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GIRSANOV FUNCTIONALS AND OPTIMAL BANG-BANG LAWS FOR FINAL VALUE STOCHASTIC CONTROL

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Abstract. Girsanov's theorem is a generalization of the Cameron–Martin formula for the derivative of a measure induced by a translation in Wiener space. It states that for φ a nonanticipative Brownian functional with $\int |\varphi|^2 \, ds < \infty$ a.s. and $d\tilde{P} = \exp[\zeta(\varphi)]\, dP$ with $\tilde{E}\{1\} = 1$, where $\zeta(\varphi) = \int \varphi \, dw - \frac{1}{2}\int |\varphi|^2 \, ds$, the translated functions $(Tw)(t) = w_t - \int_0^t \varphi \, ds$ are a Wiener process under $\tilde{P}$. The Girsanov functionals $\exp[\zeta(\varphi)]$ have been used in stochastic control theory to define measures corresponding to solutions of stochastic DEs with only measurable control laws entering the right-hand sides. The present aim is to show that these same concepts have direct practical application to final value problems with bounded control. This is done here by an example, the noisy integrator: Make $E\{x_1^2\}$ small, subject to $dx_t = u_t \, dt + dw_t$, $|u| \leq 1$, x_t observed. For each control law there is a definite cost $v(1-t, x)$ of starting at x, t and using that law till $t = 1$, expressible as an integral with respect to (a suitable) $\tilde{P}$. By restricting attention to a dense set of smooth laws, using Itô's lemma, Kac's theorem, and the maximum principle for parabolic equations, it is possible to calculate sgn v_x for a critical class of control laws, then to *compare* control laws, "solve" the Bellman–Hamilton–Jacobi equation, and thus justify selection of the obvious bang-bang law as optimal.

stochastic control	Feynman–Kac integrals
bang-bang principle	absolute continuity
transformation of measures	

1. Introduction

In many problems of optimal stochastic control anyone with a good physical or engineering intuition can correctly guess an optimal control law, but cannot justify his guess mathematically except perhaps by laborious machine calculations on examples, using say dynamic programming. Our object is to show that the exponential functionals expressing the derivatives of measures induced by translations in Wiener space provide a neat setting in which such justifications can be given, without any computations at all.

The connection between these functionals and stochastic control is provided by a seminal theorem of Girsanov [8] which generalizes an earlier formula of Cameron and Martin [4]. It states that for φ a non-anticipative Brownian functional with $\int |\varphi|^2 \mathrm{d}s < \infty$ a.s., and $\mathrm{d}\tilde{\mathrm{P}} = \exp [\zeta(\varphi)] \mathrm{dP}$ with $\zeta(\varphi) = \int \varphi \, \mathrm{d}w - \frac{1}{2} \int |\varphi|^2 \mathrm{d}s$, $\tilde{E}\ 1 = 1$, the translated functions $w_t - \int_0^t \varphi \, \mathrm{d}s$ are a Wiener process under $\tilde{\mathrm{P}}$. This result is used in stochastic control theory as follows: it is assumed that the controlled system is described by a differential-functional equation $\mathrm{d}x_t = f(t, x, u(t, x)) \, \mathrm{d}t + \mathrm{d}\tilde{w}_t$; here f represents the system and u is a particular control law; both f and u are nonanticipative with respect to (the function) x, and u need only be measurable. A "solution" of the equation is provided by the functions w_t under $\tilde{\mathrm{P}}$ with $\varphi(t, w) = f(t, w, u(t, w))$, in the sense that there is a Wiener process W_t such that

$$w_t = \int_0^t f(s, w, u(s, w)) \, \mathrm{d}s + W_t \, .$$

No more is claimed for this solution than that it has the right distributions. This idea has been exploited in stochastic control theory to give existence proofs for optimal control laws [1], counterexamples [6], and Hamilton–Jacobi conditions for optimality [5]. We show that it is also useful for justifying some natural guesses as to the identity of optimal control laws in final value problems with bounded control; indeed we solve a class of problems of this type, using only inequalities and the maximum principle.[1]

