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Cayley lattices of finite Coxeter groups are bounded

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Abstract

An *interval doubling* is a constructive operation which applies on a poset P and an interval I of P and constructs a new "bigger" poset $P' = P[I]$ by replacing in P the interval I with its direct product with the two-element lattice. The main contribution of this paper is to prove that finite *Coxeter lattices* are *bounded*, i.e., that they can be constructed starting with the two-element lattice by a finite series of interval doublings.

The boundedness of finite Coxeter lattices strengthens their algebraic property of *semidistributivity*. It also brings to light a relation between the interval doubling construction and the *reflections* of Coxeter groups.

Our approach to the question is somewhat indirect. We first define a new class $\mathcal{HH}$ of lattices and prove that every lattice of $\mathcal{HH}$ is bounded. We then show that Coxeter lattices are in $\mathcal{HH}$ and the theorem follows. Another result says that, given a Coxeter lattice L_W and a parabolic subgroup W_H of the finite Coxeter group W, we can construct L_W starting from W_H by a series of interval doublings. For instance the lattice, associated with A_n, of all the permutations on $n+1$ elements is obtained from A_{n-1} by a series of interval doublings. The same holds for the lattices associated with the other infinite families of Coxeter groups B_n, D_n and $I_2(n)$.

Keywords: Bounded lattice; Coxeter group and lattice; Doubling operation; Reflection; Semidistributivity; Zonotope

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1. Introduction

In 1984, Björner proved that the weak order defined on a finite Coxeter group is a lattice. Since then, Björner and authors like Wachs [3,4] and Le Conte de Poly-Barbut [18,19] have studied this family of groups and associated lattices and proved a number of properties, among which the pseudocomplementation and the semidistributivity [18]. Here we show that finite Coxeter lattices are bounded, which allows a new and constructive understanding of these objects.

Section 2 presents the notions of *doubling* and of *contracting* operations applied on a poset, notions which define the so-called *bounded lattices*. In Section 3 we define a new class of lattices, denoted by $\mathcal{HH}$. A lattice is in $\mathcal{HH}$ if it is finite, semidistributive and if it satisfies some given additional properties. We then prove that all the lattices of $\mathcal{HH}$ are bounded, i.e., that they can be constructed starting from the two-element lattice by a finite series of *interval doublings*.

Section 4 gives the preliminary notions and results on finite Coxeter groups and associated lattices, which allow us to prove that all finite Coxeter lattices are in $\mathcal{HH}$ and, therefore, that they are bounded. Given a Coxeter lattice L_W and a parabolic subgroup W_H of the finite Coxeter group W, it is possible to construct L_W from W_H by a series of interval doublings.

Throughout the paper, all considered structures are assumed to be *finite*, even though not explicitly mentioned.

All basic notions on graphs, posets and lattices are assumed to be known. We just recall that an element j of a lattice L is *join-irreducible* (respectively *meet-irreducible*) if it cannot be obtained as the join (respectively the meet) of elements of L distinct from j (respectively from m). Equivalently, an element j (respectively m) of L is a non-zero (respectively non-unit) join-irreducible (respectively meet-irreducible) if it covers (respectively is covered by) a unique element in L, which is then denoted by j^- (respectively m^+). The set of non-zero join-irreducibles of L is denoted by J_L or simply J and the set of its non-unit meet-irreducibles by M_L or simply M.

A lattice L is *semidistributive* if for all elements $x, y, z \in L$, $x \wedge y = x \wedge z$ implies $x \wedge y = x \wedge (y \vee z)$, and $x \vee y = x \vee z$ implies $x \vee y = x \vee (y \wedge z)$. In such a lattice L, there exists a bijection between the sets J_L of join-irreducibles of L and the set M_L of its meet-irreducibles (Geyer, 1994). This bijection associates to $j \in J_L$ (respectively to $m \in M_L$) the unique meet-irreducible m (respectively unique join-irreducible j) such that $j \not\leqslant m$, $j^- \leqslant m$ and $j \leqslant m^+$. Note that in [21] and, more generally, in the *theory of concept analysis,* an ordered pair (j, m) satisfying these three conditions is said to belong to the *double-arrow relation,* which is denoted by $j \updownarrow m$ (in [6] the expression of this bijection is given in the case of the semidistributive Permutohedron, which is a particular Coxeter lattice).

Notation. In the following and for any semidistributive lattice L, $m_j \in M_L$ will denote the bijective image of the join-irreducible j in the relation $\updownarrow$, and dually $j_m \in J_L$ will denote the bijective image of the meet-irreducible m in $\updownarrow$.

For the definitions about lattices not recalled here, we refer the reader to the literature, especially to the books by Barbut and Monjardet [1], Birkhoff [2] or Davey and Priestley [10]. For more details on the arrow relation, see [21] and [15].

2. The doubling and the contracting constructions and the class of bounded lattices

The definition of a bounded lattice uses the notion of *interval doubling*, a simple construction introduced by Day to give a simple solution to the word problem in free lattices [11]. This operation assigns to a poset P and an interval I of P a new poset $P' = P[I]$ by "doubling" in P the interval I, i.e., by replacing I in P with its direct product by the two-element lattice (Fig. 1). We recall that $+$ denotes the disjoint set union.

Definition 1 (*The doubling construction*). Let $(P, \leqslant)$ be a poset and $I \subseteq P$ an interval of P. We denote by $\mathcal{B} = (\{0, 1\}, \leqslant)$ the two-element lattice where $0 < 1$. The poset P' defined on the set $(P - I) + (I \times \mathcal{B})$ is denoted by $P' = P[I]$ and is given by the following order:

$$x' \leqslant' y' \quad \Leftrightarrow \quad \begin{cases} x', y' \in P - I \text{ and } x' \leqslant y', \text{ or} \\ x' \in P - I,\ y' = yi \in I \times \mathcal{B} \text{ and } x' \leqslant y, \text{ or} \\ x' = xi \in I \times \mathcal{B},\ y' \in P - I \text{ and } x \leqslant y', \text{ or} \\ x' = xi \in I \times \mathcal{B},\ y' = yj \in I \times \mathcal{B},\ x \leqslant y \text{ and } i \leqslant j \text{ in } \mathcal{B}. \end{cases}$$

This construction has found a number of applications in the study of finite lattices, free lattices and varieties. It has also been extended to the doubling of a *convex* set (a subset C of a set X is said to be *convex* if for all x and y in C such that $x \leqslant y$, $[x, y] \subseteq C$)—see [14] and [15] for some developments on this subject. It is easy to check that the join and the meet operations are preserved by the convex doubling operation. Thus if P is a lattice, the result P' of the doubling of C in P is also a lattice. In the paper, we will exclusively consider *interval* doublings applied on *lattices*.

Notation. For a lattice L, an interval I of L and $L' = L[I]$, the elements of the direct product interval $I' = I \times \mathcal{B}$ are partitioned in two isomorphic intervals $I_0 = I \times \{0\}$ and

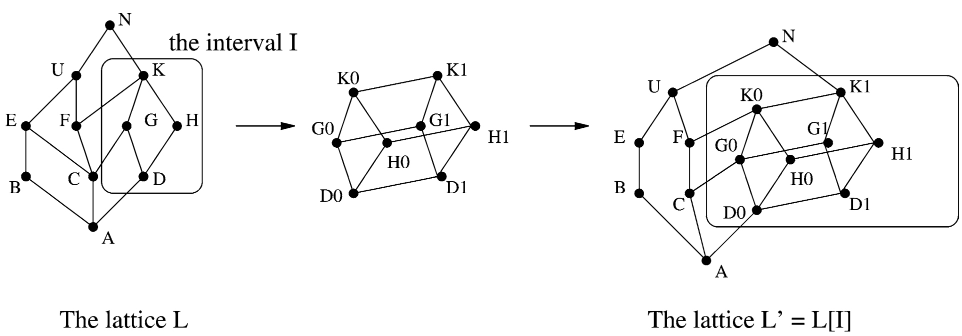

Fig. 1. An interval doubling constructing the lattice L' from the lattice L and the interval $I \subseteq L$.

$I_1 = I \times \{1\}$. An element x of I will generate the two elements $x0 \in I_0$ and $x1 \in I_1$ with $x0 \prec x1$ (where $\prec$ denotes the cover relation associated to the order relation of L).

Moreover every join-irreducible of L as well as the least element of I each induce exactly one join-irreducible of L' (and dually for meet-irreducible elements and the greatest element of I). So the following lemma is a direct consequence of Definition 1.

