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Note

Orthogonal polynomials arising from the wreath products of a dihedral group with a symmetric group

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Abstract

Some classes of orthogonal polynomials are discussed in this paper which are expressed in terms of $(n+1, m+1)$-hypergeometric functions. The orthogonality comes from that of zonal spherical functions of certain Gelfand pairs.

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1. Introduction

We discuss some discrete orthogonal polynomials arising from Gelfand pairs [3,4] of wreath products.

In the previous paper [4] one of the authors has shown that the Gelfand pairs of complex reflection group, $G(r,1,n) \cong \mathbb{Z}/r\mathbb{Z} \wr S_n$, and symmetric group, S_n, gives

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orthogonal polynomials which are expressed by the $(r, 2r)$-hypergeometric functions [1,5].

Theorem 1.1 (Mizukawa [4, Theorem 4.6]). *The zonal spherical functions of the Gelfand pair* $(G(r,1,n), S_n)$ *have* $(r, 2r)$*-hypergeometric expressions*

$$\omega^{(k_0,k_1,\dots,k_{r-1})}_{(\ell_0,\ell_1,\dots,\ell_{r-1})} = F((-\ell_1,\dots,-\ell_{r-1}),(-k_1,\dots,-k_{r-1});-n;\tilde{\Xi}_r).$$

Here $\tilde{\Xi}_r = (1-\xi^{ij})_{1\leqslant i,j\leqslant r-1}$ *with* $\xi = \exp(2\pi\sqrt{-1}/r)$ *and* ℓ_i, k_i *are non-negative integers such that* $\sum_{i=0}^{r-1} k_i = \sum_{i=0}^{r-1} \ell_i = n$.

We denote the shifted factorial of an indeterminate x by

$$(x)_m = x(x+1)(x+2)\cdots(x+m-1)$$

for $m \in \mathbb{Z}_{>0}$ and

$$(x)_0 = 1.$$

In the theorem above $F((-\ell_1,\dots,-\ell_{r-1}),(-k_1,\dots,-k_{r-1});-n;\tilde{\Xi}_r)$ is called the $(r, 2r)$- hypergeometric function which is, by definition,

$$\sum_{\substack{\sum_{i,j} a_{ij}\leqslant n\\ (a_{ij})\in M_{r-1,r-1}(\mathbb{N}_0)}} \frac{\prod_{i=1}^{r-1}(-\ell_i)_{\sum_{j=1}^{r-1}a_{ij}} \prod_{j=1}^{r-1}(-k_j)_{\sum_{i=1}^{r-1}a_{ij}}}{(-n)_{\sum_{i,j}a_{ij}}} \frac{\prod(1-\xi^{ij})^{a_{ij}}}{\prod a_{ij}!},$$

where $M_{n,m}(\mathbb{N}_0)$ is a set of $n\times m$-matrix with non-negative integer elements. Furthermore, their orthogonalities are described as follows.

Theorem 1.2 (Mizukawa [4, Corollary 4.7]). *If* $k = (k_0,\dots,k_{r-1})$, $k' = (k'_0,\dots,k'_{r-1})$ *and* $\ell = (\ell_0,\dots,\ell_{r-1})$ *satisfies* $\sum_{i=0}^{r-1} k_i = \sum_{i=0}^{r-1} k'_i = \sum_{i=0}^{r-1} \ell_i = n$, *then we have*

$$\frac{1}{r^n}\sum_{\ell_0+\cdots+\ell_{r-1}=n} \binom{n}{\ell_0,\dots,\ell_{r-1}} F(-\tilde{\ell},-\tilde{k};-n;\tilde{\Xi}_r)\overline{F(-\tilde{\ell},-\tilde{k}';-n;\tilde{\Xi}_r)} = \binom{n}{k_0,\dots,k_{r-1}}^{-1}\delta_{kk'}.$$

Here we put $\tilde{\ell} = (\ell_1,\dots,\ell_{r-1})$ *for* $\ell = (\ell_0,\ell_1,\dots,\ell_{r-1})$.

We see that these orthogonal polynomials are defined on the unit circle in the complex plane. We take great interest in other orthogonal polynomials obtained from the $(r, 2r)$-hypergeometric functions. Our purpose of this paper is to obtain the orthogonal polynomials which are defined on a real interval, and expressed in terms of the $(r, 2r)$-hypergeometric functions. For this purpose we consider the wreath products of a dihedral group with a symmetric group.

