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Characterizations of consistent marked graphs

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Abstract

A *marked graph* is a graph with a + or − sign on each vertex and is called *consistent* if each cycle has an even number of − signs. This concept is motivated by problems of communication networks and social networks. We present some new characterizations and recognition algorithms for consistent marked graphs.

1. Introduction

Data in the social sciences can often be modeled using a *signed graph*, a graph where every *edge* has a sign + or −, or a *marked graph*, a graph where every *vertex* has a sign + or −. A marked graph is called *consistent* if every cycle has an even number of − signs. The concept of consistency is analogous to the concept of balance in signed graphs: A signed graph is called *balanced* if every cycle has an even number of − signs. (For references on balance, see [12] and [16].) In this paper, we give a number of new characterizations and recognition algorithms for consistent marked graphs.

Consistent marked graphs were introduced by Beineke and Harary [3]. (An analogue for digraphs was introduced by Beineke and Harary [4].) The concept was motivated by communication networks. If binary messages are sent through a network with − vertices reversing messages and + vertices leaving them unchanged, then a consistent marked graph has the following consistency property: If a message is sent from x to y through two different vertex-disjoint paths, and if x and y have the same sign, then y will receive the same message no matter which path is followed. In a marked graph,

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let us say that the *sign* of a path or cycle is + if there is an even number of − signs and − otherwise. (The signs of the end vertices of the path are counted.) Even for consistent marked graphs, even if x and y have the same sign, there can be two paths from x to y of different signs; the hypothesis of vertex disjointness is needed to conclude that they have the same sign. Consistent marked graphs arise in a similar way in social networks, networks whose vertices represent people. If some people always lie and some always tell the truth, a consistent social network has the property that if a message is sent from x to y along two vertex-disjoint paths and x and y have the same sign, then y will receive the same message independent of the path followed.

The notion of consistency of a marked graph has proven useful in the theory of balance of signed graphs. Harary and Kabell [7,8] described an efficient algorithm for determining if a given signed graph is balanced by setting up a correspondence between marked graphs and balanced signed graphs. This correspondence has also been useful in solving the problem of counting balanced signed graphs (Harary and Kabell [8]). (The problems of enumerating both balanced signed graphs and marked graphs have also been studied by Harary et al. [9].) The problem of characterizing consistent marked digraphs was solved by Beineke and Harary [4]. Rao [15] obtained an early characterization of consistent marked graphs and gave a polynomial algorithm for their recognition and Acharya [1,2] gave other characterizations. Hoede [11] characterized consistent marked graphs in terms of fundamental cycles of a cycle basis and observed that the characterization gives rise to a polynomial algorithm for determining consistency that is considerably simpler than Rao's. Here, we shall give a still more efficient algorithm. Beineke and Harary [4] introduced the problem of determining if an unmarked digraph can be marked using at least one − sign so that the resulting marked digraph is consistent. Roberts [17] studied this problem for marked graphs.

In this paper, we use the following notation in a (marked) graph G: V_+ is the set of + vertices, V_- the set of − vertices, n the number of vertices, m the number of edges. If S is a set of vertices or edges of G, then $G[S]$ is the subgraph induced by S. If T is a tree in G, then the unique path in T between vertices x and y is denoted by T_{xy}. A special case of this notation is P_{xy} when the tree is a path P. Vertices x and y in G are said to be *k-connected* (*k-edge-connected*) if there are k (internally) vertex-disjoint (edge-disjoint) paths between x and y.

A set of cycles in a graph G is a *cycle basis* if it is a minimal set of cycles with the property that every cycle can be expressed as a symmetric difference of cycles in the set. One well-known way to construct a cycle basis is to find a spanning tree T of G. A *tree chord* relative to T is an edge not in T. Every tree chord defines a unique cycle of G, called a *fundamental cycle* relative to T or a *T-fundamental cycle*, whose remaining edges are in T. The set of fundamental cycles relative to a tree is well known to be a cycle basis for G.

2. Characterizations of consistent marked graphs

The notion of 3-connectedness between a vertex pair plays a central role in studying marked graphs, as is shown in the following theorem.

Theorem 1 (Beineke and Harary [3]). *If a marked graph is consistent, then any two* 3*-connected vertices receive the same sign.*

Let G be a marked graph and let $x, y \in V(G)$ have the same sign. If they are positive, a path joining x and y is said to be *coherent* if it contains an even number of negative vertices and is said to be *incoherent* otherwise. If they are negative, a path joining x and y is said to be *coherent* if it contains an odd number of negative vertices, and is said to be *incoherent* otherwise.

Theorem 2. *A marked graph is consistent if and only if any two vertex-disjoint paths between a pair of vertices with the same sign are either both coherent or both incoherent.*

Proof. The necessity is obvious. For the sufficiency, we note that every cycle C has at least two vertices of the same sign. Since the two vertex-disjoint paths joining these vertices on C are either both coherent or both incoherent, C is positive. □

Theorem 3. *Let G be a* 2*-connected marked graph that is not a cycle. Then G is consistent if and only if any* 3*-connected pair of vertices receive the same sign and paths between them either are all coherent or all incoherent.*

Proof. *Necessity*: By Theorem 1, a 3-connected pair of vertices u and v receive the same sign. Let P^1, P^2 and P^3 be three vertex-disjoint paths from u to v. Since G is consistent, P^1, P^2 and P^3 are all coherent or all incoherent. Suppose the former; the latter case is similar. Suppose there is also a path P from u to v which is incoherent. Choose P such that $|V(P) \setminus V(P^1 \cup P^2 \cup P^3)|$ is minimum. By consistency of G and incoherence of P, P must have common vertices with $V(P^1 \cup P^2 \cup P^3) \setminus \{u, v\}$.

