Solving coupled systems of linear second-order differential equations knowing a part of the spectrum of the companion matrix

Lucas Jódar and Enrique Navarro
Departamento de Matemática Aplicada, Universidad Politécnica de Valencia, P.O. Box 22.012, Valencia, Spain

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Abstract


In this paper an explicit closed-form solution of initial-value problems for coupled systems of time-invariant second-order differential equations is given without computing the exponential of the associated companion matrix.

Keywords: Coupled differential system, companion matrix, co-solution, algebraic matrix equation, explicit solution, Moore–Penrose pseudo-inverse.

1. Introduction

The behaviour of many physical systems in engineering can be modeled by the following systems of equations:

\[
x''(t) + A_1 x'(t) + A_0 x(t) = f(t),
\]

where \( A_0, A_1 \) are \( n \times n \) complex matrices, elements of \( \mathbb{C}^{n \times n} \), and \( x(t), f(t) \) are vectors in \( \mathbb{C}^n \). The model (1.1) can describe electrical, mechanical, thermal and other problems [3].

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It is well known [4, 7] that the solution of problems related to the system (1.1) can be treated considering the equivalent first-order extended system

\[ \begin{align*}
  y'(t) &= Cy(t) + g(t), \\
  y(t) &= \begin{bmatrix} x \\ x' \end{bmatrix},
\end{align*} \]

(1.2)

where \( g(t) = \int f(t) \) and 

\[ C = \begin{bmatrix}
  0 & I \\
  -A_0 & -A_1
\end{bmatrix}. \]

The relationship between the solutions \( x(t) \) of (1.1) and \( y(t) \) of (1.2) is

\[ \begin{align*}
  x(t) &= [I, 0] y(t), \\
  y(t) &= \exp(tC)x(0) + \int_0^t \exp((t-s)C)g(s) \, ds.
\end{align*} \]

(1.3)

This approach involves an increase of the computational cost because of the increase of the problem dimension and a lack of information derived from the reduction of the order of the original system.

In order to avoid the inconveniences of the consideration of the extended system (1.2), some work has been done using solutions and co-solutions of the associated algebraic matrix equation

\[ 2' + A_2 + A_0 = 0; \]  \hspace{1cm} (1.4)

see [4–6] for details.

The aim of this paper is to find explicit solutions of initial-value problems for the coupled system (1.1), by using only the knowledge of a part of the spectrum \( \sigma(C) \) of the companion matrix \( C \) defined in (1.2). Recall that the computation of the matrix exponential is not an easy task [9].

2. Initial-value problems

For the sake of clarity in the presentation we introduce the concept of rectangular co-solution of (1.4), recently given in [6].

**Definition 1.** We say that \((X, T)\) is an \((n, m)\) co-solution of (1.4) if \( X \in \mathbb{C}^{n \times m}, X \neq 0, \) \( T \in \mathbb{C}^{m \times m} \) and

\[ XT^2 + A_1XT + A_0X = 0. \]  \hspace{1cm} (2.1)

**Definition 2** (Jódar and Navarro [6]). Let \((X_i, T_i)\) be an \((n, m_i)\) co-solution of (1.4), \( 1 \leq i \leq k, \) with \( m_1 + m_2 + \cdots + m_k = 2n. \) We say that \( \{(X_i, T_i); 1 \leq i \leq k\} \) is a \( k\)-complete set of co-solutions of (1.4), if the block matrix

\[ W = (W_{ij}) = (X_jT_j^{-1}), \quad 1 \leq i \leq 2, \quad 1 \leq j \leq k, \]

is invertible in \( \mathbb{C}^{2n \times 2n}. \)

