int_0^t R(t, s) \varphi(s) \, ds$$

and

$$\dot{\rho}(\varphi)(t) = \int_0^t (\partial/\partial s) R(t, s) \varphi(s) \, ds.$$

Here $C = C([0, \infty), R^n)$, the space of continuous functions $\varphi: [0, \infty) \to R^n$, with the compact open topology. As $R(t, s)$ is continuous it is clear that ρ maps C into C continuously. That $\dot{\rho}$ maps C into C continuously follows immediately from (4) and the assumption on B upon using Fubini's Theorem.

In particular, we are concerned with obtaining conditions so that the linear subspaces,

$$\begin{aligned}
BC &= \{\varphi \in C: \varphi(t) \text{ is bounded for } t \geqslant 0\}, \\
BC_l &= \{\varphi \in BC: \varphi \text{ has a limit at infinity}\}, \\
BC_0 &= \{\varphi \in BC_l: \varphi \text{ has limit zero at infinity}\}, \\
A(\omega) &= \{\varphi \in BC: \varphi \text{ is asymptotically } \omega\text{-periodic}\}, \\
AAP &= \{\varphi \in BC: \varphi \text{ is asymptotically almost periodic}\},
\end{aligned}$$

endowed with the usual sup norm, $\| \ \|$, are invariant under the operators ρ and $\dot{\rho}$. Here a function φ is asymptotically ω-periodic if there is a continuous ω-periodic function $p(t)$ so that $\varphi(t) - p(t) \to 0$ as $t \to \infty$ while a function φ is asymptotically almost periodic if there are functions $p(t)$ and $q(t)$ such that $p(t)$ is almost periodic on R, $q(t)$ is continuous for $t \geqslant 0$, $q(t) \to 0$ as $t \to \infty$, and $\varphi(t) = p(t) + q(t)$.

As the topology on each of these subspaces is stronger than the topology on C it follows easily from the closed graph theorem that if any of the above subspaces are invariant under ρ and $\dot{\rho}$ then there operators are continuous in the sup norm (cf. [7, p. 252]).

Throughout this section we will impose the following condition: (H2) A is a stable matrix, i.e., all eigenvalues of A have negative real parts, and C is the positive definite symmetric solution of the matrix equation $A^TC + CA = -I$. Also, there exists a constant $M > 0$ so that

$$\int_0^t | CB(t, s)| \, ds \leqslant M$$

for all $t \geqslant 0$ and $2\beta M/\alpha < 1$ where α^2 and β^2 are, respectively, the smallest and largest eigenvalues of C. Here and henceforth x^T denotes the transpose of x, x either a vector or a matrix.

THEOREM 4. *Suppose* (H1) *and* (H2) *are true. Then if f is bounded, all solutions of* (1) *are bounded. In particular, $R(t, 0)$ is bounded and $\rho, \dot{\rho}: BC \to BC$.*

Proof. Define $V(t, x)$ by $V(t, x) = x^TCx$. As $\alpha^2 | x |^2 \leqslant V(t, x) \leqslant \beta^2 | x |^2$, we see that (i) of Theorem 1 is satisfied for this V.

Along solutions of (1) we have for $t \geqslant t_0$

$$V' = -| x(t)|^2 + 2x^T(t) \int_0^t CB(t, s) \, x(s) \, ds + 2x^T(t) \, Cf(t)$$

$$\leqslant -| x(t)|^2 + 2 \, | x(t)| \int_0^t | CB(t, s)| \, | x(s)| \, ds + 2 \, | x(t)| \, | C | \, \| f \|.$$

Now, if $h^2V(t, x(t)) > V(s, x(s))$ for $s \leqslant t$, where $h > 1$ is a constant to be determined, then

$$h^2\beta^2 \, | x(t)|^2 \geqslant h^2V(t, x(t)) \geqslant V(s, x(s)) \geqslant \alpha^2 \, | x(s)|^2$$

and

$$(h\beta/\alpha) \, | x(t)| \geqslant | x(s)|, \qquad \text{for} \quad s \leqslant t.$$

Thus,

$$V' \leqslant -| x(t)|^2 + (2h\beta/\alpha) \, | x(t)|^2 \int_0^t | CB(t, s)| \, ds + 2 \, | x(t)| \, | C | \, \| f \|$$

and as $2\beta M/\alpha < 1$, h may be chosen so that $h > 1$ and $2h\beta M/\alpha < 1$ so that

$$V' \leqslant [(2h\beta M/\alpha) - 1] \, | x(t)|^2 + 2 \, | C | \, \| f \| \, | x(t)| \leqslant 0$$

if $| x(t)| \geqslant 2 \, | C | \, \| f \| / [1 - (2h\beta M/\alpha)]$.

It now follows from Theorem 1 that if f is bounded, all solutions of (1) are bounded. From Eq. (5), the solutions of (1) with $t_0 = 0$ and $x(0) = x_0$ may be written as $x(t) = R(t, 0)\, x_0 + \rho(f)(t)$ and the conclusion that $R(t, 0)$ is bounded follows by taking $f(t)$ to be identically zero and $\rho(BC) \subset BC$ follows by taking $x_0 = 0$.

To see that $\dot{\rho}(BC) \subset BC$, let $f \in BC$ and consider

$$\dot{\rho}(f)(t) = \int_0^t (\partial/\partial x)\, R(t, s)\, f(s)\, ds.$$

From Eq. (4), we obtain

$$\begin{aligned}\dot{\rho}(f)(t) &= -\int_0^t R(t, s)\, Af(s)\, ds - \int_0^t \int_s^t R(t, u)\, B(u, s)\, f(s)\, du\, ds \\ &= -\rho(Af)(t) - \int_0^t R(t, u) \int_0^u B(u, s)\, f(s)\, ds\, du\end{aligned}$$

from Fubini's Theorem. As $\int_0^t |CB(t, s)|\, ds$ is bounded and C is invertible it follows that $\int_0^t |B(t, s)|\, ds$ is bounded. Also, from (H1) we see that $\int_0^t B(t, s)\, f(s)\, ds$ is continuous. Thus, as $\int_0^t B(t, s)\, f(s)\, ds \in BC$ and $\rho(BC) \subset BC$ we have $\dot{\rho}(BC) \subset BC$ and the proof is complete.

