Annals of Pure and Applied Logic 54 (1991) 1–15
North-Holland

Large cardinals and iteration trees of height ω

Alessandro Andretta

Department of Mathematics and Computer Science, Dartmouth College, Hanover, NH 03755, USA

Communicated by T. Jech
Received 15 August 1990

Abstract

Andretta, A., Large cardinals and iteration trees of height ω, Annals of Pure and Applied Logic 54 (1991) 1–15.

In this paper we continue the line of work initiated in "Building iteration trees". It is shown that the existence of a certain kind of iteration tree of height ω is equivalent to the existence of a cardinal δ that is Woodin with respect to functions in the next admissible.

0. Introduction

Iteration trees, first introduced in [4], have become a crucial tool in the development of inner models for large cardinals. In [1] all sorts of iteration trees are constructed, assuming the existence of a Woodin cardinal. Conversely, [5] shows that the existence of an alternating chain (a particular kind of iteration tree) with the supremum of the critical points in the well-founded parts of the branches yields a weak form of Woodin cardinal, what here will be called a *Woodin-in-the next-admissible* cardinal, a-Woodin for short.

In this paper it is shown that the existence of an a-Woodin cardinal is enough to build iteration trees of height ω, and thus, using the above mentioned result of [5], an equivalence is obtained. To achieve this, a construction quite different from the one in [1] is used, as most of the machinery (reflecting cardinals, blocks of indiscernibles) used in [1] is not available in the present set-up. To overcome this difficulty we use a 'tree argument' to show that the iteration tree has to exist. This technique, which is due to Steel, was first used in his construction of an alternating chain.

The main result of this paper differs from the one in [1] in two respects. It uses an ostensibly weaker (in fact optimal) hypothesis, and the technique used here applies to tree orderings of height ω only. In fact, Theorem 4.1 shows that one a-Woodin cardinal is not enough to construct iteration trees of height $\omega + 2$.

We have tried to make this paper self-contained but we could not perform miracles. In particular the reader is assumed to have some acquaintance with extenders and iteration trees as developed in [4], [5] or in [1]. No knowledge of inner models is required. This paper owes a lot to [4] and [5], as any reader familiar with these works will immediately recognize. Indeed the whole subject of iteration trees would not exist if it weren't for those two papers.

For standard set-theoretic facts the reader should consult [3] or [2] for the results on admissible sets. The notation is as in [3], with some exceptions. A subset b of ω will be identified with its enumerating function, $b = \{b(0) < b(1) < \cdots\}$. rank$(x)$ denotes the rank of the set x, i.e., the least α such that $x \in V_{\alpha+1}$. A tree $\mathcal{U}$ (in the sense of Descriptive Set Theory) on a set X is a family of finite sequences from X ($\mathcal{U} \subseteq {}^{<\omega}X$), closed under subsequences ($s \in \mathcal{U} \Rightarrow s \restriction n \in \mathcal{U}$), and ordered by reverse inclusion ($s \leqslant t \Leftrightarrow s \supseteq t$). If $s \in \mathcal{U}$ and $\mathcal{U}$ is well-founded, then $\mathcal{U}[s] = \{t \in {}^{<\omega}X \mid s^\frown t \in \mathcal{U}\}$ is also a well-founded tree and its rank is the rank of s in $\mathcal{U}$. The rank of a well-founded tree $\mathcal{U}$ is denoted by $\|\mathcal{U}\|$. Also, if $\mathcal{U}$ is a tree on $X \times Y$, then we will write its elements as (s, t), where $s \in {}^{<\omega}X$, $t \in {}^{<\omega}Y$ and $\ln(s) = \mathrm{lh}(t)$. A pre-well-ordering, pwo for short, is a transitive, non-reflexive, well-founded binary relation. If M is a model of set theory, wfp(M) denotes its well-founded part.

1. Iteration trees

Let us briefly recall the definitions we will be using throughout this paper. For a more complete treatment of what follows the reader is referred to [4], [5] or [1].

Suppose we are given:

(1) a tree ordering T on ω, such that 0 is the T-least element and $n\,T\,m$ implies $n < m$; also n^* denotes the T-immediate predecessor of $n+1$;

(2) models M_n and embeddings $\pi_{n,m} : M_n \to M_m$, for $n\,T\,m$, and such that if $m\,T\,k$ then $\pi_{m,k} \circ \pi_{n,m} = \pi_{n,k}$;

(3) extenders $E_n \in M_n$ and ordinals ρ_n such that the sequence $\langle \rho_n \mid n \in \omega \rangle$ is increasing and

$$M_n \vDash \text{“}E_n \text{ is an extender of strength} \geqslant \rho_n + 1\text{”}$$

and $\pi_{n^*,n+1} = i_{E_n} : M_{n^*} \to \mathrm{Ult}(M_{n^*}, E_n) = M_{n+1}$;

(4) $n^* =$ the least $m \leqslant n$ such that the critical point of $E_n \leqslant \rho_m$.

In this case $\mathcal{T} = \langle T, \langle (E_n, \rho_n) \mid n \in \omega \rangle \rangle$ is an iteration tree of height ω over the model M_0. We will only be concerned with the case $M_0 = V$. The ordinal ω is called the height (or length) of $\mathcal{T}$. Iteration trees of transfinite height are studied in [5] and [1] and will be briefly discussed in Section 4. Sometimes the notion of a *finite* iteration tree will be used. Such trees are defined just as above except for the fact that the tree ordering T is defined on some $n < \omega$.

A few elementary facts about iteration trees that will be used in the sequel are listed here.

Facts. (1) *If* $n < m$, *then the models* M_n *and* M_m *agree up to* $\rho_n + 1$, *i.e.*, $V_{\rho_n+1} \cap M_n = V_{\rho_n+1} \cap M_m$.

(2) *If* n, $m+1$, $k+1$ *are consecutive elements on* T, *i.e.*, $n = m^*$ *and* $m+1 = k^*$, *then* $\mathrm{crit}(E_k) > \rho_m$, *that is on any branch, there is no overlap between the* ρ*'s and the critical points of the* π*'s.*

(3) *If* b *and* c *are infinite branches of* T, *then* $\sup\{\rho_n \mid n \in b\} = \sup\{\rho_n \mid n \in c\}$.

(4) *If* $\mathcal{T}$ *is a finite iteration tree of length* $n+1$, *and* E_n *is an extender in* M_n, $\mathrm{crit}(E_n) \leqslant \rho_i$, *then* $\mathrm{Ult}(M_i, E_n)$ *is well-founded. Thus well-foundedness is never a problem in extending a finite iteration tree.*

It was shown in [5] that if $\mathcal{T}$ is an iteration tree of length ω over V and T its tree ordering, then T must have an infinite branch. Hence from now on we will just consider tree orderings with such property.