In order to simplify presentation and to expound the methods at their barest, we limit principal attention to an example: the noisy controlled integrator depicted in Fig. 1; natural extensions to other cases are discussed in a final section; their full extent is not yet known. In our example $f(s, x, u) = u$ and the equation to be solved is $\mathrm{d}x_t = u(t, x_t)\mathrm{d}t + \mathrm{d}w_t$. The feedback loop contains the control law, which depends only on the current value x_t and is restricted to a value between -1 and $+1$. The control problem is to pick a law $u : [0, 1] \times \mathrm{R} \to [-1, 1]$ so as to make the expected output x_1 of the integrator at time 1 small. Now it is "physically obvious" that the solution to this problem is to push x_s in the negative direction if it is positive, and in the positive direction if it is negative. That is, anyone's obvious guess is that the best $u(s, x)$ should be the bang-bang law $-\mathrm{sgn}\, x$. The interesting *mathematical* problem is to show that this guess is indeed right.

[1] We mean here the maximum principle for parabolic operators, not the maximum principle of Pontrjagin used in control theory.

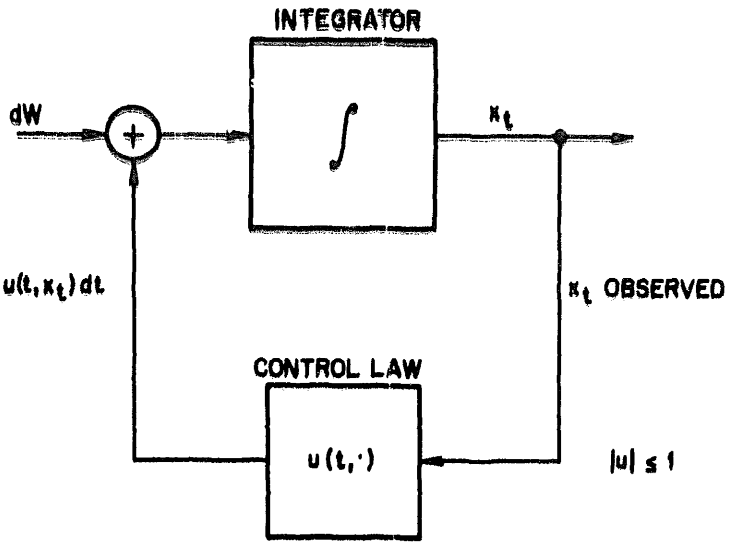

Fig. 1. Linear control of noisy integrator; $dx_t = u(t, x_t)\,dt + dW$.

2. Formulation

Let $k(x)$ be a positive even function, increasing in $x > 0$, that measures the cost of having the integrator assume the value x at the final time 1. Let the class $\mathcal{A}$ of admissible control laws consist of all functions $u : [0,1] \times \mathbf{R} \to [-1,1]$ such that u is jointly measurable. Consider the problem of choosing a control law $u \in \mathcal{A}$ so as to minimize $J[u] = \mathrm{E}\{k(x_1)\}$subject to $dx_t = u(t, x_t)\,dt + dw_t$. The question at once arises, in what sense is the equation intended to hold? For the ordinary theory of Itô stochastic differential equations is not applicable here because u is not known to be Lip. However, the problem can be formulated adequately and then solved provided we accept solutions, or rather solution measures, obtained from Girsanov's theorem as in the introduction.

To proceed, we define for each $u \in \mathcal{A}$, $x \in \mathbf{R}$ and $s \in [0,1]$ a "solution" starting at x at time s and using control law u. Let w be a Wiener process. To solve $x_s = x$, $dx_t = u(t,x_t)\,dt + dW_t$, take the functions

$$x_t = x + w_{t-s}, \qquad t \geq s \geq 0\,,$$

under the measure

$$d\tilde{\mathbf{P}} = \exp\,[\zeta_s^t(T_x u)]\ \mathrm{d}\mathbf{P}, \qquad T_x u(v,\cdot) = u(v, x+\cdot)\,,$$

$$= \exp\left[\int_s^t u(v, x + w_{v-s})\mathrm{d}w_v - \tfrac{1}{2}\int_s^t u^2(v, x + w_{v-s})\,\mathrm{d}v\right]\mathrm{d}\mathbf{P}\,.$$