Suppose P and P^1 have common vertices. Let x and y be two vertices on both P and P^1 such that at least one of x, y is not u or v and vertices between x and y on P^1 are not on P. We now note that there are three vertex-disjoint paths from x to y, one of which is P^1_{xy}. If at most one of $V(P^2)$ and $V(P^3)$ intersects $V(P_{xy})$, then the other two paths are P_{xy} and the path obtained by P_{xu}, P^2 or P^3 from u to v, then P_{vy}. Suppose both $V(P^2)$ and $V(P^3)$ intersect $V(P_{xy})$ and on P_{xy}, the first vertex on $V(P^2) \cup V(P^3)$ is a and the last is b. Without loss of generality, $a \in V(P^2)$. If b is also in $V(P^2)$, then the other two paths are P_{xa}, P^2_{ab}, P_{by} and P_{xu}, P^3_{uv}, P_{vb}. If b is in $V(P^3)$, then the other two paths are P_{xa}, P^2_{av}, P_{vb}, and P_{xu}, P^3_{ub}, P_{by}. Therefore, x and y are 3-connected and hence they receive the same sign.

The paths P_{xy} and P^1_{xy} form a cycle. Since G is consistent, the cycle is positive and the two paths either are both coherent or both incoherent. Therefore if we replace P_{xy} by P^1_{xy} on P to get P', then P' is also incoherent since P is, but $|V(P') \setminus V(P^1 \cup P^2 \cup P^3)| < |V(P) \setminus V(P^1 \cup P^2 \cup P^3)|$, which is a contradiction.

Sufficiency: Let C be a cycle of G. Since G is 2-connected and not a cycle, there are two vertices u and v on C which are 3-connected. Then u and v receive the same sign and two paths on C from u to v either are both coherent or both incoherent, which implies that C is positive. □

Theorem 4 (Acharya [2], Rao [15]). *If G is a consistent marked graph, then*

(1) $G[V_-]$ *is bipartite*;
(2) *between each connected component of* $G[V_+]$ *and each partite set in the bipartition of each component of* $G[V_-]$, *there is at most one edge.*

Condition (2) of Theorem 4 will be called *single joining* and a marked graph satisfying conditions (1) and (2) will be called a *split marked graph*.

Let G be a graph and $A \subseteq V(G)$. We use the terminology "*shrink A to a vertex a*" to mean to replace A by a single vertex a and replace an edge bv for $b \in A$ by an edge av (omitting duplications). If $G[A]$ is connected and bipartite with partite sets C and D, "*shrink A to an edge cd*" will mean to shrink C to c and D to d in G, add an edge cd, and replace any edge xv for $x \in C$ by cv and any edge yv for $y \in D$ by dv.

Given a marked graph G, suppose we shrink each component of $G[V_+]$ to a vertex to get a graph G'. Then, as indicated in [3], if G is split, then G is consistent if and only if G' is.

Theorem 5. *Let G be a split marked graph and let G' be the result of shrinking each nontrivial component of $G[V_-]$ to an edge. Then G is consistent if and only if G' is.*

Proof. Suppose u and v are in the same partite set of a component of $G[V_-]$, they have a common neighbor $x \in V_-$, and G' is obtained from G by shrinking them to a vertex w. We prove that there is a negative cycle in G if and only if there is a negative cycle in G'. This will prove the theorem, since by iterating the procedure, we shrink each nontrivial component of $V[G_-]$ to an edge.

First assume that G has a negative cycle C. If C does not contain both u and v, then clearly G' has a negative cycle. Suppose both u and v are on C, say $C = u, u_1, \ldots, u_r, v, v_1, \ldots, v_s, u$. If either r or $s = 1$, say $r = 1$, then since u and v have no common positive neighbor by single joining, u_1 is a negative vertex. After shrinking, there is a cycle $C' = w, v_1, \ldots, v_s, w$, with fewer negative vertices than C. If s is also 1, then C negative implies that v_1 is positive, which violates single joining. Therefore, $s > 1$ and so C' is a negative cycle. Now suppose both $r, s \geqslant 2$. Then G' has a negative cycle since one of the two cycles $w, u_1, \ldots, u_r, w$ and $w, v_1, \ldots, v_s, w$ must be negative.

Now assume that G' has a negative cycle C'. If C' does not contain w, then C' is also a negative cycle in G and the conclusion holds. Suppose C' contains w, say $C' = w, w_1, \ldots, w_t, w$. Suppose that the following condition **P** holds: either both uw_t and uw_1 or both vw_t and vw_1 are in G. Then there is a corresponding negative cycle in G. Note that if x is w_1 or w_t, the condition is satisfied. If condition **P** fails, we know that uw_1 or vw_1 and uw_t or vw_t are in G. If uw_1 is in G (which we may assume without loss of generality), then uw_t, w_1v are not in G. If x is not on C' in G', then $C = u, w_1, \ldots, w_t, v, x, u$ is a negative cycle of G. If x is on the cycle C', say $C' = w, w_1, \ldots, w_r, x, w_{r+2}, \ldots, w_t, w$, then there are two cycles $C_1 = u, w_1, \ldots, w_r, x, u$ and $C_2 = x, w_{r+2}, \ldots, w_t, v, x$ in G and one must be negative. □

Suppose G is a split marked graph and G' is obtained from G by shrinking each component of $G[V_+]$ to a vertex and shrinking each nontrivial component of $G[V_-]$ to an edge. We already know that G is consistent iff G' is consistent. Furthermore, as indicated by Hoede [11], if we insert a $+$ vertex on each edge of $G'[V_-]$ to get graph G'', then G' is consistent if and only if G'' is consistent.

