

Expansion properties of a random regular graph after random vertex deletions

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Abstract

We investigate the following vertex percolation process. Starting with a random regular graph of constant degree, delete each vertex independently with probability p , where $p = n^{-\alpha}$ and $\alpha = \alpha(n)$ is bounded away from 0. We show that a.a.s. the resulting graph has a connected component of size $n - o(n)$ which is an expander, and all other components are trees of bounded size. Sharper results are obtained with extra conditions on α . These results have an application to the cost of repairing a certain peer-to-peer network after random failures of nodes.

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1. Introduction

In this paper we investigate the effect of randomly deleting some vertices in a random regular graph. Take a random d -regular graph G on n vertices and independently delete each vertex with probability p . The result is a random graph \widehat{G} with maximum degree at most d . We analyse the structure of \widehat{G} , with particular focus on whether (the largest connected component of) \widehat{G} is an expander graph. Here d is fixed, n tends to infinity such that dn is even, and we take $p = n^{-\alpha}$ for some function $\alpha = \alpha(n)$. In this paper we treat only the case where α is bounded away from 0, since otherwise even the largest component of the graph is not an expander. Our work is motivated by an application in peer-to-peer networks, as described below.

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In Section 1.1 we describe our main result. Related work is described in Section 1.2. The application to a certain peer-to-peer network is explained in Section 1.3. Our calculations will be carried out in the configuration model which is described in Section 2. Then our calculations are presented in Section 3.

1.1. Notation, terminology and our main result

There are several related definitions of *expander* graphs. We will say that a graph G on n vertices is a β -*expander* if every set S of $s \leq n/2$ vertices has at least βs neighbours outside S . An alternative definition involves $d(S)$, the sum of the degrees of vertices in S , and $e(S)$, the number of edges leading out of S , and defines G to be an γ -*expander* if $e(S) \geq \gamma d(S)$ for all sets $S \subseteq V(G)$ of vertices with $d(S) \leq |E(G)|$. For bounded-degree graphs these give equivalent notions of expanders, up to a constant factor in translating γ to β .

In this paper, all asymptotics are as $n \rightarrow \infty$. We say that an event holds *asymptotically almost surely* (a.a.s.) if the probability that it holds tends to 1. We adapt the standard $O(\cdot)$, $o(\cdot)$ notation to accommodate versions which hold a.a.s., following [14, Section 8.2.1]. Specifically, let $f(n)$, $g(n)$ and $\phi(n)$ be functions such that $|f| < \phi g$. If $\phi(n)$ is bounded for sufficiently large n then we write $f = O(g)$, and if $\phi \rightarrow 0$ as $n \rightarrow \infty$ then we write $f = o(g)$. If $f/g = 1 + o(1)$ then we write $f \sim g$ and say that f and g are *asymptotically equal*. If a statement S about random variables involves the notation $O(\cdot)$ or $o(\cdot)$ then S is not an event, and we define ‘‘a.a.s. S ’’ to mean that all inequalities of the form $|f| < \phi g$ which are implicit in S hold a.a.s.

Let $\mathcal{G}_{n,d}$ denote the uniform probability space of all (simple) d -regular graphs on the vertex set $[n] = \{1, \dots, n\}$. Our main result is the following.

Theorem 1. Fix $d \geq 3$ and a constant $\eta > 0$. Suppose that $\alpha = \alpha(n)$ satisfies

$$\alpha(n) \geq \eta \tag{1}$$

for n sufficiently large. Let $G \in \mathcal{G}_{n,d}$ and let \widehat{G} be the graph obtained by independently deleting vertices of G independently with probability $n^{-\alpha}$. Then

- (a) there is a constant $\beta > 0$ such that a.a.s. \widehat{G} has a connected component of size $n - o(n)$ that is a β -*expander*, and all other components are trees of bounded size;
- (b) if $\eta > \frac{1}{2(d-1)}$ then there is a constant $\beta > 0$ such that a.a.s. \widehat{G} consists of a connected component that is a β -*expander*, together with $o(n^{(d-2)/(2d-2)})$ isolated vertices;
- (c) if $\eta \geq \frac{1}{d-1}$ then there is a constant $\beta > 0$ such that a.a.s. \widehat{G} is a β -*expander*.

The result in (a) is best possible, in the sense that if α goes to 0 in the deletion probability $n^{-\alpha}$ then there is no fixed positive expansion rate: that is, there is no fixed $\beta > 0$ as stated in the theorem. The reason for this is as follows. It can be shown by the second moment method that if $k < 1/(\alpha(d-2))$ then there are a.a.s. many paths of degree 2 vertices of length at least k in the large connected component. Any one such path causes the expansion rate to be at most $2/(k-1)$. This is explained further after Lemma 5 below.