We remark that both the solution functions and the measure depend on (s, x), as they should. According to Girsanov's result, the process

$$W_t = w_{t-s} - \int_s^t u(v, x + w_{v-s})\,\mathrm{d}v$$

is a Wiener process under $\tilde{\mathbf{P}}$, and $\mathrm{d}x_t = u(t, x_t)\,\mathrm{d}t + \mathrm{d}W_t$ in the sense that

$$x_t = x + \int_s^t u(v, x_v)\,\mathrm{d}v + W_t\,.$$

3. Representation of the cost

The cost (function) in Markovian stochastic control problems is the expected cost as a function of starting place and starting time, for a particular control law. It is convenient to use the notation $g(\cdot,\cdot)$ systematically for the "turned-around" control law $g(1 - v, x) = u(v, x)$, $0 \leqslant v \leqslant 1$. By setting $t = 1$, $\tau = 1 - s$ = "time to go", and changing variables a bit, we can write the cost function for the control law u as

$$v(\tau, x) = \mathbf{E}\{k(x_1)\mid x_{1-\tau} = x\}$$

$$= \mathbf{E}\left\{k(x + w_\tau)\exp\left[\int_0^\tau g(\tau - v, x + w_v)\,\mathrm{d}w_v - \tfrac{1}{2}\int_0^\tau g^2(\tau - v, x + w_v)\mathrm{d}v\right]\right\}$$

$$= \mathbf{E}\left\{k(x + w_\tau)\exp[\xi_0^\tau(\tau, T_x g)]\right\}, \qquad T_x g(v,\cdot) = g(v, x+\cdot)\,.$$

This is an explicit representation; the eventual cost, starting from $s = 0$, is then $J[u] = v(1, x)$. Our method will be to compare v's corresponding to different control laws, and so prove that $u(t, x) = -\operatorname{sgn} x$ *achieves*

$$\inf_{v \in \mathscr{A}} J[v]\,.$$

4. Preliminary results

Kac [9] has given what amounts to a PDE for expectations of the form

$$\gamma(\tau, x) = \mathbf{E}\left\{c(x + w_\tau) \exp\left[\int_0^\tau V(\tau - v, x + w_v)\, dv\right]\right\},$$

namely: $\gamma(0, x) = c(x)$, $\gamma_\tau = \frac{1}{2}\gamma_{xx} + \gamma V$. This equation is valid under a wide range of weak conditions [3]. For our purposes it will suffice to prove it when $c \in C^2$ is of exponential type and $V \in C^2$ is at most linear in its second argument. Under these conditions γ is C^1 in τ and C^2 in x. This can be proved by first using the scaling $w_v \to \tau^{1/2}\, w(v/\tau)$ in the expectation so that the E integration is only over a Wiener process defined over [0, 1], and then using absolute convergence of differentiated integrands to justify successive differentiations under E. Then we argue that

$$\gamma(\tau + \delta, x) = \mathbf{E}\left\{c(x + w_{\tau+\delta} - w_\delta + w_\delta)\right.$$

$$\times \exp\left[\int_\delta^{\tau+\delta} V(\tau + \delta - v, x + w_v - w_\delta + w_\delta)\, dv\right.$$

$$\left.\left. + \int_0^\delta V(\tau + \delta - v, x + w_v)\, dv\right]\right\}$$

$$= \mathbf{E}\{\gamma(\tau, x + w_\delta)\} + \delta\gamma(\tau, x)\, V(\tau, x) + o(\delta)$$

$$= \gamma(\tau, x) + \mathbf{E}\{\gamma_x(\tau, x)\, w_\delta\} + \tfrac{1}{2}\mathbf{E}\{\gamma_{xx}(\tau, x)\, w_\delta^2\} + \delta\gamma V + o(\delta),$$

and Kac's formula follows.

The cost function can be related to Kac's integral with

$$V(s, x) = -G_s - \tfrac{1}{2}(g_x + g^2),$$

$$c(x) = k(x) \exp[G(0, x)],$$

where

$$G(s, x) = \int_0^x g(s, z)\, dz.$$

A similar idea was used by L.A. Shepp and the author [2]. By Itô's differential formula, for $g \in \mathcal{A} \cap C^3$,

$$G(0, x+w_\tau) - G(\tau, x) = \int_0^\tau \int_0^{x+w_v} \frac{\partial}{\partial v} g(\tau - v, z)\,dz\,dv + \int_0^\tau g(\tau - v, x + w_v)\,dw_v + \tfrac{1}{2} \int_0^\tau g_z\Big|_{\substack{\tau-v \\ x+w_v}} dv .$$

