Discrete rotations and symbolic dynamics

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Abstract

The aim of this paper is to study local configurations issued from discrete rotations. The algorithm of discrete rotations that we consider is the discretized rotation. It simply consists in the composition of a Euclidean rotation with a rounding operation, as studied in [B. Nouvel, E. Rémila, On colorations induced by discrete rotations, in: DGCI, in: LNCS, vol. 2886, 2003, pp. 174–183; B. Nouvel, E. Rémila, Characterization of bijective discretized rotations, in: International Workshop on Combinatorial Images Analysis, 10th International Conference, IWIA 2004, Auckland, New Zealand, December 1–4, 2004, in: LNCS, vol. 3322, 2004, pp. 248–259; B. Nouvel, E. Rémila, Configurations induced by discrete rotations: Periodicity and quasiperiodicity properties, Discrete Appl. Math. 2–3 (147) (2005) 325–343]. It is possible to encode all the information concerning a discrete rotation as two multidimensional words \(C_\alpha\) and \(C'_\alpha\) that we call configurations. In this paper, we introduce two discrete dynamical systems defined by a \(\mathbb{Z}^2\)-action on the two-dimensional torus that allow us to describe the configurations \(C_\alpha\) and \(C'_\alpha\) via a suitable symbolic coding; we then deduce various combinatorial properties for both families of configurations, and in particular, results concerning densities of symbol occurrence.

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1. Introduction

Symbolic dynamics, and more generally, discrete dynamical systems, have natural and deep interactions with combinatorics on words. This interaction is particularly well-illustrated in the Sturmian case, e.g., see [10,9]. The combinatorial objects involved are the Sturmian words, while the dynamical systems are the irrational translations on the torus \(\mathbb{T}^1 = \mathbb{R}/\mathbb{Z}\) identified with the circle of perimeter 1. A Sturmian word is indeed a coding with respect to a particular two-interval partition of the one-dimensional torus \(\mathbb{T}^1\) of the orbit of a point under the action of an irrational rotation. This point of view allows one to deduce many combinatorial properties of Sturmian words, as discussed in [6], such as, e.g., the densities of occurrence of factors that can be computed thanks to the equidistribution properties of irrational rotations, or properties such as powers of factors in Sturmian words [20], or else the characterization of Sturmian words that are fixed points of substitution [4].

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Several attempts at the generalization of this fruitful interaction have been proposed. For more details, see the survey [6]. One of the first ideas which comes to mind is a rotation of the two-dimensional torus $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$. As an example, the Tribonacci word, that is, the fixed point of the substitution $1 \mapsto 12$, $2 \mapsto 13$, $3 \mapsto 1$, codes the orbit of a point of the torus $\mathbb{T}^2$ under the action of a translation with respect to a partition of $\mathbb{T}^2$ into three pieces with a fractal boundary [16,11]. More generally, fixed points of Arnoux–Rauzy sequences over $n$ letters [3] code orbits of points of the torus $\mathbb{T}^{n-1}$ under the action of a translation with respect to a partition of $\mathbb{T}^{n-1}$ into $n$ pieces with a fractal boundary [1]. See also more generally [19].

A second approach, which is dual to the previous one, consists in working with two translations of $\mathbb{T}^1$. It is indeed convenient to describe arithmetic discrete planes in the sense of [17] by using the coding with respect to a three-interval partition of a $\mathbb{Z}^2$-action by two irrational translations on $\mathbb{T}^1$ [5]. One thus gets two-dimensional words over a three-letter alphabet that can be considered as two-dimensional Sturmian words [8]. The study of the underlying dynamical system provides a way to obtain a better understanding of the combinatorial and geometric properties of arithmetic discrete planes, such as the enumeration of some local configurations, the so-called $(m, n)$-cubes, as well as their densities of occurrence, or their centroisometry properties [5].

In all these cases, connections between word combinatorics, symbolic dynamics, arithmetic, and discrete geometry prove to be natural and enlightening. In the present paper, we consider a further generalization motivated by discrete geometry, and more precisely, arithmetic discrete geometry, in the sense of [17]. We study, indeed, configurations associated with a discrete rotation. Let us note that there exist several extensions of the notion of Euclidean rotation in discrete geometry, as reviewed in [2]. We consider here discrete rotations defined as the composition of a Euclidean rotation with a rounding operation. It is possible to encode all the information concerning a discrete rotation as two multidimensional words $C_{\alpha}$ and $C_{\alpha}'$ that we call configurations. These configurations have been introduced and studied in [13,15,15]. The main purpose of the present paper is to prove that both configurations are codings of a $\mathbb{Z}^2$-action by two rotations on $\mathbb{T}^2$ with respect to a partition into a finite number of rectangles. We then deduce, in particular, results concerning the density of each symbol in $C_{\alpha}$ and $C_{\alpha}'$.