1.2. Related work

While the vertex deletion process which we analyse in this paper does not seem to appear in the literature, there are various papers [7,8,11] investigating the result of deleting edges of random regular graphs independently with some given probability. This is usually described as

edge percolation, and the resulting graph is sometimes called the *faulty graph*. These papers are also motivated by applications to communications networks. Nikolettseas et al. [11] focus on the connectivity properties of the faulty graph, and undertake a study somewhat similar to ours. Goerdts [7, Theorem 2] proves that for small constant edge deletion probability, there is a linear-sized component of the faulty graph. However, it is *not* an expander. Goerdts and Molloy [8] extend this analysis to give a threshold on the fault probability for the existence of a linear-sized k -core whenever $3 \leq k < d$. (The k -core of a graph is the unique maximal subgraph in which each vertex has degree at least k , see for example [4, p. 150].) The k -core is with high probability an expander, but only contains some proportion of the vertices. These results are considering much higher deletion probabilities than we do in the present paper, because they tolerate a very large number of disconnected vertices: linear in n .

The paper of Alon et al. [1] considers edge percolation on expander graphs, which includes random regular graphs of degree at least 3. (Though they consider graphs of high girth, this is a minor detail.) They determine the threshold at which a giant component exists. They also give a result [1, Proposition 5.1] on the expansion of the giant component when the edge deletion probability tends to 0. This involves $(1/\log n)$ expansion however, not constant rate expansion. For random regular graphs, Pittel [12] gave a more detailed analysis and determined the order of the transition window of appearance of a giant component in a random regular graph under edge percolation.

1.3. Application to a peer-to-peer network

The vertex deletion process which we study in this paper is motivated by an application to a peer-to-peer network proposed by Bourassa and Holt [5,9]. This network, called the *Swan network*, is based on random regular graphs. Under normal operating conditions, the network is given by a d -regular graph, where $d \geq 4$ is an even constant (in practice $d = 4$). Bourassa and Holt claimed that their networks quickly acquire some desirable characteristics of uniformly distributed random regular graphs, such as high connectivity and logarithmic diameter. (Note that random d -regular graphs are a.a.s. expander graphs for $d \geq 3$ [3], and as such they are connected and have logarithmic diameter. Specifically, it is well known and easy to see that if a graph G is a γ -expander then G has diameter which is bounded above by $\log_{1+\gamma}(n/2)$.)

Cooper, Dyer and Greenhill [6] gave theoretical support to these claims by defining a Markov chain to model the behaviour of the Swan networks. They showed that under certain natural assumptions about arrival and departure rates, and with a slight alteration of the mechanism of departure, the Markov chain converges rapidly to its stationary distribution, which is uniform when conditioned on a fixed number of vertices. While random d -regular graphs are a.a.s. connected for $d \geq 3$ (indeed, d -connected), a Swan network in the absence of departures is always connected.

In the context of peer-to-peer Swan networks, the random deletion of a vertex corresponds to a client failing. Edges correspond to LAN or Internet connections and so are far more robust. Individual clients fail due to lost power, shut down or log-off events, frozen applications, and similar phenomena. Hence our exclusive consideration of vertex deletions, rather than edge deletions.

Swan networks are self-administering. In particular, they are self-healing after the loss of some vertices, completing a d -regular graph among the remaining vertices. For Swan networks, there are two processes to handle lost neighbours: an inexpensive process that uses messages internal to the graph, and a more expensive process that contacts vertices using messages external to the

graph. As long as the graph remains connected, the repairs can safely use the internal repair mechanism. Hence for this application it is desirable that the majority of clients in the network remain in a connected component.

Theorem 1 models this situation and shows that the large connected component is an expander, which has three important implications for Swan networks. First, under certain constraints on the probability of node failures, Swan networks tend to remain connected under the simultaneous loss of several nodes. Second, deletions do not degrade the log-diameter of the Swan networks. Finally, **Theorem 1** implies that the current internal repair strategies could be modified, efficiently involving more of the remaining nodes in the repair.

2. The configuration model and some definitions

As is usual in this area, calculations are performed in the *configuration model* (or *pairing model*), see for example [13] or [10, Chapter 9]. A *configuration* consists of n buckets with d points each, and a perfect matching of the dn points chosen uniformly at random. The edges of the perfect matching are called *pairs*. Assume that the buckets are labelled $1, \dots, n$ and that within each bucket the points are labelled $1, \dots, d$. Denote this probability space by $\mathcal{P}_{n,d}$. Given a configuration $P \in \mathcal{P}_{n,d}$ we obtain a pseudograph $G(P)$ by shrinking each bucket down to a vertex. This pseudograph may have loops and/or multiple edges, but the probability that it is simple (with no loops or multiple edges) is bounded below by a positive constant. Moreover, conditioned on $G(P)$ being simple, it is uniformly distributed.