It follows that with V as above

$$v(\tau, x) = \exp[-G(\tau, x)] \times \mathrm{E}\left\{k(x+w_\tau)\exp[G(0,x+w_\tau)]\exp\left[\int_0^\tau V(\tau-v,x+w_v)\,dv\right]\right\}.$$

By Kac's formula, the expectation above satisfies $\varphi(0,x) = k(x)\exp[G(0,x)]$, $\varphi_\tau = \frac{1}{2}\varphi_{xx} + V\varphi$. From this we see incidentally that $v(0, x) = k(x)$, $v_\tau = \frac{1}{2}v_{xx} + gv_x$; this is precisely the equation satisfied by $\mathrm{E}\{k(x_1) \mid x_{1-\tau} = x\}$ when x_t is given by the stochastic DE $dx_t = g(1-t, x_t)dt + dw_t$, $x_{1-\tau} = x$, and g is smooth, say with bounded first partials.

Lemma 4.1. *If* $k(x) = O(\exp[\kappa|x|])$ *and* $g \in \mathcal{A} \cap C$ *with* $\sup(|g_s| + |g_x|) < \infty$ *then*

$$v(\tau, x) = O(\exp[(\kappa+2)|x|])$$

uniformly for $\tau \in [0,1]$.

Proof. $|G(s,x)| \leq |x|$, so $\zeta(\tau, T_x g) \leq 2|x| + |w_\tau| + \text{const}$.

Lemma 4.2. *With* $k \in C^2$ *and of exponential type, and* $g \in \mathcal{A} \cap C^2$ *with* $\sup(|g_s| + |g_z|) < \infty$, *the gradient* v_x *has the form*

$$\xi(\tau, x) = \mathrm{E}\left\{k'(x+w_\tau)\exp\left[\zeta_0^\tau(\tau, T_x g) + \int_0^\tau g_z\Big|_{\substack{\tau-v \\ x+w_v}} dv\right]\right\}.$$

Proof. Set

$$\eta(\tau, x) = \xi(\tau, x)\exp[G(\tau, x)]$$

η has the form of Kac's integral with $c(x) = k'(x) \exp[G(0,x)]$ and $V(s,x) = -G_s + \frac{1}{2}(g_x - g^2)$, so it satisfies

$$\eta(0,x) = k'(x) \exp[G(0,x)]\,,$$

$$\eta_\tau = \tfrac{1}{2}\eta_{xx} - \eta[G_\tau - \tfrac{1}{2}g_x + \tfrac{1}{2}g^2]\,.$$

Hence ξ satisfies

$$\xi(0,x) = k'(x)$$

$$\xi_\tau = \tfrac{1}{2}\xi_{xx} + g\xi_x + \xi g_x\,.$$

This is precisely the equation satisfied by v_x if v satisfies $v(0,x) = k(x)$, $v_\tau = \frac{1}{2}v_{xx} + gv_x$. (Just differentiate.) Thus $v_x = \xi$, because an argument similar to Lemma 4.1 shows that ξ is of exponential type, and the uniqueness theorem for parabolic operators applies.

The value of this representation will be that the sign of k' becomes relevant to that of v_x.

Lemma 4.3. $J[u]$ *is continuous in the* L_2 *topology of* $\mathcal{A}$.

Proof. Fix x and put $\Delta_t = \exp[\zeta_0^t(T_x u_1)] - \exp[\zeta_0^t(T_x u_2)]$ for $u_1, u_2 \in \mathcal{A}$. From $k(x) = O(\exp[\kappa|x|])$ and Schwarz's inequality, we find

$$|J[u_1] - J[u_2]| \le \mathrm{E}\{k(x+w_1)|\Delta_1|\} \le N\mathrm{E}\{\chi_{k(x+w_1)\le N}|\Delta_1|\}$$

$$+ \text{const} \cdot \mathrm{E}\{\chi_{|x+w_1|>\kappa^{-1}\log N} \exp[\kappa|x+w_1|]\,|\Delta_1|\} \le NE\{|\Delta_1|^2\}$$

$$+ 2\,\text{const} \cdot \sup_{u\in\mathcal{A}} \{\mathrm{E}^{1/2}\{e^{2\zeta(u)}\,\mathrm{E}^{1/2}\{\exp[2\kappa|x+w_1|]\,\chi_{|\kappa x+\kappa w_1|>\log N}\}\}\}.$$