Let T be a tree ordering and b an infinite branch. It is natural to ask whether the direct limit $M_b = \lim_{n \in b} M_n$ is well-founded. In the affirmative case the branch b is said to be well-founded, ill-founded otherwise. Does every iteration tree have a well-founded branch? Is such branch unique? The following two (still unproved!) conjectures settle the problems affirmatively.

Conjectures [5].

CBH$_\omega$ The Cofinal Branch Hypothesis for Trees of Height ω

Every iteration tree $\mathcal{T}$ of height ω over V has an infinite well-founded branch b.

UBH$_\omega$ The Unique Branch Hypothesis for Trees of Height ω

Every iteration tree $\mathcal{T}$ of height ω over V, has at most one infinite well-founded branch b.

CBH$_\omega$ and **UBH**$_\omega$ are special cases of conjectures (see [5] or [1]) dealing with iteration trees on V of arbitrary countable length. Although neither **UBH**$_\omega$ nor **CBH**$_\omega$ has been settled at the time of the writing, there are several partial results due to Martin and Steel concerning them. One of these results is of interest to us here. First some notation.

Definition. Let b be an infinite branch of the tree ordering T of an iteration tree $\mathcal{T}$ of height ω. Set $\delta_\omega = \sup\{\rho_n \mid n \in b\}$.

By Facts (1) and (2) above, δ_ω does not depend on the particular branch chosen, as $\delta_\omega = \sup\{\rho_n \mid n \in \omega\}$, and if b and c are infinite branches $M_b \cap V_{\delta_\omega} = M_c \cap V_{\delta_\omega}$.

Theorem 1.1 [5]. *Let b, c be infinite branches of the iteration tree $\mathcal{T}$ of height ω. If $\delta_\omega \in \mathrm{wfp}(M_b) \cap \mathrm{wfp}(M_c)$, then for any $\alpha \in \mathrm{wfp}(M_b) \cap \mathrm{wfp}(M_c)$, $\alpha > \delta_\omega$*

$$L_\alpha(V_{\delta_\omega} \cap M_b) \models \text{“}\delta_\omega \textit{ is a Woodin cardinal”}.$$

Hence if $\mathbf{UBH}_\omega$ fails, $L(V_{\delta_\omega} \cap M_b) \models$ “ZF + δ_ω is a Woodin cardinal”. (Choice can be added generically so to get a model of ZFC + “there is a Woodin cardinal”.) As the well-founded part of an admissible set is an admissible set, Theorem 1.1 suggests the following

Definition. δ is *Woodin-in-the-next-admissible, a-Woodin* for short, iff for every $f \in {}^\delta\delta \cap L_\alpha(V_\delta)$ there is $\kappa < \delta$ such that $f[\kappa] \subseteq \kappa$ and there is an extender $E \in V_\delta$, $\mathrm{crit}(E) = \kappa$ and such that

$$\mathrm{Ult}(V, E) \supset V_{i_E(f)(\kappa)}$$

where α is the least admissible ordinal over V_δ.

So in particular Theorem 1.1 yields,

Corollary 1.2. *Let b, c be infinite branches of an iteration tree $\mathcal{T}$ of height ω over V. If $\delta_\omega \in \mathrm{wfp}(M_b) \cap \mathrm{wfp}(M_c)$, then δ_ω is an a-Woodin cardinal.*

In Section 3 of this paper we prove a converse to 1.2, namely,

Theorem 1.3. *If δ is a-Woodin and T is a tree ordering on ω with an infinite branch, then there are E_n, $\rho_n \in V_\delta$ such that $\langle T, \langle (E_n, \rho_n) \mid n \in \omega \rangle \rangle$ is an iteration tree and for every infinite branch b, $\delta_\omega \in \mathrm{wfp}(M_b)$.*

Hence 1.2 and 1.3 establish an equivalence between a large cardinal hypothesis and the existence of certain kinds of iteration trees. Something more can be said about well-founded branches.

Definition [4]. An iteration tree $\mathcal{T}$ of height ω is *continuously ill-founded off b*, b an infinite branch of T, if there are ordinals $\{\gamma_n \mid n \in \omega \setminus b\}$ such that for $n, m \in \omega \setminus b$, $n\,T\,m$ implies $\pi_{n,m}(\gamma_n) \geqslant \gamma_m$ and if there is a $k \in b$ with $n < k < m$, then $\pi_{n,m}(\gamma_n) > \gamma_m$.

It is clear that for any infinite branch $c \neq b$, $\{\pi_{n,c}(\gamma_n) \mid n \in c\}$ witness the ill-foundedness of M_c. On the other hand M_b is well-founded.

Theorem 1.4 [4]. *If $\mathcal{T}$ is an iteration tree of height ω over V, continuously ill-founded off b, b an infinite branch of T, then M_b is well-founded.*

The proof of Theorem 1.3 can be modified to show

Theorem 1.5. *Let δ be a-Woodin, T a tree ordering on ω, b an infinite branch of T. Then there are E_n, $\rho_n \in V_\delta$ such that $\langle T, \langle (E_n, \rho_n) \mid n \in \omega \rangle \rangle$ is continuously ill-founded off b and for every branch c of T, $\delta_\omega \in \mathrm{wfp}(M_c)$.*

2. Strength and admissibility

Let us observe a few facts about the large cardinal hypothesis involved in the statements of 1.3 and 1.5. Woodin cardinals are a-Woodin, and an easy Skolem-hull argument shows that the converse is not true. Indeed the least a-Woodin cardinal has confinality ω, while a full-fledged Woodin cardinal is inaccessible, and hence regular, and there is a closed unbounded set of smaller a-Woodins. On the other hand some facts about Woodin cardinals do relativize to the admissible case. The proof of Lemma 4.2 in [4] characterizing Woodin cardinals as being 'strong' with respect to any given $A \subseteq V_\delta$, goes through verbatim here.

Lemma 2.1. *Let $M \supset V_\delta$ be an admissible set and suppose δ is a Woodin cardinal in M. Then for $A \in V_{\delta+1} \cap M$, the set of κ such that for every $\alpha < \delta$ there exists an extender $E \in V_\delta$ with $\mathrm{crit}(E) = \kappa$, $\mathrm{str}(E) = \eta \geq \alpha$ and $i_E(A) \cap V_\eta = A \cap V_\eta$, is unbounded in δ.*

Definition (Martin). Let $W \subseteq V_\delta \times V_\delta$ be a pre-well-ordering (pwo) of V_δ, let $\kappa < \delta$ and let $a \in V_\delta$. The rank of a in W is denoted by $|a|_W$.