2. Main results

Now if $-N$ is a negative integer, then we define the finite series called the $(n+1, m+1)$-*hypergeometric functions* [1,5];

$$F(\alpha, \beta; -N, X) = \sum_{\substack{\sum_{i,j} a_{ij} \leqslant N \\ (a_{ij}) \in M_{n,m-n-1}(\mathbb{N}_0)}} \frac{\prod_{i=1}^{n} (\alpha_i)_{\sum_{j=1}^{m-n-1} a_{ij}} \prod_{j=1}^{m-n-1} (\beta_j)_{\sum_{i=1}^{n} a_{ij}}}{(-N)_{\sum_{i,j} a_{ij}}} \frac{\prod x_{ij}^{a_{ij}}}{\prod a_{ij}!}$$

for $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{C}^n$, $\beta = (\beta_1, \ldots, \beta_{m-n-1}) \in \mathbb{C}^{m-n-1}$. Our purpose of this paper is to obtain the following orthogonality relations.

Theorem 2.1. *For a positive integer* $m = [r/2]$, *we assume that* $k = (k_0, \ldots, k_m)$, $k' = (k'_0, \ldots, k'_m)$ *and* $\ell = (\ell_0, \ldots, \ell_m)$ *are elements of* $\mathbb{N}_0^{m+1}$ *such that* $k_0 + \cdots + k_m = k'_0 + \cdots + k'_m = \ell_0 + \cdots + \ell_m = n$. *We put* $\tilde{\ell} = (\ell_1, \ldots, \ell_m)$ *for* $\ell = (\ell_0, \ell_1, \ldots, \ell_m)$ *and* $\tilde{\Theta}_r = (1 - \cos(2\pi ij/r))_{1 \leqslant i,j \leqslant m}$. *Then we have*

(1) *If r is an odd positive integer,*

$$\frac{1}{r^n} \sum_{\ell_0 + \cdots + \ell_m = n} 2^{n-\ell_0} \binom{n}{\ell_0, \ldots, \ell_m} F(-\tilde{\ell}, -\tilde{k}; -n; \tilde{\Theta}_r) F(-\tilde{\ell}, -\tilde{k}'; -n; \tilde{\Theta}_r) = 2^{-n+k_0} \binom{n}{k_0, \ldots, k_m}^{-1} \delta_{kk'}.$$

(2) *If r is an even positive integer,*

$$\frac{1}{r^n} \sum_{\ell_0 + \cdots + \ell_m = n} 2^{n-\ell_0-\ell_m} \binom{n}{\ell_0, \ldots, \ell_m} F(-\tilde{\ell}, -\tilde{k}; -n; \tilde{\Theta}_r) F(-\tilde{\ell}, -\tilde{k}'; -n; \tilde{\Theta}_r) = 2^{-n+k_0+k_m} \binom{n}{k_0, \ldots, k_m}^{-1} \delta_{kk'}.$$

Remark 2.2. The single variable versions of Theorem 1.2, namely $r = 2$ and $r = 3$, are the $p = 1/2$ and $2/3$ orthogonality statements for the Krawtchouk polynomials and that Dunkl [2] had given a spherical function interpretation for these cases.

Actually these relations are obtained from the orthogonality of the zonal spherical functions of the Gelfand pair of finite groups $(D(r,n), D(1,n))$. In the rest of the paper we prove Theorem 2.1 by computing the zonal spherical functions of $(D(r,n), D(1,n))$.

3. Zonal spherical functions of $(D(r,n), D(1,n))$

Fix $r \in \mathbb{Z}_+$ and $n \in \mathbb{N}_0 = \{0,1,2,\ldots\}$. Put $\xi = \exp(2\pi\sqrt{-1}/r)$. Let

$$D_r = \langle a, b; a^2 = b^r = (ab)^2 = 1 \rangle$$

be the dihedral group of order $2r$ and S_n be the symmetric group. We denote by $G = D(r,n) = D_r \wr S_n$. We define the subgroup H of G by