The sup in the second term is finite by an argument of Girsanov [8]. The second E in the second term goes to 0 as $N \to \infty$. So it is enough to show that $\mathrm{E}\{|\Delta_1|^2\} \to 0$ as $\|u_1 - u_2\|$ does the same. To this end note that by the c_2-inequality,

$$\Delta_t = \int_0^t \exp[\zeta_0^s(T_x u_1)]\,T_x(u_1 - u_2)\,\mathrm{d}w_s + \int_0^t T_x u_2 \Delta_s\,\mathrm{d}w_s\;\mathrm{E}\{|\Delta_t|^2\}$$

$$\le 2\mathrm{E}\left\{\int_0^t \exp[2\zeta_0^s(T_x u_1)\;\; T_x(u_1 - u_2)^2\,\mathrm{d}s\right\} + 2\int_0^t \mathrm{E}\{|\Delta_s|^2\}\,\mathrm{d}s\,.$$

Also,

$$\mathbf{E}\left\{\int_0^t T_x(u_1-u_2)^2\,\mathrm{d}s\right\} \le \left(\int_0^\eta + \int_\eta^1\right) \mathbf{E}\{T_x(u_1-u_2)^2\}\,\mathrm{d}s$$

$$\le (2\pi)^{-1/2} \int_0^\eta \int_{-\infty}^{\infty} \exp[-w^2/2s]\, s^{-1/2}\, T_x(u_1-u_2)^2\,\mathrm{d}s\,\mathrm{d}w$$

$$+ (2\pi\eta)^{-1/2} \, \| u_1 - u_2 \| \,.$$

Since $\| T_x u_1 - T_x u_2 \| = \| u_1 - u_2 \|$, we can pick first η so small and then $\| u_1 - u_2 \|$ so small that $\mathbf{E}\{\int_0^t T_x(u_1-u_2)^2\} < \epsilon$ (uniformly in x, in fact, although this is not needed). The desired result now follows from Gronwall's lemma, because

$$\sup_{u\in\mathcal{A}} \mathbf{E}\left\{\int_0^1 \exp[4\zeta_0^s(u)]\,\mathrm{d}s\right\} < \infty\,.$$

5. The sgn of the gradient

In this section we calculate $\operatorname{sgn} v_x$ for control laws that are smooth in t, x and odd in x. Knowing $\operatorname{sgn} v_x$ makes it possible to use the maximum principle to compare such control laws to others.

Lemma 5.1. *If $k \in C^2 \cap$ even, and $g \in \mathcal{A} \cap C^3 \cap$ odd in x, then*

$$v_x\big|_{x=0} = 0\,.$$

Proof. This follows from the even and odd properties of k and g, respectively, along with the fact that $-w$ is a Wiener process. We have, by Itô's lemma,

$$v_x = \mathbf{E}\Bigg\{\exp[\zeta_0^\tau(\tau, T_x g)]\,\Bigg\{k'(x+w_\tau) + k(x+w_\tau)\Bigg[g(0, x+w_\tau) - g(\tau, x)$$

$$- \int_0^\tau g_s(s,z)\Big|_{\substack{s=\tau-v\\ z=x+w_v}} \mathrm{d}v - \tfrac{1}{2}\int_0^\tau (2gg_z + g_{zz})_{\substack{s=\tau-v\\ z=x+w_v}}\,\mathrm{d}v\Bigg]\Bigg\}\Bigg\}$$

$$= \mathbf{E}\Bigg\{\exp[\zeta_0^\tau(\tau, T_x g)]\,\Bigg\{k'(x+w_\tau)$$

$$+ k(x + w_\tau) \int_0^\tau g_z(s, z)\Big|_{\substack{s=\tau-v \\ z=x+w_v}} [\mathrm{d}w_v - g(\tau - v, x + w_v)]\, \mathrm{d}v \Big\}\Big\}$$

