Degenerate lower-dimensional tori in Hamiltonian systems

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Abstract

We study the persistence of lower-dimensional tori in Hamiltonian systems of the form \( H(x, y, z) = \langle \omega, y \rangle + \frac{1}{2} \langle z, M(\omega)z \rangle + \varepsilon P(x, y, z, \omega) \), where \((x, y, z) \in T^n \times \mathbb{R}^n \times \mathbb{R}^{2m} \), \( \varepsilon \) is a small parameter, and \( M(\omega) \) can be singular. We show under a weak Melnikov nonresonant condition and certain singularity-removing conditions on the perturbation that the majority of unperturbed \( n \)-tori can still survive from the small perturbation. As an application, we will consider the persistence of invariant tori on certain resonant surfaces of a nearly integrable, properly degenerate Hamiltonian system for which neither the Kolmogorov nor the \( g \)-nondegenerate condition is satisfied.

MSC: primary 58F05, 58F27, 58F30

Keywords: Degeneracy; Hamiltonian systems; KAM theory; Lower-dimensional tori; Persistence

1. Introduction and main results

We consider the Melnikov persistence problem of lower-dimensional, possibly degenerate, invariant tori for Hamiltonian of the form

\[ H(x, y, z) = \langle \omega, y \rangle + \frac{1}{2} \langle z, M(\omega)z \rangle + \varepsilon P(x, y, z, \omega) \]

where \((x, y, z) \in T^n \times \mathbb{R}^n \times \mathbb{R}^{2m} \), \( \varepsilon \) is a small parameter, and \( M(\omega) \) can be singular. We show under a weak Melnikov nonresonant condition and certain singularity-removing conditions on the perturbation that the majority of unperturbed \( n \)-tori can still survive from the small perturbation. As an application, we will consider the persistence of invariant tori on certain resonant surfaces of a nearly integrable, properly degenerate Hamiltonian system for which neither the Kolmogorov nor the \( g \)-nondegenerate condition is satisfied.

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\[ H = e(\omega) + \langle \omega, y \rangle + \frac{1}{2} \langle z, M(\omega)z \rangle + \varepsilon P(x, y, z, \omega, \varepsilon), \quad (1.1) \]

where \((x, y, z) \in T^n \times R^n \times R^{2m}\), \(\omega\) is a parameter in a bounded closed region \(O \subset R^n\), \(\varepsilon \in (0, 1)\) is a small parameter, \(M\) is a real analytic, matrix-valued function on some complex neighborhood \(O(r) = \{ \omega : |\text{Im} \omega| < r \}\) of \(O\) taking values in the space of \(2m \times 2m\) symmetric matrices, and \(P\) is real analytic in a complex neighborhood \(D(r) \times O(r) \times \Delta\) of \(T^n \times \{0\} \times \{0\} \times O 	imes (0, 1)\) for some \(D(r) = \{ (x, y, z) : |\text{Im} x| < r, |y| < s^2, |z| < s \}\). The Hamiltonian \(H\) is associated with the standard symplectic form

\[ \sum_{i=1}^{n} dx_i \wedge dy_i + \sum_{j=1}^{m} dz_j \wedge dz_{m+j}. \quad (1.2) \]

Clearly, the unperturbed system associated to (1.1) admits a family of invariant \(n\)-tori \(T_\omega = T^n \times \{0\} \times \{0\}\) with linear flows which are parameterized by the toral frequency \(\omega \in O\).

The persistence problem of hyperbolic \(n\)-tori in (1.1) was studied first by Moser [24] and later by Graff [11] and Zehnder [36] for a fixed Diophantine \(n\)-torus \(T_\omega\). The study of the persistence of nonhyperbolic \(n\)-tori in (1.1) was initiated by Melnikov [22,23] who considered the persistence of the majority of the unperturbed \(n\)-tori \(T_\omega\) under certain coupling nonresonance conditions, called Melnikov conditions, between the tangential frequencies \(\omega\) and the normal ones associated to the eigenvalues of \(M(\omega)\). The Melnikov persistence problem has been extensively studied for various (normally) nondegenerate cases (i.e., \(M\) is nonsingular over \(O\)) of (1.1), for infinite-dimensional Hamiltonian systems, and also for reversible systems (see [2–7,9,10,12–15,17,19,21,26–30,32,34,35] and references therein).

A similar persistence problem was posted by Kuksin in [16] for the degenerate case (i.e., \(M(\omega)\) is singular at some points of \(O\)) of (1.1). The problem was studied in [19] under tangential nondegeneracy, i.e., the quadratic term in (1.1) has the form

\[ \frac{1}{2} \left( y \begin{bmatrix} z \\ y \end{bmatrix}, M(\omega) \begin{bmatrix} y \\ z \end{bmatrix} \right) \]

and \(M(\omega)\) is nonsingular over \(O\). The aim of this paper is to study the degenerate case without assuming tangential nondegeneracy. More precisely, we will study the Hamiltonian (1.1) and show that some nondegenerate conditions on the perturbation can remove the singularity and hence yield the persistence of the majority of invariant, quasi-periodic \(n\)-tori under a suitable nonresonance condition of Melnikov type.

For simplicity, we will use the same symbol \(|\cdot|\) to denote an equivalent vector norm (and its induced matrix norm) in an Euclidean space, absolute value of numbers, Lebesgue measure of sets, and \(l^1\) norm of integer-valued vectors. Also, \(|\cdot|_D\) will be used to denote the sup-norm of a function on a domain \(D\).

Let \(\lambda_1(\omega), \ldots, \lambda_{2m}(\omega)\) be eigenvalues of \(JM(\omega)\), where \(J\) denotes the standard \(2m \times 2m\) symplectic matrix. We assume the following conditions for (1.1):

(H1) The set

\[ \{ \omega \in O : \sqrt{-1} \langle k, \omega \rangle - \lambda_i(\omega) - \lambda_j(\omega) \neq 0, \forall k \in Z^n \setminus \{0\}, 1 \leq i, j \leq 2m \} \]

admits full Lebesgue measure relative to \(O\).
(H2) There exists a real analytic family $z_\varepsilon: \mathcal{O}(r) \to D(s) = \{z: |z| < s\}$ such that

$$M(\omega)z_\varepsilon(\omega) + \varepsilon \partial_z [P](0, z_\varepsilon(\omega), \omega, 0) = 0,$$

for all $\omega \in \mathcal{O}(r)$, where $[P](y, z, \omega) = \int_{T^n} P(x, y, z, \omega, 0) \, dx$.

(H3) There exists a constant $N_1 > 0$ such that the minimum $\lambda^\varepsilon_{\min}(\omega)$ among the absolute values of all eigenvalues of $M_\varepsilon(\omega) = M(\omega) + \varepsilon \partial^2_z [P](0, z_\varepsilon(\omega), \omega)$ satisfies $\lambda^\varepsilon_{\min}(\omega) > N_1 \varepsilon$ for all $\omega \in \mathcal{O}(r)$.

Our main result states as the following.

**Theorem 1.** Assume (H1)–(H3). Then there is an $\varepsilon_0 > 0$ and Cantor sets $\mathcal{O}_\varepsilon \subset \mathcal{O}$, $0 < \varepsilon < \varepsilon_0$, with $|\mathcal{O} \setminus \mathcal{O}_\varepsilon| \to 0$ as $\varepsilon \to 0$ such that for each $0 < \varepsilon < \varepsilon_0$ the Hamiltonian system (1.1) admits a Whitney smooth family of real analytic, quasi-periodic $n$-tori $T_\varepsilon^\omega$, $\omega \in \mathcal{O}_\varepsilon$, which also varies smoothly in $\varepsilon$.

We note that if $M(\omega)$ is nonsingular over $\mathcal{O}$, then conditions (H2), (H3) are automatically satisfied. In the case that $M(\omega)$ becomes singular, invariant $n$-tori can be destroyed if condition (H2) fails. For example, it is easy to see that the Hamiltonian

$$H(x, y, u, v) = \langle \omega, y \rangle + \frac{1}{2} u^2 \pm \varepsilon v, \quad (x, y, u, v) \in T^n \times R^n \times R^1 \times R^1$$

admits no invariant $n$-tori for any $\varepsilon > 0$ and (H2) is not satisfied for this Hamiltonian. Condition (H3) is of course not optimal for the persistence of invariant $n$-tori of Hamiltonian (1.1). In general, it should be possible to replace (H3) by a weaker nondegenerate condition. This is certainly an interesting problem worthy for a further study.

Condition (H1) is stronger than the first Melnikov nonresonance condition but is weaker than the second Melnikov nonresonance condition by allowing multiple normal eigenvalues of $JM(\omega)$. This condition was first introduced in [35] and has been employed in various studies on the persistence of lower-dimensional tori in Hamiltonian systems (see [7,19]).

