Zariski closure, completeness and compactness

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Abstract

The categorical theory of closure operators is used to introduce and study separated, complete and compact objects with respect to the Zariski closure operator naturally defined in any category \( \mathcal{X}(A, \Omega) \) obtained by a given complete category \( \mathcal{X} \) (endowed with a proper factorization structure for morphisms) and by a given \( \mathcal{X} \)-algebra \( (A, \Omega) \) by forming the affine \( \mathcal{X} \)-objects modelled by \( (A, \Omega) \).

Several basic examples are provided.

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1. Objects modelled by a given algebra

Let \( \mathcal{X} \) be a complete category and let \( A \) be an \( \mathcal{X} \)-object. The category, \( \mathcal{X}(A) \), of affine \( \mathcal{X} \)-objects modelled by \( A \) (or over \( A \)) is defined as follows. The objects are the pairs \( (X, A(X)) \) with \( X \) an \( \mathcal{X} \)-object and \( A(X) \) a subset of the set \( \mathcal{X}(X, A) = A^X \) of all \( \mathcal{X} \)-morphisms from \( X \) to \( A \). A morphism from \( (X, A(X)) \) to \( (Y, A(Y)) \) is an \( \mathcal{X} \)-morphism \( f : X \to Y \) such that \( \beta \circ f \in A(X) \) for each \( \beta \in A(Y) \).

It is clear that \( \mathcal{X}(A) \) is a topological category over \( \mathcal{X} \) via the obvious forgetful functor \( U : \mathcal{X}(A) \to \mathcal{X} \) (for every \( U \)-source \( \{f_i : X \to U(Y_i, A(Y_i))\}_{i \in I} \) just put in \( A(X) \) all the compositions \( \alpha_i \circ f_i \), \( i \in I, \alpha_i \in A(Y_i) \)). In particular \( \mathcal{X}(A) \) is complete and every...
factorization structure \((\mathcal{E}, \mathcal{M})\) of \(\mathcal{X}\) admits a lifting in \(\mathcal{X}(A)\) with first factor \(U^{-1}(\mathcal{E})\) and second factor those \(\mathcal{X}(A)\)-morphisms in \(U^{-1}(\mathcal{M})\) which are initial. We shall refer to \((\mathcal{E}(A), \mathcal{M}(A))\) for the above factorization structure of \(\mathcal{X}(A)\) associated to \((\mathcal{E}, \mathcal{M})\).

Let now \((A, \Omega)\) be an \(\mathcal{X}\)-algebra; that is: \(A\) is an \(\mathcal{X}\)-object and for a given class \(T\) of sets

\[\Omega = \{ \omega_T : A^T \to A \mid T \in T \}\]

is a family of operations in \(A\). For every object \(X\) of \(\mathcal{X}\) the set \(A^X\) become a \(\text{Set}\)-algebra in the obvious way. We will denote by \(\mathcal{X}(A, \Omega)\) the full subcategory of \(\mathcal{X}(A)\) whose objects are those \((X, A(X))\) for which \(A(X)\) is a subalgebra of the \(\text{Set}\)-algebra \(A^X\). Note that for each family \(\{f_i : X \to A\}_{i \in T}\) and morphism \(\alpha : Y \to X\) we have that

\[\omega_T \circ (f_i \circ \alpha) = \omega_T \circ (f_i) \circ \alpha.\]  

Since the intersection of subalgebras is a subalgebra, \(\mathcal{X}(A, \Omega)\) is bicoreflective in \(\mathcal{X}(A)\) and therefore it is topological over \(\mathcal{X}\) ([21,9] for \(\mathcal{X} = \text{Set}\)). Moreover it follows from (1) that every \(\mathcal{M}(A)\)-subobject of an \(\mathcal{X}(A, \Omega)\)-object is in \(\mathcal{X}(A, \Omega)\), i.e. \(\mathcal{X}(A, \Omega)\) is \(\mathcal{M}(A)\)-hereditary in \(\mathcal{X}(A)\), so that the lifting of the \((\mathcal{E}, \mathcal{M})\)-factorization structure of \(\mathcal{X}\) to \(\mathcal{X}(A)\), say \((\mathcal{E}(A, \Omega), \mathcal{M}(A, \Omega))\), is precisely the restriction to \(\mathcal{X}(A, \Omega)\) of the \((\mathcal{E}(A), \mathcal{M}(A))\)-factorization structure of \(\mathcal{X}(A)\).