$$= E\{\varphi(\tau, x, w)\} .$$

Since $g(s, x) = -g(s, -x)$, $g_x(s, x) = g_z(s, z)|_{z=-x}$, we find that

$$\zeta_0^\tau(\tau, T_x g)_w = - \int_0^\tau g(\tau - v, -x - w_s)\, \mathrm{d}w_s - \tfrac{1}{2} \int_0^\tau g^2(\tau - v, -x - w_s)\, \mathrm{d}s$$

$$= \zeta_0^\tau(\tau, T_{-x} g)_{-w} ,$$

$$\varphi(\tau, x, w) = -\varphi(\tau, -x, -w) ,$$

and since $-w$ has the same distribution as w,

$$\mathbf{E}\{\varphi(\tau, x, w)\} = -\mathbf{E}\{\varphi(\tau, -x, -w)\} = -\mathbf{E}\{\varphi(\tau, -x, w)\} ,$$

$$\mathbf{E}\{\varphi(\tau, 0, w)\} = -\mathbf{E}\{\varphi(\tau, 0, w)\} = 0 .$$

Lemma 5.2. *Suppose that the assumptions of Lemma 5.1 are satisfied, and that in addition $k'(x) \geq 0$ in $x \geq 0$. Then*

$$\operatorname{sgn} v_x = \operatorname{sgn} x .$$

Proof. We already know that $v_x = 0$ at $x = 0$, from Lemma 5.1. For $x > 0$, consider the stopping time $s = \inf v\colon\ w_v = -x$ and the decomposition, from Lemma 4.2,

$$v_x = \mathbf{E}\Big\{\chi_{s \leq \tau} \exp\Big[\zeta_0^s(\tau, T_x g) + \int_0^s g_z\Big|_{\substack{\tau-v \\ x+w_v}} \mathrm{d}v\Big]$$

$$\times\ \mathbf{E}\Big\{k'(x + w_s + w_\tau - w_s) \exp\Big[\zeta_s^\tau(\tau, T_x g) + \int_0^\tau g_z\Big|_{\substack{\tau-v \\ x+w_v}} \mathrm{d}v\Big]\Big| w_v,\ v \leq s\Big\}\Big\}$$

$$+\ \mathbf{E}\Big\{\chi_{s > \tau}\Big[\exp\ \zeta_0^\tau(\tau, T_x g) + \int_0^\tau g_z\Big|_{\substack{\tau-v \\ x+w_v}} \mathrm{d}v\Big]\, k'(x + w_\tau)\Big\} .$$

Since $w_\cdot - w_s$ is a "fresh" Brownian motion independent of $\sigma\{w_v, v \leq s\}$, and

$$\zeta_s^\tau(\tau, T_x g)_w = \zeta_s^\tau(\tau, T_{-w_s} g)_w = \zeta_0^{\tau-s}(\tau - s, g)_{w_. - w_s},$$

the conditional expectation above is independent of the first two factors when τ is given and can be replaced by $v_x(\tau - s, 0) = 0$. The second term is nonnegative because $k'(x + w_\tau) \geq 0$ if $\tau < s$. A similar argument, or one based on odd symmetry, shows that $v_x \leq 0$ for $x \leq 0$.

6. Comparison of control laws

To smooth control laws from $\mathscr{A}$, which are defined only on $[0, 1] \times \mathbf{R}$, we shall extend them to $\mathbf{R}^2$ by equating them to 0 when $s \notin [0, 1]$. For functions $f : \mathbf{R}^2 \to [-1, 1]$, we shall use the smoothings $f \to S_\delta f$ defined by

$$(S_\delta f)(y) = \frac{1}{4\delta^2} \int_C f(y + v)\, \mathrm{d}v,$$

where C = square of side 2δ centered on the origin in $\mathbf{R}^2$. $(S_\delta)^n f$ belongs to $\mathscr{A}$ if f does, approaches f in L_2 as $\delta \to 0$, and has bounded n^{th} partials.

We now show that given any control law $f \in \mathscr{A}$ there is another law $u \in \mathscr{A}$, as good as f to within ϵ, and within ϵ in norm of our natural guess candidate $\sigma(s, x) = -\operatorname{sgn} x$.