Since G'' is bipartite with one partite set having only positive vertices and the other only negative vertices, G'' is consistent if and only if the length of every cycle is a multiple of 4. In other words, after polynomial time translating, the problem of checking if a marked graph is consistent reduces to the problem of checking if a bipartite graph only contains cycles with length $4k$.

Another way to see this is as follows. Let G be a marked graph. Form graph $T(G)$ by inserting a $+$ vertex in each edge incident to two $-$ vertices and three vertices with signs $-, +, -$ in that order in each edge incident to two $+$ vertices. Then $T(G)$ is bipartite with one part of $+$ vertices and another part of $-$ vertices, and G is consistent if and only if $T(G)$ only has cycles with length $0 \bmod 4$. To prove this, use induction on $s(G) = m(G) - q(G)$, where $m(G)$ is the number of edges of G and $q(G)$ is the number of edges of G joining vertices of different sign. The inductive step follows from the observation that if $W(G)$ is obtained from G by adding vertices on one edge according to the procedures in the definition of $T(G)$, then G is consistent if and only if $W(G)$ is consistent, $T(W(G)) = T(G)$, and $s(W(G)) < s(G)$.

In fact, we have the following result.

Theorem 6. *The problem of determining if a marked graph is consistent is polynomially equivalent to the problem of determining if a bipartite graph has all cycles of length* $0 \bmod 4$.

The polynomial reduction from the second problem to the first goes as follows. Let G be a bipartite graph. Assign $+$ signs to vertices of one bipartite class and $-$ signs to vertices of the other class, getting a marked graph H. Then every cycle of G has length $0 \bmod 4$ if and only if H is consistent.

To state another characterization of consistent marked graphs, we use the terminology "the common path of a pair of T-fundamental cycles" to mean the maximal common path.

Theorem 7 (Hoede [11]). *Let G be a marked graph and let T be a spanning tree of G. Then G is consistent if and only if it satisfies the following two conditions*:

(1) *Each T-fundamental cycle is positive.*
(2) *On the common path of a pair of intersecting T-fundamental cycles, the two end vertices always have the same sign.*

Theorem 7 also provides a polynomial algorithm to check if a marked graph is consistent. There are $m - n + 1$ fundamental cycles relative to any spanning tree. Hence there are $O(m^2)$ pairs of cycles to check for condition (2). For each pair, the checking time is $O(n)$. The complexity to check condition (1) is $O(mn)$. Therefore the complexity of this simple algorithm of Hoede's is $O(m^2n)$. Algorithm 1 in Section 3 gives an $O(mn)$ algorithm.

The following theorem is equivalent to Theorem 7.

Theorem 8. *Let G be a marked graph and let T be a spanning tree. Then G is consistent if and only if it satisfies the following two conditions*:

(1) *Each T-fundamental cycle is positive.*
(2) *Each cycle which is the symmetric difference of two T-fundamental cycles is positive.*

Proof. We show that under condition (1), conditions (2) in Theorems 7 and 8 are equivalent. Suppose that condition (2) of Theorem 7 holds. Let C be a symmetric difference of two T-fundamental cycles C_1, C_2, where C_1 has a $-$ signs, C_2 has b $-$ signs, and the common path has c $-$ signs (excluding end vertices). Then C has $a+b-2c$ $-$ signs if end vertices have the $+$ sign or $a+b-2c-2$ $-$ signs otherwise. Since a and b are even, C is positive. The other direction is similar. □

Because of Theorem 6, the following theorem can be thought of as the bipartite version of Theorem 8.

Theorem 9 (Conforti and Rao [5]). *Let G be a bipartite graph and T be a spanning tree of G. All cycles in G have length* $0 \bmod 4$ *if and only if G satisfies the following two conditions*:

(1) *Each T-fundamental cycle has length* $0 \bmod 4$.
(2) *Each cycle which is the symmetric difference of two T-fundamental cycles has length* $0 \bmod 4$.

Theorem 9 leads to an algorithm with complexity $O(m^2n)$ to determine if all cycles in a graph G have length $0 \bmod 4$. In Section 3 we will provide two $O(mn)$ algorithms (Algorithms 2 and 3) to complete the same task.

Theorem 10. *Let G be a marked graph and T a spanning tree of G. The following are equivalent*:

(1) *On the common path of a pair of intersecting T-fundamental cycles, the two end vertices always have the same sign.*
(2) *Two* 3*-connected vertices always have the same sign.*
(3) *Two* 3*-edge-connected vertices always have the same sign.*

The proof of Theorem 10 will use the following two lemmas.

Lemma 1. *Let u and v be a pair of* 3*-connected vertices in graph G and let T be a spanning tree of graph G. Then there exist vertices $v_1, v_2, \ldots, v_r$ such that with $u = v_0$*

and $v = v_{r+1}$, *each pair* v_i *and* v_{i+1} *are the end vertices of the common path of two* T*-fundamental cycles.*

Proof. For any two 3-connected vertices u and v, let $\sigma(u, v)$ be the smallest k so that there are three pairwise vertex-disjoint paths in G from u to v with k tree chords on these paths. We use induction on $\sigma(u, v)$ to prove the lemma. If P^1, P^2 and P^3 are three pairwise vertex-disjoint paths from u to v, at least two of them contain tree chords. If $\sigma(u, v) = 2$, then u, v themselves are end vertices of common paths of two T-fundamental cycles.