COROLLARY 1. *Assume that the hypotheses of Theorem 4 are valid and in addition, $B(t, s) = B(t - s)$. Then $R(t, s) = R(t - s)$, $R(t)$ and $R'(t) \in L^P(R^+)$ for $1 \leqslant P \leqslant \infty$ and both $R(t)$ and $R'(t) \to 0$ as $t \to \infty$.*

Proof. That $R(t, s) = R(t - s)$ follows from Lemma 2 of [5] while a result of Corduneanu [2] states that $\rho(BC) \subset BC$ if and only if

$$\int_0^t |R(t, s)|\, ds \leqslant M < \infty$$

for some constant M and all $t \geqslant 0$. Thus, $R(t) \in L^1(R^+)$ and similarly $B(t) \in L^1(R^+)$. The rest of the corollary follows from Theorem 2.5 of [6] and standard arguments.

THEOREM 5. *Let* (H1) *and* (H2) *be valid and let*

$$\lim_{T\to\infty} \sup \left\{ \int_0^{t-T} |B(t, s)|\, ds : t \geqslant T \right\} = 0. \tag{6}$$

If $f \in BC_0$ then every solution of (1) *tends to zero as $t \to \infty$. More specifically, given $M_1 > 0$ and $\eta > 0$ there exists $T(\eta, M_1) > 0$ so that if $x(t) = x(t, t_0, \theta)$*

is a solution of (1) *with* $|x(t)| \leqslant M_1$ *for all* t *then* $|x(t)| < \eta$ *for* $t \geqslant T(\eta, M_1) + t_0$. *Also,* $\rho, \dot{\rho}: BC_0 \to BC_0$ *and* $R(t, 0) \to 0$ *as* $t \to \infty$.

Proof. Let $f \in BC_0$. As the hypotheses of Theorem 4 are satisfied we know the solutions of (1) are bounded. Let $M_1 > 0$ be given and suppose $x(t) = x(t, t_0, \theta)$ is a solution of (1) which satisfies $|x(t)| \leqslant M_1$ for all t. As in the previous theorem, we define $V(t, x) = x^T Cx$. Along $x(t)$ we have for $t \geqslant t_0$,

$$V' \leqslant -|x(t)|^2 + 2|x(t)| \int_0^t |CB(t, s)|\,|x(s)|\,ds + 2|x(t)|\,|C|\,|f(t)|.$$

Defining $r_j = j$, $j = 1, 2,\ldots$, and letting $h > 1$ we obtain as above that $h^2 V(t, x(t)) \geqslant V(s, x(s))$ for $t - r_j \leqslant s \leqslant t$, $t \geqslant r_j + t_0$, implies that $(h\beta/\alpha)|x(t)| \geqslant |x(s)|$, $t - r_j \leqslant s \leqslant t$. Thus,

$$\begin{aligned} V' &\leqslant -|x(t)|^2 + 2|x(t)| \int_{t-r_j}^t |CB(t, s)|\,|x(s)|\,ds \\ &\quad + 2|x(t)| \int_0^{t-r_j} |CB(t, s)|\,|x(s)|\,ds + 2|x(t)|\,|C|\,|f(t)| \\ &\leqslant -|x(t)|^2 + (2h\beta/\alpha)|x(t)|^2 \int_0^t |CB(t, s)|\,ds + 2M_1^2 \int_0^{t-r_j} |CB(t, s)|\,ds \\ &\quad + 2M_1 |C|\,|f(t)| \\ &\leqslant [(2hM\beta/\alpha) - 1]\,|x(t)|^2 + 2M_1 \Big(M_1 \int_0^{t-r_j} |CB(t, s)|\,ds + |C|\,|f(t)|\Big). \end{aligned}$$

Now as $2\beta M/\alpha < 1$, $h > 1$ can be chosen so that $-K = (2h\beta M/\alpha) - 1 < 0$. Now let u_k be defined to be $2/k$. For each k we may choose $v_k = r_{j_k}$ so that for $t \geqslant v_k$,

$$2M_1 \Big(M_1 \int_0^{t-v_k} |CB(t, s)|\,ds + |C|\,|f(t)|\Big) < K/2k^2.$$

Thus, for $t \geqslant v_k + t_0$ and $M_1 \geqslant |x(t)| \geqslant u_k/2 = 1/k$, we have

$$V' \leqslant -K|x(t)|^2 + K/2k^2 \equiv -w_k(|x(t)|) \leqslant -K/2k^2 < 0.$$

As the hypotheses of Theorem 2 are satisfied we see that if $f \in BC_0$, the solutions of (1) behave in the prescribed manner. As in Theorem 4, if we take $t_0 = 0$ and $x(0) = x_0$, $R(t, 0) \to 0$ as $t \to \infty$ and $\rho(BC_0) \subset BC_0$ follows from (5). To show $\rho(BC_0) \subset BC_0$ one argues as in Theorem 4 and notes that (H2) and (6) imply that $\int_0^t B(t, s) f(s)\, ds$ is in BC_0 if f is in BC_0.

It is convenient to note here that Theorems 4 and 5 have immediate generalizations to the equation

$$x'(t) = Ax(t) + \int_0^t B(t, s, \mu)\, x(s)\, ds + f_\mu(t), \tag{1μ}$$

parameterized by $\mu \in \mathcal{O}\!\ell$. We will assume that (H1) is valid for $B(t, s, \mu)$, for each $\mu \in \mathcal{O}\!\ell$ and also:

(H2μ). A is a stable matrix, C is the positive definite symmetric solution of the matrix equation $A^T C + CA = -I$, and there exists a constant $M > 0$ so that

$$\int_0^t | CB(t, s, \mu)|\, ds \leqslant M$$

for all $t \geqslant 0$ and all $\mu \in \mathcal{O}\!\ell$ and, also $2\beta M/\alpha < 1$ where α^2 and β^2 are, respectively, the smallest and largest eigenvalues of C.

The following is an extension of Theorems 4 and 5; it is stated as a corollary since it may be proved in the same fashion as these theorems are, and will be used in establishing subsequent results.

