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JOURNAL OF COMPUTATIONAL AND APPLIED MATHEMATICS
Journal of Computational and Applied Mathematics 203 (2007) 80–86
www.elsevier.com/locate/cam

Iterative methods for solving variational inclusions in Banach spaces

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Received 8 January 2006; received in revised form 16 March 2006

Abstract

In this paper, we consider a system of nonlinear variational inclusions involving H-accretive operators studied by Huang and Fang in q-uniformly smooth Banach spaces. Using resolvent operator technique, we suggest an iterative algorithm for finding an approximate solution to the system of variational inclusions. Further, we discuss convergence criteria for the approximate solution of the system of variational inclusions. The theorems presented in this paper improve and unify many known results of variational inclusions.

Keywords: q-uniformly smooth space; H-accretive operator mapping; Resolvent operator technique; A system of nonlinear variational inclusions; Iterative algorithm

1. Introduction and preliminaries

Variational inequalities and variational inclusions are among the most interesting and important mathematical problems and have been studied intensively in the past years since they have wide applications in mechanics, physics, optimization and control, nonlinear programming, economics and transportation equilibrium, and engineering sciences, etc. In the theory of variational inequalities and variational inclusions, the development of an efficient and implementable iterative algorithm is interesting and important. Various kinds of iterative algorithms to solve the variational inequalities and inclusions have been developed by many authors. For details, we can refer to [1–10] and the references therein. Among these methods, the resolvent operator techniques for solving variational inequalities and variational inclusions are interesting and important.

Recently, Huang and Fang [5] introduced a new class of maximal η-monotone mapping in Hilbert spaces, which is a generalization of the classical maximal monotone mapping, and studied the properties of the resolvent operator associated with the maximal η-monotone mapping. They also introduced and studied a new class of nonlinear variational inclusions involving maximal η-monotone mapping in Hilbert spaces. For some related works, we refer to [5] and the references therein.

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0377-0427/$ - see front matter
doi:10.1016/j.cam.2006.03.011

In this paper, we further generalize the resolvent operator technique of H-accretive operators introduced by Huang and Fang. We construct a new algorithm for solving the system of variational inclusions by using the resolvent operator technique. We discuss convergence criteria for the approximate solution of the system of variational inclusions. The theorems presented in this paper improve and unify many known results of variational.

In what follows, we always let X be a real Banach space with dual space X^*, $\langle \cdot, \cdot \rangle$ be the dual pair between X and X^*, and 2^X denote the family of all the nonempty subsets of X. The generalized duality mapping $J_q(x) : X \to 2^X$ is defined by

$$J_q(x) = \{f^* \in X^* : \langle x, f^* \rangle = \|x\|^q, \|f^*\| = \|x\|^{q-1}\},$$

where $q > 1$ is a constant. In particular, J_2 is the usual normalized duality mapping. It is known that, in general, $J_q = \|x\|^{q-2} J_2$, for all $x \in X$, and $J_q(x)$ is single-valued if X^* is strictly convex. In the sequel, unless otherwise specified, we always suppose that X is a real Banach space such that $J_q(x)$ is single-valued and H is a Hilbert space. If $X = H$, then J_2 becomes the identity mapping of H.

The modulus of smoothness of X is the function $\rho_X : [0, +\infty) \to [0, +\infty)$ defined by

$$\rho_X(t) = \sup\{\tfrac{1}{2}(\|x + y\| + \|x - y\| - 1) : \|x\| \leqslant 1, \|y\| \leqslant t\}.$$

A Banach space X called uniformly smooth if

$$\lim_{t \to \infty} \frac{\rho_X(t)}{t} = 0,$$

X is called q-uniformly smooth if there exists a constant $c > 0$, such that

$$\rho_X(t) \leqslant ct^q, \quad q > 1.$$

Note that J_q is single-valued if X is uniformly smooth. In the study of characteristic inequalities in q-uniformly smooth Banach spaces, Xu [10] proved the following lemma.