Lemma 1. *Let L be a lattice, $[a,b] = I \subseteq L$ an interval of L and $L' = L[I]$. The following holds*:

(1) $|L'| = |L| + |I|$.
(2) $J' = \{j\colon\ j \in J \cap (L - I)\} + \{j0\colon\ j \in J \cap I\} + \{a1\}$ *and* $|J'| = |J| + 1$.
(3) $M' = \{m \in M\colon\ m \in L - I\} + \{m1\colon\ m \in M \cap I\} + \{b0\}$ *and* $|M'| = |M| + 1$.
(4) $a1 \updownarrow b0$.
(5) *For any $j \in J'$ with $j \neq a1$ and any $m \in M'$ with $m \neq b0$, $j \not\updownarrow b0$ and $a1 \not\updownarrow m$.*

So the doubling of the interval I creates exactly one new join-irreducible ($a1$) and one new meet-irreducible ($b0$) in the lattice L'. For instance in Fig. 1, $|J_L| = 5$ with $J_L = \{B, C, D, F, H\}$ and $|M_L| = 6$ with $M_L = \{B, E, G, H, U, K\}$. After the doubling of the interval $I = [D, K]$, we obtain the lattice $L' = L[I]$ with $J' = \{B, C, F, D0, H0\} + \{D1\}$ and with $M' = \{B, E, U, G1, H1, K1\} + \{K0\}$. At last, $D1$ and $K0$ satisfy $D1 \updownarrow K0$ and point (5) of Lemma 1 is also verified.

The result below is implicitly proved in [13].

Lemma 2. *Semidistributivity is closed under interval doubling.*

Definition 2 [12]. A lattice L is *bounded*[1] if either L is the one-element lattice or if there exists a sequence $\mathcal{B} = L_1, \ldots, L_i, \ldots, L_{p-1}, L_p = L$ of lattices and a sequence $I_1, \ldots, I_i, \ldots, I_{p-1}$ such that I_i is an interval of L_i and $L_{i+1} = L_i[I_i]$, for every $i < p$.

A lattice is bounded if it can be obtained starting from the two-element lattice $\mathcal{B}$ by a finite sequence of interval doublings. Day and authors like Wille, Ganter and Geyer for instance, have provided a number of results on these lattices and on relative lattices. Figure 2 gives an example of the construction of a bounded lattice.

A proof of the following result is provided in [15].

Proposition 1. *Any bounded lattice is semidistributive.*

Since a bounded lattice is a lattice which can be constructed starting from $\mathcal{B}$ by a finite sequence of interval doublings, such a lattice is equivalently characterizable by the fact that it can be "discontructed" until $\mathcal{B}$ by an iteration of the operation "opposite" of the interval

[1] The original definition of a bounded lattice was introduced by McKenzie [20] in terms of a *bounded lattice homomorphism*. A few years later, Day proved that these lattices were characterized by means of the interval doubling as given in Definition 2.

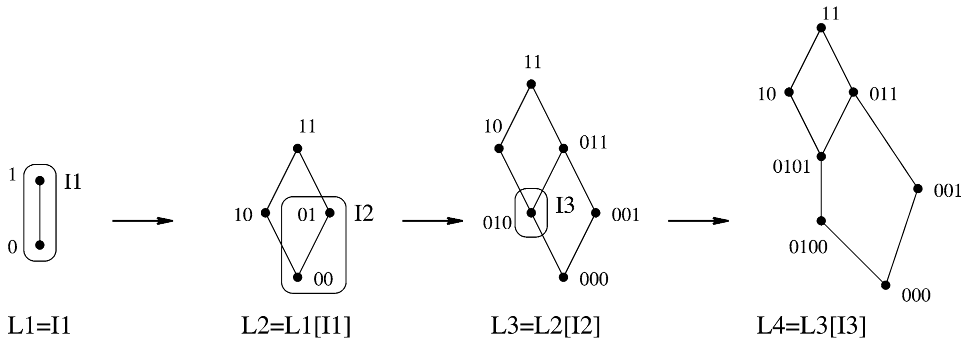

Fig. 2. A series of 3 interval doublings, starting with the two-element lattice $\mathcal{B}$. All lattices L_1 to L_4 are bounded.

doubling. We will call this operation an *interval contraction*. We first need to define the notion of *gluing conditions*:

Definition 3. Let I be an interval of a lattice L, with I equal to the direct product of an interval I_0 by $\mathcal{B}$. We denote by I_1 the interval $I - I_0$, isomorphic with I_0. We say that I satisfies the *gluing conditions* if the two following conditions are verified:

(1) $\forall(y, x1, x0) \in (L - I_1) \times I_1 \times I_0$ $(y < x1 \Rightarrow y \leqslant x0)$.
(2) $\forall(z, x1, x0) \in (L - I_0) \times I_1 \times I_0$ $(z > x0 \Rightarrow z \geqslant x1)$.

The following result directly derives from the definition of an interval doubling.

Proposition 2. *If $L' = L[I]$ is the lattice obtained by the doubling of the interval I in the lattice L and if $I' = I \times \mathcal{B}$, then I' satisfies the gluing conditions.*

We now define the notion of *contractible interval*:

Definition 4. Let L be a lattice and $I \subseteq L$ an interval of L. We say that I is *contractible* (in L) if L can be obtained from a lattice L_0 by the doubling of an interval $I_0 \subseteq L_0$ (with $I = I_0 \times \mathcal{B}$).

From now on, we shall always denote by I_0 and I_1 the two isomorphic intervals constituting the contractible interval I (with the convention that I_0 is the "lower" interval and I_1 the "upper" one, and that I is replaced by I_0 in the contraction).

Definition 5. Let L be a lattice and $I \subseteq L$ a contractible interval of L. We call *contraction of L* (*w.r.t. I*) the operation constructing the "smaller" lattice L_0 by replacing I with I_0 in L. The contraction of an interval is the converse operation of the interval doubling.

We have seen in Lemma 2 that the interval doubling preserves the semidistributivity. The interval contraction has obviously the same property.

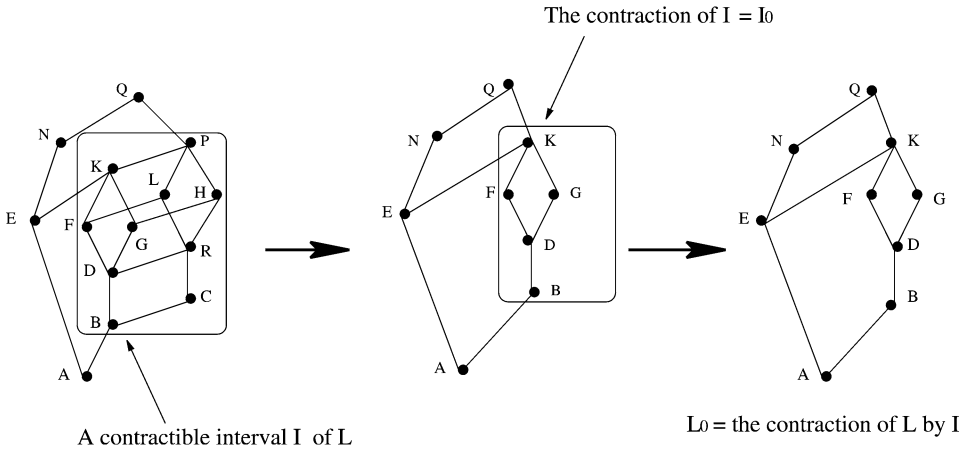

Fig. 3. The contraction of the contractible interval I of a lattice L.

Lemma 3. *Semidistributivity is closed under interval contraction.*

3. The class $\mathcal{HH}$ of lattices

In this section, we define the class $\mathcal{HH}$ of lattices ($\mathcal{HH}$ stands for **H**at and anti-**H**at) and show that all lattices of $\mathcal{HH}$ are bounded. In Section 4, we will then prove that Cayley lattices associated with finite Coxeter groups are bounded by showing that they are particular lattices of $\mathcal{HH}$.

We have to set the following definitions:

Definition 6. Let P be a poset and $x, y, z \in P$. We say that the triple (y, x, z) is a *hat* (respectively an *anti-hat*) if $y \neq z$, $y \prec x$ and $z \prec x$ (respectively if $y \neq z$, $x \prec y$ and $x \prec_P z$). A hat is denoted by $(y, x, z)^\wedge$ and an anti-hat by $(y, x, z)_\vee$.

Definition 7. Let L be a lattice and $x, y \in L$ satisfying $x < y$. The interval $I = [x, y]$ is a 2-*facet* of L if it contains only two paths that intersect only in x and y (i.e., if the diagram of I is a polygon such that there exists two distinct upper covers x_1 and x_2 of x with $y = x_1 \vee x_2$). Such a 2-facet will be denoted by F_{x_1,x,x_2} and is clearly defined by the anti-hat $(x_1, x, x_2)_\vee$.

It is clear that in any 2-facet $[x, y]$ there exists two distinct lower covers y_1 and y_2 of y such that $x = y_1 \wedge y_2$, so a 2-facet can equivalently be defined by this property. It is then defined by a hat $(y_1, y, y_2)^\wedge$ and will be denoted by F^{y_1,y,y_2} (note that x_1 and y_1 are not necessarily distinct, as well as x_2 and y_2).

Definition 8. Let L be a lattice. Let (x, y) be an arc of L such that there exists $z \in L$ with (x, y, z) a hat and with $[x \wedge z, y]$ a 2-facet. We then denote by $(x', x \wedge z, z')$ the associated

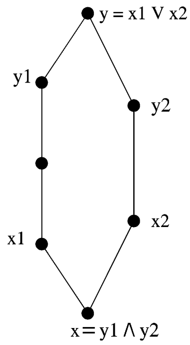

Fig. 4. An example of a 2-facet $F_{x_1,x,x_2} = F^{y_1,y,y_2}$.

anti-hat (with $x' \leqslant x$ and $z' \leqslant z$). A labelling $T = \{t_1, t_2, \ldots, t_p\}$ of the arcs of the covering relation of L is called a 2-*facet labelling* if it satisfies the following property:

If t labels (x, y) and if t' labels (z, y) then t labels $(x \wedge z, z')$ and t' labels $(x \wedge z, x')$.