This paper is organized as follows. We first introduce the definitions and conventions in Section 2. Section 3 is devoted to the dynamical study of the configuration $C_{\alpha}$, from which combinatorial properties are deduced in Section 4. A similar study for $C_{\alpha}'$ is performed in Section 5. Let us note that the results presented here extend those of [7].

2. Definitions and conventions

We work in the discrete plane $\mathbb{Z}^2$. For each point $v \in \mathbb{Z}^2$, $x_v$ stands for its horizontal coordinate and $y_v$ for its vertical coordinate.

Let $x$ be a real number. We recall that the floor function $x \mapsto \lfloor x \rfloor$ is defined as the greatest integer less or equal to $x$. The rounding function is defined as $\lfloor x \rfloor := \lfloor x + 0.5 \rfloor$ and $\{x\} := x - \lfloor x \rfloor$. These applications can be extended to vectors in $\mathbb{Z}^2$ by independent application on each component.

The discretization cell of the point $v \in \mathbb{Z}^2$ is defined as the set of elements $w$ in $\mathbb{R}^2$ which have the same image by discretization as $v$, i.e., $[v] = [w]$. Hence the discretization cell of $v$ is defined as the half-opened unit square centered at $[v]$.

We use the canonical bijection between the torus $\mathbb{T}^2 = (\mathbb{R}/\mathbb{Z})^2$ and the square $v \in \mathbb{R}^2$, $x_v \in [-\frac{1}{2}, \frac{1}{2}]$ and $y_v \in [-\frac{1}{2}, \frac{1}{2}]$, i.e., the discretization cell of 0. By slightly abusing our notation, we also denote by $[v]$ the image under the canonical projection from $\mathbb{R}^2$ onto $\mathbb{T}^2$ of a point $v \in \mathbb{R}^2$. Let us stress the fact that the map $x \mapsto \{x\}$ is thus an additive morphism from $\mathbb{R}^2$ onto $\mathbb{T}^2$.

Without loss of generality, we assume throughout this paper that $\alpha \in [0, \pi/4]$: the arguments used here can easily be extended to the case of any other octant. We denote by $r_{\alpha}$ the Euclidean rotation of angle $\alpha$:

$$r_{\alpha} : \mathbb{R}^2 \to \mathbb{R}^2, \; v \mapsto \begin{bmatrix} \cos(\alpha) & -\sin(\alpha) \\ \sin(\alpha) & \cos(\alpha) \end{bmatrix} v.$$

The discrete rotation $[r_{\alpha}]$ is defined as

$$[r_{\alpha}] : \mathbb{Z}^2 \to \mathbb{Z}^2, \; v \mapsto [r_{\alpha}(v)].$$
By \([r_\alpha]\) we mean the map
\[
\{r_\alpha\} : \mathbb{Z}^2 \to \mathcal{T}^2, \ v \mapsto \{r_\alpha(v)\}.
\]

We denote by \((i, j)\) the canonical basis of the Euclidean space \(\mathbb{R}^2\). We set \(i_\alpha := r_\alpha(i)\) and \(j_\alpha := r_\alpha(j)\).

Let \(Q\) be a finite set called alphabet. A two-dimensional word in \(Q^{\mathbb{Z}^2}\) is called a configuration over \(Q\). An application from \([0, 1, \ldots, m - 1] \times [0, 1, \ldots, n - 1]\) to \(Q\) is called a pattern of size \([m, n]\). Let \(C\) be a configuration in \(Q^{\mathbb{Z}^2}\). A pattern \(\chi\) of size \([m, n]\) occurs at position \(v\) in \(C\) if \(C(v + p) = \chi(p)\), for all \(p\) with \(x_p, y_p \in [0, 1, \ldots, m - 1] \times [0, 1, \ldots, n - 1]\). The rectangular complexity function of the configuration \(C\) is defined as the function \(p : \mathbb{N}^2 \to \mathbb{N}\), that counts the number of patterns of size \([m, n]\) in \(C\).