Finally note that \(\mathcal{X}(A, \Omega) = \mathcal{X}(A)\) for \(\Omega = \{\text{id}_A\}\).

2. Zariski closure

In this section \(\mathcal{X}\) is supposed to be a complete category endowed with a proper \((\mathcal{E}, \mathcal{M})\)-factorization structure for morphisms [14,28]. Hence \(\mathcal{E}\) is a given class of epimorphisms and \(\mathcal{M}\) is a given class of monomorphisms of \(\mathcal{X}\), both containing the isomorphisms of \(\mathcal{X}\), such that every morphism \(f\) in \(\mathcal{X}\) has a factorization \(f = m \circ e\) with \(e \in \mathcal{E}\) and \(m \in \mathcal{M}\) which is essentially unique.

Of the resulting properties of \(\mathcal{M}\), that we will use, we mention that \(\mathcal{M}\) is closed under composition and under limits; it contains the regular monomorphisms of \(\mathcal{X}\) and it is stable under pullback and left cancellable.

For every object \(X\), the class \(\mathcal{M}_X\) of \(\mathcal{M}\)-morphisms with codomain \(X\) is pre-ordered by \(m \preceq n\) if there exists \(j\) such that \(n \circ j = m\). We write \(m \cong n\) if \(m \preceq n\) and \(n \preceq m\). Under the given assumptions, \(\mathcal{M}_X\) has all set-indexed infima, given by the multiple pullback in \(\mathcal{X}\). But since \(\mathcal{M}_X\) may be large, we assume \(\mathcal{X}\) to have multiple pullbacks of arbitrarily large families of morphisms in \(\mathcal{M}\) with common codomain, with the pullbacks belonging to \(\mathcal{M}\).

In this way \(\mathcal{M}_X\) has all infima and suprema and we will refer to it as the subobject lattice of \(X\) although only \(\mathcal{M}_X/\cong = \text{Sub} X\) has the structure of a (possibly large) complete lattice. Precisely we will not distinguish in the remaining part of the paper between \(m \in \mathcal{M}_X\) and the equivalence class of \(m\) in \(\text{Sub} X\) (i.e. \(m = n\) means \(m \cong n\) when \(m\) and \(n\) are taken as subobjects).

With the above assumptions we will consider the category \(\mathcal{X}(A, \Omega)\) defined in Section 1 endowed with the factorization structure \((\mathcal{E}(A, \Omega), (\mathcal{M}(A, \Omega))\) which is proper
since $f$ is a monomorphism (respectively epimorphism) in $\mathcal{X}(A, \Omega)$ if and only if $Uf$ is a monomorphism (respectively epimorphism) in $\mathcal{X}$.

Moreover, for every $A(X) \subseteq X^X$ there is a bijective correspondence (via initiality) between $\text{Sub} X$ (in $\mathcal{X}$) and $\text{Sub}(X, A(X))$ (in $\mathcal{X}(A, \Omega)$). For this reason we will refer to $m \in \text{Sub} X$ for a subobject of $(X, A(X))$.

For $\mathcal{X} = \text{Set}$ the definition below was given in [9].

**Definition 1.** For every object $(X, A(X))$ in $\mathcal{X}(A, \Omega)$ and subobject $m$ of $(X, A(X))$, i.e. $m \in \text{Sub} X$, the formula

$$z_{(X,A(X))}(m) = \bigwedge \{ [\alpha, \beta] \mid \alpha, \beta \in A(X), [\alpha, \beta] \geq m \}$$

where $[\alpha, \beta]$ denotes the equalizer in $\mathcal{X}$ of $\alpha$ and $\beta$, defines a closure which is called the Zariski closure of $m$ in $(X, A(X))$.

Clearly, the Zariski closure is an extensive (i.e. $m \leq z(m)$) and monotone (i.e. $m \leq n \implies z(m) \leq z(n)$) operator. Moreover, it follows from the stability under pull-back of regular monomorphisms that the Zariski closure is continuous (that is: for every $\mathcal{X}(A)$-morphism $f : (X, A(X)) \to (Y, A(Y))$ and $m \in \text{Sub} X$, $f(z(m)) \leq z(f(m))$).