In these conditions, we note $(x \wedge z, z') \prec_t (x, y)$ and $(x \wedge z, x') \prec_{t'} (z, y)$. For any $t \in T$, $\prec_t$ is a binary relation defined on the arcs of the covering relation of L and which is acyclic. We denote by $\leqslant_t$ its reflexo-transitive closure which is then an order.

Remark. For convenience reasons, we will always talk about *labelling of the arcs* of a lattice L rather than a *labelling of the arcs of the covering relation* of L.

Definition 9. For any 2-facet labelling T of a lattice L, a function r from T to $\mathbb{N}$ is a 2-*facet rank function* of L if it satisfies the following properties on every 2-facet F_{x_1,x,x_2}

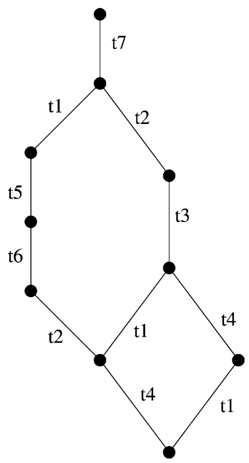

Fig. 5. An example of a 2-facet labelling on a lattice.

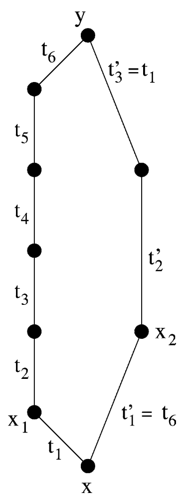

Fig. 6. $r(t_1), r(t_6) < r(t_2), r(t_5) < r(t_3), r(t_4)$; $r(t'_1), r(t'_3) < r(t'_2)$ ($k_1 = 6$ and $k_2 = 3$).

of L (with $t_1, t_2, \ldots, t_{k_1}$ the labels of the edges of one of the shortest paths from x up to $y = x_1 \vee x_2$ and $t'_1 = t_{k_1}, t'_2, \ldots, t'_{k_2} = t_1$ the labels of the edges of the other shortest path from x up to y): for $k \in \{k_1, k_2\}$,

$$r(t_1), r(t_k) < r(t_2), r(t_{k-1}) < \cdots < r(t_{(k+1)/2-1}), r(t_{(k+1)/2+1}) < r(t_{(k+1)/2}) \quad \text{if } k \text{ is odd,}$$

$$r(t_1), r(t_k) < r(t_2), r(t_{k-1}) < \cdots < r(t_{k/2-1}), r(t_{k/2+2}) < r(t_{k/2}), r(t_{k/2+1}) \quad \text{if } k \text{ is even.}$$

Definition 10. A lattice L is in the class $\mathcal{HH}$ if L is finite, semidistributive and if it satisfies the following three conditions:

(1) To every hat $(y, x, z)^\wedge$ of L is associated a unique anti-hat $(y', y \wedge z, z')_\vee$ of L such that $[y \wedge z, x]$ is a 2-facet (with $y' \leqslant y$ and $z' \leqslant z$).
(2) To every anti-hat $(y, x, z)_\vee$ of L is associated a unique hat $(y', y \vee z, z')^\wedge$ of L such that $[x, y \vee z]$ is a 2-facet (with $y \leqslant y'$ and $z \leqslant z'$).
(3) There exists a 2-facet labelling T on L and a 2-facet rank function r on T.

Theorem 1. *Let m be meet-irreducible in $L \in \mathcal{HH}$. If (m, m^+) is labelled by t, the set $E_m = \{(x, y)\colon (x, y) \leqslant_t (m, m^+)\}$ is not empty and has a least element (u, v). Moreover, v is a join-irreducible, $v^- = u$ and $v \updownarrow m$.*

Proof. Let (u, v) be a minimal element of E_m. If v is not join-irreducible there exists z in L with $z \prec_L v$ and $z \neq u$. The triple (u, v, z) is then a hat and has an associated anti-hat $(u', u \wedge z, z')_\vee$ with $u' \leqslant u$ and $z' \leqslant z$, and such that $(u \wedge z, z')$ is labelled by t. Therefore $(u \wedge z, z') \prec_t (u, v)$ and (u, v) is not minimal in E_m, a contradiction. Now since v is join-irreducible, $u = v^-$ and so $v^- \leqslant_L m$, $v \leqslant_L m^+$ and $v \not\leqslant_L m$, which implies $v \updownarrow m$. At last, since L is semidistributive, v is the unique join-irreducible satisfying $v \updownarrow m$ and so $(u, v) = (v^-, v)$ is the unique minimal element of E_m. □

This theorem naturally leads to an algorithm defined on a lattice $L \in \mathcal{HH}$ and which computes, for a given meet-irreducible m of L, the unique join-irreducible j satisfying $j \updownarrow m$. This algorithm starts with an arc (m, m^+) and constructs, when it exists, a 2-facet whose hat has the form $(m, m^+, z)^\wedge$ for some $z \in L$. We then iterate the process with the arc opposite from (m, m^+) in the 2-facet. The algorithm stops when the considered arc (x, y) does not belong to a hat. Then by construction, y is a join-irreducible satisfying $y \updownarrow m$ and, by semidistributivity of L, $y = j_m$ and does not depend on the choice of the 2-facets at each step of the algorithm.

This algorithm is directly generalizable into an algorithm which takes an arc (x, y) of the covering relation of L which is labelled by t and computes the unique ordered pair (j, m_j) such that $(j^-, j) \prec_t (x, y) \prec_t (m_j, m_j^+)$.

The existence of a 2-facet rank function on the lattices of $\mathcal{HH}$ implies the following lemma (take any 2-facet labelling T of L and any label $t \in T$ with maximum rank):

Lemma 4. *Let $L \in \mathcal{HH}$ and T a 2-facet labelling of L. There exists a label $t \in T$ such that for any hat $(y, x, z)^\wedge$ whose arc (y, x) or (z, x) is labelled by t, $F^{y,x,z}$ is a diamond.*

N.B.: The case where a label with maximum rank labels an edge which does not belong to any hat or anti-hat clearly allows the contraction of this edge, seen as a contractible interval. In the following we omit this trivial case.

For any $L \in \mathcal{HH}$ and any 2-facet labelling T of L, we denote by $\mathcal{F}_t$ the set of all the 2-facets of L whose hat and anti-hat have one edge labelled by $t \in T$. By Theorem 1,

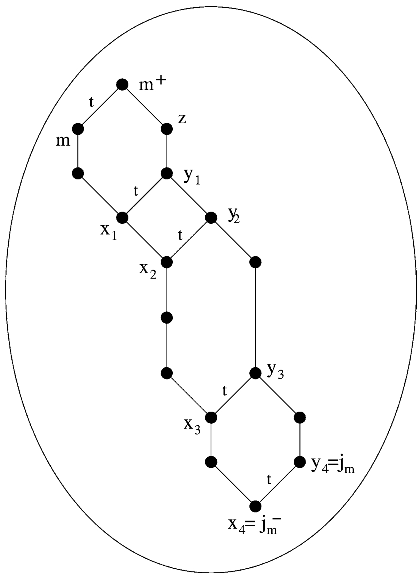

Fig. 7. The algorithm.

$\bigcup \mathcal{F}_t$ is a non-empty union of intervals, each with the form $[j^-, m_j^+]$ (for some $j \in J_L$ and $m_j \in M_L$ with $j \updownarrow m_j$).

Lemma 5. *Let $L \in \mathcal{HH}$, $j \in J_L$, m_j its bijective meet-irreducible and t the label of the arcs (j^-, j) and (m_j, m_j^+). If the 2-facets generated by our algorithm applied on (m_j, m_j^+) (i.e., those in $[j^-, m_j^+]$ having one edge labelled by t) are all isomorphic with diamonds, the interval $I_{j,m_j} = [j^-, m_j^+]$ is contractible.*

Proof. To prove that I_{j,m_j} is contractible, we have to show that $I_{j,m_j} = [j^-, m_j] \times \mathcal{B}$ and that the gluing conditions hold on it.

To prove that $I_{j,m_j} = [j^-, m_j] \times \mathcal{B}$, we start by showing that the label t induces an order isomorphism between the intervals $[j^-, m_j]$ and $[j, m_j^+]$. Consider a path $x_0 \prec x_1 \prec \cdots \prec x_i \prec \cdots \prec x_p$ from $x_0 = j^-$ to $x_p = m_j$. All elements x_i clearly belong to the interval $[j^-, m_j]$. If we note $x'_0 = j$, the triple (x_1, x_0, x'_0) is an anti-hat with the arc (x_0, x'_0) labelled by t and, therefore, $F_{x'_0, x_0, x_1}$ is a diamond and the arc (x_1, x'_1)—with $x'_1 = x'_0 \vee x_1$—is labelled by t.