If $|a|_W = 0$, then κ is *$|a|_W$-strong* if and only if for every $\alpha < \delta$ there exists an extender $E \in V_\delta$ with critical point κ and strength $\geq \alpha$.

Suppose now that $|a|_W > 0$. κ is *$|a|_W$-strong* if and only if for every $b \in V_\delta$, such that $|b|_W < |a|_W$ and for every α, there exists an extender $E \in V_\delta$ and ordinals $\bar{\kappa} > \max(\alpha, \kappa)$ and $\eta > \bar{\kappa}$ such that $\mathrm{crit}(E) = \kappa$, $\delta > \eta = \mathrm{str}(E)$, $b \in V_\eta$ and

$$\mathrm{Ult}(V, E) \models \text{“}\bar{\kappa} \text{ is } |b|_{i_E(W)}\text{-strong”}$$

and $i_E(W) \restriction V_\eta = W \restriction V_\eta$ and for all $\nu > \eta$, for all $c \in V_\eta$, if $|c|_W < |a|_W$ then

$$V \models \text{“}\nu \text{ is } |c|_W\text{-strong”} \quad \Leftrightarrow \quad \mathrm{Ult}(v, E) \models \text{“}\nu \text{ is } |c|_{i_E(W)}\text{-strong”}.$$

Some remarks are in order here. If κ is $|a|_W$-strong and $|a'|_W \leq |a|_W$, then κ is $|a'|_W$-strong. If W is a pre-well-ordering of V_δ and $W \in L_\alpha(V_\delta)$, where α is the least admissible ordinal over V_δ, then W is said to be an admissible pwo. Also the above definition involves a pre-well-ordering, rather than a well-ordering, as for any $\beta < \alpha$ there is an admissible pwo $W \subseteq V_\delta \times V_\delta$ of length β, while, as $L_\alpha(V_\delta)$ does not necessarily satisfy choice, the same need not be true of well-orderings.

Lemma 2.2. *Let δ be a-Woodin and β be the least admissible over V_δ. Let $W \in L_\beta(V_\delta)$ be a pwo. Then $\{(\kappa, a) \mid \kappa \text{ is } |a|_W\text{-strong}\} \in L_\beta(V_\delta)$.*

Proof. Let $F(a) = \{\kappa < \delta \mid \kappa \text{ is } |a|_W\text{-strong}\}$ for $a \in V_\delta$. It is enough to show that $F \in L_\beta(V_\delta)$. Unraveling the definition of F we have

$\kappa \in F(a) \iff \kappa > \delta$ and for all $\alpha, b \in V_\delta$ with $|b|_W < |a|_W$ there are an inaccessible λ, $E \in V_\lambda$, $\operatorname{crit}(E) = \kappa$, $\operatorname{str}(E) = \eta < \lambda < \delta$, and $\bar\kappa < \lambda$ with $\max(\kappa, \alpha) < \bar\kappa < \eta$ and there are j, X, Z in V_δ such that $\operatorname{Ult}(V_\lambda, E) = X$, $j = i_E^{V_\lambda}$, $Z = j(W)$, $Z \restriction V_\eta = W \restriction V_\eta$, $V_\eta \subseteq X$, $X \models \bar\kappa \in j(f)(b)$, where $f = F \restriction \{x \in V_\delta \mid |x|_W < |a|_W\}$, and for all $\nu < \eta$, for all $c \in V_\eta$, $|c|_W < |a|_W$ implies $(\nu \in f(c) \Leftrightarrow X \models \nu \in j(f)(c))$.

A quick inspection shows that

$$F(a) = \{\kappa < \delta \mid \varphi(V_\delta, a, \kappa, F \restriction \{x \in V_\delta \mid |x|_W < |a|_W\})\}$$

where φ is a $\Delta_1^{L_\beta(V_\delta)}$-formula, so by Σ_1-Recursion on the well-founded relation W in $L_\beta(V_\delta)$, $F \in L_\beta(V_\delta)$.

Lemma 2.3. *Let δ be a-Woodin and let W be an admissible pwo of V_δ. Then for any $a \in V_\delta$ the set $\{\kappa < \delta \mid \kappa \text{ is } |a|_W\text{-strong}\}$ is unbounded in δ.*

Proof. By induction on the ordinal $|a|_W$.

The case when $|a|_W = 0$ follows at once from 2.1 so we may assume $|a|_W > 0$ and that for all $b \in V_\delta$, $|b|_W < |a|_W$, the set of $|b|_W$-strong cardinals is unbounded in δ. Then 2.2 implies that $A = \{(\lambda, b) \mid \lambda \text{ is } |b|_W\text{-strong and } |b|_W < |a|_W\} \in L_\beta(V_\delta)$ where β is as in 2.2. By 2.1 there are unboundedly many $\kappa < \delta$ such that for any $\alpha < \delta$ there exists an extender $E \in V_\delta$ such that $\operatorname{crit}(E) = \kappa$, $\operatorname{str}(E) = \eta \geqslant \alpha$ and $i_E(A) \cap V_\eta = A \cap V_\eta$. We claim that any such κ is $|a|_W$-strong.

Given any $|b|_W < |a|_W$ and $\alpha < \delta$, choose $\bar\kappa > \max(\alpha, \kappa)$ to be $|b|_W$-strong. Such a $\bar\kappa$ exists by our inductive hypothesis. Let $E \in V_\delta$ be an extender with critical point κ and strength $\eta > \bar\kappa$. Then for $\nu < \eta$ and $|c|_W < |a|_W$, $c \in V_\eta$,

$$\begin{aligned} V \models \text{“}\nu \text{ is } |c|_W\text{-strong”} &\iff (\nu, c) \in A \cap V_\eta \\ &\iff (\nu, c) \in i_E(A) \cap V_\eta \\ &\iff \operatorname{Ult}(V, E) \models \text{“}\nu \text{ is } |c|_{i_E(W)}\text{-strong”}. \end{aligned}$$

Thus κ is $|a|_W$-strong and this is what we had to prove. □

The next result is the analogue of the One Step Lemma in [4] (see also 3.4 and 3.5 in [1]) for cardinals strong in a pwo W.