Lemma 1.1 (*Xu [10]*). *Let X be a real uniformly smooth Banach space. Then, X is q-uniformly smooth if and only if there exists a constant $c_q > 0$, such that for all $x, y \in X$*

$$\|x + y\|^q \leqslant \|x\|^q + q\langle y, J_q(x)\rangle + c_q\|y\|^q. \tag{1.1}$$

Definition 1.1. Let $T, H : X \to X$ be two single-valued operators. The operator T is said to be

(i) accretive if

$$\langle Tx - Ty, J_q(x - y)\rangle \geqslant 0 \quad \forall x, y \in X;$$

(ii) strictly accretive if

$$\langle Tx - Ty, J_q(x - y)\rangle \geqslant 0 \quad \forall x, y \in X$$

and the equality holds if and only if $y = x$;

(iii) strongly accretive if there exists a constant $r > 0$, such that

$$\langle Tx - Ty, J_q(x - y)\rangle \geqslant r\|x - y\|^q \quad \forall x, y \in X;$$

(iv) strongly accretive with respect to H if there exists a constant $\gamma > 0$, such that

$$\langle Tx - Ty, J_q(Hx - Hy)\rangle \geqslant \gamma\|x - y\|^q \quad \forall x, y \in X;$$

(v) Lipschitz continuous if there exists a constant $s > 0$, such that

$$\|Tx - Ty\| \geqslant s\|x - y\| \quad \forall x, y \in X.$$

Definition 1.2. Multivalued mapping $M : X \to 2^X$ is said to be

(i) accretive if $x, y \in X$

$$\langle u - v, J_q(x - y)\rangle \geqslant 0 \quad \forall x, y \in X, \quad u \in M(x), \quad v \in M(y);$$

(ii) m-accretive if M is accretive and $(I + \lambda M)X = X$ for $\forall \lambda > 0$, where I denotes the identity mapping on X.

It is well known that when $X = H$ is Hilbert space, concept of accretive is identical with monotone.

Remark 1.1. If $X = H$, J_q replaces by the operator $\eta : X \to X$ in Definitions 1.1 and 1.2, then the operators is called η-monotone type (see [8]).

Definition 1.3. Let $H : X \to X$ be a single-valued mapping and $M : X \to 2^X$ multivalued mapping. We say that M is H-accretive if M is accretive and $(H + \lambda M)X = X$ hold, for $\lambda > 0$.

Definition 1.4. Let $H : X \to X$ be a strictly accretive mapping and M be an H-accretive mapping. The resolvent mapping $R^H_{M,\lambda} : X \to X$ associated with H and M is defined by

$$R^H_{M,\lambda}(u) = (H + \lambda M)^{-1}(u) \quad \forall u \in X.$$

Lemma 1.2 (*Fand and Huang [4]*). *Let $H : X \to X$ be strongly accretive mapping with constant $r > 0$ and $M : X \to 2^X$ be an H-accretive mapping. Then, the resolvent operator $R^H_{M,\lambda} : X \to X$ is Lipschitz continuous with constant $1/r$, i.e.*

$$\|R^H_{M,\lambda}(u) - R^H_{M,\lambda}(v)\| \leqslant \frac{1}{r}\|u - v\| \quad \forall u, v \in X.$$

2. The iterative algorithm for a system of variational inclusions

In this section, we consider the following system of variational inclusions of finding $u, v \in X$ such that

$$0 \in Hg(u) - Hg(v) + \rho(T(v) + M(g(u))), \tag{2.1}$$

$$0 \in Hg(v) - Hg(u) + \nu(T(u) + M(g(v))), \tag{2.2}$$

where $\rho > 0$, $\nu > 0$, $T, H : X \to X$ is a single-valued mapping, M is multivalued mapping, $g : X \to X$ is strongly accretive single-valued mapping.

We remark that if $u = v$, $\rho = \nu$ (2.1), (2.2) reduces to a variational inclusion of finding $u \in X$ such that

$$0 \in T(u) + M(g(u). \tag{2.3}$$

Variational inclusion (2.3) is an important generalization of variational inclusion considered by Fang and Huang [4]. If $H = I$ (2.1), (2.2) reduces to a variational inclusion of finding $u, v \in X$ such that

$$0 \in g(u) - g(v) + \rho(T(v) + M(g(u))), \tag{2.4}$$

$$0 \in g(v) - g(u) + \nu(T(u) + M(g(v))). \tag{2.5}$$

This variational inclusion considered by Kazmi and Bhat [8]. For applications of such variational inclusions, see [9,6].

Some special cases:

Case I: If $X = H$ is Hilbert space $H = I$, $g = I$ $M(g(\cdot)) = \partial\phi(\cdot)$, where $\phi : H \to R \cup \{+\infty\}$ is a proper function, and $\partial\phi(\cdot)$ denotes the η-subdifferential of ϕ, then (2.4), (2.5) reduces to the following system of nonlinear variational-like inequalities. Find $u, v \in H$ such that

$$\langle T(v) - \rho^{-1}(u - v), \eta(w, u)\rangle + \phi(w) - \phi(u) \geqslant 0 \quad \forall w \in H, \quad \rho > 0, \tag{2.6}$$

$$\langle T(v) - \nu^{-1}(u - v), \eta(w, u)\rangle + \phi(w) - \phi(u) \geqslant 0 \quad \forall w \in H, \quad \nu > 0. \tag{2.7}$$