Theorem 1 has no restriction on the invariant tori type, i.e., the perturbed tori can be normally hyperbolic, elliptic or of mixed type. However, unlike the nondegenerate cases considered in [19,35], an unperturbed, persisted torus of (1.1) can change its type after perturbation in the case of normal degeneracy. Consider the following two Hamiltonians:

$$H_1 = \langle \omega, y \rangle + u^2 + \varepsilon u - \varepsilon v^2 + \varepsilon \tilde{P}_1(x, y, u, v) \equiv \langle \omega, y \rangle + \varepsilon P_1(x, y, z),,$$

$$H_2 = \langle \omega, y \rangle + \varepsilon u + \varepsilon v + \varepsilon u^2 + \varepsilon v^2 + \varepsilon \tilde{P}_2(x, y, u, v) \equiv \langle \omega, y \rangle + \varepsilon P_2(x, y, z),$$

where $x, y, \omega$ are as in (1.1), $z = (u, v) \in R^2$, and $[\tilde{P}_i] = 0$, $i = 1, 2$. Clearly,

$$M = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{for } H_1 \quad \text{and} \quad M = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{for } H_2.$$  

Hence the unperturbed $n$-tori in both cases are of degenerate elliptic types.

Since $M$ are constant matrices in both cases, (H1) is satisfied for both $H_1$ and $H_2$. Moreover, it is easy to see that $z_\varepsilon = (-\frac{\varepsilon}{2}, 0)$ for $H_1$ and $z_\varepsilon = (-\frac{1}{2}, -\frac{1}{2})$ for $H_2$, i.e., (H2) is satisfied in both
cases. Since $M_\varepsilon$ equals

$$
\begin{pmatrix}
2 & 0 \\
0 & -2\varepsilon
\end{pmatrix}
$$

and

$$
\begin{pmatrix}
2\varepsilon & 0 \\
0 & 2\varepsilon
\end{pmatrix}
$$

for $H_1$ and $H_2$, respectively, and $\lambda_{\min}^\varepsilon = 2\varepsilon$ in both cases, (H3) is also satisfied for both $H_1$ and $H_2$. Hence Theorem 1 is applicable to both $H_1$ and $H_2$ to yield the persistence of two respective families of invariant, quasi-periodic $n$-tori.

However, for $H_1$,

$$
J M_\varepsilon = \begin{pmatrix}
0 & -2\varepsilon \\
-2 & 0
\end{pmatrix}
$$

has eigenvalues $\lambda_\pm = \pm 2\sqrt{\varepsilon}$, and, for $H_2$,

$$
J M_\varepsilon = \begin{pmatrix}
0 & 2\varepsilon \\
-2\varepsilon & 0
\end{pmatrix}
$$

has eigenvalues $\lambda_\pm = \pm 2\sqrt{-1}\varepsilon$. Thus the perturbed $n$-tori are all (nondegenerate) hyperbolic for $H_1$ and are all (nondegenerate) elliptic for $H_2$.

Normal degeneracy naturally occurs in a nearly integrable, properly degenerate Hamiltonian system. As an application of Theorem 1, we consider the following properly degenerate Hamiltonian

$$
H(I, \theta, \varepsilon) = H_{00}(I_1, \ldots, I_r) + \varepsilon P(\theta, I, \varepsilon)
$$

(1.3)

associated to the symplectic form

$$
\sum_{i=1}^{d} d\theta_i \wedge dI_i,
$$

where $(I, \theta) = (I_1, \ldots, I_d, \theta_1, \ldots, \theta_d) \in G \times T^d$, $G \subset \mathbb{R}^d$ is a bounded closed region, $r < d$, $H_{00}$, $P$ are real analytic, and $H_{00}$ satisfies the Kolmogorov nondegenerate condition on $\tilde{G} = \{(I_1, \ldots, I_r): I \in G\}$, i.e.,

(H3) the Hessian $\left(\frac{\partial^2 H_{00}}{dt_i \, dI_j}\right)_{i,j=1}$ is nonsingular on $\tilde{G}$.

The unperturbed system associated to (1.3) admits a family of invariant, resonant $d$-tori $T_I$ parametrized by $I \in G$. Under the condition (H3) and some condition on the perturbation which removes the degeneracy of the unperturbed Hamiltonian, it was shown by Arnold [1] that there is a large subset of the phase space which is filled by invariant, quasi-periodic $d$-tori of the perturbed system exhibiting both fast and slow oscillations. However, if the perturbation fails to completely remove the degeneracy of the unperturbed Hamiltonian, then in general the unperturbed $d$-tori are expected to break up but some nondegenerate frictions or subtori of them can persist under certain Poincaré nondegenerate conditions on the perturbation. An extreme case is when $r$-dimensional subtori are considered. Let $y = (I_1, \ldots, I_r)$, $u = (I_{r+1}, \ldots, I_d)$, $\phi = (\theta_1, \ldots, \theta_r)$, $\psi = (\theta_{r+1}, \ldots, \theta_d)$, $z = (u, \psi)$, and $[P](y, z) = \int_{T_r} P(\phi, \psi, y, u, 0) \, d\phi$ in (1.3). We assume the following Poincaré nondegenerate condition that
(H4) \([P](y, \cdot)\) has a real analytic family of nondegenerate critical points, i.e., there exists a real analytic function \(z_\ast: \tilde{G} \to \mathbb{R}^{2m}\), where \(m = d - r\), such that \(\partial_z[P](y, z_\ast(y)) = 0\), \(\det \partial_z^2[P](y, z_\ast(y)) \neq 0\), \(y \in \tilde{G}\).

Now, for each \(I = (y, u) \in \tilde{G}\), the unperturbed, resonant \(d\)-torus \(T_I\) is foliated into invariant \(r\)-tori \(T_\bar{y} = T^r \times \{\psi\}\) with frequencies \(\omega_0(y) = \partial_y H_{00}(y)\), parameterized by \(\psi \in T^m\).

The following result is a corollary of Theorem 1.

**Theorem 2.** Assume (H3) and (H4). Then there is an \(\varepsilon_0 > 0\) and Cantor sets \(\tilde{G}_\varepsilon \subset \tilde{G}\), \(0 < \varepsilon < \varepsilon_0\), with \(|\tilde{G} \setminus \tilde{G}_\varepsilon| \to 0\) as \(\varepsilon \to 0\) such that for each \(0 < \varepsilon < \varepsilon_0\) the Hamiltonian system (1.3) admits a Whitney smooth family of real analytic, quasi-periodic \(r\)-tori \(T^\bar{y}\), \(y \in \tilde{G}_\varepsilon\), which also varies smoothly in \(\varepsilon\).

The persistence of subtori split from resonant tori of a nearly integrable Hamiltonian system has been studied in [8,18,20,31] on any \(g\)-resonant surface under Poincaré nondegenerate conditions of the perturbation and Kolmogorov or \(g\)-nondegenerate condition of the unperturbed Hamiltonian. For the properly degenerate, nearly integrable Hamiltonian (1.3), Theorem 2 gives a result along the same line when neither the Kolmogorov nor the \(g\)-nondegenerate condition of the unperturbed Hamiltonian is satisfied. We note in the present case that the group \(g\) is simply \([0] \times \mathbb{Z}^m\), where \(0\) is the zero vector in \(\mathbb{Z}^r\), and \(\tilde{G}\) is the \(r\)-dimensional \(g\)-resonant surface.

To prove Theorem 1, we will first reduce the Hamiltonian system (1.1) to the following normal form:

\[
H(x, y, z) = e_\delta(\omega) + \left\{ \Omega_\delta(\omega), y \right\} + \frac{1}{2} \left\{ z, M_\delta(\omega)z \right\} + \delta P(x, y, z, \omega, \delta)
\]  

(1.4)

associated to the symplectic form (1.2), where \((x, y, z) \in T^n \times R^r \times \mathbb{R}^{2m}\) and \(\omega \in \mathcal{O}\) and \(\delta \in [0, 1]\) are parameters with \(\mathcal{O} \subset \mathbb{R}^n\) being a bounded closed region, \(\Omega_\delta = \text{id} + O(\delta)\), and \(M_\delta(\omega)\) is a \(2m \times 2m\) symmetric matrix for each \(\delta\) and \(\omega\). Moreover, for some complex neighborhoods \(\Delta\) of \([0, 1]\), \(\mathcal{O}(r) = \{ \omega: |\text{Im } \omega| < r \}\) of \(\mathcal{O}\), \(D(r, s) = \{(x, y, z): |\text{Im } x| < r, |y| < s^2, |z| < s\}\) of \(T^n \times [0] \times [0] \subset T^n \times R^n \times \mathbb{R}^{2m}\), \(e, \Omega, M\) are real analytic on \(\Delta \times \mathcal{O}(r)\), and \(P\) is real analytic on \(D(r, s) \times \mathcal{O}(r) \times \Delta\). We assume the following condition:

(H5) There is a constant \(\sigma > 0\) such that

\[
\inf_{0 < \delta < 1} \left| \det \frac{1}{\delta} M_\delta \right| \geq \sigma > 0.
\]

Clearly, when \(P = 0\), the unperturbed system of (1.4) admits a family of invariant \(n\)-tori \(T_\omega = T^n \times \{0\} \times \{0\}\) parametrized by the toral frequency \(\omega \in \mathcal{O}\).