Lemma 2.4. *Assume M and N are admissible sets, $W \in M$ a pwo of V_δ^M, $Z \in N$ a pwo of V_δ^N, $M \cap V_{\kappa+1} = N \cap V_{\kappa+1}$ and $W \upharpoonright V_\kappa = Z \upharpoonright V_\kappa$. Suppose κ is $|a|_W$-strong in M, $|b|_W < |a|_W$ and $\alpha < \delta$. Suppose also that for all $\nu < \kappa$, for all $c \in V_\kappa$ with $|c|_W < |a|_W$*

$$M \models \text{“}\nu \textit{ is } |c|_W\textit{-strong”} \quad \Leftrightarrow \quad N \models \text{“}\nu \textit{ is } |c|_Z\textit{-strong”}.$$

Then there are E, $\bar{\kappa}$ and *η, $\operatorname{crit}(E) = \kappa$, $\operatorname{str}(E) = \eta > \max(\bar{\kappa}, \alpha)$, $b \in V_\eta$ such that, if $\bar{N} = \operatorname{Ult}(N, E)$* is well-founded and *$\bar{Z} = i_E^N(Z)$, then*

$$\bar{N} \models \text{“}\bar{\kappa} \textit{ is } |b|_{\bar{Z}}\textit{-strong”}$$

and for all $\nu < \bar{\kappa}$, for all $c \in V_\eta$, $|c|_W < |b|_W$

$$M \models \text{“}\nu \textit{ is } |c|_W\textit{-strong”} \quad \Leftrightarrow \quad \bar{N} \models \text{“}\nu \textit{ is } |c|_{\bar{Z}}\textit{-strong”}.$$

Proof. Let $\bar{\kappa}$, η, E as in the definition of κ being strong with respect to W, and assume $\bar{N}$ is well-founded. Then $\bar{N} \cap V_{i_E^N(\kappa)} = \operatorname{Ult}(M, E) \cap V_{i_E^M(\kappa)}$ and $\bar{Z} \upharpoonright V_{i_E^N(\kappa)} = i_E^N(Z \upharpoonright V_\kappa) = i_E^M(W \upharpoonright V_k)$ so $\bar{Z} \upharpoonright V_\eta = i_E^M(W) \upharpoonright V_\eta$. Let $A = \{(\nu, c) \mid \nu < \kappa$ and $c \in V_\delta \cap M$ and $(\nu$ is $|c|_W$-strong$)^M\}$. By 2.2, $A \in M$ so $i_E^N(A) \cap V_{i_E(\kappa)} = i_E^M(A) \cap V_{i_E(\kappa)}$. Thus for $\nu < \bar{\kappa} < i_E(\kappa)$ and $c \in V_\eta$, $|c|_W < |b|_W$,

$$\begin{aligned} M \models \text{“}\nu \text{ is } |c|_W\text{-strong”} \quad &\Leftrightarrow \quad (\nu, c) \in A \cap V_\eta \\ &\Leftrightarrow \quad (\nu, c) \in i_E^M(A) \cap V_\eta \\ &\Leftrightarrow \quad (\nu, c) \in i_E^N(A) \cap V_\eta \\ &\Leftrightarrow \quad \bar{N} \models \text{“}\nu \text{ is } |c|_{\bar{Z}}\text{-strong”} \end{aligned}$$

and this is what we had to prove. □

3. The construction

We are now ready to prove Theorem 1.3, namely that every tree ordering T on ω with an infinite branch can be realized as an iteration tree on V such that $\delta_\omega \in \operatorname{wfp}(M_c)$ for any branch c. For technical reasons that will be clear in the proof of 1.5, we also require that the critical points of the embeddings departing from the chosen branch b to be smaller than the critical points of the embeddings on b. Let us make this into a definition.

Definition. Let T be a tree ordering on ω, b an infinite branch of T. For $\theta \leq \omega$, $\mathcal{T} = \langle T \upharpoonright 1 + \theta, \langle (E_n, \rho_n) \mid n \in \theta \rangle, \langle \sigma_n \mid n \in b \cap \theta \rangle \rangle$ is a *b-regular iteration tree* iff, whenever $m, j < \theta$, $m = j^*$ then

(1) if $m \notin b$ or $j + 1 \in b$, then $\operatorname{crit}(E_j) = \rho_m$;

(2) if $m \in b$ and $j + 1 \notin b$, then $\operatorname{crit}(E_j) = \sigma_m < \rho_m$ and $\forall k < m$ $(\operatorname{rank}(E_k) < \sigma_m)$;

(3) if $j + 1 = k^*$ for some $k < \theta$, then there exists an inaccesible cardinal λ, $\operatorname{rank}(E_j) < \lambda < \operatorname{crit}(\pi_{j+1,k+1})$.

Lemma 3.1. *If $\mathcal{T}$ is b-regular, then for every infinite branch c of T,* $\delta_\omega \in \mathrm{wfp}(M_c)$.

Proof. The critical points on each branch are strictly increasing (see Fact (2) after the definition of iteration tree) and converge to δ_ω. Thus $\delta_\omega \subseteq M_c$. The third clause in the definition above ensures that each one of those ω many inaccessibles is not moved by any embedding on c. Hence $\pi_{0,c}(\delta_\omega) = \delta_\omega \in \mathrm{wfp}(M_c)$. $\square$

The next result shows how to construct finite b-regular iteration trees. For the sake of readability we will sometimes write $\boldsymbol{S}(\kappa, a, W)$ for "κ *is* $|a|_W$-*strong*".

Proposition 3.2. *Let δ be a-Woodin and let α be the least admissible over V_δ. Suppose $\langle T \restriction n+1,\ \langle (E_k, \rho_k) \mid k<n \rangle,\ \langle \sigma_k \mid k \in n \cap b \rangle \rangle \in V_\delta$ is a b-regular iteration tree of length $n+1$, where T is a tree ordering on ω and b an infinite branch. Suppose also that $W \in L_\alpha(V_\delta)$ is a pwo of V_δ and that we are given ordinal $\xi < \delta$, a natural number $m \geq 1$ and some $p, q \in V_\delta$, with $|p|_W \geq |q|_W + m$ such that*

$$M_n \vDash \forall k < n\ (\boldsymbol{S}(\rho_k, p, W_n) \text{ and } \boldsymbol{S}(\sigma_k, p, W_n), \text{ for } k \in b)$$

and for every $k \leq n$, $\nu < \rho_k$, $r \in V_{\rho_k}$ with $|r|_W < |p|_W$,

$$M_n \vDash \boldsymbol{S}(\nu, r, W_n) \quad \Leftrightarrow \quad M_k \vDash \boldsymbol{S}(\nu, r, W_k)$$

and

$$W_n \restriction V_{\rho_k} = W_k \restriction V_{\rho_k}$$

where $W_k = \pi_{0,k}(W)$. Then there are $E_i, \rho_i, \sigma_i \in V_\delta$ for $n \leq i < n+m$ such that $\langle T \restriction n+m+1,\ \langle (E_k, \rho_k) \mid k < n+m \rangle,\ \langle \sigma_k \mid k \in b \cap n+m \rangle \rangle$ is b-regular, $\rho_n > \xi$ and

$$M_{n+m} \vDash \forall k < n+m\ (\boldsymbol{S}(\rho_k, q, W_{n+m}) \text{ and } \boldsymbol{S}(\sigma_k, q, W_{n+m}) \text{ for } k \in b)$$

and for every $k \leq n+m$, $\nu < \rho_k$, $r \in V_{\rho_k}$ with $|r|_W < |p|_W$,

$$M_{n+m} \vDash \boldsymbol{S}(\nu, r, W_{n+m}) \quad \Leftrightarrow \quad M_k \vDash \boldsymbol{S}(\nu, r, W_k)$$

and

$$W_{n+m} \restriction V_{\rho_k} = W_k \restriction V_{\rho_k}.$$

Proof. By induction on m, and the case $m=1$ is where the heart of the matter lies. Thus suppose $m=1$. Recall that n^* is the immediate T-predecessor of $n+1$. There are two possibilities.