For any $1 \leqslant i \leqslant p$, let us denote by x'_i the join of x_i and x'_{i-1}. Every x'_i is an element of the interval $[j, m_j^+]$ and, by the argument given above, all arcs (x_i, x'_i) are labelled by t. So the label t "associates" to every x_i of $[j^-, m_j]$ the element x'_i of $[j, m_j^+]$. Dually, it is clear that t "associates" to every element x'_i of $[j, m_j^+]$ the element x_i of $[j^-, m_j]$. The label t thus describes a bijection between the elements of the intervals $[j^-, m_j]$ and $[j, m_j^+]$, which is moreover an order isomorphism. Indeed $x_i \prec x_{i+1}$ in $[j^-, m_j]$ is equivalent to $x'_i \prec x'_{i+1}$ in $[j, m_j^+]$ since every tuple $(x_i, x_{i+1}, x'_i, x'_{i+1})$ forms a diamond by hypothesis, with $x_i \prec x'_i$ and $x_{i+1} \prec x'_{i+1}$. To prove that $I_{j,m_j} = [j^-, m_j] \times \mathcal{B}$, we still have to show that the only edges existing between an element of $[j^-, m_j]$ and an element of $[j, m_j^+]$ are exactly those labelled by t, that go from an element x to its bijective image x'. Since the proof of that point uses the gluing conditions we first prove that these conditions are satisfied on I_{j,m_j}.

Let x' be an element of $[j, m_j^+]$ and x its bijective image in $[j^-, m_j]$ (so $x \prec x'$). Assume that the gluing conditions do not hold, i.e., for instance that there exists an element z of $L - [j, m_j^+]$ such that $z \leqslant_L x'$ and $z \nleqslant_L x$. There exists z_0 and y' satisfying $z \leqslant z_0 \prec y' < x'$ with $z_0 \notin [j, m_j^+]$ and $y' \in [j, m_j^+]$. The triple (z_0, y', y) is a hat whose arc (y, y') is r-labelled by t. Therefore the interval $[y \wedge z_0, y']$ is a diamond (by hypothesis on t) with the arc $(z_0 \wedge y, z_0)$ labelled by t, which implies that the arc $(z_0 \wedge y, z_0) \in \mathcal{F}_t$. Since (y, y') is an arc that belongs to a 2-facet included in $[j^-, m_j^+]$, so does $(y \wedge z_0, z_0)$ and, therefore, $z_0 \in [j, m_j^+]$, a contradiction.

We prove that I_{j,m_j} is isomorphic to the direct product $[j^-, m_j] \times \mathcal{B}$. Let $x \in [j^-, m_j]$ and x' its image in $[j, m_j^+]$. Assume there exists $y' \in [j, m_j^+]$ with $y' \neq x'$ and $x \prec y'$. By the gluing conditions, we have $x' < y'$. Now $x \prec x' \prec y'$ implies that $x \nprec y'$, a contradiction.

Assume now that there exists $y' \in [j, m_j^+]$ such that $y' \prec x$. Since $y' \in [j, m_j^+]$ there exists $y \in [j^-, m_j]$ such that $y \prec y'$, so $y \prec y' \prec x$ with $x, y \in [j^-, m_j]$. This implies $y' \in [j^-, m_j]$, a contradiction. □

Theorem 2. *The class $\mathcal{HH}$ of lattices is closed for the contraction of a contractible interval w.r.t. a label whose 2-facet rank function is maximal.*

Proof. Since the interval contraction preserves semidistributivity, we check that conditions (1) and (3) of Definition 10 are also preserved (condition (2) is dual from condition (1)).

Let $L \in \mathcal{HH}$, I a contractible interval of L and L'' the lattice obtained by the contraction of $I = I_0 + I_1$ in L. We prove that any hat $(y, x, z)^\wedge$ of L'' has been generated by a hat of L, so the interval contraction in $\mathcal{HH}$ does not generate any hat. Five cases may occur:

(1) $x, y, z \notin I_0$: then $(y, x, z)^\wedge$ was already a hat of L.
(2) $x, y, z \in I_0$: then $(y, x, z)^\wedge$ is the result of the contraction of the hats $(y_0, x_0, z_0)^\wedge$ in I_0 and $(y_1, x_1, z_1)^\wedge$ in I_1. Since $x' = y \wedge z \in I_0$, $(y', x', z')_\vee$ is the result of the contraction of $(y_0', x_0', z_0')^\vee$ and $(y_1', x_1', z_1')^\vee$.
(3) $x \in I_0$ and $y, z \notin I_0$: then y and z were elements of L and x has been generated by the contraction of $x_0 \in I_0$ and $x_1 \in I_1$. Since $x' \notin I_0$, $(y', x', z')_\vee$ was already an anti-hat of L.
(4) $x, y \in I_0$ and $z \notin I_0$: then z was an element of L and x and y have been generated by the contraction of x_0 and x_1 for x and y_0 and y_1 for y.
(5) $x, y \notin I_0$ and $z \in I_0$: then x and y were elements of L and z has been generated by the contraction of z_0 and z_1.

Note. The case $x \notin I_0$ and $y, z \in I_0$ does not exist since I_0 is an interval.

Thus the origin of every hat of L'' is well defined. The determination of the anti-hat associated with a hat of L'' follows the same arguments and is left to the reader.

Finally the interval contraction does not generate any new hat or anti-hat and conditions (1) and (2) hold. The existence of a 2-facet labelling and of a 2-facet rank function is trivially closed under interval contraction when the 2-facet rank function of the concerned label is maximal, so condition (3) also holds. □

We get the announced result:

Corollary 1. *Every lattice of $\mathcal{HH}$ is bounded.*

We end this section with an additional property of these lattices given in Proposition 3.

Definition 11. Let $L \in \mathcal{HH}$ and $<_\updownarrow$ the binary relation defined on $(J_L \times M_L)^2$ by:

$$\left(j', m_{j'}\right) <_\updownarrow (j, m_j)$$

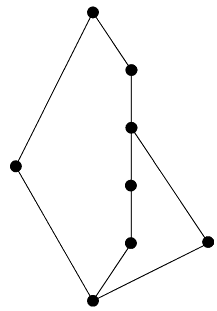

Fig. 8. A bounded lattice that does not belong to $\mathcal{HH}$.

if the fact that the interval $[j^-, m_j^+]$ is contractible implies that $[j'^-, m_{j'}^+]$ has already been contracted.

The binary relation $<_\updownarrow$ is well defined (indeed by the construction of any interval $[j^-, m_j^+]$ described in our algorithm, it is easy to point out the pairs (j', m'_j) that have to be "contracted" before; it suffices to observe the non diamond generated 2-facets and to compute the pairs $(j', m'_{j'})$ whose contraction transform these 2-facets into diamonds). Since the lattices of $\mathcal{HH}$ are bounded, there necessarily exists a linear extension of $<_\updownarrow$ that corresponds with the order of contraction of all the pairs (j, m_j) of the lattice (chosen among all possible orders of contractions of these pairs). This implies that $<_\updownarrow$ contains no cycle and so its reflexo-transitive closure is an order relation on $(J_L \times M_L)^2$.

Consider now the associated lattice $\mathcal{T}$ of all ideals of the poset $((J_L \times M_L)^2, <_\updownarrow)$. By a well-known Birkhoff's result, $\mathcal{T}$ is distributive and, by definition of $<_\updownarrow$, the elements of $\mathcal{T}$ are all the contracted lattices that can be reached from L down to $\mathcal{B}$. Moreover all the series of interval doublings that lead from $\mathcal{B}$ to L are exactly given by all the maximal paths of $\mathcal{T}$. Hence the proposition below.

Proposition 3. *Let L be a lattice of $\mathcal{HH}$. The set of all the lattices that can be obtained from L by a series of interval contractions is a distributive lattice when ordered by the following natural order relation*: $L < L'$ *if L can be obtained from L' by a series of interval contractions.*

Note at last that the lattice of Fig. 8 proves that $\mathcal{HH}$ is strictly contained in the class of bounded lattices.

4. On Coxeter lattices

4.1. Preliminaries

In this part of the section, we prove that the class of Coxeter lattices is included in $\mathcal{HH}$, which directly implies that Coxeter lattices are bounded. To do so, we recall and propose some definitions and results on these lattices. For more details, the standard references for Coxeter groups are the books by Bourbaki [5] and by Humphreys [17].

Definition 12. A group W is a *Coxeter group* if W has a set of generators $S \subset W$, subject only to relations of the form

$$(ss')^{m(s,s')} = e$$

where $m(s,s) = 1$ for any s in S (all generators have order 2), and $m(s,s') = m(s',s) \geqslant 2$ for $s \neq s'$ in S. The pair $\{W, S\}$ is called a *Coxeter system*.

It has been shown in [14] that the class of bounded lattices is closed under direct product. Therefore to prove that Coxeter lattices are bounded, we will only deal with the case of *irreducible* finite Coxeter groups (i.e., those which can not be decomposed as the direct product of two Coxeter groups).