We will prove the following result from which Theorem 1 follows.

**Theorem 3.** Assume (H5) and that (H1) holds for eigenvalues of \(JM_0(\omega)\). Then there are \(\mu = \mu(r, s) > 0, \delta > 0, \gamma > 0\) sufficiently small such that if

\[
|P|_{D(r, s) \times \mathcal{O}(r)} < \gamma^{Am^2} s^2 \mu,
\]  

(1.5)
then there exists a Cantor set $O^* \subset O$, with $|O \setminus O^*| \to 0$ as $\gamma, \delta \to 0$, for which the following holds. For each $\delta$ and $\omega \in O^*$, the unperturbed torus $T_\omega$ persists and gives rise to a slightly deformed, analytic, quasi-periodic, invariant torus of the perturbed system (1.4), and moreover, these perturbed tori form a Whitney smooth family.

The rest of paper is organized as follows. Section 2 is devoted to the proof of Theorem 3 via KAM method, in which we will give details for one KAM step, prove an iteration lemma, show convergence of KAM iterations, and conduct measure estimate. We will prove Theorems 1 in Section 3 by making a normal form reduction to (1.1) in order to remove the singularity of $M$ and to improve the order of perturbation. Theorem 2 will also be proved in this section as a corollary of Theorem 1.

2. Proof of Theorem 3

We will prove Theorem 3 in this section by using KAM method, i.e., we will construct a symplectic transformation, consisting of infinitely many successive steps, called KAM steps, of iterations, so that the $x$-dependent terms are pushed into higher-order perturbations after each step.

Initially, we set $e^0 = e_\delta$, $\Omega^0 = \Omega_\delta$, $M^0 = M_\delta$, $P_0 = P$, $O_0 = O$, $r_0 = r$, $s_0 = s$, $\mu_* = \mu$, $\gamma_0 = \gamma$, and

$$
N_0 = e^0 + \left\{ \Omega^0(\omega), y \right\} + \frac{1}{2} \left\langle z, M^0(\omega)z \right\rangle, \quad H_0 = N_0 + \delta P_0.
$$

For simplicity, we suspend the dependence of all quantities on $\delta$ in the rest of the section.

By (1.5) and Cauchy’s estimate, we have that

$$
|\partial^l_x P_0|_{D_0 \times O_0} < c_0 \gamma_0^{2m^2} s_0^2 \mu, \quad |l| \leq 4m^2,
$$

for some constant $c_0 > 0$ only depending on $r_0$. Let $\mu_0 = c_0 \mu_*$. Then

$$
|\partial^l_x P_0|_{D_0 \times O_0} < \gamma_0^{4m^2} s_0^2 \mu_0, \quad |l| \leq 4m^2.
$$

Suppose that after a $\nu$th KAM step, we arrive at a real analytic, parameter-dependent Hamiltonian

$$
H = H_v = N + \delta P, \quad N = N_v = e(\omega) + \left\{ \Omega(\omega), y \right\} + \frac{1}{2} \left\langle z, M(\omega)z \right\rangle, \quad (2.1)
$$

where $(x, y, z) \in D = D_v = D(r, s)$, $r = r_v \leq r_0, s = s_v \leq s_0$, $\omega \in O = O_v \subset O_0$, $e(\omega) = e^v(\omega)$, $\Omega(\omega) = \Omega^v(\omega)$, $M(\omega) = M^v(\omega)$, $P = P_v(x, y, z, \omega)$ are real analytic in all their arguments and also depend on $\delta \in [0, 1]$ analytically, and moreover,

$$
|\partial^l_x P|_{D \times O} < \gamma^{4m^2} s^2 \mu, \quad |l| \leq 4m^2.
$$

for some $0 < \gamma = \gamma_v \leq \gamma_0$, $0 < \mu = \mu_v \leq \mu_0$.

We will construct a symplectic transformation $\Phi_+ = \Phi_{\nu+1}$, which, in smaller frequency and phase domains, carries the above Hamiltonian into the next KAM cycle. Thereafter, quantities (domains, normal form, perturbation, etc.) in the next KAM cycle will be simply indexed by $(\nu + 1)$. Also, all constants $c_1 - c_9$ below are positive and independent of the iteration process.
2.1. One step of KAM iteration

Below, we will show detailed constructions of the KAM iteration for the Hamiltonian (2.1). First, we expand the perturbation $P$ into Taylor–Fourier series

$$P = \sum_{k \in \mathbb{Z}^n, i \in \mathbb{Z}_+, j \in \mathbb{Z}_+^m} P_{kij} y^i z^j e^{\sqrt{-1} \langle k, x \rangle},$$

and let

$$R = \sum_{|k| \leq K_+} (P_{k00} + \langle P_{k10}, y \rangle + \langle P_{k01}, z \rangle + \langle z, P_{k02}z \rangle) e^{\sqrt{-1} \langle k, x \rangle},$$

$$I = \sum_{|k| > K_+} \sum_{i, j} P_{kij} y^i z^j e^{\sqrt{-1} \langle k, x \rangle},$$

$$II = \sum_{|k| \leq K_+} \sum_{2|i| + |j| \geq 3} P_{kij} y^i z^j e^{\sqrt{-1} \langle k, x \rangle},$$

where

$$K_+ = \left(\left\lfloor \log \frac{1}{\mu} \right\rfloor + 1 \right)^3.$$

Then

$$P - R = I + II.$$

Let

$$r_+ = \frac{r}{2} + \frac{r_0}{4}.$$ 

We now estimate $\partial_{\omega} |P - R|, \omega \in \mathcal{O}, |l| \leq 4m^2$, on a smaller complex domain $D(r_*, \alpha s)$, where $\alpha = \mu^{1/3}$ and

$$r_* = r_+ + \frac{3}{4}(r - r_+).$$

For each $\omega \in \mathcal{O}, |l| \leq 4m^2$, since

$$\left| \sum_{i \in \mathbb{Z}_+, j \in \mathbb{Z}_+^m} \partial_{\omega}^l P_{kij} y^i z^j \right| \leq |\partial_{\omega}^l P|_{D} e^{-|k|r}$$

for all $|y| \leq s^2, |z| \leq s$, we have

$$|\partial_{\omega}^l I|_{D(r_*, s)} \leq \sum_{|k| > K_+} |\partial_{\omega}^l P|_{D} e^{-\frac{|k|(r - r_+)}{4}} \leq \gamma^{4m^2} s^2 \mu \sum_{l > K_+} l^n e^{-\frac{l(r - r_+)}{4}}$$

$$\leq \gamma^{4m^2} s^2 \mu \int_{K_+}^\infty \lambda^n e^{-\frac{\lambda(r - r_+)}{4}} d\lambda \leq \gamma^{4m^2} s^2 \mu^2,$$

(2.5)
provided that
\[ (C1) \quad \int_{K_+}^{\infty} \lambda^n e^{-\frac{\lambda(r - r_\ast)}{4}} d\lambda \leq \mu. \]

Hence
\[ \left| \partial_l^I (P - I) \right|_{D(r_\ast, s)} \leq \left| \partial_l^I P \right|_{D(r_\ast, s)} + \left| \partial_l^I I \right|_{D(r_\ast, s)} \leq 2\gamma^{4m^2} s^2 \mu \]
for all \( \omega \in \mathcal{O} \).