In conclusion the Zariski closure is a closure operator of $\mathcal{X}(A, \Omega)$ with respect to $(E(A, \Omega), M(A, \Omega))$ in the sense of Dikranjan and Giuli [12]. Finally $z$ is idempotent ($z(z(m)) = z(m)$), in fact $[\alpha, \beta] \geq m$ is equivalent to $[\alpha, \beta] \geq z(m)$, and it is hereditary (i.e. $z_{(Y,A(Y))}(m) = y^{-1}(z_{(X,A(X))}(y \circ m))$ for every subobject $y : (Y, A(Y)) \to (X, A(X))$ of $(X, A(X))$), in fact $A(Y)$ is given restricting $A(X)$ to $Y$. In conclusion we have proved the following

**Proposition 1.** The Zariski closure is an idempotent and hereditary closure operator of $\mathcal{X}(A, \Omega)$ with respect to $(E(A, \Omega), M(A, \Omega))$.

A subobject $m$ of $X$ is called $z$-closed if $z_X(m) = m$; a morphism $f$ is called $z$-closed if it sends $z$-closed subobjects into $z$-closed subobjects. Since $z$ is hereditary, for a subobject to being $z$-closed is equivalent to be $z$-closed as a morphism [19]. A morphism $f : X \to Y$ is called $z$-dense if $z_Y(f (\text{id}_X)) = \text{id}_Y$.

The idempotency and (hereditariness, hence) weak hereditariness of $z$ gives (see [12, Prop. 1.3])

**Corollary 1.** ([9, Theorem 5.7] for $\mathcal{X} = \text{Set}$) $(z$-dense morphism, $z$-closed $M(A, \Omega)$-morphism) is a factorization structure of $\mathcal{X}(A, \Omega)$.

As noted in Section 1, $\mathcal{X}(A, \Omega)$ is $M(A)$-hereditary in $\mathcal{X}(A)$ and it is topological over $\mathcal{X}$ so that the Zariski closure in $\mathcal{X}(A, \Omega)$ coincides with the one given in $\mathcal{X}(A)$ and restricted to $\mathcal{X}(A, \Omega)$.

A closure operator $c$ is called grounded whenever $c_X(0_X) = 0_X$ for every object $X$ (here $0_X$ is the bottom element of Sub $X$). It is called additive if it is grounded and $c_X(m \vee n) = c_X(m) \vee c_X(n)$ holds for every object $X$ and $m, n \in \text{Sub} X$.

The Zariski closure is neither additive nor grounded in general, as it is explained below.
Let \textbf{Set} be the category of sets and functions endowed with the (surjective, injective) factorization structure. In the topological category \textbf{Set}(A), \(|A| \geq 2\), constant functions need not be morphisms. Indeed in a set with one element there are \(2^A\) structures and the empty set has two structures. The indiscrete objects in \textbf{Set}(A) are the ones of the form \((X, \emptyset)\).

In particular the property \(z(\emptyset) = \emptyset\) is not fulfilled in nonempty indiscrete objects, i.e. the Zariski closure is not grounded. The (clearly) hereditarily bicoreflective subcategory of \textbf{Set}(A) consisting of all \((X, A(X))\) with \(A(X)\) containing all constant functions, that we denote by \textbf{SET}(A), has the induced Zariski closure which is grounded, but not additive (i.e., in general \(z(M \cup N) \neq z(M) \cup z(N)\)). Take \(X = \{a, b, c\}\) and for given \(a_1 \neq a_2\) in \(A\) let \(A(X) = \{\alpha, \beta, \gamma\}\), where \(\alpha(a) = \alpha(b) = \alpha(c) = a_1\), \(\beta(a) = \beta(b) = a_1\) and \(\beta(c) = a_2\), \(\gamma(a) = \gamma(c) = a_1\), \(\gamma(b) = a_2\). Then every point is closed but every two-point subset is not closed. Adding the constant map \(\delta(a) = \delta(b) = \delta(c) = a_2\) we have that \(\{b, c\}\) is not closed which shows that even in \textbf{SET}(A) the Zariski closure is not additive.