Case 1: $n^* = n$. Let $\lambda < \delta$ be the least inaccessible larger than $\max(\sup\{\mathrm{rank}(E_k) \mid k<n\},\ \xi,\ \mathrm{rank}(q))$. If $n \in b$ pick two ordinals $\rho_n > \sigma_n > \lambda$ such that, in M_n, ρ_n and σ_n are $|q|_{W_n}$-strong. If instead $n \notin b$, pick just one ordinal $\rho_n > \lambda$, $|q|_{W_n}$-strong in M_n. This is possible by 2.4. By the definition of

$|q|_{W_n}$-strong, there is an extender $E_n \in V_\delta \cap M_n$ with $\operatorname{crit}(E_n) = \rho_n$ if $n+1 \in b$, $\operatorname{crit}(E_n) = \sigma_n$ otherwise, and strength $\rho_n + 1$ such that, letting $M_{n+1} = \operatorname{Ult}(M_n, E_n)$, for all $v \leqslant \rho_n$, for all $r \in V_{\rho_n} \cap M_n$, $|r|_{W_n} < |q|_{W_n}$,

$$M_{n+1} \vDash \boldsymbol{S}(v, r, W_{n+1}) \iff M_n \vDash \boldsymbol{S}(v, r, W_n),$$

$$W_{n+1} \upharpoonright V_{\rho_{n+1}} = W_k \upharpoonright V_{\rho_k},$$

and so, in particular,

$$M_{n+1} \vDash \forall k < n+1\ (\boldsymbol{S}(\rho_k, q, W_{n+1}) \text{ and } \boldsymbol{S}(\sigma_k, q, W_n) \text{ if } k \in b).$$

Case 2: $n^* < n$. Let $\lambda < \delta$ be as in Case 1. By 3.1 applied to $M = M_n$ and $N = M_{n^*}$, $\kappa = \rho_{n^*}$ and get $E_n \in V_\delta \cap M_n$, choose $E_n \in V_\delta \cap M_n$ with $\operatorname{crit}(E_n) = \sigma_{n^*}$ if $n^* \in b$ and $n+1 \notin b$; $\operatorname{crit}(E_n) = \rho_{n^*}$ otherwise. Let $\rho_n + 1 = \operatorname{str}(E_n) > \lambda$ be $|q|_{W_{n+1}}$-strong in M_{n+1}. If $n \in b$, pick also an ordinal σ_n, $|q|_{W_{n+1}}$-strong and such that $\rho_n > \sigma_n > \lambda$. Moreover make sure that for all $k \leqslant n+1$, $v \leqslant \rho_n$, $r \in V_{\rho_k} \cap M_k$, $|r|_{W_k} < |q|_{W_k}$

$$M_{n+1} \vDash \boldsymbol{S}(v, r, W_{n+1}) \iff M_k \vDash \boldsymbol{S}(v, r, W_k),$$

$$W_{n+1} \upharpoonright V_{\rho_k} = W_k \upharpoonright V_{\rho_k},$$

and so, in particular,

$$M_{n+1} \vDash \forall k < n+1\ [\boldsymbol{S}(\rho_k, r, W_{n+1}) \text{ and } \boldsymbol{S}(\sigma_k, r, W_{n+1}) \text{ if } k \in b].$$

This completes the proof in the case when $m = 1$.

Assume now the result is true for some $m \geqslant 1$ and let us prove it for $m+1$. Let $p' \in V_\delta$ be such that $|p|_{W_n} \geqslant |p'|_{W_n} + m > |q|_{W_n}$. By inductive hypothesis, our tree can be extended up to length $n + m + 1$ using the parameters p and p' and then arguing as in the case $m = 1$ with p' and q we get the full result. □

We can now prove Theorem 1.3. We will actually prove something stronger.

Theorem 1.3. *Suppose δ is an a-Woodin cardinal, T a tree ordering on ω, b an infinite branch of T. Then there are E_n, ρ_n, $\sigma_n \in V_\delta$ such that $\mathcal{T} = \langle T, \langle (E_n, \rho_n) \mid n \in \omega \rangle, \langle \sigma_n \mid n \in b \rangle \rangle$ is a b-regular iteration tree on V and $\delta_\omega \in \operatorname{wfp}(M_c)$ for every branch c.*

Proof. Recall our notational convention: a subset b of ω is identified with its enumerating function, $b = \{b(0) < b(1) < \cdots\}$. Consider the tree (in the sense of Descriptive Set Theory) of attempts to construct $\mathcal{T}$ defined as follows:

$$\begin{aligned}(s, t) \in \mathcal{U} \iff\ & \operatorname{lh}(s) = \operatorname{lh}(t) = n,\ s, t \in V_\delta, \text{ and for } k < n,\ s(k) = \sigma_{b(k+1)},\\ & t(k) = \langle (E_i, \rho_i) \mid b(k) \leqslant i < b(k+1) \rangle, \text{ such that}\\ & \langle T \upharpoonright b(n) + 1, \langle (E_k, \rho_k) \mid k < b(n) \rangle, \langle \sigma_{b(k+1)} \mid k < n \rangle \rangle\\ & \text{is a } b\text{-regular iteration tree.}\end{aligned}$$

As usual, we set $(s, t) \leqslant (s', t')$ iff $s \supseteq s'$ and $t \supseteq t'$. Any infinite branch of $\mathcal{U}$ yields a b-regular iteration tree as desired and vice versa, so it is enough to show that $\mathcal{U}$ is ill-founded. Suppose, towards a contradiction, that this is not the case, i.e., suppose from now on that $\mathcal{U}$ is well-founded.