By Cauchy’s estimate, we also have
\[ \left| \partial_l^I H \right|_{D(r_\ast, \alpha s)} = \left| \int \frac{\partial^{|i|+|j|}}{\partial y^i \partial z^j} \partial_l^I (P - I) \, dy \, dz \right|_{D(r_\ast, \alpha s)} \leq \int \left| \frac{\partial^{|i|+|j|}}{\partial y^i \partial z^j} \partial_l^I (P - I) \right|_{D(r_\ast, \alpha s)} \, dy \, dz \]
\[ \leq c_1 \gamma^{4m^2} \alpha^2 \gamma^2 s^2 \mu \leq c_1 \gamma^{4m^2} s^2 \mu^2, \tag{2.6} \]
for all \( \omega \in \mathcal{O} \) and \(|l| \leq 4m^2\), where \(|2i| + |j| = 3\) and \( \int = \int_0^y \cdots \int_0^y \int_0^z \cdots \int_0^z \) is the \((2|i| + |j|)\)-fold integral. Thus by (2.5), (2.6),
\[ \left| \partial_l^I (P - R) \right|_{D(r_\ast, \alpha s) \times \mathcal{O}} \leq \left| \partial_l^I I \right|_{D(r_\ast, s) \times \mathcal{O}} + \left| \partial_l^I H \right|_{D(r_\ast, \alpha s) \times \mathcal{O}} \]
\[ \leq c_2 \gamma^{4m^2} s^2 \mu^2, \quad |l| \leq 4m^2. \tag{2.7} \]

It follows that
\[ \left| \partial_l^I R \right|_{D(r_\ast, \alpha s) \times \mathcal{O}} \leq c_3 \gamma^{4m^2} s^2 \mu, \quad |l| \leq 4m^2. \tag{2.8} \]

Next, we construct a Hamiltonian \( F \) of the form
\[ F = \sum_{0<|k| \leq K_+, |i| + |j| \leq 2} F_{kij} y^i z^j e^{\sqrt{-1} (k, x)} + (F_{001}, z), \tag{2.9} \]
such that the time 1-map \( \Phi = \Phi^1_F \) generated by the Hamiltonian vector field \( X_F = (F_y, -F_x, JF_z)^T \) carries \( H \) into the Hamiltonian in the next KAM cycle.

Denote \([R] = \int_{T^n} R(x) \, dx\) and \( \tilde{R} = R - [R] \). We let \( F \) be such that
\[ \{N, F\} + \delta \tilde{R} + \delta (P_{001}, z) = 0. \tag{2.10} \]
Then

\[ H_+ \equiv H \circ \Phi = H \circ \Phi^1_F = (N + \delta R) \circ \Phi^1_F + \delta (P - R) \circ \Phi^1_F \]

\[ = N + \delta[R] - \delta\langle P_{001}, z \rangle + \delta \int_0^1 \{R_t, F\} \circ \Phi^t_F \, dt + \delta (P - R) \circ \Phi^1_F \]

\[ = e^+ + \{\Omega^+, y\} + \frac{1}{2}\langle z, M^+ z \rangle + \delta P_+ \equiv N_+ + \delta P_+, \quad (2.11) \]

where

\[ e^+ = e + \delta P_{000}, \quad (2.12) \]

\[ \Omega^+ = \Omega + \delta P_{010}, \quad (2.13) \]

\[ M^+ = M + \delta P_{002}, \quad (2.14) \]

\[ R_t = (1 - t)([R] - R - \langle P_{001}, z \rangle) + R, \quad (2.15) \]

\[ P_+ = \int_0^1 \{R_t, F\} \circ \Phi^t_F \, dt + (P - R) \circ \Phi^1_F. \quad (2.16) \]

Substituting (2.2) and (2.9) into (2.10) yields

\[- \sum_{0 < |k| \leq K_+} \sqrt{1 - \langle k, \Omega \rangle} (F_{k00} + \langle F_{k10}, y \rangle + \langle F_{k01}, z \rangle + \langle z, F_{k02}z \rangle) \sqrt{1 - \langle k, x \rangle} \]

\[ + \sum_{0 < |k| \leq K_+} (\langle Mz, JF_{k01} \rangle + 2\langle Mz, JF_{k02}z \rangle) \sqrt{1 - \langle k, x \rangle} + \langle Mz, JF_{001} \rangle \]

\[ = -\delta \sum_{0 < |k| \leq K_+} (P_{k00} + \langle P_{k10}, y \rangle + \langle z, P_{k01} \rangle + \langle z, P_{k02}z \rangle) \sqrt{1 - \langle k, x \rangle} - \delta \langle P_{001}, z \rangle. \]

By comparing coefficients above, we obtain the following linear homological equations

\[ \sqrt{1 - \langle k, \Omega \rangle} F_{k00} = \delta P_{k00}, \quad (2.17) \]

\[ \sqrt{1 - \langle k, \Omega \rangle} F_{k10} = \delta P_{k10}, \quad (2.18) \]

\[ A_{1k} F_{k01} = \delta P_{k01}, \quad (2.19) \]

\[ A_{2k} F_{k02} = \delta P_{k02}, \quad (2.20) \]

\[ M^\top J F_{001} = -\delta P_{001}. \quad (2.21) \]

where

\[ A_{1k} = \sqrt{1 - \langle k, \Omega \rangle} I_{2m} - MJ, \quad A_{2k} = \sqrt{1 - \langle k, \Omega \rangle} I_{4m^2} - (MJ) \otimes I_{2m} - I_{2m} \otimes (MJ). \]

Hereafter \( \otimes \) denotes the tensor product of two matrices.
It is clear that Eqs. (2.17)–(2.20) are uniquely solvable on the new frequency domain

\[ O_+ = \{ \omega \in \mathcal{O}: \frac{\langle k, \Omega(\omega) \rangle}{|k|^2}, |\det A_{1k}(\omega)| > \frac{\gamma^2 m}{|k|^{2\tau m}}, \]

\[ |\det A_{2k}(\omega)| > \frac{\gamma^{4m^2}}{|k|^{4\tau m^2}}, 0 < |k| \leq K_+ \} \]

(2.22)
to yield the desired function \( F \).

To estimate the transformation, we let

\[ D_* = D(r_*, s), \quad D_{i\alpha} = D\left(r_* + \frac{i(r - r_*)}{2}, \frac{\alpha s}{2}\right), \quad i = 1, 2. \]

For each \( \omega \in O_+ \) and \(|l| \leq 4m^2\), since by Cauchy’s estimate,

\[ |\partial^l \omega P_{kij}| \leq |\partial^l \omega P|_{D \times \mathcal{O}} s^{-(2i + j)} e^{-|k| |r|} \leq \gamma^{4m^2} s^{2-2i-j} \mu e^{-|k| |r|}, \quad 0 \leq 2i + j \leq 2, \]

we have by (2.17)–(2.21) that

\[ \left| \frac{1}{\delta} \partial^l \omega F_{k00} \right| \leq c_4 |k|^\tau s^2 \mu e^{-|k| |r|}, \quad \left| \frac{1}{\delta} \partial^l \omega F_{k10} \right| \leq c_4 |k|^\tau \mu e^{-|k| |r|}, \]

\[ \left| \frac{1}{\delta} \partial^l \omega F_{k01} \right| \leq c_4 |k|^{2\tau m} s \mu e^{-|k| |r|}, \quad \left| \frac{1}{\delta} \partial^l \omega F_{k02} \right| \leq c_4 |k|^{4\tau m^2} \mu e^{-|k| |r|}, \]

\[ \left| \partial^l \omega F_{001} \right| \leq s \mu e^{-|k| |r|} \leq c_5 s \mu. \]

By direct differentiation, we have

\[ \left| \partial^l \omega F \right|_D \leq c_6 \mu \Gamma(r - r_*) + c_6 \mu, \quad |i| + |j| \leq 2, \ |l| \leq 4m^2 \]

(2.23)
for all \( \omega \in O_+ \), where

\[ \Gamma(r - r_*) = \sum_{k \in \mathbb{Z}^n} |k|^{4\tau m^2 + 2} e^{-\frac{|k||r - r_*|}{4}}. \]

Since

\[ \Phi^t_F = \text{id} + \int_0^t JDF \circ \Phi^\lambda_D d\lambda, \quad (2.24) \]

\[ D\Phi^t_F = I_{2(n+m)} + \int_0^t J(D^2 F) D\Phi^\lambda_D d\lambda, \quad (2.25) \]
we have by (2.23) that
\[ \Phi_t^F : D_{\frac{1}{2}\alpha} \to D_{\alpha}, \]
for each \( \omega \in \mathcal{O}_+ \), \( 0 < t \leq 1 \), provided that
\begin{align*}
(C2) \quad c_6 \mu \Gamma (r - r_+) & \leq \frac{r - r_+}{2}, \\
(C3) \quad c_6 \mu \Gamma (r - r_+) & \leq \frac{1}{4}. 
\end{align*}
Moreover,
\[ |\partial^l \omega^{D_{\frac{1}{2}\alpha}} (\Phi_t^F - \text{id})|_{D_{\frac{1}{2}\alpha} \times \mathcal{O}_+} \leq c_7 \mu \Gamma (r - r_+) + c_7 \mu, \quad |l| \leq 4m^2, \quad i = 0, 1, \]
for all \( 0 < t \leq 1 \).