$$H = \langle a \rangle \wr S_n \cong D(1,n).$$

We consider the pair of groups (G,H). Let E_n be the $n \times n$ identity matrix. We can represent G as a subgroup of GL_{2n} by taking generators

$$A = \begin{pmatrix} 0 & 1 & \\ 1 & 0 & \\ & & E_{2n-2} \end{pmatrix}, \quad B = \begin{pmatrix} \xi & & \\ & \xi^{-1} & \\ & & E_{2n-2} \end{pmatrix}$$

and

$$T_i = \begin{pmatrix} E_{2i-2} & & & \\ & 0 & E_2 & \\ & E_2 & 0 & \\ & & & E_{2n-2i-2} \end{pmatrix} \quad (1 \leqslant i \leqslant n-1).$$

We remark that $D(1,n) \cong W(B_n)$, where $W(B_n)$ is the Weyl group of type B and that $D(2,n) \cong V_4 \wr S_n$, where V_4 denotes by Kleinsche Vierergruppe. We define another subgroup K of G by

$$K = \langle b \rangle \wr S_n \cong G(r,1,n),$$

where $G(r,1,n)$ is the imprimitive complex reflection group.

Proposition 3.1. (1) *The representatives of double coset $H \backslash G / H$ are given by*

$$\left\{ \left(\underbrace{1,\ldots,1}_{\ell_0}, \underbrace{b,\ldots,b}_{\ell_1}, \ldots, \underbrace{b^m,\ldots,b^m}_{\ell_m}; e \right) \in G;\ \sum_{i=0}^{m} \ell_i = n \right\},$$

where $m = \frac{r-1}{2}$ if r is odd, $m = \frac{r}{2}$ if r is even, and e is a unit element of S_n.

(2)

$$\left| H \left(\underbrace{1,\ldots,1}_{\ell_0}, \underbrace{b,\ldots,b}_{\ell_1}, \ldots, \underbrace{b^m,\ldots,b^m}_{\ell_m}; e \right) H \right| = \begin{cases} 2^{2n-\ell_0} \binom{n}{\ell_0,\ldots,\ell_m} n!, & \text{if } r = 2m+1 \\ 2^{2n-\ell_0-\ell_m} \binom{n}{\ell_0,\ldots,\ell_m} n!, & \text{if } r = 2m. \end{cases}$$

Proof. Since (2) follows from (1), we only prove (1). Let d be an element of D_r. For $0 \leqslant t \leqslant m$, any element of G is written uniquely as $d = a^s b^t a^u$ for suitable $s, u \in \{0,1\}$.

For $(d_1, \ldots, d_n; \sigma) = (a^{s_1} b^{t_1} a^{u_1}, \ldots, a^{s_n} b^{t_n} a^{u_n}; \sigma) \in G (d_i \in D_r, \sigma \in S_n)$, we have

$$(a^{s_{\tau(1)}}, \ldots, a^{s_{\tau(n)}}; \tau^{-1})(a^{s_1} b^{t_1} a^{u_1}, \ldots, a^{s_n} b^{t_n} a^{u_n}; \sigma)(a^{u_{\sigma(1)}}, \ldots, a^{u_{\sigma(n)}}; \sigma^{-1}\tau)$$
$$= (b^{t_{\tau(1)}}, \ldots, b^{t_{\tau(n)}}; e).$$

Since $\tau \in S_n$ can be chosen suitably, the representatives of double coset are determined by the number of b^t for $0 \leqslant t \leqslant m$. $\square$

Let $(a^{s_1} b^{t_1}, \ldots, a^{s_n} b^{t_n}; \sigma)$ be an element of G, where $s_i \in \{0, 1\}, t_i \in \{0, 1, \ldots, r-1\}$ $(1 \leqslant i \leqslant n)$ and $\sigma \in S_n$. The group G acts on the Laurent polynomial ring $\mathbb{C}[x_1, \ldots, x_n, x_1^{-1}, \ldots, x_n^{-1}]$ as

$$(a^{s_1} b^{t_1}, \ldots, a^{s_n} b^{t_n}; \sigma) f(x_1, \ldots, x_n) = f(\xi^{-t_{\sigma(1)}} x_{\sigma(1)}^{\varepsilon_1}, \ldots, \xi^{-t_{\sigma(n)}} x_{\sigma(n)}^{\varepsilon_n}),$$

where $\varepsilon_i = (-1)^{s_{\sigma(i)}}$ for $1 \leqslant i \leqslant n$. We define the map from $\mathbb{N}_0^{m+1}$ to the set of all partitions as follows;

$$\psi(k_0, k_1, \ldots, k_m) = (0^{k_0} 1^{k_1} \cdots m^{k_m}),$$

where k_i is the multiplicity of i for $0 \leqslant i \leqslant m$ [3, I-1] .