Assume the conclusion holds for all 3-connected vertex pairs u, v with $\sigma(u, v) \leqslant k$ and consider a 3-connected vertex pair x, y with $\sigma(x, y) = k + 1$. Let P^1, P^2 and P^3 be three pairwise vertex-disjoint paths from x to y such that the total number of tree chords on these paths is $\sigma(x, y) = k + 1$. There is a unique path P on T joining x and y. Suppose edge xw is on P. We claim xw is on one of P^1, P^2 and P^3. If not, let z be the first vertex on P from x to y and on one of P^1, P^2 or P^3. Assume z is on P^1. We replace P^1_{xz} by P_{xz} to get three paths with fewer tree chords, which contradicts the definition of P^1, P^2 and P^3.

Suppose without loss of generality that xw is on P^2. Then P^1 and P^3 must contain tree chords. Let the first tree chords on P^1 and P^3 from x to y be e' and e'' respectively. The removal of edge xw divides T into two parts and the two tree chords e' and e'' join these two parts. Consider the fundamental cycles with tree chords e' and e'', respectively. One end vertex of their common path is x. Let the other one be u. We claim that u, y are 3-connected and $\sigma(u, y) \leqslant k$, which will complete the proof.

Case 1: u is on P^2. In the fundamental cycle containing e'', suppose the path from u to P^3 hits P^3 at b. Then there are three paths from u to y, i.e., $P^2_{uy}, P^2_{ux}P^1$ and $T_{ub}P^3_{by}$. Since the numbers of tree chords on P^2 and P^2_{uy} are the same, on P^1 and on $P^2_{ux}P^1$ are the same, and on $T_{ub}P^3_{by}$ is less than on P^3, we have $\sigma(u, y) \leqslant k$.

Case 2: u is not on P^2. In the common path on the two fundamental cycles, let b be the last vertex on P^2. In the fundamental cycle containing e', suppose the path from u to P^1 hits P^1 at a, and in the fundamental cycle containing e'', suppose the path from u to P^3 hits P^3 at c. Then $T_{ua}P^1_{ay}$, $T_{ub}P^2_{by}$ and $T_{uc}P^3_{cy}$ are three paths from u to y. By similar reasoning as in Case 1, we have $\sigma(u, y) \leqslant k$. □

Lemma 2. *Let* u *and* v *be a pair of* 3*-edge-connected vertices in graph* G*. Then there exist vertices* $v_1, v_2, \ldots, v_r$ *such that with* $u = v_0$ *and* $v = v_{r+1}$*, each pair* v_i *and* v_{i+1} *is* 3*-connected.*

Proof. If u and v are 3-connected, the conclusion obviously holds. Suppose u and v are not 3-connected and select three edge-disjoint paths P^1, P^2 and P^3 from u to v with minimum total length. Let $X = \{v_1, \ldots, v_r\}$ be the set of vertices other than u and v on at least two of P^1, P^2 and P^3. If two paths, say P^1 and P^2, each contain vertices $v_i \in X$ and $v_j \in X$, and v_i precedes v_j on path P^1 from u to v, then v_i precedes v_j on path P^2 from u to v. For, if v_j precedes v_i on P^2, then $\overline{P^1} = P^1_{uv_i}P^2_{v_iv}, \overline{P^2} = P^2_{uv_j}P^1_{v_jv}$ and P^3 are three edge-disjoint paths with total length less than the original ones.

Let $X' = X \cup \{u, v\}$. By the previous observation, we can arrange the vertices in X' in order according to their appearance on three paths from u to v. Suppose the set in order is $X' = \{v_0, v_1, \ldots, v_r, v_{r+1}\}$. We show that v_i and v_{i+1} are 3-connected. Take $i = 0$. If v_1 is on all three paths, then v_0 and v_1 are 3-connected. If v_1 is on P^1 and P^2 but not on P^3, there is a vertex v_i, $i > 1$, on P^3 and P^1 since v is on P^3 and P^1. Take the smallest i. Then $P^1_{v_0 v_1}, P^2_{v_0 v_1}$ and $P^3_{v_0 v_i} P^1_{v_i v_1}$ are three vertex-disjoint paths from v_0 to v_1. The case $i = k$ is analogous.

Suppose $0 < i < k$. If v_i and v_{i+1} are both on the three paths, they are 3-connected. If one is on the three paths, the proof is as above, so we may assume neither is on the three paths. Now suppose both v_i and v_{i+1} are on P^1 and P^2. Let v_s be on P^1 and P^3 with the biggest $s < i$ and v_t be on P^1 and P^3 with the smallest $t > i + 1$. Then $P^1_{v_i v_{i+1}}, P^2_{v_i v_{i+1}}$ and $P^1_{v_i v_s} P^3_{v_s v_t} P^1_{v_t v_{i+1}}$ are three vertex-disjoint paths from v_i to v_{i+1}. Suppose v_i is on P^1 and P^2 and v_{i+1} is on P^2 and P^3. Let v_s be on P^1 and P^3 with the biggest $s < i$ and v_t be on P^1 and P^3 with the smallest $t > i + 1$. Then $P^2_{v_i v_{i+1}}$, $P^1_{v_i v_s} P^3_{v_s v_{i+1}}$ and $P^1_{v_i v_t} P^3_{v_t v_{i+1}}$ are three vertex-disjoint paths from v_i to v_{i+1}. □

To complete the proof of Theorem 10, note that Lemmas 1 and 2 show that (1) implies (2) and (2) implies (3). Since the common path of a pair of fundamental cycles has end vertices that are 3-edge-connected, (3) implies (1).