As $(\mathcal{U}, \leqslant) \in L_\alpha(V_\delta)$, where α is the least admissible ordinal over V_δ, there is $\mathbf{r} \in L_\alpha(V_\delta)$, $\mathbf{r}: \mathcal{U} \to \{\nu < \theta \mid \nu = \bigcup \nu\}$ and such that $(s, t) < (s', t')$ implies $\mathbf{r}(s, t) < \mathbf{r}(s', t')$.

Let $W \in L_\alpha(V_\delta)$ be a pre-well-ordering of V_δ of length $\theta + \omega$. The map $\mathbf{r}$ will now be used together with 2.4 to define an infinite descending sequence in $\mathcal{U}$, $(\emptyset, \emptyset) = (s_0, t_0) > (s_1, t_1) > (s_2, t_2) > \cdots$, with $\mathrm{lh}(s_n) = n$, together with sets $p_n \in V_\delta$ such that

$(1)_n$ $M_{b(n)} \models \forall m < b(n)[\boldsymbol{S}(\rho_m, p_n, W_{b(n)})$ and $\boldsymbol{S}(\sigma_m, p_n, W_{b(n)})$ if $m \in b]$,

and for $m < b(n)$, $\nu \leqslant \rho_{b(n)}$, $q \in W_{\rho_{b(n)}}$

$(2)_n$ $M_{b(n)} \models \boldsymbol{S}(\nu, q, W_{b(n)}) \Leftrightarrow M_m \models \boldsymbol{S}(\nu, q, W_m)$,

$(3)_n$ $W_{b(n)} \restriction V_{\rho_m} = W_m \restriction V_{\rho_m}$,

$(4)_n$ for $n \geqslant 1$, $\pi_{0,b(n)}(\mathbf{r}(s_{n-1}, t_{n-1})) = |p_n|_{W_{b(n)}}$

where $W_m = \pi_{0,m}(W)$.

Choose $p_0, p_1 \in V_\delta$ with $|p_0|_W = \theta + \omega$, $|p_1|_W = \theta$ and $s_0 = t_0 = \emptyset$. By Lemma 3.2 with $p_0 = p$, $p_1 = q$, $n = 0$ and $m = b(1)$ we get a b-regular tree $\langle T \restriction (b(1) + 1), t_1, s_1 \rangle$ satisfying $(1)_1$, $(2)_1$, $(3)_1$ and $(4)_1$. Suppose now $s_n t_n$ and p_n have been defined so that $(1)_n$, $(2)_n$, $(3)_n$ and $(4)_n$ hold. Pick $p_{n+1} \in V_\delta$ such that $|p_{n+1}|_{W_{b(n)}} = \pi_{0,b(n)}(\mathbf{r}(s_n, t_n))$. Then $(4)_n$ and $\mathbf{r}(s_n, t_n) < \mathbf{r}(s_{n-1}, t_{n-1})$ imply $|p_n|_{W_{b(n)}} > |p_{n+1}|_{W_{b(n)}}$. By Lemma 3.2 again, using $p_n = p$, $p_{n+1} = q$ and $\xi = \mathrm{rank}(p_{n+1})$ there are s_{n+1} and t_{n+1} that extend the iteration tree up to length $b(n + 1) + 1$. It is immediate to show that $(1)_{n+1}$, $(2)_{n+1}$, $(3)_{n+1}$ hold. For $(4)_{n+1}$ note that

$$\begin{aligned} \pi_{0,b(n+1)}(\mathbf{r}(s_n, t_n)) &= \pi_{b(n),b(n+1)}(\pi_{0,b(n)}(\mathbf{r}(s_n, t_n))) \\ &= \pi_{b(n),b(n+1)}(|p_{n+1}|_{W_{b(n)}}) = |p_{n+1}|_{W_{b(n+1)}} \end{aligned}$$

where the last equality holds because every extender in t_{n+1} (and hence the embedding $\pi_{b(n),b(n+1)}$) have critical points larger than $\xi = \mathrm{rank}(p_{n+1})$.

This completes the definition of the infinite descending sequence $\langle (s_n, t_n) \mid n \in \omega \rangle$, witnessing that $(\mathcal{U}, \leqslant)$ was ill-founded, after all. This is what we had to prove. □

Theorem 1.3 does not tell us anything about the well-foundedness of the various branches of the iteration tree $\mathcal{T}$. On the other hand the construction of $\mathcal{T}$ revolves around the branch b, so it would be natural to expect $\mathcal{T}$ to be continuously ill-founded off b. Note also that we did not make any use of the ordinals σ_n's. So far, if anything, they were only a burden. Their presence is vindicated, though, in the proof of the next result, Theorem 1.5, where they are instrumental in showing continuity of ill-foundedness. Recall the statement of

Theorem 1.5. *Let δ be a-Woodin and let T be a tree ordering on ω, b an infinite branch of T. Then there are ρ_n, $E_n \in V_\delta$, such that $\mathcal{T} = \langle T, \langle (E_n, \rho_n) \mid n \in \omega \rangle \rangle$ is an iteration tree on V, continuously ill-founded off b and $\delta_\omega \in \mathrm{wfp}(M_c)$, for any infinite branch c. Hence M_b is well-founded, while all the other M_c's are ill-founded.*

Proof. As in the proof of 1.3 we will construct a $\mathcal{T}$ that is actually b-regular. Let $(\mathcal{U}, \leqslant)$ be as in the proof of 1.3. If (s, t) and (s', t') are elements of $\mathcal{U}$, we say that $(s, t) \lhd (s', t')$ iff there is $n < \min(\mathrm{lh}(s), \mathrm{lh}(s'))$ such that

$$(s \restriction n, t \restriction n) = (s' \restriction n, t' \restriction n) \quad \text{and} \quad s(n) < s'(n).$$

Similarly for (F, G) and (F', G') branches of $\mathcal{U}$,

$$(F, G) \lhd (F', G') \quad \Leftrightarrow \quad \exists n\, ((F, G) \restriction n = (F', G') \restriction n \text{ and } F(n) < F'(n)).$$

So a b-regular iteration tree $\langle T, \langle (E_n, \rho_n) \mid n \in \omega \rangle, \langle \sigma_n \mid n \in b \rangle \rangle$ given by (F, G) is $\lhd$ than $\langle T, \langle (E_n', \rho_n') \mid n \in \omega \rangle, \langle \sigma_n' \mid n \in b \rangle \rangle$ given by (F', G') if for some $m \in b$, for $i < m$, $j \in b \cap m$

$$E_i = E_i', \qquad \rho_i = \rho_i', \qquad \sigma_j = \sigma_j' \quad \text{and} \quad \sigma_m < \sigma_m'.$$

Claim 1. *There exists a $\lhd$-minimal branch (F, G) of $\mathcal{U}$.*

Proof of Claim 1. Suppose $(F, G) \restriction n$ has been defined so that $\mathcal{U}[(F, G) \restriction n]$ is ill-founded. Choose $t = \langle (E_i, \rho_i) \mid b(n) \leqslant i < b(n+1) \rangle \in V_\delta$ and $\sigma < \delta$ so that $\mathcal{U}[(F \restriction n)^\frown\langle \sigma \rangle, (G \restriction n)^\frown t]$ is ill-founded and σ is least such. Set $F(n) = \sigma$ and $G(n) = t$.