COROLLARY 2. *Suppose* (H1) *is valid for* $B(t, s, \mu)$ *for each* $\mu \in \mathcal{O}\!\ell$ *and that* (H2μ) *holds. If there exists* $K > 0$ *so that* $|f_\mu(t)| \leqslant K$ *for* $t \geqslant 0$ *and all* $\mu \in \mathcal{O}\!\ell$, *then the solutions of* (1μ) *are uniform bounded. If further,*

$$\lim_{T\to\infty} \sup \left\{\int_0^{t-T} | B(t, s, \mu)|\, ds : t \geqslant T, \mu \in \mathcal{O}\!\ell\right\} = 0 \tag{6μ}$$

and $f_\mu(t) \to 0$ *as* $t \to \infty$ *uniformly in* μ, *given* $M_1 > 0$ *and* $\eta > 0$ *there exists* $T(\eta, M_1) > 0$ *so that if* $x(t)$ *is a solution of* (1μ) *with* $| x(t)| \leqslant M_1$ *for all* t *then* $| x(t)| < \eta$ *for* $t \geqslant T(\eta, M_1) + t_0$ *uniformly for* $\mu \in \mathcal{O}\!\ell$.

Remark Although we could continue with the initial value problem as stated for (1) and (1μ), the purpose of this general initial value problem was to obtain Theorem 5 in the case where $t_0 = 0$, and so henceforth we shall always assume $t_0 = 0$ and $x(0) = x_0$ is the initial value problem for (1).

We now consider some results involving the function spaces BC_l, $A(\omega)$, and AAP.

THEOREM 6. *Suppose that* (H1), (H2), *and* (6) *are valid. Suppose in addition,*

$$\lim_{t\to\infty} \int_0^t B(t, s)\, ds = \mathscr{B}(\infty) \tag{7}$$

exists and $A + \mathscr{B}(\infty)$ *is nonsingular. Then if* $f \in BC_l$ *every solution* $x(t)$ *of* (1) *satisfies*

$$\lim_{t\to\infty} x(t) = -(A + \mathscr{B}(\infty))^{-1} \lim_{t\to\infty} f(t). \tag{8}$$

Also, $\rho, \dot{\rho}: BC_l \to BC_l$.

Proof. Let $f \in BC_l$ *and* $f(\infty)$ *be defined by*

$$f(\infty) = \lim_{t\to\infty} f(t).$$

Let $x_1 = -(A + \mathscr{B}(\infty))^{-1} f(\infty)$ and $w(t) = x(t) - x_1$, where $x(t)$ is a solution of (1). Then $w(t)$ satisfies

$$w'(t) = Aw(t) + \int_0^t B(t, s)\, w(s)\, ds + \left(A + \int_0^t B(t, s)\, ds\right) x_1 + f(t).$$

From the definition of x_1 it follows that $(A + \int_0^t B(t, s)\, ds)\, x_1 + f(t)$ tends to zero as $t \to \infty$ and so by Theorem 5, $w(t)$ tends to zero as $t \to \infty$. That $\rho, \dot{\rho}: BC_l \to BC_l$ follows from (4), (5), (6), and (7). Q.E.D.

THEOREM 7. *Suppose* (H1) *and* (H2) *are valid and that* (6) *holds. Suppose in addition, for* $n, m = 0, 1, 2, \ldots$,

$$\int_0^t | B(t + m\omega, s + m\omega) - B(t + n\omega, s + n\omega)|\, ds \to 0 \tag{9}$$

as $t \to \infty$ *uniformly in* n *and* m. *Then if* $f(t)$ *is asymptotically* ω*-periodic, the solutions of* (1) *are asymptotically* ω*-periodic. Also,* $\rho, \dot{\rho}: A(\omega) \to A(\omega)$.

Proof. Suppose $f(t)$ is asymptotically ω-periodic. As our problem is linear, it follows from Theorem 5 that we may assume with no loss of generality that $f(t)$ is ω-periodic.

Let $x(t)$ be a solution of (1) and consider the uniformly bounded family of functions $y_{m,n}(t)$, $m, n = 0, 1, 2, \ldots$, defined by

$$y_{m,n}(t) = x(t + m\omega) - x(t + n\omega).$$

The functions $y_{m,n}(t)$ satisfy the equations

$$\begin{aligned} y'_{m,n}(t) = {} & Ay_{m,n}(t) + \int_0^t B(t, s)\, y_{m,n}(s)\, ds \\ & + \int_0^t [B(t + m\omega, s + m\omega) - B(t, s)]\, y_{m,n}(s)\, ds \\ & + \int_0^t [B(t + m\omega, s + m\omega) - B(t + n\omega, s + n\omega)]\, x(s + n\omega)\, ds \\ & + \int_0^{m\omega} B(t + m\omega, s)\, x(s)\, ds - \int_0^{n\omega} B(t + n\omega, s)\, x(s)\, ds. \end{aligned}$$

Now, as

$$\int_0^{n\omega} |B(T+n\omega, s)|\, ds = \int_0^{T+n\omega-T} |B(T+n\omega, s)|\, ds$$
$$\leqslant \sup \left\{ \int_0^{t-T} |B(t, s)|\, ds : t \geqslant T \right\}$$

it follows from (6) that

$$\int_0^{n\omega} |B(t+n\omega, s)|\, ds \to 0$$

as $t \to \infty$ uniformly in n. As $y_{m,n}(t)$ and $x(t+n\omega)$ are uniformly bounded for $m, n = 0, 1,\ldots$, and since (9) holds, it follows that $y_{m,n}(t)$ satisfies

$$y'_{m,n}(t) = Ay_{m,n}(t) + \int_0^t B(t, s)\, y_{m,n}(s)\, ds + g_{m,n}(t),$$

where $g_{m,n}(t) \to 0$ as $t \to \infty$ uniformly in m, n. It follows now from Corollary 2 that the functions $y_{m,n}(t)$ tend to zero as $t \to \infty$ uniformly in m and n; that is, given $\epsilon > 0$ there exists $N\omega$ so that $t \geqslant N\omega$ implies $|y_{m,n}(t)| < \epsilon$ for all m and n.

If we now consider the functions $x(t+n\omega)$ for $0 \leqslant t \leqslant \omega$ we see that for $n, m \geqslant N$,

$$\begin{aligned} &| x(t+n\omega) - x(t+m\omega)| \\ &\quad = | x(t+N\omega+(n-N)\omega) - x(t+N\omega+(m-N)\omega)| \\ &\quad = | y_{n-N,m-N}(t+N\omega)| < \epsilon. \end{aligned}$$

Thus, the sequence of functions $\{x(t+n\omega)\}$ is uniformly Cauchy on $0 \leqslant t \leqslant \omega$ and must converge uniformly to a continuous function $x_0(t)$ on $[0, \omega]$. Clearly, $x_0(0) = x_0(\omega)$ and so $x_0(t)$ may be extended periodically to $[0, \infty)$ and $x(t) - x_0(t) \to$ as $t \to \infty$.