Proposition 3.2. *The induced representation* 1_H^G *is decomposed as follows.*

$$1_H^G \cong \bigoplus_{\sum_{i=0}^m k_i = n} W^{(k_0, k_1, \ldots, k_m)}.$$

Each $W^{(k_0, k_1, \ldots, k_m)}$ *is an irreducible G-module which is realized as follows:*

$$W^{(k_0, k_1, \ldots, k_m)} = \bigoplus_{f \in L_n(\psi(k_0, k_1, \ldots, k_m))} \mathbb{C} f.$$

Here, in the case that $r = 2m + 1$,

$$L_n(\lambda) = \{x_{\sigma(1)}^{\varepsilon_1 \lambda_1} x_{\sigma(2)}^{\varepsilon_2 \lambda_2} \cdots x_{\sigma(n)}^{\varepsilon_n \lambda_n}; \varepsilon_i \in \{\pm 1\}, \sigma \in S_n\},$$

and if $r = 2m$,

$$L_n(\lambda) = \Big\{ \left(x_{\sigma(1)}^{\lambda_1} + x_{\sigma(1)}^{-\lambda_1}\right)\left(x_{\sigma(2)}^{\lambda_2} + x_{\sigma(2)}^{-\lambda_2}\right) \cdots \left(x_{\sigma(k_m)}^{\lambda_{k_m}} + x_{\sigma(k_m)}^{-\lambda_{k_m}}\right)$$
$$\times x_{\sigma(k_m+1)}^{\varepsilon_{k_m+1} \lambda_{k_m+1}} x_{\sigma(k_m+2)}^{\varepsilon_{k_m+2} \lambda_{k_m+2}} \cdots x_{\sigma(n)}^{\varepsilon_n \lambda_n}; \varepsilon_i \in \{\pm 1\}, \sigma \in S_n \Big\}$$

for $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n), \lambda_i \geqslant \lambda_{i+1} \geqslant 0$.

Proof. It is easy to see that each $W^{(k_0, \ldots, k_m)}$ is G-invariant. From definition we have

$$dim\, W^{(k_0, k_1, \ldots, k_m)} = \begin{cases} \binom{n}{k_0, k_1, \ldots, k_m} 2^{n-k_0} & \text{if } r = 2m+1, \\ \binom{n}{k_0, k_1, \ldots, k_m} 2^{n-k_0-k_m} & \text{if } r = 2m. \end{cases}$$

We define the inner product on each $W^{(k_0,\dots,k_m)}$ as follows:

$$[\alpha x_1^{\varepsilon_1\lambda_1}\cdots x_n^{\varepsilon_n\lambda_n}|\beta x_1^{\eta_1\mu_1}\cdots x_n^{\eta_n\mu_n}] = \frac{\alpha\bar{\beta}}{\binom{n}{k_0,k_1,\dots,k_m}2^{n-k_0}}\prod_{i=1}^{n}\delta_{\varepsilon_i\lambda_i,\eta_i\mu_i}.$$

Here $\alpha,\beta\in\mathbb{C},\varepsilon_i,\eta_i\in\{\pm 1\}$, and $(\lambda_1,\lambda_2,\dots,\lambda_n)=(0^{k_0}1^{k_1}\cdots m^{k_m})$. It is easy to see that this inner product is G-invariant on $W^{(k_0,\dots,k_m)}$. For $\lambda=(\lambda_1,\dots,\lambda_n)=(0^{k_0}1^{k_1}\cdots m^{k_m})$, we define the polynomial of n variables by

$$f_\lambda(x) = \frac{2^{-k_0}}{k_0!k_1!\cdots k_m!}\sum_{\sigma\in S_n}\left(x_{\sigma(1)}^{\lambda_1}+x_{\sigma(1)}^{-\lambda_1}\right)\left(x_{\sigma(2)}^{\lambda_2}+x_{\sigma(2)}^{-\lambda_2}\right)\cdots\left(x_{\sigma(n)}^{\lambda_n}+x_{\sigma(n)}^{-\lambda_n}\right).$$