Similarly if $\mathbf{d} = (d_1, \dots, d_n)$ is the degree sequence of a graph, then $\mathcal{P}_{n,\mathbf{d}}$ denotes the configuration model where the j th bucket contains d_j points, and a perfect matching of the $2m = \sum_{j=1}^n d_j$ points is chosen uniformly at random. Here we assume that the buckets are labelled $1, \dots, n$ and that the points in the j th bucket are labelled $1, \dots, d_j$.

We can now define the bucket deletion process for configurations. For the remainder of the paper, assume that (1) holds for some positive constant η , for n sufficiently large. Given $P \in \mathcal{P}_{n,d}$, form a new configuration \hat{P} by independently deleting each bucket with probability p . Specifically:

- choose a random subset R of buckets such that $b \in R$ with probability $p = n^{-\alpha}$, independently for each bucket b ,
- delete all buckets in R ,
- delete every pair with an endpoint in a bucket in R , together with the other endpoint of the pair if it lies outside R ,
- relabel the surviving buckets with the labels $1, 2, \dots$, preserving the relative ordering of the buckets,
- relabel the points within each surviving bucket in the same way.

Note that the same distribution on \hat{P} will result if the set R of buckets to delete is chosen first, and then $P \in \mathcal{P}_{n,d}$ is selected.

We now give some definitions which we will need. A connected component of a graph which is a tree will be called an *isolated tree*, and a connected component of a graph which is a cycle will be called an *isolated cycle*.

The *2-core* of a graph G , denoted by $\text{cr}(G)$, is obtained from G by the following process: let $G_0 = G$ and for $t \geq 0$, if G_t contains a vertex v of degree 0 or 1 then let $G_{t+1} = G_t - v$, otherwise stop. The final graph is $\text{cr}(G)$. From the 2-core $\text{cr}(G)$ of G we obtain the *kernel* of G , denoted by $\text{ker}(G)$, by suppressing all vertices of degree 2. That is, if v is a vertex of degree 2 in G' with neighbours $\{a, b\}$ then delete v and replace these two edges by the edge $\{a, b\}$.

Given a graph G , an edge of G is a *cyclic edge* if it belongs to a cycle, or to a path joining two cycles. The cyclic edges are precisely those of the 2-core. The subgraph M of G induced by the non-cyclic edges, called the *mantle*, is a union of some number of components. Each of these components is a tree which is an induced subgraph of G . We define a *bush* to be a connected induced subgraph of M , so that a maximal bush is exactly a component of M . If a bush B has a vertex which is incident with at least one cyclic edge of G then this vertex is called the *root* of B . Following from these definitions, a bush can have at most one root, and maximal bushes are pairwise disjoint.

We will say that a configuration P has some property if the corresponding graph $G(P)$ has that property. This allows us to speak of paths and cycles in a configuration P , as well as subconfigurations of P which are trees, bushes and so on. In particular we can define the 2-core and kernel of a configuration.

We will need the following lemma which has a very straightforward proof and can be found in [4, p. 54].

Lemma 1. *Let k be a fixed positive integer and let $\mathbf{d} = (d_1, \dots, d_n)$ be a degree sequence satisfying $0 \leq d_i \leq d$ for all i . Then the probability that a random element of $\mathcal{P}_{n,\mathbf{d}}$ contains k specified pairs is $(1 + o(1))(2m)^{-k}$, where $m = (d_1 + \dots + d_n)/2$ is the number of pairs in the configuration.*

If an event is a.a.s. true for $G(P)$ when $P \in \mathcal{P}_{n,d}$, then it is also a.a.s. true conditional on the event that $G(P)$ is simple. This comes immediately from the fact that the probability that $G(P)$ is simple for $P \in \mathcal{P}_{n,d}$ is bounded below by a positive constant (see for example [4, p. 55]). This is the way that many results about $\mathcal{G}_{n,d}$ have been proved using $\mathcal{P}_{n,d}$.

3. The details

Let $P \in \mathcal{P}_{n,d}$ and let R be the random set of buckets chosen for deletion. Write $r = |R|$. By the well known sharp concentration of binomials, since α is bounded away from 0, a.a.s.

$$r \sim n^{1-\alpha} \tag{2}$$

provided $n^{1-\alpha} \rightarrow \infty$. Until we come to the proof of [Theorem 1](#) we will assume that the latter condition holds, so that (2) holds. The other case is easily handled afterwards.

Let \widehat{P} be the result of deleting the buckets in R from P (and performing the necessary relabellings of buckets and points). Then \widehat{P} has $n - r$ buckets. Let d_j denote the number of points in the j th bucket of \widehat{P} , and say that bucket j has degree d_j . Thus $0 \leq d_j \leq d$. The degree sequence of \widehat{P} is (d_1, \dots, d_{n-r}) and number of pairs in \widehat{P} is $(d_1 + \dots + d_{n-r})/2$. Let N_j be the number of buckets of \widehat{P} with degree j , for $0 \leq j \leq d$. The following result shows that we can use $\mathcal{P}_{n,\mathbf{d}}$ to model \widehat{P} , conditional upon it having degree sequence \mathbf{d} .