The category \(\mathcal{X}\) is identified with \(\mathcal{X}(A, \Omega)\), the full subcategory consisting of all the objects of the form \((X, A^X)\). It is in general not \(\mathcal{M}(A, \Omega)\)-hereditarily bicoreflective. When it is hereditary the Zariski closure operator of \(\mathcal{X}(A, \Omega)\) can be restricted to \(\mathcal{X}\) and in \(\mathcal{X}\) it coincides with the regular closure operator induced by \(A\). It may happen in this case that the Zariski closure operator of \(\mathcal{X}(A, \Omega)\) is not additive while it is so when restricted to \(\mathcal{X}\). In order to see this let \textbf{Top} be the category of topological spaces endowed with the (surjective, embedding) factorization structure and let \(S = \{0, 1\}\) be the Sierpinski space \((1)\) is the only proper open set). Since \textbf{Top} is hereditarily bicoreflective in \textbf{Set}(S), then the Zariski closure restricted to \textbf{Top} is the regular closure induced by \(S\), which is the so-called \(b\)-closure (see Example 1 in Section 6). While the \(b\)-closure is additive, the Zariski closure in the category \textbf{Top}(S) fails to be additive (use the three-point set, endowed with discrete topology, of the above example) but it remains grounded.

In general the Zariski closure operator defined by a space \(A\) fails to be additive even when it is restricted to \textbf{Top} (e.g., let \(A\) be a strongly rigid non trivial space).

3. Separated object

The notion of separation with respect a closure operator was introduced in [12] (see also [11]): in a category endowed with a closure operator \(c\) an object \(X\) is called \(c\)-separated if the diagonal \(\Delta_X\) is \(c\)-closed in \(X \times X\). It was shown in [9, Theorem 6.1], that for categories of the form \textbf{Set}(A, \Omega) and \(c = z\) the above definition coincides with:

**Definition 2.** \(X \in \mathcal{X}(A, \Omega)\) is called \textit{separated} if every initial morphism \(f : X \to Y\) belongs to \(\mathcal{M}\). \(\text{Sep} \mathcal{X}(A, \Omega)\) will denote the full subcategory of \(\mathcal{X}(A, \Omega)\) consisting of all separated objects.

For a more general equivalence between the previous two notions the Proposition 3 below and the diagonal theorem [18, Theorem 1.1]. For a given algebra \((A, \Omega)\) let us denote by \(\text{id}_A\) the subalgebra of \(A^X\) generated by \(\text{id}_A\). \(A = (A, \text{id}_A)\) is separated. In fact, if \(f : A \to (Y, A(Y))\) is initial, then \(\beta \circ f = \text{id}_A\) for some \(\beta \in A(Y)\). Thus \(f\) being a (section hence) regular monomorphism belongs to \(\mathcal{M}\).
If \( m : (X, A(X)) \rightarrow A^I \), for any index set \( I \) belongs to \( \mathcal{M}(A, \Omega) \) (i.e. \( m \in \mathcal{M} \) and it is initial in \( \mathcal{X}(A, \Omega) \) then \( (X, A(X)) \) is separated. In fact, let \( f : (X, A(X)) \rightarrow (Y, A(Y)) \) be an initial morphism. For each \( j \in I \), if \( p_j : A^I \rightarrow A \) is the \( j \)th projection, then the \( \mathcal{X} \)-morphism \( p'_j = p_j \circ m \) belongs to \( A(X) \) so that, by initially of \( f \), there exists a \( \mathcal{X} \)-morphism, say \( \beta_j : Y \rightarrow A \) in \( A(Y) \) such that \( \beta_j \circ f = p'_j \). Now \( \langle \beta_j \rangle \circ f = m \) (in fact, for every \( j \in I \), \( p_j \circ \langle \beta_j \rangle \circ f = \beta_j \circ f = p'_j = p_j \circ m \)) so that \( f \in \mathcal{M} \) by left cancellation property of \( \mathcal{M} \).

**Theorem 1.** Sep\( \mathcal{X}(A, \Omega) \) is epireflective in \( \mathcal{X}(A, \Omega) \).

**Proof.** For every \( \mathcal{X}(A, \Omega) \)-object \( (X, A(X)) \), let us consider the canonical morphism \( \phi : (X, A(X)) \rightarrow A^{A(X)} \) and let \( (e : (X, A(X)) \rightarrow (SX, A(SX))), m : (SX, A(SX)) \rightarrow A^{A(X)} \) be a \( (\mathcal{E}(A, \Omega), \mathcal{M}(A, \Omega)) \) factorization of \( \phi \). For every morphism \( f : (X, A(X)) \rightarrow (Y, A(Y)) \), with \( (Y, A(Y)) \) separated and \( \psi : (Y, A(Y)) \rightarrow A^{A(Y)} \) the canonical morphism, let \( \hat{f} \) be the morphism from \( A^{A(X)} \) to \( A^{A(Y)} \) associated to \( f \). Then \( \hat{f} \circ \phi = \psi \circ f \) and in the commutative square

\[
\begin{array}{ccc}
(X, A(X)) & \longrightarrow & (SX, A(SX)) \\
\downarrow f & & \downarrow \text{fom} \\
(Y, A(Y)) & \longrightarrow & A^{A(Y)} \\
\end{array}
\]