Since f_λ is an H-invariant element in $W^{(k_0,\dots,k_m)}$, we see that $W^{(k_0,\dots,k_m)}$ includes some irreducible components of 1_H^G. We define the map $\phi(x_1^{\varepsilon_1\lambda_1}x_2^{\varepsilon_2\lambda_2}\cdots x_n^{\varepsilon_n\lambda_n}) = \phi(x_1^{\varepsilon_1\lambda_1})\phi(x_2^{\varepsilon_2\lambda_2})\cdots\phi(x_n^{\varepsilon_n\lambda_n})$ by

$$\phi : W^{(k_0,\dots,k_m)}\to\mathbb{C}[x_1,\cdots,x_n],\quad x_i^k\mapsto\begin{cases} x_i^k & \text{if } k>0,\\ 1 & k=0,\\ x_i^{r+k} & \text{if } k<0.\end{cases}$$

We remark K acts on $\mathbb{C}[x]$ (cf. [4]). Since ϕ is a K-homomorphism, we obtain that

$$W^{(k_0,k_1,\dots,k_m)} \cong_K \begin{cases} \bigoplus\limits_{\substack{s_i+s_{r-i}=k_i\\ 1\leqslant i\leqslant m}} V^{(k_0,s_1,\dots,s_{r-1})} & \text{if } r=2m+1,\\ \bigoplus\limits_{\substack{s_i+s_{r-i}=k_i\\ 1\leqslant i\leqslant m-1}} V^{(k_0,s_1,\dots,s_{m-1},k_m,s_{m+1},\dots,s_{r-1})} & \text{if } r=2m.\end{cases}$$

Here $V^{(k_0,\dots,k_{r-1})} = Span_{\mathbb{C}}\{x_{\sigma(1)}^{\lambda_1}x_{\sigma(2)}^{\lambda_2}\cdots x_{\sigma(n)}^{\lambda_n}|\sigma\in S_n\}$ for $\lambda=(0^{k_0}1^{k_1}\cdots(r-1)^{k_{r-1}})$ and $\sum_{i=0}^{r-1}k_i=n$. Note that $V^{(k_0,s_1,\dots,s_{r-1})}$ are irreducible K-modules and not equivalent to each other, and this decomposition is multiplicity free. The inverse image of $V^{(k_0,s_1,\dots,s_{r-1})}$ contains a monomial in $L_n(\lambda)$. Since G acts transitively on $L_n(\lambda)$ and this decomposition is unique, we find that $W^{(k_0,\dots,k_m)}$ are irreducible and not equivalent to each other as G-module. Finally an easy calculation shows that

$$\sum_{\sum_{i=0}^{m}k_i=n} dim\, W^{(k_0,\dots,k_m)} = r^n = dim\, 1_H^G. \qquad \square$$

By this proposition we see that the induced representation 1_H^G is multiplicity free. Then we say that $(G,H)=(D(r,n),D(1,n))$ is a *Gelfand pair*. Since 1_H^G is multiplicity free, $f_{(0^{k_0}1^{k_1}\cdots m^{k_m})}$ is unique H-invariant element in irreducible component $W^{(k_0,k_1,\dots,k_m)}$. Note that f_λ satisfies

$$[f_\lambda(x)|f_\mu(x)] = \delta_{\lambda\mu}.$$

Then the function $[f_\lambda(x)|(gf_\lambda)(x)]$ is called *the zonal spherical function*. Let $g=(a^{s_1}b^{t_1},\dots,a^{s_n}b^{t_n};\sigma)\in G$. Since f_λ is H-invariant, it follows that

$$\begin{aligned}(gf_\lambda)(x) &= (a^{s_1},\dots,a^{s_n};\sigma)(b^{t_{\sigma(1)}},\dots,b^{t_{\sigma(n)}};e)f_\lambda(x)\\ &= (b^{t_{\sigma(1)}},\dots,b^{t_{\sigma(n)}};e)f_\lambda(x)\\ &= f_\lambda(\xi^{-t_{\sigma(1)}}x_1,\dots,\xi^{-t_{\sigma(n)}}x_n).\end{aligned}$$