Case II: If in case I $\eta(w, u) = w - u$, $\forall w, u \in H$, and $\partial\phi$ be the subdifferential of a proper convex lower semicontinuous function $\phi : H \to R \cup \{+\infty\}$, then (2.6), (2.7) reduces to the following system of nonlinear variational inequalities: find $u, v \in H$ such that

$$\langle T(v) - \rho^{-1}(u - v), w - u\rangle + \phi(w) - \phi(u) \geqslant 0 \quad \forall w \in H, \quad \rho > 0, \tag{2.8}$$

$$\langle T(v) - v^{-1}(u - v), w - u\rangle + \phi(w) - \phi(u) \geqslant 0 \quad \forall w \in H, \quad v > 0. \tag{2.9}$$

Case III: If, in case II, we take $\partial\phi = \delta_K$, the indicator function on a nonempty closed convex set $K \subset H$, then system (2.8), (2.9) reduces to the following system of finding $u, v \in H$ such that

$$\langle \rho T(v) - u + v, w - u\rangle \geqslant 0 \quad \forall w \in H, \quad \rho > 0, \tag{2.10}$$

$$\langle v T(v) - u + v, w - u\rangle \geqslant 0 \quad \forall w \in H, \quad v > 0 \tag{2.11}$$

which is same as the system of nonlinear variational inequalities considered by many authors. For the suitable choices of the mappings T, g, and M, (2.1), (2.2) includes, as special cases, various classes of variational inclusions and variational inequalities, see [1–10] and the references therein.

In this section, we mainly discuss iterative methods for the system of variational inclusions (2.1), (2.2).

First, we give the following lemma, the proof of which is a direct consequence of the definition of $R^H_{M,\lambda}$ and hence, is omitted.

Lemma 2.1. *Let $H : X \to X$ be a strictly accretive and let $M : X \to 2^X$ be H-accretive, then (u, v) is the solution of variational inclusions* (2.1), (2.2) *if and only if it satisfies*

$$g(u) = R^H_{M,\rho}[Hg(v) - \rho T(v)], \quad \rho > 0, \tag{2.12}$$

where

$$g(v) = R^H_{M,v}[Hg(u) - vT(u)], \quad v > 0. \tag{2.13}$$

Iterative Algorithm I. For an arbitrarily chosen $u_0 \in X$, compute $\{u_n\}$, $\{v_n\}$ by the iterative schemes

$$u_{n+1} = u_n - g(u_n) + R^H_{M,,\rho}[Hg(v_n) - \rho T(v_n)], \quad \rho > 0, \tag{2.14}$$

where

$$g(v_n) = R^H_{M,v}[Hg(u_n) - vT(u_n)], \quad v > 0. \tag{2.15}$$

If g is invertible, then (2.14), (2.15) can be rewritten as

$$\begin{aligned} g(u) &= R^H_{M,\rho}[HR^H_{M,v}(Hg(u) - vT(u)) - \rho Tg^{-1}R^H_{M,v}(Hg(u) - vT(u))] \\ &= R^H_{M,\rho}(H - \rho Tg^{-1})R^H_{M,v}(H - vTg^{-1})g(u). \end{aligned} \tag{2.16}$$

This fixed-point formulation allows us to suggest the following iterative method which is known as modified resolvent method. Thus we have the following Iterative Algorithm II.

Iterative Algorithm II. For an arbitrarily chosen $u_0 \in X$, compute $\{u_n\}$ by the iterative scheme

$$g(u_{n+1}) = R^H_{M,\rho}(H - \rho Tg^{-1})R^H_{M,v}(H - vTg^{-1})g(u_n), \quad \rho > 0, \quad v > 0, \quad n = 0, 1, 2, \ldots. \tag{2.17}$$

If, $\rho = v$ and $u_n = v_n$ for all $n > 0$, then the Iterative Algorithm I reduces to the following iterative algorithm.

Iterative Algorithm III. For an arbitrarily chosen $u_0 \in X$, compute $\{u_n\}$ by the iterative scheme

$$u_{n+1} = u_n - g(u_n) + R^H_{M,\rho}[Hg(u_n) - \rho T(u_n)], \quad \rho > 0. \tag{2.18}$$

We remark that Iterative Algorithm III gives the approximate solution to the variational inclusion (2.3).