The density of the symbol \(p \in Q\) in the configuration \(C \in Q^{\mathbb{Z}^2}\) is defined as the following limit (if it exists):
\[
\eta_C(p) = \lim_{N \to \infty} \frac{\text{Card}\{v \in \mathbb{Z}^2, x_v, y_v \in \{-N, \ldots, N\} \text{ and } C(v) = p\}}{(2N + 1)^2}.
\]

We similarly define the density of a pattern \(\chi\) in the configuration \(C\) as the following limit (if it exists):
\[
\eta_C(\chi) = \lim_{N \to \infty} \frac{\text{Card}\{v \in \mathbb{Z}^2, x_v, y_v \in \{-N, \ldots, N\} \text{ and } \chi \text{ occurs at position } v\}}{(2N + 1)^2}.
\]

A dynamical system \((X, T)\) is defined as the action of a continuous and onto map \(T\) on a compact space \(X\). Given two continuous and onto maps \(T_1\) and \(T_2\) acting on \(X\) and satisfying \(T_1 \circ T_2 = T_2 \circ T_1\), the \(\mathbb{Z}^2\)-action by \(T_1\) and \(T_2\) on \(X\), that we denote as \((X, T_1, T_2)\), is defined by
\[
\forall (m, n) \in \mathbb{Z}^2, \ \forall x \in X, \ (m, n) \cdot x = T_1^m \circ T_2^n(x).
\]

It is natural to associate a two-dimensional symbolic dynamical system with the triple \((X, T_1, T_2)\) by coding the orbits of the points of \(X\) under the \(\mathbb{Z}^2\)-action as follows: given \(x_0 \in X\), a finite partition \(\{X_q\}_{q \in Q}\) of \(X\), and a labelling function \(l\) defined on \(X\) with values in the finite set \(Q\) that takes the constant value \(q\) on each of the subsets \(X_q\), then the configuration \(C\) is defined by
\[
\forall (m, n) \in \mathbb{Z}^2, \ C(m, n) = l(T_1^m \circ T_2^n(x_0))
\]
and is called the coding of the orbit of \(x_0\) under the \(\mathbb{Z}^2\)-action \((X, T_1, T_2)\) with respect to the labelling function \(l\).

3. Dynamical system associated with \(C_\alpha\)

According to [13], we associate a first configuration \(C_\alpha\) with the discrete rotation \([r_\alpha]\) that encodes local information concerning the discrete rotation: the configuration \(C_\alpha\) is defined at point \(v \in \mathbb{Z}^2\) according to the action of the discrete rotation on the 4-neighbours of \(v\) (see Fig. 1); furthermore, there exists a planar transducer that uses the configuration \(C_\alpha\) as input and gradually computes the action of the discrete rotation [15].

More precisely, for a given \(v \in \mathbb{Z}^2\), we denote by \(\mathcal{V}_\beta(v)\) the set of 4-neighbours of \(v\), that is, \(\mathcal{V}_\beta(v) = \{v + i, v + j, v - i, v - j\}\). The configuration \(C_\alpha\) maps each point \(v\) of \(\mathbb{Z}^2\) to the set \([r_\alpha](\mathcal{V}_\beta(v)) - [r_\alpha](v)\), that is,
\[
C_\alpha(v) := \{a_0, a_1, a_2, a_3\} \text{ with } (a_k = [r_\alpha(v + r_{x/2}^k(i))] - [r_\alpha(v)], \text{ for } k = 0, \ldots, 3).
\]

One can easily check that \(C_\alpha\) contains either 3 or 4 non-zero elements; for a detailed proof, see [13]. Let \(Q_\alpha\) stand for the finite set of values taken by \(C_\alpha\).

We define a frame of the torus \(\mathbb{T}^2 \equiv [-\frac{1}{2}, \frac{1}{2}] \times [-\frac{1}{2}, \frac{1}{2}]\) as a rectangle of the form \([a, b] \times [c, d]\), with \(-\frac{1}{2} \leq a \leq b < \frac{1}{2}\) and \(-\frac{1}{2} \leq c \leq d < \frac{1}{2}\). The interpretation of \(C_\alpha\) as a coding of a \(\mathbb{Z}^2\)-action is based on the following result:

**Theorem 1** ([15]). There exists a partition \(P_\alpha = \{I_p, p \in Q_\alpha\}\) of the torus \(\mathbb{T}^2\) into a finite number of frames such that
\[
\forall v \in \mathbb{Z}^2, \ C_\alpha(v) = p \iff \{r_\alpha(v)\} \in I_p.
\]
can then be reformulated as follows: according to the directions of the vectors of Fig. 2.