In [6] it is proved that any equation of the type (1.4) admits a \( k\)-complete set of co-solutions, and note that if \( \{(X_i, T_i); 1 \leq i \leq k\} \) is a \( k\)-complete set of co-solutions, then the rank of the matrix \( [X_1, X_2, \ldots, X_k] \) is equal to \( n. \) From [6], to obtain a \( k\)-complete set of co-solutions we need the knowledge of the set \( \sigma(C) \) of all eigenvalues of the companion matrix \( C. \) Now we introduce a new concept that permits us to solve problems related to (1.1) knowing only a part of \( \sigma(C). \)
Definition 3. Let $\Omega = \{X_i, T_i\}; 1 \leq i \leq h$ be a set of rectangular co-solutions of (1.4). We define the rank of the set $\Omega$ as the rank of the matrix $[X_1, X_2, \ldots, X_h]$. We say that the set $\Omega$ is a minimal set of co-solutions of (1.4), if its rank is $n$ and any proper subset of co-solutions has a rank strictly smaller than $n$.

Let $(X_i, T_i)$ be an $(n, m_i)$ co-solution of (1.4) such that $\Omega = \{(X_i, T_i); 1 \leq i \leq r\}$ is a minimal set of co-solutions, and let us consider the matrices

$$X_0 = [X_1, X_2, \ldots, X_r], \quad T_0 = \text{diag}(T_1, T_2, \ldots, T_r), \quad (2.2)$$

and let

$$m = m_1 + m_2 + \cdots + m_r \geq n. \quad (2.3)$$

An easy computation shows that for any vector $p$ in $\mathbb{C}^m$, the vector function

$$x_1(t) = X_0 \exp(tT_0)p$$

is a solution of

$$x''(t) + A_1x'(t) + A_0x(t) = 0. \quad (2.4)$$

Let us look for solutions of (1.1) of the form

$$x_2(t) = X_0 \exp(tT_0)v(t), \quad (2.5)$$

where $v(t)$ is a differentiable $\mathbb{C}^n$-valued function to be determined. From (2.5) it follows that $x_2(t)$ satisfies (1.1) if $v(t)$ verifies

$$X_0 \exp(tT_0)v'(t) + (A_1X_0 + 2X_0T_0)\exp(tT_0)v'(t) = f(t). \quad (2.6)$$

If we consider the change defined by

$$u(t) = \exp(tT_0)v'(t), \quad (2.7)$$

then $v(t)$ satisfies (2.6) if and only if $u(t)$ solves the equation

$$X_0u'(t) + (A_1X_0 + X_0T_0)u(t) = f(t). \quad (2.8)$$

If we denote by $X_0^+$ the Moore-Penrose pseudo-inverse of $X_0$, since the rank of the matrix $X_0$ is $n$, from [1, Theorem 3.2.6], a set of solutions of (2.8) is given by

$$u(t) = X_0^+ \exp(-t(A_1 + X_0T_0X_0^+))\left\{q + \int_0^t\exp(s(A_1 + X_0T_0X_0^+))f(s)\, ds\right\}, \quad q \in \mathbb{C}^n. \quad (2.9)$$

From (2.7) and (2.9) by integration it follows that

$$v(t) = \int_0^t\exp(-sT_0)X_0^+\exp(-s(A_1 + X_0T_0X_0^+))s) \times \left\{q + \int_0^s\exp(u(A_1 + X_0T_0X_0^+))f(u)\, du\right\}\, ds, \quad (2.10)$$

and from (2.5) and (2.10) it follows that

$$x_2(t) = X_0\int_0^t\exp(T_0(t - s))X_0^+\exp(-s(A_1 + X_0T_0X_0^+)) \times \left\{q + \int_0^s\exp(u(A_1 + X_0T_0X_0^+))f(u)\, du\right\}\, ds. \quad (2.11)$$
Thus a set of solutions of (1.1) is given by

\[ x(t) = x_1(t) + x_2(t) \]

\[ = X_0 \exp(tT_0)p + X_0 \int_0^t \exp(T_0(t-s))X_0^+ \exp(-s(A_1 + X_0T_0X_0^+)) \]

\[ \times \left\{ q + \int_0^s \exp(u(A_1 + X_0T_0X_0^+))f(u) \, du \right\} \, ds, \]  

(2.12)

where \( p, q \) are arbitrary vectors in \( \mathbb{C}^n \).