By Theorem 10, the following theorem is equivalent to Theorem 7. However, we give a direct proof.

Theorem 11. *Let T be a spanning tree of a marked graph G. G is consistent if and only if it satisfies the following two conditions*:

(1) *Each T-fundamental cycle is positive.*
(2) *Each 3-connected vertex pair has the same sign.*

Proof. The necessity follows from Theorem 1. To prove the sufficiency, we prove that every cycle is positive by induction on the number of non-T edges in the cycle. If there is one such edge, condition (1) gives the result. If there are at least two non-T edges in cycle C and all cycles with fewer non-T edges than C are positive, find vertices u and v in C such that $T_{u,v}$ has no vertices in common with C except u and v. If C_1 and C_2 are the two paths in C from u to v, then $C_1 T_{uv}$ and $C_2 T_{uv}$ form cycles with fewer non-T edges than C, and therefore each is positive. Moreover, C_1, C_2, and T_{uv} show that u and v are 3-connected and therefore have the same sign. We conclude that C is positive. □

To generalize Theorem 11, we ask: can we replace "each T-fundamental cycle" by "each cycle in a basis", i.e., replace tree basis by any basis. The answer is given in Theorem 14.

Let G be a graph. We can get a basis by repeating the following process until we can no longer do so. Pick an edge e. If there is a cycle containing e, put the cycle in the basis. Delete e. A basis which is obtained by this method is called an *ordering basis*. (In [10], it is called a *fundamental basis*.) Note that a tree basis is an ordering basis. We first note that Theorem 11 holds for ordering bases. In the proof, we use the term *block* in a graph to mean a maximal 2-connected subgraph.

Theorem 12. *Let G be a marked graph and $\mathscr{B}$ be an ordering basis of G. G is consistent if and only if G satisfies the following two conditions*:

(1) *Each cycle in $\mathscr{B}$ is positive.*
(2) *Each 3-connected vertex pair has the same sign.*

Proof. The necessity follows from Theorem 1. To prove the sufficiency, we let cycles in $\mathscr{B}$ be $C_n,\dots,C_m$ according to the order they are found in the construction of the ordering basis and arrange edges such that e_i, $i=n,\dots,m$, is the edge we are using to find C_i and e_i, $i=1,\dots,n-1$, are the other edges. For $i \geqslant n$, let G_i be the subgraph induced by edges $e_1,\dots,e_{n-1}$ and all edges in cycles C_j, $j=n,n+1,\dots,i$. Note that $G_m=G$. We use induction on $i \geqslant n$ to prove that G_i is consistent. By condition (1), this holds for $i=n$. Suppose it holds for $i=k-1$. To prove it for $i=k$, we only have to prove that the block of G_k which contains edge e_k is consistent, so we can suppose G_k itself is a block. There is a spanning tree T in G_{k-1} which contains $C_k \setminus e_k$. Since G_{k-1} is consistent, all T-fundamental cycles of G_{k-1} are positive. T is also a spanning tree of G_k and since C_k is positive by condition (1), all T-fundamental cycles of G_k are positive. This plus condition (2) implies that G_k is consistent, by Theorem 11. □

We now generalize the conclusion of Theorems 11 and 12 to any basis. An *even graph* is a graph in which each vertex has even degree. An even graph is an edge-disjoint union of cycles. Let G be a marked graph and G' a subgraph of G. Denote by $d^-(v)$ the number of $-$ vertices adjacent to v in G' and let $\lambda(G')=\Sigma_{v\in V(G')}d^-(v)$. Suppose G' is an even subgraph. Then G' is said to be *coherent* if $\lambda(G') \equiv 0 \bmod 4$ and *incoherent* if $\lambda(G') \equiv 2 \bmod 4$. Notice that $\lambda(G')$ is always even if G' is even. We can also generalize the definition to paths. Let P be a path whose end vertices have the same sign. Then P is said to be *coherent* if $\lambda(P) \equiv 0 \bmod 4$ and *incoherent* if $\lambda(P) \equiv 2 \bmod 4$. Notice that $\lambda(P)$ is even if and only if the end vertices of P have the same sign. This definition of coherent is the same as that in the beginning of this section. The following theorem is easy to prove.

Theorem 13. *A marked graph is consistent if and only if each of its even subgraphs is coherent.*

The following theorem is the desired generalization of Theorem 11.

Theorem 14. *Let G be a marked graph and let $\mathscr{B}$ be a basis of G. Then G is consistent if and only if it satisfies the following two conditions.*

(1) *Each cycle in $\mathscr{B}$ is positive.*
(2) *Each 3-connected vertex pair has the same sign.*

Proof. The necessity of conditions (1) and (2) follows by Theorem 1. The sufficiency follows by Theorem 13 if we can prove that each even subgraph of G is coherent. Since an even subgraph is an edge-disjoint union of cycles, each even subgraph G' is the symmetric difference of cycles in $\mathscr{B}$. We prove the claim by induction on the number k of these cycles. If $k=1$, G' is just a cycle in $\mathscr{B}$ and is positive by condition (1) and therefore coherent. Arguing by induction on k, we suppose that each even subgraph which is the symmetric difference of at most $k-1$ cycles in $\mathscr{B}$ is coherent, let G' be an even subgraph which is the symmetric difference of k cycles in $\mathscr{B}$, let C be one of these cycles, and let G'' be the symmetric difference of the remaining $k-1$ cycles. Clearly, G'' is again even and hence is coherent by the induction hypothesis.