This defines a $\lhd$-minimal (F, G) as desired and proves Claim 1. $\square$

Fix, from now on, a $\lhd$-minimal branch (F, G) of $\mathcal{U}$ as of Claim 1.

Our goal is to show that the iteration tree given by (F, G) is continuously ill-founded off b. Hence ordinals γ_n, for $n \in \omega \setminus b$, must be defined so that if $n\, T\, m$ then $\pi_{n,m}(\gamma_n) \geqslant \gamma_m$ and if there is $n < b(k) < m$, then the inequality is strict. The γ_n's will be ranks of certain well-founded trees defined in terms of $\mathcal{U}$ and the embeddings $\pi_{n,m}$.

Definition. For $n \in \omega \setminus b$ let

$\bar{n}$ = the largest i such that $b(i) < n$,
$\underline{n}$ = the largest i such that $b(i)\, T\, n$.

See Fig. 1.

Thus $\underline{n} \leqslant \bar{n}$. Notice that it is enough to define the γ_n's for those $n \in \omega \setminus b$ such that $\underline{n} < \bar{n}$: for the remaining $n \in \omega \setminus b$ set

$$\gamma_n = \sup\{\gamma_m + 1 \mid (m-1)^* = n\}.$$

So assume from now on that $n \in \omega \setminus b$ and $\underline{n} < \bar{n}$.

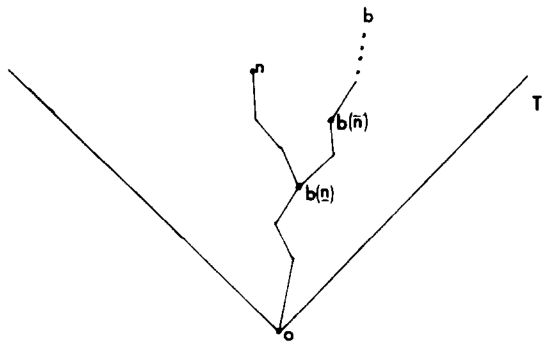

Fig. 1.

Claim 2. $\pi_{0,b}(F, G) \restriction \bar{n} = \pi_{0,b(\bar{n})}(F, G) \restriction \bar{n} \in M_n \cap V_{\sigma_{b(n)}}$.

Proof of Claim 2. In order to simplify the notation a bit, let $m = b(\bar{n})$ and $\sigma = \sigma_m$. By the second clause in the definition of b-regularity, $(F, G) \restriction \bar{n} \in V_\sigma$ and as $\pi_{0,m}$ does not move σ, $\pi_{0,m}(F, G) \restriction \bar{n} \in V_\sigma \cap M_m$. As $\mathrm{crit}(\pi_{m,b}) = \rho_m > \sigma$,

$$\pi_{0,b}(F, G) \restriction \bar{n} = \pi_{m,b}(\pi_{0,m}(F, G) \restriction \bar{n}) = \pi_{0,m}(F, G) \restriction \bar{n}.$$

Hence $(F, G) \restriction \bar{n} \in M_m \cap V_\sigma = M_n \cap V_\sigma$, as the two models agree up to $\rho_m > \sigma$.
This proves Claim 2. $\square$

Claim 3. $\pi_{0,b}(F, G) \restriction \underline{n} + 1 \lhd \pi_{0,n}(F, G) \restriction \underline{n} + 1$.

Proof of Claim 3. As $\underline{n} \leq \bar{n}$, Claim 2 and b-regularity imply that $\pi_{0,m}(F, G) \restriction \underline{n} = \pi_{0,b}(F, G) \restriction \underline{n} \in V_\sigma$, where $m = b(\underline{n})$ and $\sigma = \sigma_m$. Using the fact that $\mathrm{crit}(\pi_{m,b}) = \rho_m > \sigma = F(m) = \mathrm{crit}(\pi_{m,n})$, it follows at once that

$$\pi_{0,n}(F(m)) = \pi_{m,n} \circ \pi_{0,m}(F(m)) = \pi_{m,n}(F(m)) > F(m)$$

and

$$\pi_{0,b}(F(m)) = \pi_{m,b} \circ \pi_{0,m}(F(m)) = F(m)$$

hence $\pi_{0,b}(F, G) \restriction \underline{n} + 1 \lhd \pi_{0,n}(F, G) \restriction \underline{n} + 1$ as required, proving Claim 3. $\square$

As $(F, G) \restriction \underline{n} + 1$ is $\lhd$-minimal among the pairs (s, t) such that $\mathcal{U}[(s, t)]$ is ill-founded, the same is true of $\pi_{0,n}(F, G) \restriction \underline{n} + 1$ and $\pi_{0,n}(\mathcal{U})$ in M_n. Hence in M_n, $\pi_{0,n}(\mathcal{U})[\pi_{0,b}(F, G) \restriction \underline{n} + 1]$ is well-founded. By Claim 1 and $\underline{n} + 1 \leq \bar{n}$,

$$M_n \vDash \pi_{0,n}(\mathcal{U})[\pi_{0,b}(F, G) \restriction \underline{n} + 1] \text{ is well-founded.}$$

Definition. For $n \in \omega \backslash b$ such that $\underline{n} < \bar{n}$, let

$$\gamma_n = \|\pi_{0,n}(\mathcal{U})[\pi_{0,b}(F, G) \restriction \bar{n}]\|.$$

We must now check that the γ_n's work. Pick $l, n \in \omega \backslash b$, $l T n$ and $\underline{l} < \bar{l}$. This implies that $\underline{n} < \bar{n}$ too, as $\underline{n} = \underline{l}$ and $\bar{l} < \bar{n}$.

Claim 4. $\pi_{l,n}(\pi_{0,b}(F, G) \restriction \bar{l}) = \pi_{0,b}(F, G) \restriction \bar{l}$.

Proof of Claim 4. Observe that, by Claim 2, $\pi_{0,b}(F, G) \restriction \bar{l} \in V_{\sigma_{b(\bar{l})}}$, and that $\mathrm{crit}(\pi_{l,n}) \geqslant \rho_l > \sigma_{b(\bar{l})}$, as $b(\bar{l}) < l$. The Claim follows at once. □

We will be done by proving.