We now estimate the new Hamiltonian. It is clear from (2.12)–(2.14) that
\begin{align*}
|\partial^l (e^\omega + e^-) |_{\mathcal{O}_+} & \leq c_8 \delta \gamma^4 m^2 s^2 \mu, \\
|\partial^l (\Omega - \Omega^+) |_{\mathcal{O}_+} & \leq c_8 \delta \gamma^4 m^2 s \mu, \\
|\partial^l (M - M^+) |_{\mathcal{O}_+} & \leq c_8 \delta \gamma^4 m^2 s^2 \mu 
\end{align*}
for all \( |l| \leq 4m^2 \).

To estimate the new frequency domain, we let
\[ \gamma_+ = \frac{\gamma_0}{4} + \frac{\gamma}{2}. \]
If we choose \( \mu \) sufficiently small such that
\[ (C4) \quad 3c_8 \delta \mu K_+^{4m^2 \tau + 4m^2} < \min \left\{ \frac{\gamma - \gamma_+}{\gamma_0}, \frac{\gamma^{2m} - \gamma_+^{2m}}{\gamma_0^{2m}}, \frac{\gamma^{4m^2} - \gamma_+^{4m^2}}{\gamma_0^{4m^2}} \right\}, \]
then by (2.13), (2.14),
\begin{align*}
|\langle k, \Omega^+ \rangle | & \geq |\langle k, \Omega \rangle | - \delta |\langle k, P_{010} \rangle | \geq \frac{\gamma}{|k|^\tau} - \delta c_8 \gamma^4 m^2 \mu |K_+| \geq \frac{\gamma_+}{|k|^\tau}, \\
|\det A_{1k}^+ | & \geq |\det A_{1k} | - \left| \sqrt{-1} \langle k, \delta P_{010} \rangle I_{2m} + \delta P_{002} J \right| \\
& > \frac{\gamma^{2m}}{|k|^{2m \tau}} - \left( \delta c_7 \gamma^{4m^2} \mu K_+ \right)^{2m} - \delta c_8 \gamma^{4m^2} \mu \\
& > \frac{\gamma^{2m}}{|k|^{2m \tau}} - 2c_8 \delta \gamma^{4m^2} \mu K_+^{2m} > \frac{\gamma_+^{2m}}{|k|^{2m \tau}}. \\
|\det A_{2k}^+ | & \geq |\det A_{2k} | - \left| \sqrt{-1} \langle k, \delta P_{010} \rangle I_{4m^2} - (\delta P_{002} J) \otimes I_{2m} - I_{2m} \otimes (\delta P_{002} J) \right|
\end{align*}
\[
\gamma^{4m^2} - (\delta c_8 \gamma^{4m^2} \mu K) + 2(\delta c_8 \gamma^{4m^2} \mu)^2 m
\]

for all \(0 < |k| \leq K_+\), \(\omega \in O_+\).

To estimate the new perturbation, we let

\[s_+ = \frac{1}{2} \alpha s, \quad D_+ = D(r_+, s_+).\]

Then by (2.7), (2.8), (2.15), (2.16), (2.23) and Cauchy’s estimate, we have

\[
\left| \partial^l \omega P_+ \right|_{D_+ \times O_+} \leq \int_0^1 \left| \partial^l \omega \right|_{\mathcal{F}} (R_t, F) \circ \Phi_t^F \, dt + \left| \partial^l \omega (P - R) \circ \Phi_1^F \right|_{D_+ \times O_+}
\]

\[
= \int_0^1 \left| \partial^l \omega \left( \frac{\partial R_t}{\partial x} \frac{\partial F}{\partial y} - \frac{\partial R_t}{\partial y} \frac{\partial F}{\partial x} + \frac{\partial R_t}{\partial z} \frac{\partial F}{\partial z} \right) \right| \circ \Phi_t^F \, dt \bigg|_{D_+ \times O_+}
\]

\[
+ \left| \partial^l \omega (P - R) \circ \Phi_1^F \right|_{D_+ \times O_+} \leq c \gamma^{4m^2} s^2 \mu^2 \left( \Gamma (r - r_+) + 1 \right)^{4m^2 + 1}
\]

\[
\leq c_9 s^2 \gamma^{4m^2} \left( \Gamma (r - r_+) + 1 \right)^{4m^2 + 1} \mu^\frac{4}{3}
\]

(2.33)

for all \(|l| \leq 4m^2\).

Finally, let

\[\mu_+ = (16c_0 \alpha)^\frac{1}{6} \mu,\]

where

\[c_0 = \max\{c_1, \ldots, c_9\}.
\]

If

(C5) \[c_9 \mu^\frac{4}{3} \left( \Gamma (r - r_+) + 1 \right)^{4m^2 + 1} \leq \mu_+,\]

then

\[
\left| \partial^l \omega P_+ \right|_{D_+ \times O_+} \leq \gamma^{4m^2} s^2 \mu_+^\frac{4}{3}, \quad |l| \leq 4m^2.
\]

(2.34)

This completes one step of KAM iterations.
2.2. Iteration lemma

For each \( \nu = 1, 2, \ldots \), we index all index-free quantities above by \( \nu \) and index all \( \nu \)-indexed quantities above by \( \nu + 1 \). This yields the following sequences

\[
\begin{align*}
  r_\nu, \gamma_\nu, s_\nu, \alpha_\nu, \mu_\nu, H_\nu, N_\nu, P_\nu, e_\nu, \Omega_\nu, M_\nu, D_\nu, K_\nu, \mathcal{O}_\nu, \Phi_\nu.
\end{align*}
\]

In particular,

\[
\begin{align*}
  r_\nu &= r_0 \left( 1 - \sum_{i=1}^{\nu} \frac{1}{2i+1} \right), \quad \gamma_\nu = \gamma_0 \left( 1 - \sum_{i=1}^{\nu} \frac{1}{2i+1} \right), \quad s_\nu = \frac{1}{4} \alpha_{\nu-1} s_{\nu-1}, \\
  \alpha_\nu &= \frac{1}{\nu}, \quad \mu_\nu = (16c_0 \alpha_{\nu-1}) \frac{1}{\nu} \mu_{\nu-1}, \quad H_\nu = N_\nu + \delta P_\nu, \\
  N_\nu &= e_\nu + (\Omega_\nu, y) + \frac{1}{2} (z, M_\nu z), \quad D_\nu = D(r_\nu, s_\nu), \quad K_\nu = \left( \log \frac{1}{\mu_{\nu-1}} + 1 \right)^{\frac{3}{2}}, \\
  \mathcal{O}_\nu &= \left\{ \omega \in \mathcal{O}_{\nu-1} : \frac{|\langle k, \Omega_\nu \rangle|}{|k|^{\tau}}, |\det A_{1k}^v(\omega)| > \frac{C^{2m}}{|k|^{2\tau m}}, \right. \left| \det A_{2k}^v(\omega) \right| > \frac{C^{2m}}{|k|^{2\tau m}}, 0 < |k| < K_\nu \right\},
\end{align*}
\]

where

\[
\begin{align*}
  A_{1k}^v &= \sqrt{-1} (k, \Omega_\nu)^{I_{2m}} (M_\nu)^{I_{2m}} J, \\
  A_{2k}^v &= \sqrt{-1} (k, \Omega_\nu)^{I_{4m}} - (M_\nu)^{I_{4m}} \otimes I_{2m} - (M_\nu)^{I_{2m}} \otimes (M_\nu^v J)
\end{align*}
\]

for \( 0 < |k| < K_\nu \).

The following iteration lemma ensures the validity of the KAM iteration for all steps.