Lemma 2. *The pairing \widehat{P} is uniformly random conditioned on its degree sequence $\mathbf{d} = (d_1, \dots, d_{n-r})$.*

Proof. First notice that the set R determines an injection $\varphi : [n - r] \rightarrow [n]$ which is the inverse of the relabelling operation performed when \widehat{P} is constructed. The probability of a particular \widehat{P} with degree sequence $\mathbf{d} = (d_1, \dots, d_{n-r})$ is given by

$$\binom{n}{r} \left(\prod_{j=1}^{n-r} \binom{d}{d_j} \right) n^{-\alpha r} (1 - n^{-\alpha})^{n-r} N_{\mathbf{d}} / |\mathcal{P}_{n,d}|$$

where

- $\binom{n}{r}$ is the number of order-preserving injections $\varphi : [n - r] \rightarrow [n]$, giving the labels of the buckets from \widehat{P} in P ,
- $\binom{d}{d_j}$ is the number of order-preserving injections from $[d_j]$ to $[d]$, giving the labels of the points from bucket j of \widehat{P} in bucket $\varphi(j)$ of P ,
- $n^{-\alpha r} (1 - n^{-\alpha})^{n-r}$ is the probability that the buckets chosen for deletion are exactly the r buckets of P which do not correspond to buckets of \widehat{P} ,
- $\widehat{\mathbf{d}} = (\widehat{d}_1, \dots, \widehat{d}_n)$ is the degree sequence given by

$$\widehat{d}_i = \begin{cases} d & \text{if } i \notin \varphi([n - r]), \\ d - d_j & \text{if } i = \varphi(j), \end{cases}$$

- $N_{\widehat{\mathbf{d}}}$ is the number of configurations with degree sequence $\widehat{\mathbf{d}}$, giving the number of ways to complete the configuration P .

Since the above expression depends only on \mathbf{d} and not on the particular structure of \widehat{P} , it follows that \widehat{P} is uniformly random conditioned on its degree sequence \mathbf{d} . \square

For $0 \leq j \leq d$ let

$$\mu_j = \binom{d}{j} n^{1-(d-j)\alpha}.$$

Lemma 3. Assume that r satisfies (2). Form \widehat{P} from $P \in \mathcal{P}_{n,d}$ by deleting the buckets in R as described in Section 2. Then, for $0 \leq j \leq d$, we have $\mathbf{E}N_j \sim \mu_j$ and a.a.s.

$$\begin{aligned} N_j &\sim \mu_j && \text{if } \mu_j \rightarrow \infty, \\ N_j &= O(\log \log n) && \text{if } \mu_j = O(1), \\ N_j &= 0 && \text{if } \mu_j = o(1). \end{aligned} \tag{3}$$

In all cases, a.a.s. $N_j = o(\mu_\ell)$ for $0 \leq j < \ell \leq d$.

Proof. Fix $j \in \{0, \dots, d\}$. Choose a random configuration $P \in \mathcal{P}_{n,d}$. The probability that a given bucket $b \notin R$ is incident with exactly $d - j$ pairs which are incident with points in R is asymptotically equal to

$$\binom{d}{j} \binom{dr}{d-j} (d-j)! (dn)^{-(d-j)} \sim \binom{d}{j} n^{-(d-j)\alpha} = \mu_j/n.$$

(The first factor chooses $d - j$ points in b and the second factor chooses $d - j$ points in R . There are $(d - j)!$ ways to match up these points using pairs, and the probability that a random element of $\mathcal{P}_{n,d}$ contains these pairs is asymptotic to $(dn)^{-(d-j)}$, by Lemma 1.) Therefore by linearity of expectation,

$$\mathbf{E}N_j \sim \mu_j,$$

proving the first statement.

Now suppose that $\mu_j \rightarrow \infty$. Similar calculations for an ordered pair of buckets $b, c \notin R$ show that $\mathbf{E}[N_j]_2 \sim (\mathbf{E}N_j)^2$. This establishes the sharp concentration of N_j whenever $\mu_j \rightarrow \infty$. The other two statements in (3) follow from Markov’s inequality, as does the final statement of the lemma. \square

Now fix a positive integer K such that

$$K > \frac{2}{(d-2)\eta},$$

where η is the constant from (1). Recall the definition of a bush given before the statement of Lemma 1. Note that a bucket v of a bush B has the same degree in B as it does in \widehat{P} unless v is a root.

Lemma 4. Let $\mathbf{d} = (d_1, d_2, \dots, d_{n-r})$ be a degree sequence such that $0 \leq d_i \leq d$ for all i , r satisfies (2), and N_j satisfies (3) for $0 \leq j \leq d$. Let $\widehat{P} \in \mathcal{P}_{n,\mathbf{d}}$. Then a.a.s. \widehat{P} has no bushes with more than K buckets (including isolated trees).