Taking $x_0 = x(0) = 0$, it follows from (5) that $\rho: A(\omega) \to A(\omega)$. Also, for $f \in A(\omega)$

$$\dot{\rho}(f)(t) = -\rho(Af)(t) - \int_0^t R(t, u) \int_0^u B(u, s)\, f(s)\, ds\, du$$

and as $\dot{\rho}: BC_0 \to BC_0$ we may assume without loss of generality that $f(t)$ is ω-periodic. Arguing as above using (6) and (9) it is easily shown that the family of functions

$$\int_0^{t+n\omega} B(t+n\omega, s)\, f(s)\, ds$$

is uniformly Cauchy on $[0, \omega]$ and so the function $\int_0^t B(t, s) f(s)\, ds$ is in $A(\omega)$. As $\rho\colon A(\omega) \to A(\omega)$ it now follows that $\dot{\rho}\colon A(\omega) \to A(\omega)$ and the proof is complete.

Before considering Eq. (1) when $f(t)$ is asymptotically almost periodic (hereafter, a.a.p.) we state the following characterization of a function being a.a.p. which may be found in Fink [4, Chap. 9, Sect. 3] or in Yoshizawa [13, Chap. 1, Sect. 3].

LEMMA 1. *A function $f(t)$ is* a.a.p. *if and only if for each $\epsilon > 0$ there exist $l(\epsilon) > 0$ and $T(\epsilon) \geqslant 0$ such that in each interval $[t_0, t_0 + l(\epsilon)]$, $t_0 \geqslant 0$, there exists $\tau = \tau(\epsilon, t_0)$ so that*

$$|f(t+\tau) - f(t)| \leqslant \epsilon, \qquad t \geqslant T(\epsilon). \tag{10}$$

THEOREM 8. *Let* (H1), (H2), *and* (6) *hold and suppose for each $\epsilon > 0$ there exists $l(\epsilon) > 0$ and $T(\epsilon) \geqslant 0$ such that in each subinterval of R^+ of length $l(\epsilon)$, there exists τ such that*

$$\int_0^t | B(t+\tau, s+\tau) - B(t, s)|\, ds \leqslant \epsilon, \qquad t \geqslant T(\epsilon), \tag{11}$$

and that there exists $\delta = \delta(\epsilon) > 0$ so that (11) *holds for $0 \leqslant \tau < \delta$. Then if $f(t)$ is* a.a.p., *every solution of* (1) *is* a.a.p. *and ρ, $\dot{\rho}$: $AAP \to AAP$.*

Proof. For each $\epsilon > 0$, we denote the set of τ for which (10) holds by $E(\epsilon, f)$ and the set of τ for which (11) holds by $E(\epsilon, B)$. It can be shown that under the above hypotheses on f and B that $E(\epsilon, f) \cap E(\epsilon, B)$ is relatively dense in R^+; that is, there exists $L = L(\epsilon) > 0$ such that each interval $[t_0, t_0 + L]$, $t_0 \geqslant 0$, contains points of $E(\epsilon, f) \cap E(\epsilon, B)$. The proof of this is analogous to the corresponding proof that the intersection of analogous sets for almost periodic functions is relatively dense on R, (cf. for example Besicovitch [1, pp. 1–5]), and although somewhat more complicated, is omitted here.

Let $x(t)$ be a solution of (1). It follows from Theorem 4 that there exists B_0 so that $| x(t)| \leqslant B_0$ for $t \geqslant 0$. Also, as $\rho(BC) \subset BC$, it follows from a result of Corduneanu [2] that there exists N such that $\int_0^t | R(t, s)|\, ds \leqslant N$ for $t \geqslant 0$. For any fixed $\tau \geqslant 0$, define $y_\tau(t) = x(t + \tau) - x(t)$. It follows that

$$y_\tau'(t) = Ay_\tau(t) + \int_0^t B(t, s)\, y_\tau(s)\, ds + g(t, \tau), \tag{12}$$

where

$$g(t, \tau) = \int_0^t (B(t+\tau, s+\tau) - B(t, s))\, x(s+\tau)\, ds$$

$$+ \int_0^\tau B(t+\tau, s)\, x(s)\, ds + f(t+\tau) - f(t).$$

From the above remarks, given $\epsilon > 0$, there exists $L = L(\epsilon) > 0$ and $T_1(\epsilon) \geqslant 0$ so that each subinterval of R^+ of length L contains a number τ such that

$$\int_0^t |B(t+\tau, s+\tau) - B(t, s)|\, ds \leqslant \epsilon/(16B_0N) \tag{13}$$

and

$$|f(t+\tau) - f(t)| \leqslant \epsilon/(8N) \tag{14}$$

for $t \geqslant T_1(\epsilon)$. From (6), it follows that there exists $T_2(\epsilon) \geqslant T_1(\epsilon)$ such that

$$\int_0^\tau |B(t+\tau, s)|\, ds \leqslant \epsilon/(16B_0N) \tag{15}$$

for $t \geqslant T_2(\epsilon)$ and any $\tau \geqslant 0$. Finally, from Theorem 5 we see that there exists $T_3(\epsilon) \geqslant T_2(\epsilon)$ such that $T_3(\epsilon) > 1$ and

$$|R(t, 0)| \leqslant \epsilon/(4B_0) \tag{16}$$

for $t \geqslant T_3(\epsilon)$. We find then that given $\epsilon > 0$ there exists $L(\epsilon) > 0$ such that in each subinterval of R^+ of length $L(\epsilon)$, there exists τ such that for $t \geqslant T_3(\epsilon)$, (13), (14), (15), and (16) hold. Using the estimates (13)–(15) and the fact that $|y_\tau(t)| \leqslant 2B_0$ for $t \geqslant 0$, we find that $|g(t, \tau)| \leqslant 3\epsilon/(8N)$ for $t \geqslant T_3(\epsilon)$. Also, from the definition of $g(t, \tau)$ we see that there exists a constant $K > 0$ so that $|g(t, \tau)| \leqslant K$ for $0 \leqslant t \leqslant T_3(\epsilon)$. We now decompose $g(t, \tau)$ by letting $g(t, \tau) = g_1(t, \tau) + g_2(t, \tau)$ where

$$g_1(t, \tau) = \begin{cases} g(t, \tau), & t \geqslant T_3(\epsilon) \\ g(T_3(\epsilon), \tau)(t + 1 - T_3(\epsilon)), & T_3(\epsilon) - 1 \leqslant t \leqslant T_3(\epsilon) \\ 0 & 0 \leqslant t \leqslant T_3(\epsilon) - 1. \end{cases}$$