Lemma 6.1. *$f \in \mathscr{A}$, $\epsilon > 0 \Rightarrow \exists\, u \in \mathscr{A}$, $\| u - \sigma \| < \epsilon$ and*

$$J[u] \leq J[f] + \epsilon .$$

Proof. Choose δ by Lemma 4.3 so that $\|(S_\delta)^4 f - f\|$ is so small that

$$J[(S_\delta)^4 f] \leq J[f] + \tfrac{1}{2}\epsilon .$$

Let now $u, h \in \mathscr{A}$ be C^4 functions such that (see Fig. 2),

$$h = u^2 (S_\delta)^4 f,$$

$$u(s, z) = -\operatorname{sgn} z, \quad |z| \geq \delta,$$

$$u(s, z) = -u(s, -z),$$

$$u(s, z) \leq 0 \quad \text{for } z \geq 0,$$

$$\sup_{s,z} (|u_s| + |u_z|) < \infty .$$

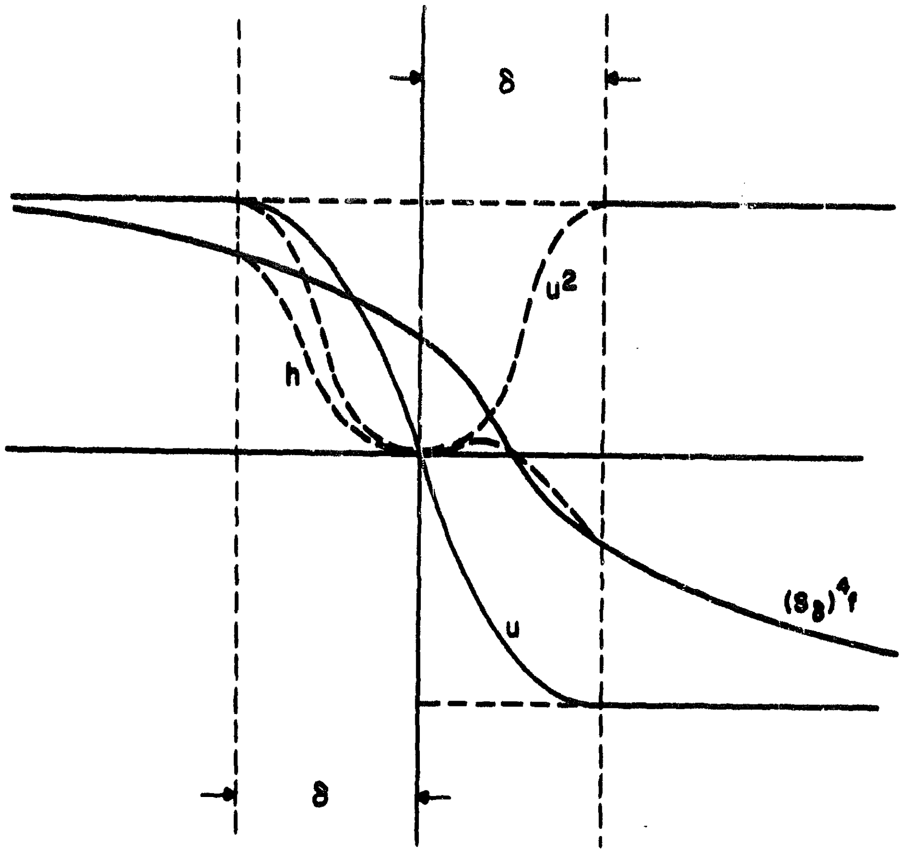

Fig. 2. Section of functions h, u^2 and $(S_\delta)^4 f$ at a fixed time.

It is clear that δ can be further reduced, if necessary, so that both

$$\|u-\sigma\| < \epsilon ,$$

$$J[h] \leq J[(S_\delta)^4 f] + \tfrac{1}{2}\epsilon .$$

Note that $h \geq u$ in $z \geq 0$ and $h \leq u$ in $z \leq 0$. Define the parabolic operator $L[g]$ by

$$L[g] = \frac{1}{2}\frac{\partial^2}{\partial x^2} + g(\tau, x)\frac{\partial}{\partial x} - \frac{\partial}{\partial \tau},$$

so that with $\xi(\tau, x) = J_{\tau,x}[u]$, $\eta(\tau, x) = J_{\tau,x}[h]$,

$$L[u]\xi = 0 , \qquad L[h]\eta = 0 .$$

By Lemma 5.2, $\operatorname{sgn} \xi_x = \operatorname{sgn} x$, so that

$$h \operatorname{sgn} x \geq u \operatorname{sgn} x ,$$

$$L[u]\,\xi = \tfrac{1}{2}\xi_{xx} + u\,\xi_x - \xi_\tau$$

$$= \tfrac{1}{2}\xi_{xx} + u \operatorname{sgn} x\,|\xi_x| - \xi_\tau$$

$$\leq \tfrac{1}{2}\xi_{xx} + h \operatorname{sgn} |\xi_x| - \xi_\tau = L[h]\,\xi .$$