Proof. First observe that every tree on $k \geq 2$ buckets has at least $k/2 + 1$ buckets of degree 1 or 2. (This can be proved using induction.) Suppose that \widehat{P} contains a bush B with at least $K + 1$ buckets. Then \widehat{P} contains a bush with exactly $k + 1$ buckets, for some k satisfying $K \leq k \leq 2K$. To see this, suppose that B is a bush in \widehat{P} with at least $2K + 2$ buckets. If B has a root then let b be the root bucket of B , otherwise let b be any bucket of B . Then at least one neighbour of b , say b' , is the root of a (smaller) bush B' in \widehat{P} with more than K buckets. By induction on B' , the result follows.

So now let B be a bush in \widehat{P} with $k + 1$ vertices, where $K \leq k \leq 2K$, and let S be the set of buckets in B . Ignoring the root bucket if present (which may have higher degree in \widehat{P} than it does in B), it follows that there are at least $k/2$ buckets in S with degree 1 or 2 in \widehat{P} . Moreover there are k pairs in \widehat{P} between points in buckets of S .

Now we prove that a.a.s. there are no such sets S of buckets in a random element of $\mathcal{P}_{n,\mathbf{d}}$. There are $N_1 + N_2$ buckets in \widehat{P} of degree 1 or 2, and by (3), a.a.s.

$$N_1 + N_2 = \begin{cases} O(\mu_2) & \text{if } \mu_2 \rightarrow \infty \text{ or } \mu_2 = o(1), \\ O(\log \log n) & \text{if } \mu_2 = O(1). \end{cases}$$

Here if $\mu_2 \rightarrow \infty$ and $\mu_1 = O(1)$ then we use the fact that a.a.s. $N_1 = o(\mu_2)$ (which follows from the last statement of Lemma 3), rather than the arbitrary upper bound of $O(\log \log n)$ from (3). Hence there are a.a.s. at most

$$O(1) n^{k/2+1} g(n)^{k/2}$$

ways to choose the buckets belonging to the set S , where

$$g(n) = \begin{cases} \mu_2 = n^{1-(d-2)\alpha} & \text{if } \mu_2 \rightarrow \infty \text{ or } \mu_2 = o(1), \\ \log \log n & \text{if } \mu_2 = O(1). \end{cases} \tag{4}$$

There are $O(1)$ ways to choose locations for the k pairs between points of S , and the probability that a random element of $\mathcal{P}_{n,\mathbf{d}}$ contains these pairs is $O(n^{-k})$, by Lemma 1. Therefore the expected number of such sets S in \widehat{P} is a.a.s.

$$O(1) n^{k/2+1} g(n)^{k/2} n^{-k}.$$

This is clearly $o(1)$ if $g(n) = \log \log n$, and otherwise

$$\begin{aligned} O(1) n^{k/2+1} g(n)^{k/2} n^{-k} &= O(n^{1-(d-2)\alpha k/2}) \\ &= O(n^{1-(d-2)\eta K/2}) \\ &= o(1) \end{aligned}$$

by choice of K . Hence by Markov’s inequality in either case there are a.a.s. no such sets S , for $K \leq k \leq 2K$. The lemma follows. \square

To create the 2-core $\text{cr}(\widehat{P})$ of \widehat{P} , start with \widehat{P} and delete all buckets of degree 0. Then while any buckets of degree 1 remain, delete one at each time step until none remain. Finally, relabel the remaining buckets and the points within the remaining buckets, respecting the relative ordering. This process is equivalent to deleting all isolated trees and “pruning” all maximal bushes of \widehat{P} (where pruning involves deleting all buckets of the maximal bush except the root, and deleting all pairs incident with any non-root bucket of the bush), followed by relabelling. Denote the number of buckets in $\text{cr}(\widehat{P})$ by t and let $\mathbf{d}' = (d'_1, \dots, d'_t)$ be the degree sequence of $\text{cr}(\widehat{P})$. This defines N'_j , the number of buckets in $\text{cr}(\widehat{P})$ with degree j , for $2 \leq j \leq d$ (since $\text{cr}(\widehat{P})$ has no buckets of degree 0 or 1).

Lemma 5. *Let $\mathbf{d} = (d_1, \dots, d_r)$ be as in Lemma 4. Let $\widehat{P} \in \mathcal{P}_{n,\mathbf{d}}$. Then the 2-core $\text{cr}(\widehat{P})$ of \widehat{P} has the following properties:*

- (i) a.a.s. $t \sim n - r$ and $N'_j \sim N_j$ for $2 \leq j \leq d$,
- (ii) $\text{cr}(\widehat{P})$ is uniformly random conditioned on its degree sequence,
- (iii) a.a.s. $\text{cr}(\widehat{P})$ has no paths of length at least $K + 1$ where all internal vertices have degree 2,
- (iv) a.a.s. $\text{cr}(\widehat{P})$ has no isolated cycles.