Thus, $|g_1(t, \tau)| \leqslant 3\epsilon/(8N)$ for $t \geqslant 0$, $|g_2(t, \tau)| \leqslant K$ for $t \geqslant 0$ and $g_2(t, \tau) \to 0$ as $t \to \infty$ uniformly in $\tau \in E(\epsilon_1, f) \cap E(\epsilon_1, B)$ where $\epsilon_1 = \min\{(\epsilon/(8N)), (\epsilon/(16B_0N))\}$. From Corollary 2, taking $f_\mu(t) = g_2(t, \tau)$, where $\tau = \mu$, and $x_0 = 0$ in (1μ), we see that for the solution $\int_0^t R(t, s)\, g_2(s, \tau)\, ds$

there exists $T_4(\epsilon) \geqslant T_3(\epsilon)$ so that $| \int_0^t R(t, s) g_2(s, \tau) \, ds | < \epsilon/4$ for $t \geqslant T_4(\epsilon)$ and all $\tau \in E(\epsilon_1, f) \cap E(\epsilon_1, B)$. From (5) we see that

$$y_\tau(t) = R(t, 0) \, y_\tau(0) + \int_0^t R(t, s) \, g(s, \tau) \, ds,$$

and, hence, $| y_\tau(t)| \leqslant \epsilon$ for $t \geqslant T_4(\epsilon)$.

From this estimate and Lemma 1, we conclude that $x(t)$ is a.a.p. and if we take $x(0) = x_0 = 0$, $\rho: AAP \to AAP$ is immediate. To see that $\dot{\rho}: AAP \to AAP$ we need only consider $\int_0^t B(t, s) f(s) \, ds$ when f is almost periodic. However, with the aid of (6) and (11) it is easy to show that $\int_0^t B(t, s) f(s) \, ds$ satisfies condition (10) of Lemma 1 and so $\dot{\rho}: AAP \to AAP$ and the proof is complete.

Remark. It is clear that (9) and (11) are satisfied in the convolution case when $B(t, s) = B(t - s)$. Another case of interest is when $B(t, s) = C(s) \, B_1(t - s)$. If $C(s)$ is asymptotically ω-periodic, say $C(s) = p(s) + q(s)$ with $p(s)$ ω-periodic and $q(s) \to 0$ as $s \to \infty$, then

$$\begin{aligned} | B(t + m\omega, s + m\omega) &- B(t + n\omega, s + n\omega)| \\ &\leqslant | q(s + m\omega) - q(s + n\omega)| \, | B_1(t - s)| \\ &\leqslant 2q_1(s) \, | B_1(t - s)|, \end{aligned}$$

where

$$q_1(s) = \sup\{| q(t)|: t \geqslant s\}.$$

Now, if $B_1 \in L^1(0, \infty)$, $\int_0^t 2q_1(s) \, | B_1(t - s)| \, ds \to 0$ as $t \to \infty$ since the convolution of an L^1 function with a function in BC_0 is a function in BC_0. Hence, $B(t, s) = C(s) \, B_1(t - s)$ satisfies (9) if $B_1 \in L^1(0, \infty)$ and $C(s)$ is asymptotically ω-periodic. In a similar manner, one can show that if $B_1 \in L^1(0, \infty)$ and $C(s)$ is a.a.p. then $B(t, s) = C(s) \, B_1(t - s)$ satisfies (11).

Before proceeding further, we now will consider equation (1) when $B(t, s) = B(t - s)$ and examine the results we have obtained in the light of some results of Grossman and Miller. In this case (1) takes the form

$$x'(t) = Ax(t) + \int_0^t B(t - s) \, x(s) \, ds + f(t). \tag{17}$$

For Eq. (17), Grossman and Miller proved:

THEOREM 9 [6, Theorem 3.5]. *Suppose $B(t) \in L^1(R^+)$. Then the resolvent $R(t)$ is in $L^1(R^+)$ if and only if*

$$\operatorname{Det}(sI - A - B^*(s)) \neq 0 \tag{18}$$

for $\operatorname{Re} s \geqslant 0$ *where $B^*(s)$ is the Laplace transform of $B(t)$.*

We first note that the condition $\int_0^t |CB(t, s)|\, ds \leqslant M$ for all $t \geqslant 0$, of (H2) is just the statement that $B(t) \in L^1(R^+)$ in the convolution case. In Theorems 5 and 6, hypotheses (6) and (7) are again just the requirement that $B(t) \in L^1(R^+)$ and the requirement of Theorem 6 that $A + \mathscr{B}(\infty)$ is nonsingular is the determinant condition (18) when $s = 0$.

Now suppose that (17) is a scalar equation with A a negative constant and $B(t) \geqslant 0$ for $t \geqslant 0$. In this case, $C = -(2A)^{-1}$, $\alpha^2 = \beta^2$, and

$$\begin{aligned}\int_0^t |CB(t, s)|\, ds &= -(2A)^{-1} \int_0^t B(s)\, ds \\ &\leqslant -(2A)^{-1} B^*(0).\end{aligned}$$

Thus, we may choose $M = -(2A)^{-1}B^*(0)$ in (H2) and the requirement $2\beta M/\alpha < 1$ is seen to be $B^*(0) < -A$ or $-A - B^*(0) > 0$. In this case, this is precisely condition (18) and we see that the inequality $2\beta M/\alpha < 1$ cannot be relaxed in general.

We must note, however, that the result obtained by Grossman and Miller does not require A to be a stable matrix.

To broaden the scope of applicability of our results to a larger class of equations we now present a transformation for the convolution case. As the transformation consists of changing Eq. (17) into an integral equation and then transforming the integral equation into another integrodifferential equation of the form of (17), the transformation will be presented in two parts, one of which may be of independent interest in the study of integral equations.

We consider first the integral equation

$$x(t) = \int_0^t B_1(t - s)\, x(s)\, ds + f(t). \tag{19}$$

Associated with (19) is the resolvent equation

$$r(t) = -B_1(t) + \int_0^t B_1(t - s)\, r(s)\, ds \tag{20}$$

and the variation of constants formula

$$x(t) = f(t) - \int_0^2 r(t - s)\, f(s)\, ds. \tag{21}$$

For Eq. (19) we first note the following result of Paley and Wiener.