We recall that the *right* (respectively *left*) *Cayley graph* associated with a group W and a set S of generators of W is the graph whose vertices are the elements of W and where there is an edge from w to w' if there exists $s \in S$ such that $w' = ws$ (respectively $w' = sw$). Several partial order relations can be defined on Coxeter groups. Among them, the *right* (respectively *left*) *weak order* is the transitive closure of the right (respectively left) Cayley graph directed with respect to the increasing length, so starting from the neutral element e. The right and the left weak orders are trivially isomorphic, so and unless explicitly said otherwise, we will always use the *right* weak order throughout the paper. When no possible confusion may arise, we will simply denote it by $<$.

Any Coxeter group has a remarkable subset of elements which are called the *reflections* of the group (this denomination is due to the strong properties of these objects in the geometrical interpretation of Coxeter groups).

Definition 13. The elements of the set $T_W = \{t \in W\colon\ \exists s \in S, \exists w \in W$ such that $t = wsw^{-1}\}$ are called the *reflections* of the Coxeter group W. These elements are the conjugates of the generators of W and thus have order 2.

There exists two useful labellings of the edges of the Cayley graph of a Coxeter group. The first one labels each edge with a generator: the edge $\{w, w'\}$ with $w' = ws$ is labelled with s and we talk about *g-labelling*. The other interesting labelling uses reflections: the edge $\{w, w'\}$ with $w' = ws$ is labelled with the reflection $t = wsw^{-1}$. So if $w = s_{i_1}s_{i_2}\dots s_{i_r}$ is a reduced expression of w, the edge $\{w, w'\}$ is labelled by $t = s_{i_1}s_{i_2}\dots s_{i_r}ss_{i_r}\dots s_{i_2}s_{i_1} = wsw^{-1}$ and we then talk about *r-labelling*. We will show that the set of all the reflections of a Coxeter group W constitute a 2-facet labelling (Proposition 7) and that the length function ℓ applied on the r-labelling of the edges of the Cayley graph of W is a 2-facet rank function (Theorem 5).

The result below is easily shown by a simple computation of the expression of the reflections $t_1, t_2, \dots, t_{r-1}, t_r$ by the generators $s_1, s_2, \dots, s_{r-1}, s_r$.

Lemma 6. *Let $w = s_1s_2\dots s_r \in W$ with W a Coxeter group. Let $t_1, t_2, \dots, t_r$ be the reflections labelling the arcs (e, s_1), (s_1, s_1s_2), $\dots$, $(s_1s_2\dots s_{r-1}, s_1s_2\dots s_{r-1}s_r)$ respectively. The following holds*:

(1) $w = s_1 s_2 \dots s_r = t_r t_{r-1} \dots t_2 t_1$.
(2) *For every* $i \leqslant r$, $s_i = t_1 t_2 \dots t_i t_{i-1} \dots t_2 t_1$.

The following proposition directly derives from a result by [5].

Proposition 4. *Let* W *be a Coxeter group,* $w \in W$ *and consider in the Cayley graph of* W *oriented by the right weak order, a shortest path between* e *and* w*. The reflections* $t \in T$ *that label the arcs of this path are all distinct and do not depend on the path but only on the element* w*. We shall denote the set of these reflections by* T_w*.*

Remark. A classical corollary of this result is that any two elements w and w' of W satisfy $w \leqslant w'$ if and only if $T_w \subseteq T_{w'}$. Moreover the set T_w can equivalently be defined as the set of all the reflections t such that $\ell(tw) < \ell(w)$.

Corollary 2. *If* w_0 *denotes the unique element of maximal length in a Coxeter group* W*, then* T_{w_0} *is equal to the set* T_W *of all reflections of* W*. The number of the reflections of a Coxeter group is then equal to the length of* w_0*.*

Theorem 3 (Björner). *The weak order defined on a finite Coxeter group is a lattice, which moreover is self-dual.*

This result generalizes a Guilbaud and Rosenstiehl's result for the permutations lattice [16].

From now on, any lattice defined on a finite Coxeter group W by the right weak order will simply be called a *Coxeter lattice*, and will be denoted by L_W.

We recall that the *left* (respectively *right*) *translation* of $w \in W$ by $w' \in W$ is the element $w'w$ (respectively ww') of W. The notion of translation of an element can naturally be extended to the notion of translation of a set $X \subseteq W$ as follows: the left translation of X by an element $w' \in W$ is equal to the set $\{w'x\colon x \in X\}$.

Now if we define a function d on W^2 by $d(w, w') = \ell(w^{-1}w')$, then d is a distance relation on the elements of W. Indeed $d(w, w) = \ell(e) = 0$ and the symmetry and the triangle inequality conditions are known to be satisfied.

Proposition 5. *The distance* d *defined on* W^2 *by* $d(w, w') = \ell(w^{-1}w')$ *is invariant for the left translation by any element of the group.*

Proof. Let $d(w, w')$ be the distance between w and w' in W and consider $a \in W$. $d(aw, aw') = \ell(w^{-1}a^{-1}aw') = \ell(w^{-1}w') = d(w, w')$. $\square$

The following assertions are classical and their proof is given in [3,4] and [5].

Lemma 7.

(1) *The left translation on a Coxeter group preserves the distance and the* g*-labelling.*

(2) *Any interval* $[w, w']$ *with* $w \leqslant w'$ *is order isomorphic with the interval* $[e, w^{-1}w']$ *obtained as the left translation of* $[w, w']$ *by* w^{-1}.
(3) *Two edges labelled by the same reflection* t *are transformed by left translation by* w *in two edges labelled with the same reflection* $t' = wtw^{-1}$ (*the left translation permutes the reflections of the group*).

Proposition 4 and Lemma 7 together lead to the following property:

Lemma 8. *A chain between two elements of the Cayley graph of a Coxeter group* W *is a shortest chain between these two elements if and only if the reflections labelling its edges are all distinct.*

We recall the definition of a *parabolic subgroup*.

Definition 14. For a Coxeter system $\{W, S\}$ and any subset H of S, the *parabolic subgroup* W_H is the subgroup of W generated by the elements of H.

The group W_H is also a Coxeter group, which is always an interval for the right weak order. Moreover if $H = \{s_1, s_2\}$ (i.e., if $|H| = 2$), the Cayley graph of W_H is a polygon with $2m(s_1, s_2)$ elements and as many edges.

4.2. Coxeter lattices are bounded

One of the authors has proved the following important result:

Theorem 4 (LCPB). *All* (*finite*) *Coxeter lattices are semidistributive.*

Let L_W be a Coxeter lattice and $(ws_1, w, ws_2)_\vee$ an anti-hat of L_W (with $s_1, s_2 \in S$). Let $Max = ws_1 \vee ws_2$ and consider the interval $\mathcal{I} = [w, Max]$. By Lemma 7, $\mathcal{I}$ is isomorphic with the interval $w^{-1}\mathcal{I} = [w^{-1}w, w^{-1}Max] = [e, w^{-1}Max] = [e, s_1 \vee s_2]$, which is the parabolic subgroup of W generated by s_1 and s_2. So the interval $\mathcal{I}$ is a $2m(s_1, s_2)$-sided polygonal graph and, since the left translation preserves the g-labelling, the edges of $\mathcal{I}$ are alternatively labelled by s_1 and s_2.

Proposition 6. *Every interval* $\mathcal{I}$ *of* L_W *with the form* $\mathcal{I} = [w, ws_1 \vee ws_2]$ *where* $s_1, s_2 \in S$ *and where* $(ws_1, w, ws_2)_\vee$ *is an anti-hat of* L_W *is a* 2*-facet of* L_W. *Moreover a* 2*-facet of* L_W *is always the left translation of a parabolic subgroup of* W *generated by two generators* (*the converse is almost true*: *the left translation of a parabolic subgroup of* W *generated by two generators by an element* w *of the group is always a* 2*-facet* F_{xs_1,x,xs_2}, *but where* $x \neq w$ *in general*).

If we call *k-facet* any interval with the form $[w, ws_1 \vee ws_2 \vee \cdots \vee ws_k]$ (where the ws_i's are all upper covers of w) then a k-facet is always a *left coset* of $W_{s_1,s_2,\ldots,s_k}$ which is order isomorphic with $W_{s_1,s_2,\ldots,s_k}$. This isomorphism preserves the g-labelling.

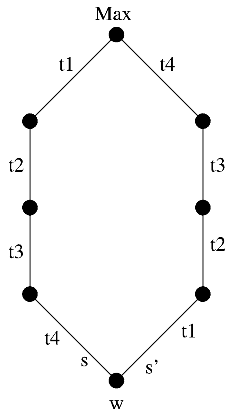

Fig. 9. Two "opposite" edges of a 2-facet are labelled by the same reflection.

The following result is particularly important since it implies that the r-labelling of the edges of a Coxeter lattice is a 2-facet labelling.