Therefore the zonal spherical functions (cf. [3,4]) are

$$\begin{aligned}[f_\lambda(x)|(gf_\lambda)(x)] &= [f_\lambda(x)|f_\lambda(\xi^{-t_{\sigma(1)}}x_1,\dots,\xi^{-t_{\sigma(n)}}x_n)]\\ &= [x_1^{\varepsilon_1\lambda_1}\cdots x_n^{\varepsilon_n\lambda_n}|x_1^{\varepsilon_1\lambda_1}\cdots x_n^{\varepsilon_n\lambda_n}]f_\lambda(\xi^{t_{\sigma(1)}},\dots,\xi^{t_{\sigma(n)}})\\ &= f_\lambda(\xi^{t_1},\dots,\xi^{t_n})/f_\lambda(1,\dots,1).\end{aligned}$$

Theorem 3.3. *For* $(k_0,\dots,k_m)\in\mathbb{N}_0^{m+1}$ *and* $\sum_{i=0}^m k_i=n$, *we have the zonal spherical function* $\omega^{(k_0,k_1,\dots,k_m)}$ *of the Gelfand pair* (G,H);

$$\begin{aligned}&\omega^{(k_0,k_1,\dots,k_m)}\Bigg(\underbrace{1,\dots,1}_{\ell_0},\underbrace{b,\dots,b}_{\ell_1},\dots,\underbrace{b^m,\dots,b^m}_{\ell_m};e\Bigg)\\ &\quad= f_\lambda\Bigg(\underbrace{1,\dots,1}_{\ell_0},\underbrace{\xi,\dots,\xi}_{\ell_1},\dots,\underbrace{\xi^m,\dots,\xi^m}_{\ell_m}\Bigg)\Bigg/ f_\lambda(1,\dots,1),\end{aligned}$$

where $\lambda=(0^{k_0}1^{k_1}\cdots m^{k_m})$.

4. Hypergeometric expression of zonal spherical functions

For $\lambda=(\lambda_1,\lambda_2,\dots,\lambda_n)=(0^{k_0}1^{k_1}\cdots m^{k_m})$, we define

$$f^{(k_0,k_1,\dots,k_m)}_{(\ell_0,\ell_1,\dots,\ell_m)}=\frac{1}{2^{n-k_0}}f_\lambda\Bigg(\underbrace{1,\dots,1}_{\ell_0},\underbrace{\xi,\dots,\xi}_{\ell_1},\dots,\underbrace{\xi^m,\dots,\xi^m}_{\ell_m}\Bigg)$$

and

$$\begin{aligned}\omega^{(k_0,k_1,\dots,k_m)}_{(\ell_0,\ell_1,\dots,\ell_m)} &= \omega^{(k_0,k_1,\dots,k_m)}\Bigg(\underbrace{1,\dots,1}_{\ell_0},\underbrace{b,\dots,b}_{\ell_1},\dots,\underbrace{b^m,\dots,b^m}_{\ell_m};e\Bigg)\\ &= \frac{1}{\binom{n}{k_0,k_1,\dots,k_m}}f^{(k_0,k_1,\dots,k_m)}_{(\ell_0,\ell_1,\dots,\ell_m)}.\end{aligned}$$

From definition of f_λ, the value of zonal spherical functions on each double coset can be described by the term of trigonometric functions,

$$c(z) = \cos(2\pi z/r).$$

Namely, for $\mu = (\mu_1, \mu_2, \ldots, \mu_n) = (0^{\ell_0} 1^{\ell_1} \cdots m^{\ell_m})$, it follows that

$$\begin{aligned} f^{(k_0,k_1,\ldots,k_m)}_{(\ell_0,\ell_1,\ldots,\ell_m)} &= \frac{1}{k_0!k_1!\cdots k_m!} \sum_{\sigma \in S_n} c(\lambda_1 \mu_{\sigma(1)}) c(\lambda_2 \mu_{\sigma(2)}) \cdots c(\lambda_n \mu_{\sigma(n)}) \\ &= \sum_{I_n^{k_0 k_1 \cdots k_m}} c(0\mu_{i_1^{(0)}}) \cdots c(0\mu_{i_{k_0}^{(0)}}) c(1\mu_{i_1^{(1)}}) \cdots c(1\mu_{i_{k_1}^{(1)}}) \cdots c(m\mu_{i_1^{(m)}}) \cdots c(m\mu_{i_{k_m}^{(m)}}), \end{aligned}$$