\( e \in \mathcal{E}(A) \) and, by \( (Y, A(Y)) \in \text{Sep}\mathcal{X}(A, \Omega) \), \( \psi \in \mathcal{M}(A) \). Then there exists a (unique) morphism \( d : (SX, A(SX)) \rightarrow (Y, A(Y)) \) such that \( d \circ e = f \) and \( \psi \circ d = \hat{f} \circ m \). Now \( (SX, A(SX)) \) is separated so Sep\( \mathcal{X}(A, \Omega) \) is epireflective in \( \mathcal{X}(A, \Omega) \) with reflection \( e : (X, A(X)) \rightarrow (SX, A(SX)) \) for every \( (X, A(X)) \in \mathcal{X}(A, \Omega) \). \( \square \)

**Proposition 2.** Every separated object is an \( \mathcal{M}(A, \Omega) \)-subobject of a product of copies of \( A \), i.e. Sep\( \mathcal{X}(A, \Omega) \) is simply cogenerated by \( A \).

In \( \mathcal{X}(A, \Omega) \) the regular closure operator induced by \( A \) or, equivalently, by Sep\( \mathcal{X}(A, \Omega) \) (the epireflective hull of \( A \)) is defined, for every subobject \( m \) of \( (X, A(X)) \), as the intersection of all equalizers pairs of morphisms from \( (X, A(X)) \) to \( A \) and containing \( m \) [12]. Now clearly \( f : (X, A(X)) \rightarrow A \) is a \( \mathcal{X}(A, \Omega) \)-morphism if and only if \( f \in A(X) \), thus we have

**Proposition 3.** The Zariski closure operator of \( \mathcal{X}(A, \Omega) \) coincides with the regular closure operator induced by \( A \) or, equivalently, by Sep\( \mathcal{X}(A, \Omega) \).

The notion of regular closure operator induced by a class \( \mathcal{P} \) of objects was introduced in [12] (see also [11]) just because it is useful to describe the epimorphisms of the epireflective hull \( \mathcal{E}(\mathcal{P}) \) of \( \mathcal{P} \) and it is also useful to describe the regular monomorphism of \( \mathcal{E}(\mathcal{P}) \) whenever the regular closure is weakly hereditary (when restricted to \( \mathcal{E}(\mathcal{P}) \)). Since the Zariski closure is (hereditary hence) weakly hereditary we have:
Corollary 2. ([9, Proposition 6.4] for $\mathcal{X} = \text{Set}$.) In Sep$\mathcal{X}(A, \Omega)$ epimorphisms coincide with $z$-dense morphisms and regular monomorphisms coincide with $z$-closed morphisms of $\mathcal{M}(A, \Omega)$.

4. Complete objects

The categorical theory of completeness was developed in [4,3]. Here we show the existence of complete objects in categories of the form Sep$\mathcal{X}(A, \Omega)$.

Definition 3. A separated object $X$ is called complete (algebraic in [9,10], absolutely $\mathcal{M}(A, \Omega)$-closed in [4]) if a morphism $f : X \to Y$ is $z$-closed whenever $Y$ is separated and $f \in \mathcal{M}(A, \Omega)$.

Definition 4. $X$ is called $z$-injective if it is injective with respect to the class $\{z$-dense$\} \cap \mathcal{M}(A, \Omega)$ (weakly $\mathcal{M}(A, \Omega)$-injective in [4]).

Proposition 4. The cogenerator $A$ of Sep$\mathcal{X}(A, \Omega)$ is $(\mathcal{M}(A, \Omega)$-injective, hence) $z$-injective.

Proof. Let $f : X \to A$ be a $\mathcal{X}(A, \Omega)$-morphism and let $m : X \to Y$ be in $\mathcal{M}(A, \Omega)$. Then $f \in A(X)$, so that, by initially of $m$, there exists $f' \in A(Y)$ with $f' \circ m = f$. Since $f' \in A(Y)$ implies that $f' : Y \to A$ is a $\mathcal{X}(A, \Omega)$-morphism, the proof is complete. \(\square\)

Proposition 5. A separated object is complete if and only if it is $z$-injective.