Proposition 7. *Two "opposite" edges of a 2-facet of a Coxeter lattice are labelled by the same reflection.*

Proof. Let $F_{ws,w,ws'}$ be the 2-facet generated by $(ws, w, ws')_\vee$, *Max* its maximum element and assume that $m(s,s') = p$. This implies that $F_{ws,w,ws'}$ has $2p$ edges. Let us denote by $e_1, e_2, \ldots, e_p$ the edges of one of the two paths of $F_{ws,w,ws'}$ going up from w to *Max*, and $e_{p+1}, e_{p+2}, \ldots, e_{2p}$ the edges of the other path going down from *Max* to w. Let t_i be the reflection labelling the edge e_i of $F_{ws,w,ws'}$. Consider now the edge $e_{(i+p) \bmod 2p}$ (i.e., the opposite edge of e_i in $F_{ws,w,ws'}$) which is labelled by the reflection $t_{(i+p) \bmod 2p} = t$. Without loss of generality, assume that

$$t_i = w \underbrace{ss's\ldots s's}_{2i-1 \text{ generators}} w^{-1}.$$

So

$$t = w \underbrace{ss's\ldots s'ss's}_{2i-1+2p \text{ generators}} w^{-1}.$$

Therefore

$$t_i t = w \underbrace{ss's\ldots s's}_{2i-1 \text{ generators}} w^{-1} w \underbrace{ss's\ldots s'ss's}_{2i-1+2p \text{ generators}} w^{-1} = w \underbrace{ss's\ldots s's}_{2p \text{ generators}} w^{-1} = wew^{-1} = e$$

and $t_i = t$. $\square$

Corollary 3. *For every Coxeter group W, the r-labelling on the edges of L_W is a 2-facet labelling.*

We recall the definition and a characterization of a *left quotient* of a Coxeter system.

Definition 15. For every $H \subset S$, the *left quotient* W^H of the Coxeter group W is the set $W^H = \{w \in W\colon\ \ell(sw) > \ell(w)$ for any $s \in H\}$.

The *right quotient* of a Coxeter group associated with a subset H of S is defined dually. We will deal only with *left* quotients and will simply call them *quotients*.

Proposition 8. *For every $H \subseteq S$, $w \in W^H$ if and only if for any $s \in H$, $s \nleq w$ in L_W.*

The following result is a consequence of the properties of the length function in Coxeter groups.

Corollary 4. *If F is a k-facet of a Coxeter group W, generated by $H \subseteq S$, the following conditions are equivalent*:

(1) $w = \min F$.
(2) $w^{-1} \in W^H$.

Let s_1 and s_2 be two distinct generators of the Coxeter group W. The parabolic subgroup $W_{\{s_1,s_2\}}$ and the quotient $W^{\{s_1,s_2\}}$ will simply be written W_{s_1,s_2} and W^{s_1,s_2} respectively.

The proof of the following result can be found in [17].

Proposition 9. *Let $\{W, S\}$ be a Coxeter system and L_W its associated lattice. For every $w \in W$ and all $s_1, s_2 \in S$, there exists a unique ordered pair $(u, v) \in W_{s_1,s_2} \times W^{s_1,s_2}$ such that $w = uv$. Moreover $\ell(w) = \ell(u) + \ell(v)$.*

Proposition 10 (Björner). *For a finite Coxeter system $\{W, S\}$ and every subset $H \subseteq S$, W^H is an interval.*

Propositions 9 and 10 together imply the corollary below.

Corollary 5. *The set of all uW^{s_1,s_2} with $u \in W_{s_1,s_2}$ constitute a partition of the elements of W into order isomorphic intervals and every set uW^{s_1,s_2} with $u \in W_{s_1,s_2}$ will then be called a* class *of the partition. Given two classes uW^{s_1,s_2} and vW^{s_1,s_2} (with $u, v \in W_{s_1,s_2}$) the isomorphism associates to $ux \in uW^{s_1,s_2}$ the element $vx \in vW^{s_1,s_2}$. This isomorphism preserves the g-labelling.*

Lemma 9. *Consider $H \subset S$, $u \in W_H$, $w = uv \in uW^H$ and $s \in S$.*

(1) *The three following conditions are equivalent*:
 (a) $ws \notin uW^H$,

(b) $vs \notin W^H$,
(c) $\exists s' \in H, vs = s'v$.

(2) *The arcs* (w, ws) *and* (u, us') *are labelled by the same reflection.*

Proof. (1) (a) and (b) are trivially equivalent and the equivalence between (b) and (c) is proved in Bourbaki [5]. (2) Since $vsv^{-1} = s'$, we find $t = uvsv^{-1}u^{-1} = us'u^{-1} = t'$. □

Lemma 9 induces the corollary below:

Corollary 6. *Let* $u \in W_{s_1,s_2}$. *The class* uW^{s_1,s_2} *has exactly two "adjacent" classes, that is to say classes* C *such that there exist* $w \in uW^{s_1,s_2}$ *and* $w' \in C$ *satisfying* $w' = ws$ *for some* $s \in S$. *These two adjacent classes are* $us_1W^{s_1,s_2}$ *and* $us_2W^{s_1,s_2}$.

Every shortest path between two elements $w \in uW^{s_1,s_2}$ *and* $w'' \in u''W^{s_1,s_2}$ (*with* $u, u'' \in W_{s_1,s_2}$) *successively goes once and only once through the classes* $us_1W^{s_1,s_2}$, $us_1s_2W^{s_1,s_2}$, $us_1s_2s_1W^{s_1,s_2}, \ldots, \underbrace{us_1s_2s_1 \ldots s_i}_{u''} W^{s_1,s_2}$ (*with* $s_1s_2s_1 \ldots s_i$ *a shortest path from* u *to* u'').

Theorem 5 (LCPB). *For every Coxeter lattice* L_W, *the length function* ℓ *is a* 2*-facet rank function when defined on the* r*-labelling of the edges of* L_W.

Proof. Let F_{ws_1,w,ws_2} be a 2-facet. We note $t_1, \ldots, t_i, \ldots, t_p$ the reflections labelling the edges of one of the two paths going from w up to $ws_1 \vee ws_2$ (so the edges of the second path from w up to $ws_1 \vee ws_2$ are r-labelled by $t_p, \ldots, t_i, \ldots, t_1$ in this order). We give the proof in the case where $m(s_1, s_2)$ is odd. The even case would be treated similarly.

If $m(s_1, s_2) = q = 2p + 1$ we only have to show that $\ell(t_i) < \ell(t_{i+1}), \ell(t'_{i+1})$ for every $i \leqslant p$. The other requested inequalities are then immediate by Proposition 7.

The distance between two elements is preserved by left translation (Proposition 5) so for every $i \leqslant q$, $\ell(t_i) = d(e, t_i)$ is equal to $d(w^{-1}, w^{-1}t_i)$.

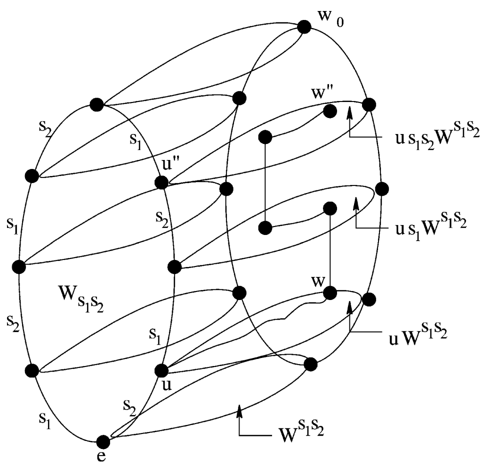

Fig. 10. Corollary 6.

Now for every $i \leqslant q$,

$$w^{-1}t_i = w^{-1}w \underbrace{s_1s_2\ldots s_1}_{2i-1 \text{ generators}} w^{-1} = \underbrace{s_1s_2\ldots s_2s_1}_{2i-1 \text{ generators}} w^{-1}.$$

To prove that $\ell(t_i) < \ell(t_{i+1}), \ell(t'_{i+1})$ for every $i \leqslant p$, we first consider the element

$$x_i = w \underbrace{s_1s_2\ldots s_1s_2}_{2i \text{ generators}} w^{-1}$$

and its converse element

$$x_i^{-1} = w \underbrace{s_2s_1\ldots s_2s_1}_{2i \text{ generators}} w^{-1}$$

that we define for $i \leqslant p$. We know that two converse elements of a Coxeter group have the same length so $\ell(x_i) = \ell(x_i^{-1})$. Thus if we show that $\ell(t_i) < \ell(x_i) < \ell(t_{i+1})$ and $\ell(t'_i) < \ell(x_i^{-1}) < \ell(t'_{i+1})$ for every $i \leqslant p$, the result will directly follow.

Since

$$\ell(x_i) = d(e, x_i) = d\big(w^{-1}, w^{-1}x_i\big) = d\big(w^{-1}, \underbrace{s_1s_2\ldots s_1s_2}_{2i \text{ generators}} w^{-1}\big),$$

our aim is to prove that

$$d\big(w^{-1}, \underbrace{s_1s_2\ldots s_2s_1}_{2i-1 \text{ generators}} w^{-1}\big) < d\big(w^{-1}, \underbrace{s_1s_2\ldots s_1s_2}_{2i \text{ generators}} w^{-1}\big) < d\big(w^{-1}, \underbrace{s_1s_2\ldots s_2s_1}_{2i+1 \text{ generators}} w^{-1}\big)$$

(the inequalities

$$d\big(w^{-1}, \underbrace{s_2s_1\ldots s_1s_2}_{2i-1 \text{ generators}} w^{-1}\big) < d\big(w^{-1}, \underbrace{s_2s_1\ldots s_2s_1}_{2i \text{ generators}} w^{-1}\big) < d\big(w^{-1}, \underbrace{s_2s_1\ldots s_1s_2}_{2i+1 \text{ generators}} w^{-1}\big)$$

are obtained by duality).