THEOREM 10 [10, p. 60]. *Suppose $B_1(t) \in L^1(R^+)$. Then $r(t) \in L^1(R^+)$ if and only if*

$$\det(I - B_1{}^*(s)) \neq 0$$

for $\operatorname{Re} s \geqslant 0$, *where $B_1{}^*(s)$ is the Laplace transform of $B_1(t)$.*

THEOREM 11. *Suppose $B_1(t)$ is absolutely continuous and $B_1(t)$ and $B_1'(t)$ are in $L^1(R^+)$ and let D be a stable matrix. Then the resolvent $r(t)$ associated with Eq.* (19) *is in $L^1(R^+)$ if and only if the resolvent $R(t)$ associated with the equation*

$$y'(t) = [D + B_1(0)]\, y(t) + \int_0^t [B_1' - B_1 D](t - s)\, y(s)\, ds + f(t) \quad (22)$$

is in $L^1(R^+)$.

Proof. Although one may prove this result by defining $y(t)$ by $y'(t) = Dy(t) + r(t)$ and substituting into equation (20) to obtain an equation of the form of (22) and argue from there, we will make use of the determinant criteria of Grossman and Miller and Paley and Wiener.

From Theorem 9 we see that the resolvent $R(t)$ associated with Eq. (22) is $L^1(R^+)$ if and only if

$$\det(sI - [D + B_1(0)] - [B_1' - B_1 D]^*(s)) \neq 0$$

for $\operatorname{Re} s \geqslant 0$. However,

$$sI - D - B_1(0) - (B_1')^*(s) + B_1{}^*(s) D = (I - B_1{}^*(s))(sI - D)$$

and the conclusion follows from Theorem 10 since D is a stable matrix.

Of course, one could differentiate (19) to obtain an integrodifferential equation and one would obtain Eq. (22) with $D = 0$. However, as one sees from the proof of Theorem 11, the resolvent of the integrodifferential equation will not be in $L^1(R^+)$.

If in addition we assume that D is chosen so that $D + B_1(0)$ is stable, we have the following corollary of Theorem 11 and Corollary 1.

COROLLARY 3. *Suppose $B_1(t)$ is absolutely continuous and that $B_1(t)$ and $B_1'(t)$ are in $L^1(R^+)$ and let D be a stable matrix so that $D + B_1(0) = A$ is stable. Suppose C is the positive definite solution of the equation $A^T C + CA = -I$ and*

$$(2\beta/\alpha) \int_0^\infty | CB_1' - CB_1 D | (s)\, ds < 1,$$

where $\alpha^2 | x |^2 \leqslant x^T C x \leqslant \beta^2 | x |^2$. *Then the resolvent* $r(t)$ *of Eq.* (19) *is in* $L^1(R^+)$.

The following result may be useful when the matrix A in Eq. (17) is not stable.

THEOREM 12. *Suppose* $B(t)$ *is in* $L^1(R^+)$. *Let* D *and* E *be stable matrices with the property that* $A + D - E$ *is stable and let* $F(t)$ *be the principal matrix solution of the differential equation*

$$y' = Ey. \tag{23}$$

Then the resolvent associated with Eq. (17) *is in* $L^1(R^+)$ *if and only if the resolvent associated with the equation*

$$x'(t) = [A + D - E]\, x(t) + \int_0^t [B_1' - B_1 D](t - s)\, x(s)\, ds \tag{24}$$

is in $L^1(R^+)$. *Here* $B_1(t) = [FA + F * B - F'](t)$, *where* $F * B$ *is the convolution of* F *and* B.

Proof. As $F(t)$ is the principal matrix solution of (23), $F(0) = I$, $F^{(k)}(t) \in L^1(R^+)$ for $k = 0, 1, 2,\ldots$, and $\det F^*(s) \neq 0$ when $\operatorname{Re} s \geqslant 0$. Following Miller [9, Theorem 6], if we convolution multiply Eq. (17) by $F(t)$ and integrate by parts we obtain the equation

$$x(t) = [FA + F * B - F'] * x + F * f, \tag{25}$$

where $*$ denotes convolution. Letting $B_1(t) = [FA + F * B - F'](t)$ we see that

$$I - B_1^*(s) = F^*(s)(sI - A - B^*(s))$$

and the resolvent $r(t)$ of (25) is in $L^1(R^+)$ if and only if the resolvent $R(t)$ of (17) is in $L^1(R^+)$. We now apply Theorem 11 to complete the proof.

To illustrate the use of the above transformation, we consider the equation

$$x'(t) = \int_0^t B(t - s)\, x(s)\, ds, \tag{26}$$

where $B(t) \in L^1(R^+)$. Here we set $D = -\gamma I$ and $E = -\lambda I$ where $\gamma > \lambda > 0$ are constant and $F(t) = e^{-\lambda t} I$. In this case Eq. (24) takes the form

$$x'(t) = (\lambda - \gamma)\, Ix(t) + \int_0^t [B + (\gamma - \lambda)\, F * B + (\gamma - \lambda)\, \lambda F](t - s)\, x(s)\, ds \tag{27}$$

The Liapunov matrix C is obvious in this case, $C = \frac{1}{2}(1/(\gamma - \lambda))I$ and

$\alpha^2 = \beta^2$. From Theorem 12 and Corollary 1 we obtain that the resolvent $R(t)$ associated with Eq. (26) is in $L^1(R^+)$ if

$$\int_0^\infty | B + (\gamma - \lambda) F * B + (\gamma - \lambda) \lambda F | (u)\, du < (\gamma - \lambda), \tag{28}$$

for some constants $\gamma > \lambda > 0$. As B, λ, and F are independent of γ we may let $\gamma \to \infty$ to obtain that $R(t)$ is in $L^1(R^+)$ if

$$\int_0^\infty | F * B + \lambda F | (u)\, du < 1 \tag{29}$$

for some $\lambda > 0$.