**Lemma 2.1.** If \( \mu_0 = \mu_0(r_0, s_0, \gamma_0) \) is sufficiently small, then the following holds for all \( |l| \leq 4m^2 \) and \( \nu = 1, 2, \ldots \):

1. \( \Phi_\nu : D_\nu \to D_{\nu-1} \) are real analytic, symplectic, and \( C^{4m^2} \) depend on \( \omega \in \mathcal{O}_\nu \). Moreover,

\[
\begin{align*}
  H_\nu &= H_{\nu-1} \circ \Phi_\nu = N_\nu + \delta P_\nu, \\
  |\partial_\omega^l D^l (\Phi_\nu - \text{id})|_{D_\nu \times \mathcal{O}_\nu} &\leq \frac{\mu_0^{1/8}}{2^v}, \quad |l| \leq 4m^2, \quad i = 0, 1. \tag{2.35}
\end{align*}
\]

2. On \( D_\nu \times \mathcal{O}_\nu \),

\[
\begin{align*}
  |\partial_\omega^l (e_\nu - e_0^0)|, \quad |\partial_\omega^l (\Omega_\nu - \Omega^0)|, \quad |\partial_\omega^l (M_\nu - M_0^v)| &\leq c_0 \gamma_0^{4m^2} \mu_0, \tag{2.36} \\
  |\partial_\omega^l (e_\nu - e_\nu^{-1})|, \quad |\partial_\omega^l (\Omega_\nu - \Omega^0)|, \quad |\partial_\omega^l (M_\nu - M^{-1}_\nu)| &\leq \frac{c_0 \gamma_0^{4m^2}}{2^v} \mu_0, \tag{2.37} \\
  |\partial_\omega^l P_\nu| &\leq \gamma_0^{4m^2} s_{\nu}^2 \mu_\nu, \quad |l| \leq 4m^2. \tag{2.38}
\end{align*}
\]
(3) $O_v = O_{v-1} \setminus \bigcup_{K_{v-1}<|k| \leq K_v} R_{v_k}^v(\gamma_0)$, where

$$R_{v_k}^v(\gamma_0) = \left\{ \omega \in O_{v-1} : \frac{|k, \Omega_{v-1}(\omega)|}{|k|^\tau} \leq \frac{\gamma_{v-1}}{|k|} \right\},$$

or

$$|\det A_{1_k}^{v-1}(\omega)| \leq \frac{\gamma_{v-1}^{2m}}{|k|^{2\tau m}},$$

or

$$|\det A_{2_k}^{v-1}(\omega)| \leq \frac{\gamma_{v-1}^{4m^2}}{|k|^{4\tau m^2}}$$

for all $k \in \mathbb{Z}^n \setminus \{0\}$.

**Proof.** We need to verify the conditions (C1)–(C5) for all $\nu = 0, 1, \ldots$.

First, we choose $\mu_0$ sufficiently small such that

$$\frac{1}{2^{v+2}} \log \frac{1}{\mu_v} > 1.$$

Then

$$\log(n+1)! + n(\nu + 2) \log 2 + n \log K_{v+1} - K_{v+1} \frac{1}{2^{v+2}}$$

$$= \log(n+1)! + n(\nu + 2) \log 2 + 3n \log \left( \log \left[ \frac{1}{\mu_v} \right] + 1 \right) - \frac{1}{2^{v+2}} \left( \log \left[ \frac{1}{\mu_v} \right] + 1 \right)^3$$

$$\leq \log(n+1)! + n(\nu + 2) \log 2 + 3n \log \left( \log \frac{1}{\mu_v} + 2 \right) - \left( \log \frac{1}{\mu_v} \right)^2$$

$$\leq - \log \frac{1}{\mu_v}.$$

Hence

$$\int_{K_{v+1}} \lambda^n e^{-\lambda(r_v - r_{v+1})} d\lambda \leq (n+1)! \frac{K_{v+1}^n}{(r_v - r_{v+1})^n} e^{-K_{v+1}(r_v - r_{v+1})} \leq \mu_v,$$

i.e., (C1) holds.

Next, we note that

$$\Gamma(r_v - r_{v+1}) = \Gamma \left( \frac{1}{2^{v+2}} \right) \leq \int_1^\infty \lambda^{n+4\tau m^2+2} e^{-\lambda \frac{1}{2^{v+5}}} d\lambda \leq (n+4\tau m^2 + 2)! 2^{(v+5)(n+4\tau m^2)}.$$

(2.39)

Thus, to prove (C2), it is sufficient to show that

$$c_0 \mu_v (n+4\tau m^2 + 2)! 2^{(v+5)(n+4\tau m^2)} \leq \frac{1}{2^{v+2}},$$

(2.40)
which clearly holds for \( \nu = 0 \) if \( \mu_0 \) is sufficiently small. We now consider \( \nu \geq 1 \). Since

\[
\mu_v = (16c_0\alpha_{\nu-1})^{\frac{1}{5}} \mu_{\nu-1} = (16c_0)^{\frac{\nu-1}{6}} \mu_0 \frac{19(\nu-1)}{18},
\]

(2.40) is equivalent to

\[
\left(2^{\frac{5}{11}+n+4\tau m^2+2}c_0^\frac{1}{6}\mu_0^\frac{19}{18}\right)^{\nu-1} c_0(n+4\tau m^2+2)!2^{5(n+4\tau m^2)} \leq 1,
\]

which also holds if \( \mu_0 \) sufficiently small. This proves (C2).

(C3) follows from (2.39) and a similar argument as above.

To prove (C4), we note that for any constant \( \beta > 0, \xi > 1,\mu\beta(\log\frac{1}{\mu}+1)^\xi \to 0 \) as \( \mu \to 0 \).

Hence as \( \mu_0 \) (hence \( \mu_v \)) sufficiently small, we have

\[
3c_0\delta\mu K_{\nu+1}^{4m^2\tau+4m^2} = 3c_0\delta\mu_v \left(\log\frac{1}{\mu_v}\right) + 1 \left(4m^2\tau+4m^2\right) < \left(1 - \frac{1}{2^{4m^2}}\right),
\]

i.e., (C4) holds.

Note that (C5) is equivalent to

\[
\mu_0 \left(\Gamma(r_v - r_{\nu+1}) + 1\right)^{4m^2+1} < \frac{1}{16} \left(16c_0\right)^{\frac{1}{6}}.
\]

Since, by (2.39),

\[
\left(\Gamma(r_v - r_{\nu+1}) + 1\right)^{4m^2+1} \leq \left((n+4\tau m^2+2)!2^{(\nu+5)(n+4\tau m^2)+1}\right)^{4m^2+1},
\]

it is sufficient to show that

\[
\mu_0 \left(n+4\tau m^2+2\right)!^{1/6} \frac{(n+4\tau m^2+2)+1}{16^{1/6}} < c_0 \frac{1}{16^n/(6(n+4\tau m^2+1))},
\]

(2.43)

Let \( \lambda \gg 1 \) be such that

\[
\mu_0 < \frac{1}{(16c_0\lambda^{6\times 18/5})^3} \leq 1.
\]

Then by induction

\[
\mu_v = (16c_\mu^{\frac{1}{3}})^{\frac{1}{6}} \mu_{\nu-1} < \cdots < \frac{1}{(\lambda^{18/5})^\nu} \mu_0.
\]

(2.44)

Hence (2.43) holds if \( \mu_0 \) is sufficiently small.

It follows that the KAM step is valid for all \( \nu = 0, 1, \ldots \), from which (1) follows. In particular, (2.35) follows from (2.26), (2.42) and (2.44). Moreover, (2) follows from (2.27)–(2.29) and (2.44), and (3) follows from (2.30)–(2.32).
2.3. Convergence and measure estimate

Applying Lemma 2.1 inductively we obtain the following sequences:

\[ \Psi_\nu = \Phi_0 \circ \Phi_1 \circ \cdots \circ \Phi_\nu : D_\nu \times O_\nu \to D_0, \]
\[ H_0 \circ \Psi_\nu = H_\nu = N_\nu + \delta P_\nu, \quad N_\nu = e^v(\omega) + \langle \Omega^v(\omega), y \rangle + \frac{1}{2} \langle z, M^v(\omega) z \rangle, \quad \nu = 1, 2, \ldots. \]

By (2.38) and Cauchy’s estimate, we also have

\[ |DP_\nu|_{D(r_\nu, s_\nu/2) \times O_\nu} \leq 2 \gamma^2 m^2 \mu_\nu, \quad \nu = 1, 2, \ldots. \quad (2.45) \]

Let

\[ O_\ast = \bigcap_{\nu=0}^{\infty} O_\nu. \]

Then by Lemma 2.1 and (2.45), \( \Psi_\nu, H_\nu, N_\nu, e^v, \Omega^v, M^v, P_\nu \) converge uniformly on \( D(\frac{r_0}{2}, 0) \times O_\ast, \) say, to \( \Psi_\infty, H_\infty, N_\infty, e^\infty, \Omega^\infty, M^\infty, P_\infty, \) respectively, and moreover,

\[ H_\infty = N_\infty = e^\infty(\omega) + \langle \Omega^\infty(\omega), y \rangle + \frac{1}{2} \langle z, M^\infty(\omega) z \rangle. \]

Since \( H_0 \circ \Psi_\nu = H_\nu, \)

\[ \Phi^t_{H_0} \circ \Psi_\nu = \Psi_\nu \circ \Phi^t_{H_\nu}. \]

It follows that

\[ \Phi^t_{H_0} \circ \Psi_\infty = \Psi_\infty \circ \Phi^t_{H_\infty}, \]

on \( D(\frac{r_0}{2}, 0) \times O_\ast. \) This implies for each \( \omega \in O_\ast \) and \( 0 < \delta < 1, \) (1.4) admits an invariant, quasi-periodic \( n \)-torus with the Diophantine frequency \( \omega. \) The Whitney smoothness of these tori follows from a standard argument using the Whitney extension theorem (see [19,25] and references therein).