Since $w^{-1} \in W^{s_1,s_2}$ (by Corollary 4) then $w^{-1}t_1 = s_1w^{-1} \in s_1W^{s_1,s_2}$, $w^{-1}t_2 = s_1s_2s_1w^{-1} \in s_1s_2s_1W^{s_1,s_2}, \ldots$ and $w^{-1}t_p \in w^{-1}w'W^{s_1,s_2}$ with $w^{-1}w' = \max W_{s_1,s_2}$.

By Proposition 9, $w^{-1}t_i = u_iv$ with $u_i \in W_{s_1,s_2}$, $v \in W^{s_1,s_2}$ and $\ell(w^{-1}t_i) = \ell(u_i) + \ell(v)$ (note that $\ell(v)$ does not depend on i).

We prove that $\ell(t_i) < \ell(x_i)$ for every $i \leqslant p$. Indeed u_i has a unique reduced decomposition using s_1 and s_2, which starts either with s_1 or with s_2. In other words if $u_i = \underline{s_1}s_2\ldots s_1$, every shortest path from w^{-1} to $w^{-1}t_i$ will go only through copies of

W^{s_1,s_2} with the form $\underline{s_1} s_2 \ldots s_j W^{s_1,s_2}$ (indeed by Lemma 8, every path from w^{-1} to $w^{-1}t_i$ going successively through

$$s_2 W^{s_1 s_2}, s_2 s_1 W^{s_1,s_2}, \ldots, \underbrace{s_2 \ldots s_2}_{u_i} W^{s_1,s_2}$$

will have at least two distinct arcs labelled by the same reflection, and will not be a shortest path).

Now if C is a shortest path from w^{-1} to $w^{-1}x_i$, C can be written

$$C = I_1 s_1' I_2 s_2' \ldots s_{2i}' I_{2i+1},$$

with s_1' labelling the unique edge of C going from W^{s_1,s_2} to $s_1 W^{s_1,s_2}$, s_2' labelling the unique edge of C going from $s_1 W^{s_1,s_2}$ to $s_1 s_2 W^{s_1,s_2}$, s_3' labelling the unique edge of C going from $s_1 s_2 W^{s_1,s_2}$ to $s_1 s_2 s_1 W^{s_1,s_2}$, $\ldots$, etc. From C we deduce the path $C' = I_1 s_1' I_2 s_2' \ldots I_{2i-1} s_{2i-1}' I_{2i} I_{2i+1}$ (obtained from C by removing s_{2i}') that goes from w^{-1} to $w^{-1}t_i$ and which is shorter (of exactly one unit) than C. Every shortest path from w^{-1} to $w^{-1}t_i$ will be shorter than C' so it will also be shorter than a shortest path from w^{-1} to $w^{-1}x_i$, which implies $\ell(t_i) < \ell(x_i)$.

The same arguments applied on x_i and t_{i+1} prove that $\ell(x_i) < \ell(t_{i+1})$. We would also prove that way that $\ell(t_i') < \ell(x_i^{-1}) < \ell(t_{i+1}')$ and since $\ell(x_i) = \ell(x_i^{-1})$, we have the theorem. □

By Theorem 4 together with Propositions 6 and 7, we deduce the announced result:

Theorem 6. *Every Coxeter lattice is in the class $\mathcal{HH}$ and therefore is bounded.*

4.3. The contraction of a given Coxeter lattice into the lattice of any of its parabolic subgroups

We begin with a useful result:

Proposition 11. *The following are satisfied for any parabolic subgroup W_H of a Coxeter group W.*

(1) *For every left translation uW^H of W^H (with $u \in W_H$), the set T^{uW^H} of all the reflections labelling an edge of the class uW^H is equal to the set T^{W^H} of all the reflections labelling an edge of W^H.*
(2) *The set T_W of all the reflections of W is partitioned in two classes: the class T_{W_H} of all the reflections labelling an edge of W_H and T^{W^H}.*

Proof. (1) If we note w_H and w^H the greatest elements of W_H and W^H respectively, then $w_0 = \max W$ is the greatest element of $w_H W^H$ (indeed $\ell(w_H w^H) = \ell(w_H) + \ell(w^H)$, which is the greatest length for an element of W). Consider the element $w_H s s_1$ with $w_H s \in W_H$ and $w_H s s_1 \notin W_H$. We have $(w_H s s_1 \vee w_H) \in w_H W^H = \{x \in W\colon\ w_H \leqslant x\}$,

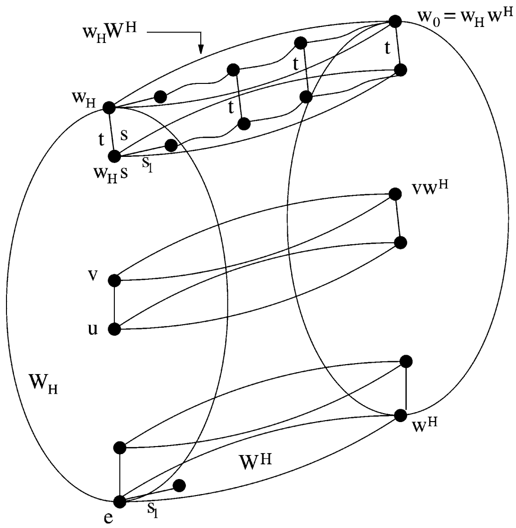

Fig. 11. Proposition 11.

so max F_{w_H,w_Hs,w_Hss_1} belongs to $w_H W^H$. Moreover the arc (w_Hs, w_H) and its opposite arc in F_{w_H,w_Hs,w_Hss_1} are labelled by the same reflection t. Since $(w_Hss_1 \vee w_H) \in w_H W^H$ there exists an arc (x, y) of F_{w_H,w_Hs,w_Hss_1}, distinct from (w_Hs, w_H), that leads up from w_HsW^H to w_HW^H. This arc (x, y) and (w_Hs, w_H) are the only arcs of F_{w_H,w_Hs,w_Hss_1} that go from w_HsW^H to w_HW^H (otherwise F_{w_H,w_Hs,w_Hss_1} would not be a 2-facet). Clearly (x, y) is the opposite arc of (w_Hs, w_H) in F_{w_H,w_Hs,w_Hss_1} since otherwise there would exist two distinct arcs with the same r-label on a shortest path from w_Hs to max F_{w_H,w_Hs,w_Hss_1}.

Now (w_Hs, w_H) and (x, y) have the same r-label t and the iteration of these arguments on the anti-hat $(y, x, xs')_\vee$ with $xs' \in w_HsW^H$ and $x \prec xs'$ constructs another 2-facet with the same properties. We prove that the last possible iteration of this operation constructs a 2-facet whose greatest element is w_0. If the last anti-hat that appears in this construction is $(w_1, w_2, w_3)_\vee$ with $w_1 \in w_HW^H$ and $w_2, w_3 \in w_HsW^H$, the opposite arc of (w_2, w_1) in the generated 2-facet F is (w_Hsw^H, w_0). Indeed if w_Hsw^H is not covered by w_0 there exists a new anti-hat $(w_1', w_2', w_3')_\vee$ that generates a 2-facet following the same rules. In every constructed 2-facet any pair of edges which belong to a copy of W^H and that "face" each other (i.e., that are isomorphic in the sense of Corollary 5) have the same g-labels. In the whole progression of these 2-facets the corresponding paths from w_Hs to w_Hsw^H and from w_H to w_Hw^H have the same g-labelling $s_1's_2'\dots s_l'$, with $s_1's_2'\dots s_l' = w^H$. Thus if (x, y) denotes the opposite arc of (w_2, w_1) and if $x = w_Hsw^H$ then $y = w_Hw^H = w_0$. The left translation of $[w_Hs, w_0]$ by $(w_Hs)^{-1} = sw_H$ is the isomorphic—for the order and the g-labelling—interval $[e, sw^H]$. Since (w_Hsw^H, w_0) exists, so does (w^H, sw^H) and it follows that the interval $[w_Hs, w_0]$ contains exactly the elements of the union of $[w_Hs, w_Hsw^H]$ and $[w_H, w_Hw_0]$.

Consider at last an arc (u, v) of W_H and the interval $[u, vw^H]$. To prove that the arc (uw^H, vw^H) exists, we translate $[u, vw^H]$ on the left by u^{-1} and find $[e, sw^H]$ for some

$s \in H$. Since (w^H, sw^H) exists, so does (uw^H, vw^H). Finally the greatest elements of the copies of W^H together form an interval isomorphic with W_H, and this isomorphism preserves the r-labelling. Now let $z = uw^H$ and consider a reduced decomposition of z such that u is a prefix of this decomposition. Let C_1 denote the corresponding path and C_2 be another shortest path from e to z, admitting w^H as a prefix. The arcs of u in C_1 are labelled by some reflections of T_{W_H} and the arcs of C_1 from u to z are labelled by some reflections distinct from the previous ones (since C_1 is a shortest path). We have seen that the arcs of C_2 from w^H to z are labelled by the r-labels of u. It follows that the set of r-labels of the arcs of C_1 from u to z is equal to the set of r-labels of the arcs of C_2 from e to w^H. Finally the set of r-labels of every copy of the quotient W^H is equal to the set T^{W^H} of the reflections labelling an edge of W^H.