If $E(C)\cap E(G'')=\emptyset$, the conclusion holds. So we suppose $E(C)\cap E(G'')\neq\emptyset$. First assume that G'' is connected. Let $P=v_1,\dots,v_k$ be a path of C such that $v_1,v_k\in V(G'')$ and $v_2,\dots,v_{k-1}\notin V(G'')$. If v_1 and v_k are in the same block of G'', then v_1 and v_k are 2-connected in the block and 3-connected in $G''\cup P$. Therefore v_1 and v_k receive the same sign and $\lambda(P)$ is even.

Suppose v_1 and v_k are not in the same block of G''. Pick a path Q from v_1 to v_k in G'' and let $u_1,\dots,u_s$ be the cut vertices of G'' in order on Q. Then v_1 and u_1,u_i and u_{i+1}, $i=1,\dots,s-1,u_s$ and v_k are pairwise 2-connected in G''. Vertices v_1 and u_1 are 3-connected in G because we can find a third path vertex-disjoint from the first two by using P followed by $Q_{v_ku_1}$. The other cases are proved in the same way using P followed by $Q_{v_ku_{i+1}}$ and P followed by $Q_{v_ku_s}$. Therefore, v_1 and v_k receive the same sign and $\lambda(P)$ is even.

C is the union of paths, which we can partition into two subsets, $C\cap G''$ and $C\setminus G''$. By the above conclusion, $\lambda(C\setminus G'')$ is even. Since $\lambda(C)\equiv 0 \bmod 4$, $\lambda(C\cap G'')\equiv\lambda(C\setminus G'') \bmod 4$. G' is obtained from G'' by replacing $C\cap G''$ by $C\setminus G''$. Therefore $\lambda(G')\equiv\lambda(G'') \bmod 4$, i.e., G' is coherent.

Now assume that G'' is disconnected. If $E(C)$ intersects only one of the components of G'', the proof is the same as for the case where G'' is connected. So suppose that there are at least two components in G'', G_1 and G_2, such that $E(C)\cap E(G_i)\neq\emptyset$ for $i=1,2$. Let $C=v_0,v_1,\dots,v_k,v_{k+1},\dots$. Suppose that $P=v_1,\dots,v_k$ such that v_1 and v_k are in the same component and v_0 and v_{k+1} are not in this component. Then v_1 and v_k are 3-connected in G' and therefore in G, and receive the same sign by condition (2). We can partition C into two subgraphs F_1 and F_2, where F_2 is a union of paths whose end vertices are in different components of G'' but whose middle vertices are not in G''. By the above discussion, $\lambda(F_1)$ is even, and so therefore is $\lambda(F_2)$.

By the proof of the case when G'' is connected, $\lambda(F_1\setminus G'')$ is even. Hence, so is $\lambda(C\setminus G'')$, since $\lambda(F_2)$ is even and $F_2\cap G''=\emptyset$, so

$$\lambda(C\setminus G'')=\lambda((F_1\cup F_2)\setminus G'')=\lambda(F_1\setminus G''\cup F_2).$$

Moreover, $\lambda(C\cap G'')=\lambda(F_1\cap G'')$ is also even since $\lambda(F_1)$ and $\lambda(F_1\setminus G'')$ are even. Thus, since $\lambda(C)\equiv 0 \bmod 4$, it follows that $\lambda(C\cap G'')\equiv\lambda(C\setminus G'') \bmod 4$, as before, and then, as before, $\lambda(G')\equiv\lambda(G'') \bmod 4$, i.e., G' is coherent. □

3. Algorithms

The above theorems provide an algorithm to test the consistency of a marked graph. By Theorems 10 and 14, it suffices to check if there is a *positive basis*, a basis with all cycles positive, and each 3-edge-connected vertex pair has the same sign. Checking the former can be done by constructing any basis; either all cycles in it are positive or there is a negative cycle in it and G is inconsistent. Checking the latter can be limited to vertices of degree at least 3 because, obviously, a vertex of degree at most 2 cannot be 3-edge connected to any other vertex. Let $p(u,v)$ be the largest number of pairwise edge-disjoint paths between u and v. The key observation we shall use is the following.

Theorem 15 (Even [6]). *Given a graph G of m edges, checking whether or not $p(u,v) \geqslant k$ can be accomplished in $O(km)$ time.*

Here is a sketch of the proof of Theorem 15. Given G, define a directed network N with the same vertex set as G, by taking arcs from x to y and y to x whenever xy is an edge of G, and by putting a capacity of 1 on each arc. Then $p(u,v)$ is the maximum flow from u to v in N. (See the proof of Theorems 6.4, 6.8 in [6].) Thus, we can test whether or not $p(u,v) \geqslant k$ by running a network flow algorithm until the flow reaches k or cannot be continued. A standard network flow algorithm ([13,14]) proceeds by searching for augmenting chains. Each augmenting chain increases the total flow by at least 1, so we need at most k searches for augmenting chains to test if $p(u,v) \geqslant k$. Each such search requires a number of steps on the order of the number of arcs in the network, so it is $O(m)$. We conclude that checking whether or not $p(u,v) \geqslant k$ can be accomplished in $O(km)$ time (see [6, p. 132]). We shall apply this result for the case $k = 3$, so we have an $O(m)$ algorithm.

We now get the following algorithm.

Algorithm 1

Input: A 2-connected marked graph G.

Output: The answer to the question: Is G consistent?