For measure estimate, we need the following lemma from [19].

**Lemma 2.2.** Let \( M(\omega), \omega \in O, \) be a family of symmetric, \( 2m \times 2m \) matrices and \( \lambda_1(\omega), \ldots, \lambda_{2m}(\omega) \) be the eigenvalues of \( JM(\omega) \) satisfying the Melnikov nonresonant condition (H1). Denote

\[ A_{1k}(\omega) = \sqrt{-1} (k, \omega) I_{2m} - M(\omega) J, \]
\[ A_{2k}(\omega) = \sqrt{-1} (k, \omega) I_{4m^2} - (M(\omega) J) \otimes I_{2m} - I_{2m} \otimes (M(\omega) J), \quad \omega \in O, \ k \in \mathbb{Z}^n \setminus \{0\}. \]

Then the following hold:
For every $k \in \mathbb{Z}^n \setminus \{0\}$,
\[
\det A_{1k} = \prod_{i=1}^{2m} (\sqrt{-\lambda_i} (k, \omega) - \lambda_i), \quad \det A_{2k} = \prod_{i,j=1}^{2m} (\sqrt{-\lambda_i} (k, \omega) - \lambda_i - \lambda_j).
\]

The set
\[
\{ \omega \in \mathcal{O}: \langle k, \omega \rangle \neq 0, \quad \det A_{1k}(\omega) \neq 0, \det A_{2k}(\omega) \neq 0, \quad \forall k \in \mathbb{Z}^n \setminus \{0\}\}
\]
admits full Lebesgue measure relative to $\mathcal{O}$.

We are now ready to estimate the measure $|\mathcal{O} \setminus \mathcal{O}_*|$. Since
\[
\mathcal{O}_{v+1} = \mathcal{O}_v \setminus \bigcup_{|k| \leq K_{v+1}} R_{k}^{v+1}(\gamma), \quad v = 0, 1, \ldots,
\]
we have
\[
\mathcal{O} \setminus \mathcal{O}_* = \bigcup_{v=0}^{\infty} \bigcup_{K_v < |k| \leq K_{v+1}} R_{k}^{v+1}(\gamma).
\]

By (2.36), we have that
\[
|\partial_\omega^{2m} A_{1k}^v(\omega)| = |k|^{2m} \left( (2m)! + O\left( \frac{1}{|k|+1} \right) + O(\delta + \mu) \right),
\]
\[
|\partial_\omega^{4m^2} A_{2k}^v(\omega)| = |k|^{4m^2} \left( (4m^2)! + O\left( \frac{1}{|k|+1} \right) + O(\delta + \mu) \right),
\]

where $O\left( \frac{1}{|k|+1} \right)$ and $O(\delta + \mu)$ are independent of $v, \omega$. It follows from [33], Lemma 2.2 that there is a positive integer $n_0$ and a positive constant $c$ such that
\[
|R_{k}^{v+1}(\gamma)| \leq c \frac{\gamma}{|k|^\tau},
\]

for all $v$ and $|k| \geq n_0$. Let $v_0$ be such that $K_v > n_0$ as $v \geq v_0$. Then
\[
\left| \bigcup_{v=1}^{\infty} \bigcup_{K_v < |k| \leq K_{v+1}} R_{k}^{v+1}(\gamma) \right| \leq c \gamma \sum_{v=0}^{\infty} \sum_{K_v < |k| \leq K_{v+1}} \frac{1}{|k|^\tau} = O(\gamma). \tag{2.46}
\]

To estimate $R_{k}^{v+1}$ for $0 < |k| \leq K_v$, $v \leq n_0$, we let $\gamma, \delta$ be sufficiently small such that
\[
R_{k}^{v+1}(\gamma) \subset R_{k}^v(\gamma) = \left\{ \omega \in \mathcal{O}_v: \left| \langle k, \omega \rangle \right| \leq \frac{2\gamma v}{|k|^\tau}, \text{ or } \left| \det A_{1k}^0(\omega) \right| \leq \frac{2\gamma v^{2m}}{|k|^{2m}}, \text{ or } \right.
\]
\[
\left. \left| \det A_{2k}^0(\omega) \right| \leq \frac{2\gamma v^{4m^2}}{|k|^{4m^2}} \right\}.
\]
for all $0 < |k| \leq K_\nu$, $\nu \leq n_0$. Then by (H1) and Lemma 2.2, $|R_k^{\nu+1}(\gamma)| \leq |R_k^\nu(\gamma)| \to 0$ as $\gamma, \delta \to 0$, uniformly for all $0 < |k| \leq K_\nu$, $\nu \leq n_0$. Consequently,

$$\left| \bigcup_{\nu=0}^{v_0} \bigcup_{0 < |k| \leq K_\nu} R_k^{\nu+1}(\gamma) \right| \to 0,$$

as $\gamma, \delta \to 0$. Combining this with (2.46), we have that

$$\left| \mathcal{O}_0 \setminus \mathcal{O}_e \right| \leq \left| \bigcup_{\nu=0}^{v_0} \bigcup_{0 < |k| \leq K_\nu} R_k^{\nu+1}(\gamma) \right| + \left| \bigcup_{\nu=0}^{v_0} \bigcup_{K_\nu < |k| \leq K_{\nu+1}} R_k^{\nu+1}(\gamma) \right| \to 0,$$

as $\gamma, \delta \to 0$.

The proof of Theorem 3 is now completed.

3. Proof of Theorems 1 and 2

3.1. Reduction to normal form

Consider Hamiltonian (1.1). In order to apply Theorem 3, we need to first remove the singularity of $M(\omega)$ by considering $M_\varepsilon(\omega)$ as in (H3).

Let $z_\varepsilon(\omega)$ be as in (H2) and consider the translation $\phi: x = x, y = y, z \to z + z_\varepsilon$. Then

$$\tilde{H} = H \circ \phi(x, y, z) = \tilde{e}_\varepsilon(\omega) + \langle \Omega_\varepsilon, y \rangle + \frac{1}{2} \langle z, M_\varepsilon(\omega) \rangle + \varepsilon \tilde{P}(x, y, z, \omega), \quad (3.1)$$

where

$$\tilde{e}_\varepsilon(\omega) = \varepsilon [P](0, z_\varepsilon(\omega)),$$

$$\Omega_\varepsilon(\omega) = \omega + \varepsilon \partial_y [P](0, z_\varepsilon(\omega)),$$

$$M_\varepsilon(\omega) = M(\omega) + \varepsilon \partial^2 z [P](0, z_\varepsilon(\omega)),$$

$$\tilde{P}(x, y, z, \omega) = O\left( \left( |y| + |z| \right)^2 \right) + \sum_{k \neq 0} \sum_{i, j} P_{kij} y^i z^j e^{\sqrt{-1} \langle k, x \rangle},$$

where $O\left( \left( |y| + |z| \right)^2 \right)$ is independent of $x$.

Hamiltonian (3.1) is in the form (1.4) when $\delta = \varepsilon$ but the order of $\tilde{P}$ needs to be improved in order for condition (1.5) to satisfy. To improve the order of $\tilde{P}$, a crucial idea is to perform one step of KAM iteration similar to that in Section 2. Write

$$R = \sum_{0 < |k| < K_1, 2|i| + |j| \leq 2} P_{kij} y^i z^j e^{\sqrt{-1} \langle k, x \rangle},$$

$$I = \sum_{0 < |k| < K_1, 2|i| + |j| \geq 3} P_{kij} y^i z^j e^{\sqrt{-1} \langle k, x \rangle}, \quad II = \sum_{|k| \geq K_1, i, j} P_{kij} y^i z^j e^{\sqrt{-1} \langle k, x \rangle},$$

for some $K_1 > 0$ to be determined later. Then

$$\tilde{P} = O\left( \left( |y| + |z| \right)^2 \right) + R + I + II.$$
Consider re-scaling \( y \to \varepsilon^{\frac{1}{3}} y, \ z \to \varepsilon^{\frac{1}{6}} z, \ \tilde{H} \to \varepsilon^{\frac{1}{3}} \tilde{H} \). Then the re-scaled Hamiltonian reads

\[
\bar{H} = \frac{\tilde{H}(x, \varepsilon^{\frac{1}{3}} y, \varepsilon^{\frac{1}{6}} z)}{\varepsilon^{1/3}} = \tilde{N} + \tilde{P}, \quad \tilde{N} = \tilde{e}_e + (\Omega_e, y) + \frac{1}{2}[z, M_e(\omega)z]
\]

\[
\tilde{P} = \varepsilon^{\frac{7}{6}} \mathcal{O}(\|y\| + \|z\|)^2 + \varepsilon^{\frac{2}{3}} \tilde{R} + \varepsilon^{\frac{7}{6}} \tilde{I} + \varepsilon^{\frac{2}{3}} \tilde{II},
\]

where \( \tilde{e}_e, \tilde{R}, \tilde{I}, \tilde{II} \) are obtained from their respective terms above via re-scaling. We choose \( K_1 \) such that

\[
|\tilde{I}|_{D(r,s) \times O(r)} \leq c \varepsilon.
\] (3.2)

Then there is a constant \( c > 0 \) such that

\[
|\tilde{P} - \varepsilon^{\frac{2}{3}} \tilde{R}|_{D(r,s) \times O(r)} \leq c \varepsilon^{\frac{7}{6}}
\] (3.3)

for some constant \( c > 0 \). We note that \( K_1 \to \infty \) as \( \varepsilon \to 0 \).