Proof. By Lemma 4, a.a.s. all bushes and isolated trees in \widehat{P} have at most K buckets (including the root). Hence the total number of buckets of \widehat{P} contained in bushes is a.a.s. $O(\mu_1)$ unless $\lim_{n \rightarrow \infty} \mu_1 = O(1)$ and $\mu_1 \neq o(1)$, in which case an upper bound is given by $O(\log \log n)$. Note also that by Lemma 3, a.a.s.

$$N_1 = o(\mu_d) \quad \text{and} \quad \mu_d \sim n - r.$$

It follows that $\text{cr}(\widehat{P})$ has t buckets where a.a.s.

$$t = n - r - o(n - r) \sim n - r.$$

By Lemma 3 again it follows that a.a.s. $N'_j \sim N_j$ for $2 \leq j \leq d$. This proves (i).

The proof of (ii) is similar to the argument given in the proof of Lemma 3 and for similar statements in papers on cores of random graphs, so we do not include it here.

Let $m = (d'_1 + \dots + d'_t)/2$ be the number of pairs in $\text{cr}(\widehat{P})$. By (ii) we know that, conditioned on having degree sequence \mathbf{d}' , $\text{cr}(\widehat{P})$ has the distribution of $\mathcal{P}_{n,\mathbf{d}'}$. For the remainder of the proof, assume that $t \sim n - r$ and that $\mathbf{d}' = (d'_1, \dots, d'_t)$ is a degree sequence such that $2 \leq d'_i \leq d$ for $1 \leq i \leq t$ and (3) holds with N'_j in place of N_j on the right hand side, for $2 \leq j \leq d$. Using Lemma 1, the expected number of paths in $\mathcal{P}_{n,\mathbf{d}'}$ of length $K + 1$ with K internal buckets of degree 2 is at most

$$O(1) t^2 \binom{N'_2}{K} K! (2m)^{-(K+1)} = O(n) \left(\frac{N'_2}{n} \right)^K.$$

Since (3) holds by assumption, this expression is

$$O(n^{1-K}) g(n)^K$$

where $g(n)$ is defined in (4). Using calculations as in Lemma 4, this bound is $o(1)$. Hence by (i) and (ii), applying Markov’s inequality establishes (iii).

To prove (iv), let k be an integer satisfying $2 \leq k \leq K + 1$. Then using Lemma 1, the expected number of isolated k -cycles in $\mathcal{P}_{n,\mathbf{d}'}$ is at most

$$O(1) \binom{N'_2}{k} (k - 1)! (2m)^{-k} = O(1) \left(\frac{N'_2}{m}\right)^k.$$

Therefore the expected number of isolated cycles in $\mathcal{P}_{n,\mathbf{d}'}$ of length at most $K + 1$ is at most

$$O(1) \sum_{k=2}^{K+1} \left(\frac{N'_2}{m}\right)^k = O(1) \left(\frac{N'_2}{m}\right)^2 \frac{1}{1 - N'_2/m}.$$

Since (3) implies that $N'_2 = o(m)$, we see that the expected number of isolated cycles of length at most $K + 1$ in $\mathcal{P}_{n,\mathbf{d}'}$ is $o(1)$. By (i), (ii) and Markov’s inequality, there are a.a.s. no such cycles in $\text{cr}(\widehat{P})$. Finally, any cycle of length $K + 2$ or more contains a path of length $K + 1$ where all vertices have degree 2. Hence by (iii) there are a.a.s. no cycles of length $K + 2$ or more in $\text{cr}(\widehat{P})$. This completes the proof. \square

Following the calculations in this proof, we can now see why the result of Theorem 1 is best possible, in the sense outlined in the introduction. Suppose that $p = n^{-\varepsilon}$ where $\varepsilon > 0$ may be arbitrarily small. Choose a positive integer $k \geq 2$ such that $\varepsilon(d - 2)k < 1$. (By choosing small enough ε we may choose k to be arbitrarily large.) With this deletion probability we have

$$\mu_2 > \binom{d}{2} n^{1-1/k} \rightarrow \infty,$$

so the expected number of paths in $\text{cr}(\widehat{P})$ with length at least k and with at least $k - 1$ internal vertices of degree 2 is at least

$$\Omega(1) \frac{(t - k)^2}{2m} \left(\frac{N'_2 - k}{2m}\right)^{k-1} \geq \Omega(n) \left(\frac{d - 1}{2n^{1/k}}\right)^{k-1} = \Omega(n^{1/k})$$

which tends to infinity. Standard variance calculations show that the number of such paths is sharply concentrated, so there is a.a.s. at least one such path in $\text{cr}(\widehat{P})$. This implies that the expansion constant of $\text{cr}(\widehat{P})$ is at most $2/(k - 1)$. Conditioning on the event that $G(P)$ is simple, we have the same conclusion (see the end of Section 2). Hence when $p = n^{-\varepsilon}$ where $\varepsilon = o(1)$, we may take $k \rightarrow \infty$, there is no fixed positive expansion rate β , and the conclusion of Theorem 1 does not hold.