Theorem 2.1. *Let X be a q-uniformly smooth Banach space and $H : X \to X$ be a strongly accretive and Lipschitz continuous operator with constants r and τ, respectively. $T : X \to X$ be Lipschitz continuous and strongly accretive with respect to g with constants s and t, respectively. $g : X \to X$ is a strongly accretive and Lipschitz continuous operator with constants σ and ξ, respectively. Assume that $M : X \to 2^X$ is an H-accretive operator and there exist $\rho > 0$, $v > 0$, such that*

$$0 \leqslant \theta_1 + \frac{1}{\sigma r^2}(\theta_2+\theta_3)(\theta_2+\theta_4) < 1,$$

where

$$\theta_1 = (1 - q\sigma + c_q\xi^q)^{1/q}, \quad \theta_2 = [\xi(1 - qr + c_q\tau^q)]^{1/q},$$
$$\theta_3 = (\xi - q\rho s + c_q\rho^q t^q)^{1/q}, \quad \theta_4 = (\xi - qvs + c_q v^q t^q)^{1/q}. \tag{2.19}$$

Then, the iterative sequence $\{u_n\}$, $\{v_n\}$ generated by Algorithm I converges strongly to the unique solution (u, v) of variational inclusions problem (2.1), (2.2).

Proof. From Algorithm I and Lemma 1.2, we have

$$\begin{aligned}\|u_{n+1}-u_n\| &= \|u_n - g(u_n) + R^H_{M,\rho}(Hg(v_n) - \rho T(v_n)) \\ &\quad - u_{n-1} + g(u_{n-1}) - R^H_{M,\rho}(Hg(v_{n-1}) - \rho T(v_{n-1}))\| \\ &\leqslant \|u_{n+1} - u_n + [g(u_n) - g(u_{n-1})]\| \\ &\quad + \|R^H_{M,\rho}(Hg(v_n) - \rho T(v_n)) - R^H_{M,\rho}(Hg(v_{n-1}) - \rho T(v_{n-1}))\| \\ &\leqslant \|u_{n+1} - u_n + [g(u_n) - g(u_{n-1})]\| + \frac{1}{r}\|Hg(v_n) - Hg(v_{n-1}) - \rho T(v_n) + \rho T(v_{n-1})\| \\ &\leqslant \|u_{n+1} - u_n + [g(u_n) - g(u_{n-1})]\| + \frac{1}{r}\|Hg(v_n) - Hg(v_{n-1}) \\ &\quad - g(v_n) + g(v_{n-1})\| + \frac{1}{r}\|g(v_n) - g(v_{n-1}) - \rho T(v_n) + \rho T(v_{n-1})\|.\end{aligned} \tag{2.20}$$

By Lemma 1.1 and strongly accretive of H, one has

$$\begin{aligned}&\langle Hg(v_n) - Hg(v_{n-1}), J_q(g(v_n) - g(v_{n-1}))\rangle \geqslant r\|g(v_n) - g(v_{n-1})\|^q, \\ &\|Hg(v_n) - Hg(v_{n-1}) - g(v_n) + g(v_{n-1})\|^q \\ &\quad = \|g(v_n) - g(v_{n-1})\|^q + c_q\|Hg(v_n) - Hg(v_{n-1})\|^q - q\langle Hg(v_n) - Hg(v_{n-1}), J_q(g(v_n) - g(v_{n-1}))\rangle \\ &\quad \leqslant (1 - qr + c_q\tau^q)\|g(v_n) - g(v_{n-1})\|^q \leqslant \xi(1 - qr + c_q\tau^q)\|v_n - v_{n-1}\|^q.\end{aligned} \tag{2.21}$$

Again, since $T : X \to X$ is strongly accretive with respect to g, we have the following estimate:

$$\begin{aligned}\|g(v_n) - g(v_{n-1}) - \rho T(v_n) + \rho T(v_{n-1})\|^q &= \|g(v_n) - g(v_{n-1})\|^q + c_q\rho^q\|T(v_n) - T(v_{n-1})\|^q \\ &\quad - q\rho\langle T(v_n) - T(v_{n-1}), J_q(g(v_n) - g(v_{n-1}))\rangle \\ &\leqslant (\xi - q\rho s + c_q\rho^q t^q)\|v_n - v_{n-1}\|^q,\end{aligned} \tag{2.22}$$

$$\begin{aligned}\|u_n - u_{n-1} + [g(u_n) - g(u_{n-1})]\|^q &= \|u_n - u_{n-1}\|^q + c_q\|g(u_n) - g(u_{n-1})\|^q \\ &\quad - q\langle g(u_n) - g(u_{n-1}), J_q(u_n - u_{n-1})\rangle \\ &\leqslant (1 - q\sigma + c_q\xi^q)\|u_n - u_{n-1}\|^q.\end{aligned} \tag{2.23}$$