Claim 5. (1) *If there is no i such that* $l < b(i) < n$, *then* $\pi_{l,n}(\gamma_l) = \gamma_n$.
(2) *If there is an i such that* $l < b(i) < n$, *then* $\pi_{l,n}(\gamma_l) > \gamma_n$.

Proof of Claim 5. By Claim 4,

$$\pi_{l,n}(\gamma_l) = \|\pi_{0,n}(\mathcal{U})[\pi_{0,b}(F, G) \restriction \bar{l}]\|.$$

Now if case (1) holds, then $\bar{l} = \bar{n}$, so $\pi_{l,n}(\gamma_l) = \gamma_n$.
If case (2) holds, then $\bar{l} < \bar{n}$, so $\pi_{0,b}(F, G) \restriction \bar{n} \lhd \pi_{0,b}(F, G) \restriction \bar{l}$, hence

$$\gamma_n = \|\pi_{0,n}(\mathcal{U})[\pi_{0,b}(F, G) \restriction \bar{n}]\| < \|\pi_{0,n}(\mathcal{U})[\pi_{0,b}(F, G) \restriction \bar{l}]\| = \pi_{l,n}(\gamma_l).$$

This proves Claim 5 and also Theorem 1.5. □

4. Epilogue

It is tempting to ask whether the results of Section 3 can be extended into the transfinite. In order to formulate this precisely, we must recall what it means for an iteration tree to have length $\lambda > \omega$.

Start from a tree ordering T on λ with 0 as least element, such that if $\alpha T \beta$ then $\alpha < \beta$ and if ξ is a T-limit, then ξ is a limit ordinal. (Hence if we are given a tree ordering T on ω, one and only one of the branches of T will be extended in the transfinite.) We must also have $\langle (E_\alpha, \rho_\alpha) \mid \alpha + 1 < \lambda \rangle$ and transitive models $\langle M_\alpha \mid \alpha < \lambda \rangle$, $M_0 = V$, such that:

(1) $M_\alpha \models E_\alpha$ is an extender with strength $\geqslant \rho_\alpha + 1$ and the sequence $\langle \rho_\alpha \mid \alpha + 1 < \lambda \rangle$ is increasing.

(2) If α is the T-immediate predecessor of $\beta + 1$, $\alpha = \beta^*$, then

$$\pi_{\alpha,\beta+1} : M_\alpha \to M_{\beta+1} = \mathrm{Ult}(M_\alpha, E_\beta)$$

and $\alpha =$ the least ν such that $\mathrm{crit}(E_\beta) \leqslant \rho_\nu$.

(3) $M_\xi = \lim \langle M_\nu \mid \nu T \xi \rangle$ if ξ is limit.

(4) The embeddings $\pi_{\alpha,\beta}: M_\alpha \to M_\beta$ are defined for $\alpha\, T\, \beta$, in the obvious way, so that they commute.
(See [5] or [1] for more on iteration trees of transfinite length.)

Theorem 1.5 then shows every tree ordering on $\omega+1$ can be realized as an iteration tree on V. Obviously one a-Woodin is enough to build *certain* iteration trees of transfinite length, e.g. if $\alpha < \lambda$ limit then $\alpha\, T\, \alpha + n$ for every n. Conversely the next result will show that one a-Woodin is not enough, in general, to construct an iteration tree of length $\omega+2$.

Lemma 4.1. *Suppose $\mathcal{T}$ is an iteration tree of height $\omega+2$, such that*
(1) *for some branch b of T cofinal in ω, $b \neq \{n \in \omega \mid n\, T\, \omega\}$, $\delta_\omega \in \mathrm{wfp}(M_b)$;*
(2) *$\omega^* = n_0 < \omega$, i.e., $M_{\omega+1} = \mathrm{Ult}(M_{n_0}, E_\omega)$.*
Then there is an a-Woodin cardinal δ and a measurable $\kappa < \delta$ limit of a-Woodins.

Proof. Theorem 1.1 and (1) yield immediately that δ_ω is a-Woodin in M_ω. As $(V_{\rho_\omega+1})^{M_{\omega+1}} = (V_{\rho_\omega+1})^{M_\omega}$, and $\rho_\omega \geqslant \sup\{\rho_n \mid n \in \omega\} = \delta_\omega$, δ_ω is a-Woodin in $M_{\omega+1}$ too. Let κ be the critical point of $E_\omega = \mathrm{crit}(\pi_{n_0,\omega+1})$. Then

$$\pi_{n_0,\omega+1}(\kappa) \geqslant \rho_{\omega+1} > \rho_\omega \geqslant \delta_\omega .$$

Let $A = \{\nu < \kappa \mid \nu \text{ is } a\text{-Woodin}\}$. $A \in M_{n_0} \cap M_\omega$ and $M_\omega \models$ "κ is measurable". A standard reflection argument shows that A has order type κ in M_{n_0} hence,

$$M_\omega \models \exists \delta\, \exists \kappa < \delta\ (\delta \text{ is } a\text{-Woodin and } \kappa \text{ is measurable and limit of } a\text{-Woodins}).$$

The Lemma follows immediately by elementarity of $\pi_{0,\omega}$. $\square$

The argument in the above Lemma can be iterated to obtain lower bounds for the existence of longer iteration trees. For example, if T is a tree ordering on $\omega+\omega$ such that $\delta_\omega \in M_b$ for every branch b confinal in ω, and $\{n \mid \exists m\, ((\omega + m)^* = n\}$ is unbounded in ω, then there is an a-Woodin limit of a-Woodins.

As far as upper bounds go, [1] shows that one Woodin cardinal δ is enough to realize as an iteration tree any tree ordering T on any $\lambda < \delta$, provided T has a cofinal branch. Of course the gap between Woodin cardinals and their admissible siblings is a large one, but the techniques in this paper seem to be of little help in suggesting the right hypothesis on how to build an iteration tree of height say, $\omega \cdot 2$.

Acknowledgement

This paper is a reworking of the fourth chapter of the author's doctoral dissertation written under the supervision of Professor J.R. Steel at UCLA. We would like to thank him for his invaluable guidance and advice. We would also like to thank Professor D.A. Martin for his help.

References

[1] A. Andretta, Building iteration trees, J. Symbolic Logic, to appear.
[2] J. Barwise, Admissible Sets and Structures (Springer, Berlin, 1975).
[3] T. Jech, Set Theory, (Academic Press, New York, 1978).
[4] D.A. Martin and J.R. Steel, A proof of projective determinacy, J. Amer. Math. Soc. 2, 71–126.
[5] D.A. Martin and J.R. Steel, Iteration trees, to appear.