For example, in the special case

$$B(t) = \sum_{j=1}^\infty B_j e^{-\sigma_j t} \tag{30}$$

with $\sigma_j > 0, j = 1, 2, \ldots, \sigma_k \neq \sigma_j$, for $k \neq j$, if we let $\lambda = \sigma_1$,

$$F * B(t) = B_1 t e^{-\sigma_1 t} + \sum_{j=2}^\infty B_j (e^{-\sigma_j t} - e^{-\sigma_1 t})/(\sigma_1 - \sigma_j) \tag{31}$$

we see that (29) is satisfied if

$$\int_0^\infty | B_1 t + \sigma_1 I | e^{-\sigma_1 t}\, dt + \sum_{j=2}^\infty | B_j |/(\sigma_1 \sigma_j) < 1.$$

Although the transformation of Theorem 12 is motivated by the desire to apply our results to equations where the matrix A is not stable, there is no reason why the transformation cannot be used when A is stable. If A is a stable matrix, a transformation that is suggested by the form of (24) is to let $E = A$ and $D = -\gamma I$, where $\gamma > 0$ is a constant. This choice yields $F(t) = e^{At}$ and $B_1(t) = F * B(t)$ and Eq. (24) becomes

$$x'(t) = -\gamma I x(t) + \int_0^t [B + \gamma F * B + AF * B](t - s)\, x(s)\, ds. \tag{32}$$

One advantage of making this transformation is that the Liapunov matrix $C = (2I\gamma)^{-1}$ is now obvious. From Theorem 12 and Corollary 1 one obtains now that the resolvent $R(t)$ associated with Eq. (17) is in $L^1(R^+)$ if for some $\gamma > 0$,

$$\int_0^\infty | B + \gamma F * B + AF * B | (u)\, du < \gamma. \tag{33}$$

Letting $\gamma \to \infty$ as before yields the result that $R(t) \in L^1(R^+)$ if

$$\int_0^\infty |F * B| (u) \, du < 1, \tag{34}$$

where$F(t) = e^{At}$.

Although we have considered only linear Volterra integrodifferential equations thus far, Theorems 4–8 may be applied to obtain results for perturbed versions of Eq. (1) where the perturbing terms are of higher order in the following sense:

DEFINITION. Let E be a Banach subspace of C with a stronger topology and let $g(E) \subset E$. Then g is of higher order with respect to E if and only if $g(0) = 0$ and for each $\epsilon > 0$, there exists $\delta > 0$ such that $\| g\varphi_1 - g\varphi_2 \| \leqslant \epsilon \| \varphi_1 - \varphi_2 \|$ when φ_1 and φ_2 are in E and $\| \varphi_1 \|, \| \varphi_2 \| \leqslant \delta$. The perturbed version of (1) that we consider is

$$x'(t) = Ax(t) + \int_0^t B(t, s)\{x(s) + (g_1x)(s)\} \, ds + (g_2x)(t) + f(t), \tag{2}$$
$$x(0) = x_0 ,$$

where A and B are as before, g_i maps C into C continuously and has the property that if $\varphi_1(t) = \varphi_2(t)$ on $[0, T]$ then $g_i(\varphi_1)(t) = g_i(\varphi_2)(t)$ on $[0, T]$ for $i = 1, 2$.

For Eq. (2), Grossman and Miller proved the following result which can be used if E is BC, BC_0, BC_l, $A(\omega)$, or AAP to obtain perturbation results corresponding to each of the Theorems 4–8.

THEOREM 13 [5, Theorem 5]. *Suppose g_i is of higher order with respect to E for $i = 1, 2$ and the resolvent $R(t, s)$ associated with (1) is such that both ρ, $\dot{\rho}$ map E into E. If $f \in E$ and $R(t, 0) \in E$, then for each $\epsilon > 0$, there exists an $\eta > 0$ such that if $| x_0 | \leqslant \eta$ and $\| f \| \leqslant \eta$ then Eq. (2) has a unique solution x in E with $\| x \|_E \leqslant \epsilon$.*

The techniques used in the proofs of Theorems 4, 5, 7, and 8 can also be applied directly to nonlinear equations which are almost linear in a certain sense. In particular, consider the equations

$$x'(t) = Ax(t) + \int_0^t B(t, s) \, g(x(s)) \, ds + f_i(t), \tag{35.i}$$

$i = 1, 2$, where A, B, and f_i are as before, and g is Lipschitz continuous with

$g(0) = 0$. Given two points $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$ in R^n, we define the matrix $G(x, y) = (\theta_{ij})$ by

$$\theta_{ij} = \begin{cases} \dfrac{g_i(y_1, \ldots, y_{j-1}\ \ x_j, \ldots, x_n) - g_i({}_1, \ldots, y_j, x_{j+1}, \ldots, x_n)}{x_j - y_j}, \\ \qquad \text{if} \quad x_j \neq y_j \\ 0, \qquad \text{if} \quad x_j = y_j . \end{cases}$$

We will require:

$$\text{(H3)} \qquad\qquad | G(x, y)| \leqslant 1$$

for all $x, y \in R_T$.

We remark that the requirement that g be globally Lipschitzian and that (H3) is valid for all x, y in R^n is for convenience only. If one desires local results or if it is already known by other means that every solution of (35.i) eventually enters and remains in some set $D \subset R^n$, $0 \in D$, it is clear that g need be Lipschitzian only on D and that (H3) need be satisfied only on D.

To examine the solutions of (35.i), $i = 1, 2$, let $x(t)$ be a solution of (35.1) and $y(t)$ be a solution of (35.2). Then $z(t) = x(t) - y(t)$ satisfies the linear equation

$$z'(t) = Az(t) + \int_0^t B(t, s)\, G(x(s), y(s))\, z(s)\, ds + f_1(t) - f_2(t), \qquad (36)$$

and if $B(t, s)$ satisfies (H1) it is easy to see that $B_1(t, s) = B(t, s)G(x(s), y(s))$ satisfies (H1).

We now can apply Theorems 4 and 5 to Eq. (36) to obtain the parallel results for Eqs. (35.i) which in turn are used to obtain results corresponding to Theorems 7 and 8 for Eqs. (35.i).

THEOREM 14. *Suppose* (H1), (H2), *and* (H3) *are valid and that* $f_1(t) - f_2(t)$ *is bounded. Then if* $x(t)$ *and* $y(t)$ *are solutions of* (35.1) *and* (35.2), *respectively, then* $x(t) - y(t)$ *is bounded. In particular, if* $f_1(t)$ *is bounded, every solution of* (35.1) *is bounded.*

If in addition (6) *is valid and* $f_1(t) - f_2(t) \to 0$ *as* $t \to \infty$, $x(t) - y(t) \to 0$ *as* $t \to \infty$. *In particular, if* $f_1(t) \in BC_0$ *then every solution of* (35.1) *is in* BC_0.