Step 1: *Construct a basis of G and check if it is positive. If not, G is not consistent.*

Step 2: *Let V_0 be the set of vertices of degree at least* 3. *Check if any two vertices of V_0 with different signs are* 3-*edge-connected. Do this as follows*:

Step 2a: *Pick vertices $u \in V_+ \cap V_0$ and $v \in V_- \cap V_0$. Use Even's algorithm to check if $p(u,v) \geqslant 3$. If so, then by Theorem* 14, *G is not consistent.*

Step 2b: *If $p(u,v) < 3$, then there is an edge cut in the graph consisting of two edges, and this divides the graph into two subgraphs. Repeat Step* 2a *for each of these subgraphs while there are u, v of different signs in V_0.*

Step 2c: *If inconsistency has not been concluded for any subgraph, conclude that G is consistent.*

The complexity of Algorithm 1 is $O(mn)$. To see why, note that we can build a tree basis by finding a spanning tree T using an $O(m)$ algorithm such as Kruskal's and then we can check if each tree chord produces a positive cycle. There are $O(m)$ tree chords. For each, since there are only $n-1$ edges in T, checking that the corresponding T-fundamental cycle is positive requires $O(n)$ steps. So, the overall complexity of Step 1 is $O(mn)$. (Instead, we can use ordering basis. Pick an edge xy. Use an $O(n)$ shortest path algorithm to find a path between x and y. If there is no such path, delete xy and continue. If there is a path, this plus xy forms a cycle of length at most n and we can check whether or not it is positive in $O(n)$ steps. Since we must repeat the procedure for each edge, the total complexity of the procedure is $O(mn)$.) Step 2a uses an $O(m)$ algorithm. It is easy to see by induction on n that the number of subgraphs for which Step 2a must be repeated in Step 2b is $O(n)$. Therefore the overall complexity of Algorithm 1 is $O(mn)$, as claimed.

As noted in Theorem 6, the problem of determining if a marked graph is consistent and the problem of determining if a bipartite graph has the property that every cycle has length $0 \bmod 4$ are polynomially equivalent. In the following we give two algorithms to test if a bipartite graph has the latter property. Let G be a 2-connected graph. A *starjoin path* or sj-*path* for short is a path whose inner vertices have degree 2 and whose end vertices have degree at least 3. Let G be a 2-connected graph and suppose that $\{uv, xy\}$ is an edge cut whose edges are not on the same sj-path. We contract u and v into one vertex and insert a vertex in edge xy to get a graph G'. This operation is called *sliding* (relative to the first edge).

Lemma 3. *If G' is obtained from graph G by sliding, then every cycle of G has length $0 \bmod 4$ iff every cycle of G' has length $0 \bmod 4$ and, moreover, G is bipartite iff G' is bipartite.*

Proof. There is a one-to-one, length-preserving correspondence between cycles of G and cycles of G'. □

Lemma 4. *In any graph, 3-edge-connectedness is a transitive relation.*

Proof. Let G be a graph and let u and v be 3-edge-connected and v and w be 3-edge-connected. If u and w are not 3-edge-connected, there is a minimal cut E_0 separating u and w with at most two edges. Suppose v is in the same part with u in $G \setminus E_0$. Then the cut is also a cut separating v and w, a contradiction. □

It is easy to see that in a 2-connected graph, a vertex v is 3-edge-connected to some vertex if and only if the degree of v is at least 3. Hence, if we define $u \sim v$ to mean that $u = v$ or u and v are 3-edge connected, then Lemma 4 implies that $\sim$ defines an equivalence relation on V_0, the set of vertices of degree at least 3. We use the notation $r(G)$ for the number of equivalence classes.

Algorithm 2

Input: A 2-connected bipartite graph G, which is not a cycle, with bipartition $V(G) = V' \cup V''$.

Output: The answer to the question: Does every cycle have length $0 \bmod 4$?

Step 0: *Set* $k \leftarrow 0$, $V'_k \leftarrow V'$, $V''_k \leftarrow V''$, *and* $G^{(k)} \leftarrow G$.

Step 1: *Let* $V_0^{(k)}$ *be the set of vertices of* $G^{(k)}$ *with degree at least* 3. *Do the following for graph* $G^{(k)}$.

Step 1a: *Check if one of the vertex classes in the bipartition of* $G^{(k)}$ *has no vertices of* $V_0^{(k)}$. *If so, let* $G* = G^{(k)}$ *and* $V_0^* = V_0^{(k)}$ *and go to Step* 3.

Step 1b: *Pick* $u \in V_0^{(k)} \cap V_k'$, $v \in V_0 \cap V_k''$, *and check if u and v are* 3-*edge-connected using Even's procedure. If they are, then because u and v are in different partite classes, we get the conclusion that some cycle of G does not have length* 0 mod 4.

Step 1c: *Otherwise we find an edge cut* $\{e_1, e_2\}$ *with cardinality* 2 *separating u and v. Then edges* e_1 *and* e_2 *are on* sj-*paths* P_1 *and* P_2 *respectively. Say* P_1 *goes from vertex x to vertex y. By repeatedly sliding using an edge of* P_1 *and edge* e_2, *we eventually merge x and y and obtain graph* G'. (*Note that in* G', *the vertex y is no longer present and it is replaced by a vertex of degree* 2 *in* P_2. *The equivalence classes under* $\sim$ *remain unchanged except that the equivalence classes containing x and y in* $G^{(k)}$ *are merged to form one equivalence class, omitting y. Thus,* $r(G') = r(G^{(k)}) - 1$.) *By Lemma* 3, G' *is bipartite with bipartite classes* U_1' *and* U_2'. *Set* $k \leftarrow k+1$, $G^{(k)} \leftarrow G'$, $V_k' \leftarrow U_1'$, *and* $V_k'' \leftarrow U_2$.