For practical applications such as the Swan networks, a constant but very small deletion probability is the most natural assumption. For the range of n of interest in the applications, the probability would be at most $n^{-\varepsilon}$ for some small positive constant ε that is not extremely small. For values of the parameters determined in this way, we would expect the asymptotic trends studied in this paper to be accurate.

Proof of Theorem 1. Fix a positive integer $d \geq 3$ and a constant $\eta > 0$ such that (1) holds for n sufficiently large. Let $P \in \mathcal{P}_{n,d}$ and form \widehat{P} from P as described in Lemma 3.

We first treat the case that $n^{1-\alpha} \rightarrow \infty$, so that (2) holds a.a.s., and we prove the conclusions of the theorem for the multigraph $G(P)$. Only at the end do we remove this assumption and translate the result to $G \in \mathcal{G}_{n,d}$. We have by Lemma 2, $\widehat{P} \in \mathcal{P}_{n,\mathbf{d}}$ where \mathbf{d} is the degree sequence after deletion. Let $\text{cr}(\widehat{P})$ be the 2-core of \widehat{P} . Then a.a.s. the conclusion of Lemmas 3–5 all hold.

Condition on the event that all these conclusions hold, and let $\text{ker}(\widehat{P})$ be the kernel of \widehat{P} . Then $\text{ker}(\widehat{P})$ is obtained from $\text{cr}(\widehat{P})$ by suppressing the degree-2 buckets. That is, if $b = \{p_1, p_2\}$ is a

degree-2 bucket in $\text{cr}(\widehat{P})$ involved in pairs $\{p_1, x\}, \{p_2, y\}$, then delete b , remove these pairs and add the pair $\{x, y\}$. Since $\text{cr}(\widehat{P})$ has no isolated cycles, $\text{ker}(\widehat{P})$ has exactly N'_j buckets of degree j for $3 \leq j \leq d$ (and no buckets of degree less than 3). For the reasons given in Lemma 5(ii), we omit the arguments that show that $\text{ker}(\widehat{P})$ is uniformly random conditioned on its degree sequence. Let $H = G(\text{ker}(\widehat{P}))$ be the multigraph obtained from $\text{ker}(\widehat{P})$ by shrinking buckets to vertices and replacing pairs by edges.

We know that for some constant $\delta > 0$ the multigraph H is a.s. a δ -expander. (This is well known: for example, [2, Lemma 5.3] states that for some $\alpha > 0$, the random multigraph arising from the pairing model $\mathcal{P}_{n,d}$ with degree sequence satisfying $3 \leq \min d_i \leq \max d_i \leq n^{0.02}$ is a.s. an α -expander. A version of the expansion result is also mentioned in [7] without proof.) The constant δ depends only on η . At this point we further condition on this asymptotically almost sure expansion event holding.

Let $G(\widehat{P})$ be the multigraph corresponding to the pairing \widehat{P} . We obtain $G(\widehat{P})$ from H by performing the following steps:

- replace some edges by paths of length at most K ,
- glue on some maximal bushes of size at most K by identifying their roots with distinct vertices,
- introduce some isolated trees of size at most K ,
- perform the appropriate relabellings of vertices.

Since we are conditioning on the event that the conclusions of Lemma 3 hold, $G(\widehat{P})$ consists of $O(N_1) = o(n)$ vertices in isolated trees of size at most K , together with a large component having $n - O(N_1)$ vertices. Let U be the large component, and let $u = |U|$. Note that $|H| = u - o(u)$. We now show that U is an expander.

Fix any subset $S \subseteq V(U)$ with $|S| \leq u/2$. By an *object* we mean any maximal bush which has been added to H , or path replacing an edge of H , or edge of H not replaced by a path, in the process of creating U from H . An object includes the vertex or vertices of H where it is attached. An object is *partially occupied* if it has some vertices in S and some not in S , and it is *fully occupied* if all its vertices belong to S .