Now, we have

$$\|g(v_n) - g(v_{n-1})\| \cdot \|v_n - v_{n-1}\|^{q-1} \geqslant \langle g(v_n) - g(v_{n-1}), J_q(v_n - v_{n-1})\rangle \geqslant \sigma\|v_n - v_{n-1}\|^q,$$

which implies

$$
\begin{aligned}
\|v_n - v_{n-1}\| &\leqslant \frac{1}{\sigma}\|g(v_n) - g(v_{n-1})\| \\
&= \frac{1}{\sigma}\|R^H_{M,v}(Hg(u_n) - vT(u_n)) - R^H_{M,v}(Hg(u_{n-1}) - vT(u_{n-1}))\| \\
&\leqslant \frac{1}{r\sigma}\|Hg(u_n) - Hg(u_{n-1}) - vT(u_n) + vT(u_{n-1}))\| \\
&\leqslant \frac{1}{r\sigma}\{[\xi(1 - qr + c_q\tau^q)]^{1/q} + (\xi - q\rho s + c_q\rho^q t^q)^{1/q}\}\|u_n - u_{n-1}\|, \qquad (2.24)
\end{aligned}
$$

where (2.15) has been used.

Combining (2.20)–(2.24), we have

$$
\|u_{n+1} - u_n\| \leqslant \left(\theta_1 + \frac{1}{\sigma r^2}(\theta_2 + \theta_3)(\theta_2 + \theta_4)\right)\|u_n - u_{n-1}\| \leqslant k\|u_n - u_{n-1}\|, \qquad (2.25)
$$

where

$$
\begin{aligned}
&\theta_1 = (1 - q\sigma + c_q\xi^q)^{1/q}, \quad \theta_2 = [\xi(1 - qr + c_q\tau^q)]^{1/q}, \\
&\theta_3 = (\xi - q\rho s + c_q\rho^q t^q)^{1/q}, \quad \theta_4 = (\xi - qvs + c_q v^q t^q)^{1/q}, \qquad (2.26) \\
&k = \theta_1 + \frac{1}{\sigma r^2}(\theta_2 + \theta_3)(\theta_2 + \theta_4).
\end{aligned}
$$

Since $0 \leqslant k < 1$ by condition (2.19). Now, (2.25) implies that $\{u_n\}$ is *Cauchy* sequence in X. Also, (2.24) implies that $\{u_n\}$ is *Cauchy* sequence in X. Hence, there exist $u, v \in X$ such that $u_n \to u$, $(n \to \infty)$ $v_n \to v$, $(n \to \infty)$. Since $T, H, g, R^H_{M,\lambda}$ are continuous, then it follows Iterative Algorithm I that $u, v \in X$ satisfy (2.1), (2.2), i.e. $\{u_n\}, \{v_n\}$ strongly convergence the solution of variational inclusions problem. □

Remark 2.1. It is clear that $r < \tau$, $s < t\xi^{q-1}$, $\sigma \leqslant \xi$. If $\xi > 1$, $q = 2$, i.e. X is 2-uniformly smooth, the following relation hold for some suitable values $v > 0$,

$$
\left|v - \frac{s}{c_2 t^2}\right| < \frac{\sqrt{s^2 - c_2 t^2\theta_2(2 - \theta_2)}}{c_2 t^2}, \quad s > t\sqrt{c_2\theta_2(2 - \theta_2)}, \quad \theta_2 < 1,
$$

and it ensures that $\theta_2 + \theta_4 < 1$. The following relation hold for some suitable values $\rho > 0$,

$$
\left|\rho - \frac{s}{c_2 t^2}\right| < \frac{\sqrt{s^2 - \xi c_2 t^2 + c_2 t^2(\sigma r^2 - \sigma r^2\theta_1 - \theta_2)^2}}{c_2 t^2},
$$

$$
t\sigma r^2\sqrt{c_2} > \sqrt{\xi c_2 t^2 - s^2} + t\sqrt{c_2}(\sigma r^2\theta_1 + \theta_2)
$$

and it ensures that $\theta_2 + \theta_3 < 1$. Hence, for some suitable values $\rho > 0$, $v > 0$, above conditions ensures that $0 \leqslant k < 1$. For example, $c_2 = 1$, $r = 0.98$, $\tau = 1.02$, $v = 0.7$, $\sigma = 1$, $s = 0.8$, $t = 1.02$, $\xi = 1.001$, $\rho = 0.8$, $\theta_1 = 0.04$, $\theta_2 = 0.29$, $\theta_3 = 0.62$, $\theta_4 = 0.63$, $k = 0.91$, then (2.19) hold. We note that Hilbert spaces and L_q (or l_q) are 2-uniformly smooth.

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