where the summation runs over

$$\begin{aligned} I_n^{k_0 k_1 \cdots k_m} = \{ & \{ i_1^{(0)}, \ldots, i_{k_0}^{(0)}, i_1^{(1)}, \ldots, i_{k_1}^{(1)}, \ldots, i_1^{(m)}, \ldots, i_{k_m}^{(m)} \}; \\ & \bigcup_{j=0}^{m} \{ i_1^{(j)}, \ldots, i_{k_j}^{(j)}; 1 \leqslant i_1^{(j)} < \cdots < i_{k_j}^{(j)} \leqslant n \} = \{1, 2, \ldots, n\} \}. \end{aligned}$$

Proposition 4.1.

$$f^{(k_0,k_1,\ldots,k_m)}_{(\ell_0,\ell_1,\ldots,\ell_m)} = \sum_{a \in \mathscr{A}} \prod_{i=0}^{m} \binom{\ell_i}{a_{i0}, a_{i1}, \ldots, a_{im}} \prod_{0 \leqslant i, j \leqslant m} \left(\cos\left(\frac{2\pi}{r} ij \right) \right)^{a_{ij}},$$

where

$$\mathscr{A} = \mathscr{A}^{(k_0,k_1,\ldots,k_m)}_{(\ell_0,\ell_1,\ldots,\ell_m)} = \left\{ a = (a_{ij}) \in M_{m+1}(\mathbb{N}_0); \sum_{i=0}^{m} a_{ij} = k_j, \sum_{j=0}^{m} a_{ij} = \ell_i \right\}.$$

The proof is similar to that of [4, Proposition 4.1]. For $(\mu_1, \mu_2, \ldots, \mu_n) = (0^{\ell_0} 1^{\ell_1} \cdots m^{\ell_m})$, we put $\mathscr{L}_i = \{k; \mu_k = i\}$ and

$$a_{ij} = |\mathscr{L}_i \cap \{ i_1^{(j)}, \ldots, i_{k_j}^{(j)} \}|.$$

Then we see that a_{ij} satisfy

$$\sum_{i=0}^{m} a_{ij} = k_j \text{ and } \sum_{j=0}^{m} a_{ij} = \ell_i.$$

And we obtain the claim of Proposition 4.1.

The proof of the following lemma is similar to [4, Proposition 4.4].

Lemma 4.2.

$$f^{(k_0,k_1,\ldots,k_m)}_{(\ell_0,\ell_1,\ldots,\ell_m)} = \sum_{a\in\mathscr{A}} \binom{n-\sum_{1\leqslant i,\,j\leqslant m} a_{ij}}{k_0,a_{01},\ldots,a_{0m}} \prod_{i=1}^{m} \binom{\ell_i}{a_{i0},a_{i1},\ldots,a_{im}} \times \prod_{1\leqslant i,j\leqslant m} \left(\cos\left(\frac{2\pi}{r} ij\right) - 1\right)^{a_{ij}}.$$

Proof. We compute that

$$f^{(k_0,\ldots,k_m)}_{(\ell_0,\ldots,\ell_m)} = \sum_{I_n^{k_0\cdots k_m}} \prod_{j=0}^{m} \prod_{s=1}^{k_j} c(j\mu_{i_s^{(j)}}) = \sum_{I_n^{k_0\cdots k_m}} \prod_{j=1}^{m} \prod_{s=1}^{k_j} (1 + (c(j\mu_{i_s^{(j)}}) - 1))$$
$$= \sum_{I_n^{k_0\cdots k_m}} \sum_{\substack{0\leqslant t_i\leqslant k_i \\ (1\leqslant i\leqslant m)}} \prod_{j=1}^{m} e_{t_j}(c(j\mu_{i_1^{(j)}}) - 1, \ldots, c(j\mu_{i_{k_j}^{(j)}}) - 1).$$