Proof. ($\Rightarrow$) Every product of copies of $A$ is $z$-injective by Proposition 4 and standard arguments. Let now $Z$ be complete, $m : X \to Y$ a $z$-dense morphism in $\mathcal{M}(A, \Omega)$, $f : X \to Z$ any morphism and $\phi : Z \to A^Z$ the canonical morphism. Then there exists a morphism $g : Y \to A^Z$ such that $g \circ m = \phi \circ f$, by $z$-injectivity of $A^Z$. On the other hand $m$ is a $z$-dense morphism and $\phi$ is a $z$-closed morphism in $\mathcal{M}(A, \Omega)$, by completeness of $Z$. Then, by Corollary 1 there is a (unique) morphism $d : Y \to Z$ such that (in particular) $d \circ m = f$ which says that $Z$ is $z$-injective.

($\Leftarrow$) Let $X \in \text{Sep}\mathcal{X}(A, \Omega)$ be $z$-injective and $f : X \to Y$ be a morphism in $\mathcal{M}(A, \Omega)$ with $Y \in \text{Sep}\mathcal{X}(A, \Omega)$. Then, denote by $(e, m)$ the ($z$-dense, $z$-closed in $\mathcal{M}(A, \Omega)$) factorization of $f$ (see Corollary 1), there exists a (unique) morphism $m'$ from the domain of $e$ to $X$ such that $m' \circ e = \text{id}_X$. Thus $e$ is both a section and an epimorphism, consequently $e$ is an isomorphism which says that $f$ is $z$-closed. Hence the proof is complete. \(\square\)

Alg$\mathcal{X}(A, \Omega)$ will denote the full subcategory of Sep$\mathcal{X}(A, \Omega)$ consisting of all complete objects.

Theorem 2. Alg$\mathcal{X}(A, \Omega)$ is $(\{z$-dense$\} \cap \mathcal{M}(A, \Omega))$-reflective in Sep$\mathcal{X}(A, \Omega)$ and if $f : X \to Y$ belongs to $(\{z$-dense$\} \cap \mathcal{M}(A, \Omega))$ and $Y \in \text{Alg}\mathcal{X}(A, \Omega)$ then $f$ is
the Alg \(X(A, \Omega)\)-reflection of \(X\) (i.e. Alg \(X(A, \Omega)\) is \(M(A, \Omega)\)-firm epireflective in Sep \(X(A, \Omega)\) in the terminology of [4]).

**Proof.** Since the cogenerator \(A\) of Sep \(X(A, \Omega)\) is \(z\)-injective (see Proposition 4), it follows from Proposition 5 and Corollary 1.7 of [4] that Alg \(X(A, \Omega)\) is \(M(A, \Omega)\)-firm epireflective in Sep \(X(A, \Omega)\). A direct proof can be made following the lines of the proof of Theorem 1 and then, for the firmness, using the fact that in Alg \(X(A, \Omega)\) consists of \(z\)-injective separated objects (Proposition 5).

**Corollary 3.** \(X\) is complete if and only if it is a \(z\)-closed subobject of a power of \(A\).

5. **Compact objects**

For the theory of compactness with respect a closure operator we refer to [7] (see also [13]). Here we study the compactness with respect to the Zariski closure.

**Definition 5.** \(X \in \mathcal{X}(A, \Omega)\) is called \(z\)-compact if for every \(\mathcal{X}(A, \Omega)\)-object \(Y\) the projection

\[ p_Y : X \times Y \to Y \]

is \(z\)-closed (\(z\)-preserving in [7]).

\(\text{Comp} \mathcal{X}(A, \Omega)\) will denote the class of all \(z\)-compact objects and \(\text{Sep Comp} \mathcal{X}(A, \Omega)\) the class of all separated \(z\)-compact objects. The basic results below follow from the general theory of compact objects in categories developed in [7] and from the idempotency and hereditariness of the Zariski closure operator.

**Proposition 6.** \(\text{Comp} \mathcal{X}(A, \Omega)\) is closed under finite products in \(\mathcal{X}(A, \Omega)\).

**Proposition 7.** If \(X\) is \(z\)-compact and \(m : M \to X\) in \(M(A, \Omega)\) is \(z\)-closed then \(M\) is \(z\)-compact.

**Proposition 8.** If \(X\) is \(z\)-compact and \(m : X \to Y\) belongs to \(M(A, \Omega)\) with \(Y \in \text{Sep} \mathcal{X}(A, \Omega)\) then \(Y\) is \(z\)-closed.