Now we prove that any solution of (1.1) may be obtained from (2.12) for appropriate vectors \( p, q \) in \( \mathbb{C}^n \). In fact, let \( z(t) \) be a solution of (1.1) with \( z(0) = c_0, \, z'(0) = c_1 \).

Note that from (2.12) it follows that

\[ x'(t) = X_0T_0 \exp(tT_0)(p + v(t)) + X_0 \exp(tT_0)v'(t). \]  

(2.13)

Hence and from (2.12), if we impose the conditions \( x(0) = c_0, \, x'(0) = c_1 \), it follows that vectors \( p, q \) must verify

\[ c_0 = x(0) = X_0p, \quad c_1 = x'(0) = X_0T_0p + q. \]  

(2.14)

Since the rank of \( X_0 \) is \( n \), from [2, Theorem 1.2.2], a solution of (2.14) is given by

\[ p = X_0^+, \quad q = c_1 - X_0T_0X_0^+ c_0. \]  

(2.15)

Taking \( x(t) \) defined by (2.12) with vectors \( p, q \) defined by (2.15), one gets a solution of (1.1) which satisfies the same initial conditions as \( z(t) \) and from uniqueness \( x(t) \) coincides with \( z(t) \).

In consequence, (2.12) represents the general solution of (1.1) when \( f(t) \) is continuous.

**Theorem 4.** Let \( \Omega = \{X_i, \, T_i\}; \, 1 \leq i \leq r \) be a minimal set of co-solutions of (1.4) and let \( X_0, \, T_0 \) be defined by (2.2). Then the general solution of the coupled differential system (1.1), where \( f(t) \) is continuous, is given by (2.12) for arbitrary vectors \( p, q \) in \( \mathbb{C}^n \). The unique solution of (1.1) satisfying the initial conditions \( x(0) = c_0, \, x'(0) = c_1 \), is given by (2.12) with vectors \( p, q \) defined by (2.15).

**Remark 5.** We recall that effective computation of the Moore–Penrose pseudo-inverse \( X_0^+ \) is available with MATLAB [8]. From [6], a \( k \)-complete set of co-solutions is always available and any \( k \)-complete set of co-solutions contains a subset of minimal co-solutions. The next example shows that in some cases a proper subset of a \( k \)-complete set of co-solutions is sufficient and thus we do not need all the spectral information of the companion matrix.

**Example 6.** Let us consider the coupled system

\[ x''(t) - \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} x'(t) + \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} x(t) = 2 \begin{bmatrix} \exp(-t) \\ \exp(-t) \end{bmatrix}. \]

The companion matrix takes the form

\[ C = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -1 & 1 & 1 \\ -1 & 0 & 1 & 1 \end{bmatrix}. \]
Easy computations show that $\lambda = 1$ is an eigenvalue of $C$ and 
\[
\dim \ker(C - I) = 2, \quad \dim \ker(C - I)^2 = \dim \ker(C - I)^3 = 3.
\]
The Jordan blocks associated to the eigenvalue $\lambda = 1$ are
\[
T_1 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \quad T_2 = (1).
\]
If we denote by
\[
X_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \quad X_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix},
\]
then $\Omega = \{(X_1, T_1), (X_2, T_2)\}$ is a minimal set of co-solutions of the equation
\[
Z^2 - \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}Z + \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = 0.
\]
Thus
\[
X_0 = [X_1, X_2] = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}, \quad T_0 = \begin{bmatrix} T_1 & 0 \\ 0 & T_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.
\]
From Theorem 4, the general solution of the differential system is given by
\[
x(t) = \begin{bmatrix} \exp(t) & t \exp(t) & \exp(t) \\ \exp(t) & t \exp(t) & 0 \end{bmatrix} p + \frac{1}{2} \begin{bmatrix} t \exp(t) + \sinh t & t \exp(t) - \sinh t \\ t \exp(t) - \sinh t & t \exp(t) + \sinh t \end{bmatrix} q
\]
\[+ (t \exp(t) - \sinh t) \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad p, q \in \mathbb{C}^2.
\]

References