Proof. It follows from (H3) that (H2) is valid for the kernel $B_1(t, s) = B(t, s)\, G(x(s), y(s))$ and the first conclusion follows from Theorem 4. By taking $y(t) = f_2(t) = 0$ we see that $f_1(t)$ bounded implies that $x(t)$ is bounded. The rest of the theorem follows from Theorem 5 in a similar manner.

THEOREM 15. *Suppose* (H1), (H2), *and* (H3) *are valid and that* (6) *and* (9) *hold. If* $f_1(t)$ *is asymptotically* ω*-periodic, every solution of* (35.1) *is asymptotically* ω*-periodic.*

Proof. It follows from the previous theorem that we may assume $f_1(t)$ is ω-periodic. As in the proof of Theorem 7, let

$$y_{m,n}(t) = x(t + m\omega) - x(t + n\omega), \qquad m, n = 0, 1, 2,...,$$

where $x(t)$ is any solution of (35.1). Then $y_{m,n}(t)$ satisfies the equation

$$y'_{m,n}(t) = Ay_{m,n}(t) + \int_0^t B(t, s)\, G(x(s + m\omega), x(s + n\omega))\, y_{m,n}(s)\, ds + h_{m,n}(t),$$

where $h_{m,n}(t)$ is given by

$$\begin{aligned} h_{m,n}(t) = \int_0^t & [B(t + m\omega, s + m\omega) \\ & - B(t, s)]\, G(x(s + m\omega), x(s + n\omega))\, y_{m,n}(s)\, ds \\ & + \int_0^t [B(t + m\omega, s + m\omega) - B(t + n\omega, s + n\omega)]\, g(x(s + n\omega))\, ds \\ & + \int_0^{m\omega} B(t + m\omega, s)\, g(x(s))\, ds - \int_0^{n\omega} B(t + n\omega, s)\, g(x(s))\, ds. \end{aligned}$$

The argument of Theorem 7 may now be used to finish the proof after noting that (H2) and (H3) imply that (H2μ) with $\mu = (m, n)$ is valid for $B(t, s)\, G(x(s + m\omega), x(s + n\omega))$.

THEOREM 16. *Suppose* (H1), (H2), *and* (H3) *are valid and, in addition, conditions* (6) *and* (11) *hold. If* $f_1(t)$ *is* a.a.p. *then every solution of* (35.1) *is* a.a.p.

Proof. Here one proceeds as in the proof of Theorem 8 to obtain the equation

$$y_\tau'(t) = Ay_\tau(t) + \int_0 B(t, s)\, G(x(s + \tau), x(s))\, y_\tau(s)\, ds + h_\tau(t), \qquad (37)$$

where

$$\begin{aligned} h_\tau(t) = & \int_0^t [B(t + \tau, s + \tau) - B(t, s)]\, g(x(s + \tau))\, ds \\ & + \int_0^\tau B(t + \tau, s)\, g(x(s))\, ds + f(t + \tau) - f(t) \qquad (38) \end{aligned}$$

and $y_\tau(t) = x(t + \tau) - x(t)$.

Here one shows that the validity of (H2μ) with $\mu = \tau$ for the kernel $B(t, s)\, G(x(s + \tau), x(s))$ follows from (H2) and (H3) and proceeds as in the proof of Theorem 8.

In the case where $f_1(t) \in BC_l$ a somewhat different approach than that used in Theorem 6 seems to be necessary.

THEOREM 17. *Suppose* (H1), (H2), (H3), *and* (6) *hold. Suppose also, for each* $\epsilon > 0$ *there exists* $T(\epsilon)$ *so that*

$$\int_0^t | B(t + \tau, s + \tau) - B(t, s)| \, ds \leqslant \epsilon \tag{39}$$

for $t \geqslant T(\epsilon)$ *and all* $\tau \geqslant 0$. *Then if* $f_1(t) \in BC_l$, *every solution of* (35.1) *is in* BC_l.

Proof. Let $x(t)$ be a solution of (35.1) and $\tau > 0$ be any fixed constant and define $y_\tau(t) = x(t + \tau) - x(t)$. Then $y_\tau(t)$ satisfies Eq. (37) with $h_\tau(t)$ defined by (38). Here, one argues as in the previous theorem and as in Theorem 8 that given $\epsilon > 0$, there exists $T_1(\epsilon) > 0$ so that $| y_\tau(t)| < \epsilon$ if $t \geqslant T_1(\epsilon)$ uniformly in τ. However this is the statement that $| x(t+\tau)-x(t)| < \epsilon$ if $t \geqslant T_1(\epsilon)$ for all $\tau \geqslant 0$ or equivalently that $| x(t_1) - x(t)| < \epsilon$ if $t, t_1 \geqslant T_1(\epsilon)$. This last statement is equivalent to $x(t) \in BC_l$.

We note that condition (39) with (6) implies condition (7) of Theorem 6 and, also, that if $B(t, s) = C(s)\, B_1(t - s)$ where $B_1(t) \in L^1(R^+)$ and $C(s)$ has a limit at infinity then

$$\int_0^t | C(s + \tau) - C(s)| \, | B_1(t - s)| \, ds \leqslant \int_0^t | B_1(t - s)| \, q(s) \, ds,$$

where $q(s) = \sup\{| C(s + \tau) - C(s)| : \tau \geqslant 0\} \to 0$ as $s \to \infty$. As $B_1(t) \in L^1(R^+)$,

$$\int_0^t | B_1(t - s)| \, q(s) \, ds \to 0$$

as $t \to \infty$ and (39) is verified in this case.

4. FINAL REMARKS

As we noted earlier, our results from Theorem 6 on could have been obtained for the more general initial value problem. It is clear that a number of other generalizations and extensions of our results can also be obtained.

For example, the matrix in Eq. (1) need not be constant as the proofs we have presented require very minor modifications if the equation

$$x'(t) = [A + A_1(t)]\, x(t) + \int_0^t B(t, s)\, x(s)\, ds + f(t),$$

where $A_1(t) \to 0$ as $t \to \infty$ were considered in place of (1). Of more interest would be results which ensure that the solutions of (1) are in $L^p(0, \infty)$ if $f(t)$ is in $L^p(0, \infty)$ and that ρ and $\dot{\rho}$ map $L^p(0, \infty)$ into itself. The question of how the hypotheses of Theorem 5 can be modified to obtain such results seems open.

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