**Proposition 9.** Assume that \(\mathcal{E}\) is stable under pullback in \(\mathcal{X}\). If \(f : X \to Y\) belongs to \(\mathcal{E}\) and \(X\) is \(z\)-compact, so is \(Y\).

**Proposition 10.** \(\text{Sep Comp} \mathcal{X}(A, \Omega)\) is closed under finite limits in \(\mathcal{X}(A, \Omega)\).

According to [7, 2.8] we say that the Zariski closure operator has the finite structure property of products (FSPP) if in every product of \(\mathcal{X}(A, \Omega)\)-objects \(z\) can be obtained as inverse limit of \(z\)-closures in finite subproducts, i.e.

\[ z(\prod_{i \in I} X_i)(m) = \lim_{\overrightarrow{p}} z(\prod_{i \in F} X_i)(m) \]
with $F$ running in the family of all finite subsets of $I$.

According to [7, 1.6] we say that $E$ in $X$ is a surjectivity class if there exists a class $P$ of $X$-objects such that a morphism $f : X \to Y$ belongs to $E$ exactly when every morphism $y : P \to Y$, $P \in P$, factors $f \circ x = y$.

Note that $E(A, \Omega)$ is a surjectivity class in $X(A, \Omega)$ whenever $E$ is so in $X$.

The results below are Theorem 6.4 and Corollary 6.5 of [7] for $c = z$ (taking in account that $z$ is hereditary).

**Theorem 3.** If $E$ is a surjectivity class in $X$ and $z$ has FSPP, under the Axiom of Choice $\text{Comp}X(A, \Omega)$ is closed under direct products in $X(A, \Omega)$.

**Corollary 4.** Under the assumption of the above theorem Sep $\text{Comp}X(A, \Omega)$ is closed under limits in $X(A, \Omega)$.

**Proposition 11.** Every $z$-compact separated object is complete.

**Proof.** This follows directly from Proposition 8 of this section. $\square$

In general the converse of the above proposition is false (see Example 1 below).

6. Examples

(1) Topological spaces as sets modelled by a frame.

Let $\mathcal{X} = \text{Set}$ and $(\mathcal{E}, \mathcal{M}) = (\text{surjective}, \text{injective})$. On the two-point set $A = \{0, 1\}$ let us consider the following operations:

- For an arbitrary set $T$ we let
  $$\omega_T(a_t)_{t \in T} = \max_{t \in T} a_t$$

- For a finite set $T$ we let
  $$\omega^*_T(a_t)_{t \in T} = \min_{t \in T} a_t$$

Identifying the open sets in a topological space with their characteristic functions it is clear that Top is isomorphic to $\text{Set}(A, \Omega)$ [21,9]. $A$ is in this case the Sierpinski space.

According to Proposition 3, the Zariski closure operator is the closure (well known as $b$-closure or Skula closure [25]) defined by

$$z_X(M) = \{ x \in X \mid U_x \cap M \cap \bar{x} \neq \emptyset \text{ for every neighborhood of } x \}.$$

The separated objects are the $T_0$-spaces and a $T_0$-space is complete if and only if every irreducible closed set is the closure of a point (sober $T_0$-space, see, e.g. [20], [4, Ex. 1.8(3)]).

The $z$-compact objects are the topological spaces for which the topology induced by the Zariski closure is ordinarily compact [13] or, equivalently, the hereditarily compact spaces in which every closed set is a finite union of point closures [25]. Clearly this example shows that the converse of Proposition 11 is false.
(2) In the previous example delete the second family of operations except the 0-ary one. We obtain the category $\text{CS}$ of closure spaces. The complete closure spaces are the ones in which every closed set is the closure of an unique (uniqueness ensures separation) point [8].

(3) On the unit interval $A = [0, 1]$ let us consider the following operations: for an arbitrary set $T$ we let
\[ \omega_T(a_t)_{t \in T} = \sup_{t \in T} a_t \]
for a finite set $T$ we let
\[ \omega'_T(a_t)_{t \in T} = \min_{t \in T} a_t. \]
Then it is clear that $\text{Set}(A, \Omega)$ is isomorphic to the category $\text{Fuz}$ of fuzzy topological spaces in the sense of [5]. Here $A$ coincides with the fuzzy Sierpinski space $([0, 1], \{0, 1, \text{id}\})$ considered in [26]. The Zariski closure operator of $\text{Fuz}$, being the regular closure operator induced by the fuzzy Sierpinski space by Proposition 3, coincides with the one considered in [1, Remark 2.8]. The separated objects coincide with the usual fuzzy $T_0$ spaces ($(X, \tau)$ is called $T_0$ if for every $x \neq y$ in $X$ there is $V \in \tau$ such that $V(x) \neq V(y)$). In [27] it was shown that the complete fuzzy $T_0$ spaces are the sober fuzzy $T_0$-spaces of [24] characterized in [22] in the same terms of sober $T_0$ topological spaces.