First suppose that there are at most $\varepsilon|S|/K$ partially occupied objects, where $\varepsilon > 0$ is a constant. Then at most $\varepsilon|S|$ vertices of S are in partially occupied objects. For each fully occupied object there are at most $K - 1$ vertices not in H and at least one vertex in H . Each of these vertices is involved in at most d objects, so the number of vertices in $V(H) \cap S$ is at least $1/(d(K - 1) + 1) > 1/dK$ times the number of vertices in fully occupied objects. Since all vertices of S are in either partially or fully occupied objects, it now follows that

$$|V(H) \cap S| \geq \frac{1 - \varepsilon}{dK} |S|.$$

Let A be the set of vertices in $V(H) \setminus S$ which have a neighbour in $V(H) \cap S$. We claim that there exists a constant $\gamma > 0$ such that $|A| \geq \gamma |V(H) \cap S|$. If $|V(H) \cap S| \leq |H|/2$ then the claim follows immediately because H is a δ -expander. So we may assume that $|V(H) \cap S| > |H|/2$. Then $|B| < |H|/2$, where $B = V(H) \setminus (S \cup A)$, so the expansion of H implies that $|A| \geq \delta|B|$ and hence

$$|A| \geq \frac{\delta}{1 + \delta} |A \cup B| = \frac{\delta}{1 + \delta} |V(H) \setminus S| \geq \frac{\delta(1/2 + o(1))}{1 + \delta} |H|$$

as $|H| \sim u$ and $|S| \leq u/2$. Thus, the claim holds with $\gamma = \delta/(3 + 3\delta)$. The claim implies that there are at least

$$\gamma |V(H) \cap S| \geq \frac{\gamma(1 - \varepsilon)}{dK} |S|$$

partially occupied objects.

So we may suppose that for some $\varepsilon > 0$, there are more than $\varepsilon|S|/K$ partially occupied objects. Each partially occupied object contains an element of S with a neighbour in $V(U) \setminus S$. Since each vertex in U has degree at most d , each of these neighbours can be incident with at most d partially occupied objects. Therefore S has at least $\varepsilon|S|/dK$ neighbours outside S . It follows from this that the large component U is a constant rate expander, under our assumptions. The conclusion of part (a) of the theorem now follows for the initial random multigraph $G(P)$ in place of $G \in \mathcal{G}_{n,d}$, under the assumption that $n^{1-\alpha} \rightarrow \infty$.

For (b), we need to show further that a.a.s. the only isolated trees in \widehat{P} are isolated vertices. By Lemmas 1 and 3, the expected number of isolated trees with at least two leaves and $k - 2$ other vertices is

$$n^{k-2} n^{2(1-(d-1)\alpha)} O\left(n^{-(k-1)}\right) = O(n^{1-2(d-1)\alpha}) = o(1).$$

Hence the isolated trees are a.a.s. isolated vertices, as required. Also, the number of isolated vertices is N_0 and if $\mu_0 \rightarrow \infty$ then a.a.s.

$$N_0 \sim \mu_0 = n^{1-\alpha d} = o(n^{1-d/(2d-2)})$$

for the given bound on η . In the other cases we still have $N_0 = o(n^{(d-2)/(2d-2)})$ a.a.s., using Lemma 3.

For (c), the conclusion of (b) still applies, but in addition, in this case $\mathbb{E}N_1 = o(1)$. So there are a.a.s. no isolated trees or bushes of any size, and the conclusion of (c) follows for the multigraph $G(\widehat{P})$.

This completes the proof of the theorem except for two aspects. First, we transfer the conclusions from the initial random multigraph $G(P)$ to $G \in \mathcal{G}_{n,d}$. This is done by conditioning on the event that $G(P)$ is simple. As explained at the end of Section 2, the truth of these asymptotically almost sure results is not affected. In the conditional space, $G(\widehat{P})$ becomes \widehat{G} as in the statement of Theorem 1.

Finally, we only need to dispense with the assumption that $n^{1-\alpha} \rightarrow \infty$. Assume that $n^{1-\alpha} = O(1)$. We may apply the version of (c) already proved, to conclude that the deletion of vertices from $G \in \mathcal{G}_{n,d}$ with probability $n^{-3/4}$ a.a.s. produces a β -expander G' . If we then reinstate each deleted vertex (and incident edges) independently with probability $1 - n^{3/4-\alpha}$ (noting this is positive for n sufficiently large) then the result is the same as deleting each vertex of the original graph with probability $n^{-\alpha}$, that is, it produces \widehat{G} . The vertices deleted from G to produce G' are, by easy first moment considerations, a.a.s. of distance at least 3 from each other. In this case, reinstating them cannot create any new components, and it is easy to see that after reinstating them, the resulting graph \widehat{G} is a $(\beta/2)$ -expander when $\beta \leq 2$. The only nontrivial case is when $S \subseteq V(\widehat{G})$ satisfies $|S - W| < |S \cap W|$, where W is the set of reinstated vertices. Here each vertex of $S \cap W$ has d neighbours outside W , giving $d|S \cap W|$ distinct neighbours of $S \cap W$ outside W . At most $|S - W|$ of these can lie in S , so S has at least

$$d|S \cap W| - |S - W| \geq (d - 1)|S \cap W| \geq \frac{d - 1}{2} |S|$$

neighbours outside S in \widehat{G} . This gives $\beta/2$ -expansion when $\beta \leq 2$. \square

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