Step 2: *Repeat Step* 1 (*with the new* $G^{(k)}, V_k', V_k''$) *until* $G^{(k)}$ *and* $V_0^{(k)}$ *are such that in the bipartition of* $G^{(k)}$, *one of the two classes has no vertices of* $V_0^{(k)}$. (*This will eventually happen since it happens when we get to* $G^{(k)}$ *with* $r(G^{(k)}) = 1$.) *Let* $G* = G^{(k)}$ *and* $V_0^* = V_0^{(k)}$.

Step 3: *By construction,* $G*$ *is bipartite and one of the two classes in the bipartition has no vertices of* V_0^*. *Find a spanning tree T of* $G*$ *and check if each T-fundamental cycle has length* 0 mod 4. *If there is a T-fundamental cycle with length not* 0 mod 4, *output the answer* "*no*". *Otherwise, output the answer* "*yes*".

To see that Algorithm 2 is correct, say that after Steps 1 and 2, V^1 and V^2 are the two classes in the bipartition of $G*$ and one of these, say V^2, has no vertices of degree $\geqslant 3$. Suppose that in Step 3, we conclude that all T-fundamental cycles have length 0 mod 4. Any sj-path must go between vertices in V^1 and therefore must have even length. Any common path of two T-fundamental cycles is a path on T and joins two vertices in V_0^* and all its middle vertices have degree 2. Thus, the path is an sj-path and so has even length. This together with the conclusion that all fundamental cycles have length 0 mod 4 implies that any cycles which are a symmetric difference of fundamental cycles have length 0 mod 4. By Theorem 9, all cycles have length 0 mod 4.

The complexity of this algorithm can be calculated as follows. Step 1, using Even's method, requires $O(m)$ steps. Step 2 must be carried out at most $O(n)$ times. Step 3 has complexity $O(mn)$, as in Algorithm 1. Thus, the overall complexity of the algorithm is $O(mn)$.

There is a variant on Algorithm 2, which we call **Algorithm 3**, that is the same except that we replace Step 3 by

Step 3′: *By construction,* $G*$ *is bipartite and one of the two classes in the bipartition has no vertices of* V_0^*. *Check if we can partition* V_0^* *into two sets* V_1 *and* V_2 *such* *that any* sj-*path joining one vertex in* V_1 *and one in* V_2 *has length* $2 \bmod 4$ *and any* sj-*path joining two vertices both in* V_1 *or both in* V_2 *has length* $0 \bmod 4$. *If so, output the answer* "*yes*". *If no*, *output the answer* "*no*".

Let T be a spanning tree of $G*$. We show that after Steps 0, 1, and 2, Step 3 of Algorithm 2 has a positive answer iff Step 3′ of Algorithm 3 has a positive answer. This will show that Algorithm 3 is correct. Suppose that Step 3 has a positive answer. We divide V_0^* into two parts V_1 and V_2 as follows. Pick a vertex $v \in V_0^*$ and put it into V_1. For any other vertex $u \in V_0^*$, if T_{uv} has length $0 \bmod 4$, we put it into V_1, otherwise into V_2. We show that this partition has the property stated in Step 3′. It is clear that if u and w are in V_0^*, then T_{uw} has length $0 \bmod 4$ if u and w are in the same class V_i and length $2 \bmod 4$ otherwise. Let P be an sj-path from u to w. If P has no tree chords, then $P = T_{uw}$ and so has the desired property. If P has a tree chord, then P plus T_{uw} form a fundamental cycle. We know that P has even length since it joins two vertices of the same class in the bipartition. Since all T-fundamental cycles have length $0 \bmod 4$ and T_{uw} has the property required in Step 3′, it follows that P does as well.

Conversely, suppose that Step 3′ has a positive answer and V_1, V_2 are the sets in the partition. Since G is not a cycle, $G*$ is also not a cycle. Thus, any T-fundamental cycle of $G*$ has at least two vertices of V_0^* and we can look at the fundamental cycle as consisting of a sequence of sj-paths. Each sj-path that has endpoints in the same set V_i has length $0 \bmod 4$ and each sj-path that has endpoints in different sets has length $2 \bmod 4$. Since following around the cycle returns us to the starting set V_i, there are an even number of sj-paths that have endpoints in different sets. We conclude that the cycle has length $0 \bmod 4$.

We note that the complexity of Algorithm 3 is $O(mn)$. By the discussion of the complexity of Algorithm 2, it suffices to check that Step 3′ can be completed in $O(mn)$ time. Once we reach Step 3′, one of the classes in the bipartition V^1, V^2 of $G*$, say V^2, has no vertices of degree $\geqslant 3$. Let $H*$ be obtained from $G*$ by replacing each path x, y, z with y in V^2 by an edge xz. Then V_0^* is contained in the vertex set of $H*$ and an sj-path P in $G*$ corresponds to an sj-path Q in $H*$ of half the length. We can partition the vertices of V_0^* in $G*$ to satisfy the condition in step 3′ if and only if we can partition the vertices of V_0^* in $H*$ so that sj-paths joining vertices in different classes have length $1 \bmod 2$ and sj-paths joining vertices in the same class have length $0 \bmod 2$. This is essentially the question of whether or not $H*$ is bipartite, and bipartiteness can be checked by any of a number of well-known algorithms in $O(mn)$ time.

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