Therefore

$$L[h](\eta - \xi) = -L[h]\,\xi \leq -L[u]\,\xi = 0 .$$

It follows from Lemma 4.1 that $\eta - \xi$ is of exponential type. The maximum principle for parabolic operators (see e.g. [9, p. 43]) then gives $\eta - \xi \geq 0$, i.e., $J_{\tau,x}[h] \geq J_{\tau,x}[u]$. It follows that

$$J[u] \leq J[h] \leq J[f] + \epsilon .$$

Theorem 6.2. *The control law* $u(s, x) = -\operatorname{sgn} x$ *achieves*

$$\inf_{v \in \mathcal{A}} J_{\tau,x}[v] \qquad \textit{for any } \tau, x \in [0,1] \times \mathbf{R} .$$

Proof. Take $\epsilon > 0$ and $v \in \mathcal{A}$. By Lemmas 4.3 and 6.1, there is a law $g \in \mathcal{A}$ such that

$$J[g] \leq J[v] + \tfrac{1}{2}\epsilon ,$$

$$|J[g] - J[u]| \leq \tfrac{1}{2}\epsilon ,$$

so that $J[u] \leq J[v] + \epsilon$. Since ϵ is arbitrary, the theorem follows.

7. Extensions

If the drift term of the equation to be solved is $f(t, x, u(t, x))\,\mathrm{d}t$ rather than merely $u(t, x)\,\mathrm{d}t$, we can still proceed so long as the odd and domination properties used above are valid. In proving $v_x|_{x=0} = 0$ in Lemma 5.1, only the evenness of k, and the oddness of u in x were used. Clearly, if

$$f(t, x, u) = -f(t, -x, -u) \,, \tag{7.1}$$

then $f(t, x, u(t, x))$ will be odd in x if $u(t, x)$ is, and the same arguments can be used. To get the inequality needed for applying the maximum principle, it is convenient to take f monotone in the control in the sense that increasing the magnitude of control increases the drift in the return direction, i.e.,

$$x \geq 0, \; u \geq v \Rightarrow f(t, x, u) \geq f(t, x, v) \,. \tag{7.2}$$

With assumptions (7.1) and (7.2) on f it again follows that the best control law is $-\operatorname{sgn} x$. Note that there is no assumption that $f(t, x, u) < 0$ for $x > 0$; thus f may not even be pushing the right way; the important thing is that it be *more* right for a larger control than for a smaller. (7.1) and (7.2) are met, e.g., if $f(t, x, u) = a(t, x) + b(t) u$ with $b > 0$ and $a(t, x) = -a(t, -x)$.

Moreover, it is easily seen that the assumption (7.2) can be dropped at the expense of losing the simple form $-\operatorname{sgn} x$ for an optimal law, and incurring other minor complications. One simply proves that if u_1 is odd in x and satisfies

$$f(t, x, u_1(t, x)) \underset{\geq}{\leq} f(t, x, u_2(t, x)) \,, \qquad x \underset{\leq}{\geq} 0 \,,$$

then u_1 is no worse than u_2. Setting

$$U(t, x) = \{u \in [-1, 1] : \; u \text{ minimizes } f(t, x, u)\}, \qquad x \geq 0 \,,$$

$$U(t, x) = \{u \in [-1, 1] : \; u \text{ maximizes } f(t, x, u)\}, \qquad x \leq 0 \,,$$

it follows that any $u(\cdot, \cdot) \in \mathcal{A}$ with $u(t, x) \in U(t, x)$ is optimal. Note that here $U(t, x) = -U(t, -x)$, so that $U(t, 0) = -U(t, 0) = 0$, and the definitions are consistent at $x = 0$.

Generalizations to integral criteria, to n dimensions, and to measures other than Wiener's are all possible. Since these are either quite straightforward, or else involve wholly new principles, they are not pursued here.

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