Next, similar to Section 2, we eliminate \( \varepsilon^{\frac{2}{3}} \tilde{R} \) by the symplectic transformation \( \Phi^1_F \), where

\[
F(x, y, z) = \sum_{0 < |k| < K_1, 2|j| + |j| \leq 2} F_{kij} y^i z^j e^{\sqrt{-1} \langle k, x \rangle}
\] (3.4)

satisfies

\[
\{\tilde{N}, F\} + \varepsilon^{\frac{2}{3}} \tilde{R} = 0.
\] (3.5)

Similar to (2.17)–(2.20), Eq. (3.5) is equivalent to the following system of homological equations

\[
\sqrt{-1} \langle k, \Omega_e \rangle F_{k00} = \varepsilon^{\frac{3}{2}} \tilde{P}_{k00}, \quad \sqrt{-1} \langle k, \Omega_e \rangle F_{k10} = \varepsilon^{\frac{3}{2}} \tilde{P}_{k10},
\]

\[
A_{1k} F_{k01} = \varepsilon^{\frac{3}{2}} \tilde{P}_{k01}, \quad A_{2k} F_{k02} = \varepsilon^{\frac{3}{2}} \tilde{P}_{k02},
\]

which can be uniquely solved on the open domain

\[
\mathcal{O}_0 = \left\{ \omega \in \mathcal{O} : \left| \langle k, \Omega_e(\omega) \rangle \right| > \frac{\gamma}{|k|^\tau}, \left| \det A_{1k}^\varepsilon(\omega) \right| > \frac{\gamma^{2m}}{|k|^{2\tau m}}, \left| \det A_{2k}^\varepsilon(\omega) \right| > \frac{\gamma^{4m^2}}{|k|^{4\tau m^2}}, 0 < |k| \leq K_1 \right\},
\]

where

\[
A_{1k}^\varepsilon = \sqrt{-1} \langle k, \Omega_e \rangle I_{2m} - M_e J,
\]

\[
A_{2k}^\varepsilon = \sqrt{-1} \langle k, \Omega_e \rangle I_{4m^2} - (M_e J) \otimes I_{2m} - I_{2m} \otimes (M_e J).
\]

This yields a real analytic function \( F \) of the form (3.4) which also depends on \( \omega \) real analytically. We note by (H1) and Lemma 2.2 that \( |\mathcal{O} \setminus \mathcal{O}_0| \to 0 \) as \( \varepsilon \to 0 \).
Similar to (2.23), we also have
\[ |\partial^i_x \partial^j_y F|_{D(3r_4, s) \times \mathcal{O}_0(r_4)} \leq c(r) \frac{\varepsilon^{\frac{2}{3}}}{\gamma^{4m^2}}, \quad |i| + |j| \leq 2, \]  
for some continuous function \( c(r) > 0 \). It follows from (2.24), (2.25) that if \( \varepsilon \) is sufficiently small, then
\[ \phi^t_F : D\left(\frac{r}{4}, \frac{s}{2}\right) \times \mathcal{O}_0\left(\frac{r}{4}\right) \rightarrow D\left(\frac{3r}{4}, s\right), \quad 0 < t \leq 1, \]
and moreover,
\[ |\{R, F\}|_{D(3r_4, s) \times \mathcal{O}_0(r_4)} \leq c \frac{\varepsilon^{\frac{2}{3}}}{s^2 \gamma^{4m^2}} \]  
for some constant \( c > 0 \).

Now,
\[ H_0 \equiv \bar{H} \circ \Phi^1_F = N_0 + \varepsilon P_0, \]  
where
\[ N_0 = \bar{N}, \quad P_0 = \frac{1}{\varepsilon^{1/3}} \int_0^1 \{i R, F\} \circ \phi^t_F \ dt + (\bar{P} - \varepsilon^{\frac{2}{3}} R) \circ \phi^1_F. \]
If we let \( \delta = \varepsilon, r_0 = \frac{r}{4}, s_0 = \frac{s}{2} \), then (3.8) is in the normal form (1.4), and by (3.3), (3.7),
\[ |P_0|_{D(r_0, s_0) \times \mathcal{O}_0(r_0)} \leq c \left( \varepsilon^{\frac{1}{3}} + \frac{\varepsilon^{\frac{1}{3}}}{s_0^{2} \gamma^{4m^2}} \right) \]  
for some constant \( c > 0 \).

3.2. Proof of Theorem 1

Let \( 0 < a < \frac{1}{12}, 0 < b < \frac{1}{6} - 2a, 0 < \beta < \frac{1}{6} - 2a - b \) be fixed constants and let \( \varepsilon \) be small such that \( s_0 \geq \varepsilon^a \). Define \( \gamma = \varepsilon^{4m^2} \) and \( \mu = 2ce^\beta \). Then
\[ |P_0|_{D(r_0, s_0) \times \mathcal{O}_0(r_0)} \leq \gamma^{4m^2} s_0^2 \mu. \]

Since (H3) implies (H5), all conditions of Theorem 3 are satisfied. Applying Theorem 3, we obtain a subset \( \mathcal{O}_* \) of \( \mathcal{O}_0 \), with \( |\mathcal{O}_0 \setminus \mathcal{O}_*| \rightarrow 0 \) as \( \varepsilon \rightarrow 0 \), which parametrizes a Whitney smooth family of quasi-periodic \( n \)-tori of (1.1). Since \( |\mathcal{O} \setminus \mathcal{O}_0| \rightarrow 0 \) as \( \varepsilon \rightarrow 0 \), we have \( |\mathcal{O} \setminus \mathcal{O}_*| \rightarrow 0 \) as \( \varepsilon \rightarrow 0 \). This proves Theorem 1.
3.3. Proof of Theorem 2

Let
\[ y = (I_1, \ldots, I_r), \quad x = (\theta_1, \ldots, \theta_r), \quad z = (I_{r+1}, \ldots, I_d, \theta_{r+1}, \ldots, \theta_d). \]

We associate \( \omega \in \mathcal{O} \) with \( y_0 \in G \) by \( \omega = \partial_y H_{00}(y_0) \) through the diffeomorphism between \( G \) and \( \mathcal{O} \equiv \partial_y H_{00}(G) \). Then up to a constant the Hamiltonian (1.3) under the translation \( y \to y + y_0 \) reads
\[ H = \langle \omega, y \rangle + \varepsilon P(x, y + y_0, z, \varepsilon) + O\left( |y|^2 \right). \]

After re-scaling \( y \to \varepsilon^{\frac{3}{2}} y, \) \( H \to \varepsilon^{\frac{3}{2}} H \), we have
\[ H = \langle \omega, y \rangle + \varepsilon \frac{1}{3} P(x, y, \omega, \varepsilon), \]

where
\[ P(x, y, \omega, \varepsilon) = P\left( x, \varepsilon^{\frac{3}{2}} y + y_0, z, \varepsilon \right) + \varepsilon^{\frac{3}{2}} O\left( |y|^2 \right). \]

Replacing \( \varepsilon^{\frac{1}{3}} \) by a parameter, again called \( \varepsilon \), we obtain the Hamiltonian
\[ H = \langle \omega, y \rangle + \varepsilon P(x, y, \omega, \varepsilon), \]
\[ P(x, y, \omega, \varepsilon) = P\left( x, \varepsilon^{2} y + y_0, z, \varepsilon \right) + \varepsilon^2 O\left( |y|^2 \right) \]

which is in the form (1.1) with \( M \equiv 0 \). Hence the Melnikov condition (H1) holds automatically, and (H2), (H3) are implied by (H4), (H5), respectively. Applying Theorem 1, Theorem 2 follows.

References