(4) On the set $A = [0, \infty]$ let us consider the two kind of operations given in Example 3 and the following additional operations: for every $a \in [0, \infty]$ we consider the 1-ary operation
\[ \omega_a(x) = a + x \]
for every $a \in [0, \infty]$ we consider the 1-ary operation
\[ \omega'_a(x) = (x - a) \lor 0. \]
It is shown in [23] that $\text{Set}(A, \Omega)$ coincides with the category $\text{AP}$ of approach spaces defined by R. Lowen. An exhaustive study of complete (separated) approach spaces can be found in the recent paper [2].

(5) Let $A$ be any set. In the category $\text{Set}(A)$

- $(X, A(X))$ is separated if and only if $A(X)$ separates the points of $X$;
- $(X, A(X))$ is complete if and only if $X$ is a subset of $A^I$ for which there exists a subset $P \subseteq I^2$ such that $(a_i) \in X$ if and only if $a_p = a_q$ for every $(p, q) \in P$ and $A(X)$ consists of the projections.

Here $A = (A, \{\text{id}_A\})$.

**Proposition 12.** $A$ is $z$-compact.

**Proof.** Let $M \subseteq A \times Y$ and $y \notin p_Y(z(M))$, equivalently $y$ such that $(a, y) \in z(M)$ for no $a$. If for a given (and then for all) $a \in A$ there exist $\beta_1, \beta_2 \in A(Y)$ such that $(\beta_1 \circ p_Y)(m_1, m_2) = (\beta_2 \circ p_Y)(m_1, m_2)$ for every $m_1, m_2 \in M$ and $(\beta_1 \circ p_Y)(a, y) \neq
(β_2 ◦ p_Y)(a, y) then we have β_1(m_2) = β_2(m_2) for every m_2 ∈ p_Y(M) and β_1(y) ≠ β_2(y).

Thus y ∈ z(p_Y(M)). If not, for all a ∈ A there exists β_a ∈ A(Y), such that, for all (m_1, m_2) ∈ M, m_1 = β_a(m_2) (in fact p_Y(m_1, m_2) = β_a(p_Y(m_1, m_2)) and β_a(y) = a' ≠ a).

Let us consider β_a'. Then for every m_2 ∈ p_Y(M), β_a(m_2) = m_1 = β_a'(m_2) and β_a(y) = a' ≠ β_a'(y). Thus y ∈ z(p_Y(M)).

**Proposition 13.** If |A| ≥ 2 and 0 ∈ A (A, {id_A, 0}) is not compact.

**Proof.** For simplicity assume A = {0, 1} and let Y = A and A(Y) = {id_A, τ} where τ is the twist. Then in A × Y, M = [0, τ o p_Y] = {(0, 1), (1, 1)} so that p_Y(M) = {1} which is not closed.

This paper is the final version of a preprint in Mathematik-Arbeitspapiere No. 54 (CatMAT 2000). After the time of publication several new contributions on the subject appeared:

An extensive study of completions in subcategories of the category \textbf{Set}({0, 1}) can be found in [15]. There it is shown that every subcategory consisting of all separated objects of a hereditary coreflective subcategory of \textbf{Set}({0, 1}) admits completions.

In [8] the complete objects in the category of \textit{T}_0 closure are characterized internally (see example above) and externally.

In [16] the structure of the above complete objects is explained. The crucial steps are the construction of a pointwise extension and the introduction of a so-called X-compatible family.

In [17] it is shown that these X-compatible families are in fact the points of the completion as the homomorphisms in the known spectrum construction for soberification of spaces [20], fuzzy spaces [27] and approach spaces [2], amongst others.

Veerle Claes and Eva Colebunders [6] found sufficient conditions for the equality z-compact = complete showing in particular that the equality holds in Sep\textit{SET}\{0, 1\} and in Sep\textit{CS}.

**References**


Further reading