Here e_t denotes the elementary symmetric function. Putting

$$t_0 = n - \sum_{i=1}^{m} t_i = \sum_{i=1}^{m} (k_i - t_i) + k_0,$$

it follows that

$$f^{(k_0,\ldots,k_m)}_{(\ell_0,\ldots,\ell_m)} = \sum_{\substack{0\leqslant t_i\leqslant k_i \\ (1\leqslant i\leqslant m)}} \sum_{I_n^{t_0\cdots t_m}} \prod_{j=1}^{m} \prod_{s=1}^{t_j} (c(j\mu_{i_s^{(j)}}) - 1) \times \binom{t_0}{k_1-t_1,\ldots,k_m-t_m,k_0}.$$

Analogous to the previous proposition, we can compute

$$\sum_{I_n^{t_0\cdots t_m}} \prod_{j=1}^{m} \prod_{s=1}^{t_j} (c(j\mu_{i_s^{(j)}}) - 1) = \sum_{a\in\mathscr{A}'} \prod_{i=0}^{m} \binom{\ell_i}{a_{i0},\ldots,a_{im}} \prod_{j=1}^{m} (c(ji) - 1)^{a_{ij}}$$
$$= \sum_{\substack{a\in\mathscr{A}' \\ a_{01},\ldots,a_{0m}=0}} \prod_{i=1}^{m} \binom{\ell_i}{a_{i0},\ldots,a_{im}} \prod_{1\leqslant i,j\leqslant m} (c(ij) - 1)^{a_{ij}}.$$

Here $\mathscr{A}' = \{a = (a_{ij}) \in M_{m+1}(\mathbb{N}_0); \sum_{i=0}^{m} a_{ij} = t_j, \sum_{j=0}^{m} a_{ij} = \ell_i\}$. By combining these computations, we obtain the claim of lemma. □

Using this lemma, the zonal spherical functions are written as

$$\begin{aligned}\omega^{(k_0,\dots,k_m)}_{(\ell_0,\dots,\ell_m)} &= \sum_{a\in\mathscr{A}} \frac{(n-\sum_{1\leqslant i,j\leqslant m}a_{ij})!k_1!\cdots k_m!\ell_1!\cdots\ell_m!}{n!a_{01}!\cdots a_{0m}!a_{10}!\cdots a_{m0}!} \prod_{1\leqslant i,j\leqslant m}\frac{(c(ij)-1)^{a_{ij}}}{a_{ij}!}\\ &= \sum_{\sum_{1\leqslant i,j\leqslant m}a_{ij}\leqslant n} \frac{\prod_{j=1}^m(-k_j)_{\sum_{i=1}^m a_{ij}}\prod_{i=1}^m(-\ell_i)_{\sum_{j=1}^m a_{ij}}}{(-n)_{\sum_{1\leqslant i,j\leqslant m}a_{ij}}} \prod_{1\leqslant i,j\leqslant m}\frac{(1-c(ij))^{a_{ij}}}{a_{ij}!}\\ &= F((-\ell_1,\dots,-\ell_m),(-k_1,\dots,-k_m);-n;(1-\cos(2\pi ij/r)).\end{aligned}$$

Therefore we have a hypergeometric expression of the zonal spherical functions.

Theorem 4.3.

$$\omega^{(k_0,k_1,\dots,k_m)}_{(\ell_0,\ell_1,\dots,\ell_m)} = F((-\ell_1,\dots,-\ell_m),(-k_1,\dots,-k_m);-n;(1-\cos(2\pi ij/r))_{1\leqslant i,j\leqslant m}).$$

Proof of Theorem 2.1. From Proposition 3.1 the cardinality of each double coset is

$$|H(\underbrace{1,\dots,1}_{\ell_0},\underbrace{b,\dots,b}_{\ell_1},\dots,\underbrace{b^m,\dots,b^m}_{\ell_m};e)H| = \begin{cases} 2^{n-\ell_0}\binom{n}{\ell_0,\dots,\ell_m}n!, & \text{if } r=2m+1\\ 2^{n-\ell_0-\ell_m}\binom{n}{\ell_0,\dots,\ell_m}n!, & \text{if } r=2m,\end{cases}$$

and we already know the dimension of each irreducible component of 1_H^G from Proposition 3.2. From the orthogonality of the zonal spherical functions [3,4] we have Theorem 2.1.

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Further reading

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