(2) By point (1) together with the fact that $w_0 = w_H w^H$. □

We want to show that, for any Coxeter lattice L_W and any parabolic subgroup W_H of W, it is possible to "contract" all the double-arrows associated with all the reflections of the quotient W^H, before contracting any double-arrow associated with the reflections of T_{W_H}. More precisely, we will see that for every copy of W^H, the contractions relative to the reflections of T^{W^H} agglutinate all the elements of this copy on its least element, which is the corresponding element of W_H. Moreover the edges labelled by a reflection of T_{W_H}—i.e., the edges joining two adjacent copies of W^H—will be identified by contraction with the corresponding edges of W_H.

Theorem 7. *Let L_W be a Coxeter lattice and W_H a parabolic subgroup of W. There exists a series of interval contractions that lead from L_W to the lattice L_{W_H} of its parabolic subgroup W_H.*

Proof. Let t be a reflection of T^{W^H} whose length is maximal in T^{W^H}. Let $(y, x, z)_\vee$ be an anti-hat whose arc (x, y) is labelled by t and belongs to a copy uW^H (with $u \in W_H$) of W^H, i.e., such that $x, y \in uW^H$. Two cases may occur:

(1) The arc (x, z) belongs to uW^H: the whole anti-hat is in the interval uW^H. So the maximum element $y \vee z$ of the generated 2-facet $F_{y,x,z}$ is also in uW^H and $F_{y,x,z}$ is contained in uW^H. Since t has maximum length in uW^H, $F_{y,x,z}$ is a diamond (Theorem 5).

(2) The arc (x, z) does not belong to uW^H: then z belongs to the copy usW^H of W^H for some $s \in H$ satisfying $\ell(us) = \ell(u) + 1$. Since y and z are least than or equal to $\max usW^H$, so is their join $y \vee z$. Now $y \vee z \in usW^H$: indeed if $(x', y \vee z)$ denotes the opposite edge of (x, z) in $F_{y,x,z}$, then we know that $(x', y \vee z)$ is labelled by the same reflection of T_{W_H} as (x, z), which implies that $(x', y \vee z)$ goes from uW^H to usW^H (T_{W_H} and T^{W^H} are disjoint). All the edges of the 2-facet, except for (x, z) and $(x', y \vee z)$, are in copies of W^H and, since t has maximum length in T^{W^H}, $F_{y,x,z}$ is a diamond.

In both cases the generated 2-facet is a diamond and the opposite edge of (x, y) is also in a copy of W^H. Therefore we can iterate the application of the above arguments on the opposite edge of (x, y), which is also labelled by t. It finally follows that the reflection t "constructs" only diamonds so we can start by contracting relatively to this reflection. The lattice obtained after contraction is in $\mathcal{HH}$ and, the presented arguments holding for any lattice of $\mathcal{HH}$, we can repeat the operation on the remaining part of the copies of W^H. When all interval contractions have been made relatively to the reflections of W^H, the remaining lattice is isomorphic with W_H and the theorem follows. □

This result proves the existence, for instance, of a series of interval contractions leading from the lattice L_{A_n} associated with A_n to $L_{A_{n-1}}$. The same holds within every of the three other infinite families of finite Coxeter groups, $(B_n)_{n\geqslant 2}$, $(D_n)_{n\geqslant 4}$ and $(I_2(n))_{n\geqslant 3}$ and for the isolated finite Coxeter groups E_6, E_7 and E_8 on the one hand, and H_3 and H_4 on the other hand. In terms of doublings, this gives:

Proposition 12. *There exists a particular interval doubling series from a given Coxeter lattice generated by n generators to the Coxeter lattice of the same family, generated by $n+1$ generators.*

Moreover, since A_n is a parabolic subgroup of B_{n+1} and D_{n+1}, these two lattices can be obtained from A_n by a series of interval doublings.

5. Concluding remarks

The geometrical aspect of Coxeter groups has not been treated here. Let us yet point out that finite Coxeter groups are *zonotopes* (i.e., polytopes with *zones* [9]) in which every zone corresponds to a reflection. The elements of the group, that constitute the vertices of the zonotope, are ordered as a lattice (starting from the neutral element).

The contraction relatively to a reflection of the lattice can be seen as the deletion of the zone of the zonotope, associated to the reflection, and the obtained zonotope is still a lattice. It is interesting to add that the main result of this paper finds expression in terms of zonotope as follows:

Theorem 8. *A zonotope, associated to a finite Coxeter group and oriented as a lattice from the neutral element, is transformed by successive contractions of the successive reflections* (*w.r.t. their decreasing length*)*, into a family of zonotopes that are still lattices.*

More precisely, to every Coxeter lattice L_W is associated a *lines arrangement* of reflections, where every element of W is represented by a 2-*biconvex* subset of this arrangement. A reflection t of a Coxeter lattice L_W has maximal length if and only if t is never the *end* of a line (except from the lines with only two reflections) in the lines arrangement associated to L_W. So the contraction associated to such a reflection in L_W corresponds to the deletion of t in the lines arrangement. Moreover the new lines arrangement obtained by deleting t, corresponds to the contracted lattice. This process can be iterated by deleting in the new lines arrangement a reflection which is never the end of a line. The properties of these lines arrangements are studied in [8].

Also, in 1992, Geyer used some tools of concept analysis to prove that the lattice T_n of binary bracketings on $n+1$ symbols—also called *Tamari lattice*—is bounded for every $n>0$. We can use the class $\mathcal{HH}$ to rediscover this result by proving that all T_n belong to $\mathcal{HH}$ [7]. So the class $\mathcal{HH}$ generalizes Coxeter lattices, Tamari lattices and distributive lattices and we are therefore very interested in these lattices, defined by combinatorial conditions and admitting strong properties. We have also produced a bounded lattice that does not belong to $\mathcal{HH}$, which leads us to raise the question of a characterization of all bounded lattices not in $\mathcal{HH}$.

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References

[1] M. Barbut, B. Monjardet, Ordre et classification, vols. 1 and 2, Hachette, Paris, France, 1970.
[2] G. Birkhoff, Lattice Theory, 1st ed., Amer. Math. Soc., Providence, RI, 1940.
[3] A. Björner, Orderings of Coxeter groups, in: Contemp. Math., vol. 34, Amer. Math. Soc., Providence, RI, 1984, pp. 175–195.
[4] A. Björner, M. Wachs, Generalized quotients in Coxeter groups, Trans. Amer. Math. Soc. 308 (1) (1988).
[5] N. Bourbaki, Groupes et Algèbres de Lie, Chapitres 4, 5, 6, in: Éléments de Mathématiques, vol. 34, Hermann, Paris, 1968.
[6] N. Caspard, The lattice of permutations is bounded, Internat. J. Algebra Comput. 10 (4) (2000) 481–489.
[7] N. Caspard, C. Le Conte de Poly-Barbut, Tamari lattices are bounded: a new proof, Preprint, 2003.
[8] N. Caspard, H. Crapo, C. Le Conte de Poly-Barbut, Lines arrangements of reflections in Coxeter groups, Preprint, 2003.
[9] H.S.M. Coxeter, The classification of zonohedra by means of projective diagrams, J. Math. Pures Appl. 41 (1962) 137–156.
[10] B.A. Davey, H.A. Priestley, Introduction to Lattices and Order, Cambridge University Press, 1992.
[11] A. Day, A simple solution to the word problem for lattices, Canad. Math. Bull. 13 (1970) 253–254.
[12] A. Day, Characterizations of finite lattices that are bounded-homomorphic images or sublattices of free lattices, Canad. J. Math. 31 (1979) 69–78.
[13] A. Day, J.B. Nation, S. Tschantz, Doubling convex sets in lattices and a generalized semidistributivity condition, Order 6 (1989) 175–180.
[14] A. Day, Congruence normality: the characterization of the doubling class of convex sets, Algebra Universalis 31 (1994) 397–406.
[15] W. Geyer, The generalized doubling construction and formal concept analysis, Algebra Universalis 32 (1994) 341–367.
[16] G. Guilbaud, P. Rosenstiehl, Analyse algébrique d'un scrutin, Math. Sci. Hum. 4 (1963) 9–33.
[17] J.E. Humphreys, Reflection Groups and Coxeter Groups, Cambridge University Press, Cambridge, 1990.
[18] C. Le Conte de Poly-Barbut, Sur les treillis de Coxeter finis, Math. Inf. Sci. Hum. 125 (1994) 45–57.
[19] C. Le Conte de Poly-Barbut, Treillis de Cayley des groupes de Coxeter finis. Constructions par récurrence et décompositions sur des quotients, Math. Inf. Sci. Hum. 140 (1997) 11–33.
[20] R. McKenzie, Equational bases and non-modular lattice varieties, Trans. Amer. Math. Soc. 174 (1972) 1–43.
[21] R. Wille, Subdirect decomposition of concept lattices, Algebra Universalis 17 (1983) 275–287.