Let \( C_1 \)

One has

\[ \phi_0 \]

where \( P \)

We then associate with the partition \( P_\alpha \)

More precisely, the partition \( P_\alpha \) is defined as follows: if \( \alpha \in [0, \pi/6] \) (resp. \( [\pi/6, \pi/4] \)), then the torus is divided into at most 25 frames, delimited by 10 lines (at most) with the equation \( x = -\frac{1}{2}, x = \frac{1}{2} - \cos(\alpha), x = \sin(\alpha) - \frac{1}{2}, \)

\( x = \frac{1}{2} - \sin(\alpha), x = \cos(\alpha), x = \frac{1}{2} - \cos(\alpha), y = \frac{1}{2} - \sin(\alpha), y = \frac{1}{2}, y = \sin(\alpha) - \frac{1}{2}, y = \frac{1}{2}, y = \cos(\alpha) - \frac{1}{2}, (\text{resp.} x, y = -\frac{1}{2}, x = \frac{1}{2} - \cos(\alpha), \frac{1}{2} - \sin(\alpha), \sin(\alpha) - \frac{1}{2}, \cos(\alpha) - \frac{1}{2}, \).

More precisely, the alphabet \( Q_\alpha \) has exactly 25 elements if \( \alpha \neq 0, \pi/4, \pi/6, \) 16 elements if \( \alpha = \pi/6, \) and 9, if \( \alpha = \pi/4. \)

Consider now the following two actions

\[ T_{k_\alpha} : \mathbb{T}^2 \rightarrow \mathbb{T}^2, \ x \mapsto x + \{i_{k_\alpha}\}, \ T_{j_\alpha} : \mathbb{T}^2 \rightarrow \mathbb{T}^2, \ x \mapsto x + \{j_{j_\alpha}\}. \]

One has

\[ \forall \mathbf{v} \in \mathbb{Z}^2, \ \{r_{\alpha}\} \mathbf{v} = T_{k_\alpha}^{x_{\mathbf{v}}} \circ T_{j_\alpha}^{y_{\mathbf{v}}} \circ 0. \]

We then associate with the partition \( P_\alpha \), the labelling function

\[ l_{C_\alpha} : \mathbb{T}^2 \rightarrow Q_\alpha, \ \mathbf{v} \mapsto \phi_\alpha(f_{C_\alpha}(x_{\mathbf{v}}), f_{C_\alpha}(y_{\mathbf{v}})), \]

where \( \phi_\alpha : \{0, 1, 2, 3, 4\}^2 \rightarrow Q_\alpha \) if \( \alpha \in [0, \pi/6] \) (resp. \( \phi_\alpha : \{0, 1, 3, 4, 5\}^2 \rightarrow Q_\alpha \) if \( \alpha \in [\pi/6, \pi/4] \)) is described in Fig. 2, and \( f_{C_\alpha} : [−1/2, 1/2] \rightarrow \{0, 1, 2, 3, 4, 5\} \) is defined by

\[
\begin{array}{c|c}
\alpha \in [0, \pi/6]: & \alpha \in [\pi/6, \pi/4]: \\
[−\frac{1}{2}, \frac{1}{2} - \cos(\alpha)] & [−\frac{1}{2}, \frac{1}{2} - \cos(\alpha)] \\
[\frac{1}{2} - \cos(\alpha), \sin(\alpha) - \frac{1}{2}] & [\frac{1}{2} - \cos(\alpha), \frac{1}{2} - \sin(\alpha)] \\
[\sin(\alpha) - \frac{1}{2}, \frac{1}{2} - \sin(\alpha)] & [\frac{1}{2} - \sin(\alpha), \sin(\alpha) - \frac{1}{2}] \\
[\frac{1}{2} - \sin(\alpha), \cos(\alpha) - \frac{1}{2}] & [\sin(\alpha) - \frac{1}{2}, \cos(\alpha) - \frac{1}{2}] \\
[\cos(\alpha) - \frac{1}{2}, \frac{1}{2}] & [\cos(\alpha) - \frac{1}{2}, \frac{1}{2}]
\end{array}
\]

The values taken by \( C_\alpha \), i.e., the elements of \( Q_\alpha \) are depicted in Fig. 2 according to the directions of the vectors of \( C_\alpha(\mathbf{v}) \), for \( \mathbf{v} \in \mathbb{Z}^2. \)

**Theorem 1** can then be reformulated as follows:

**Corollary 2.** Let \( C_\alpha \) be the configuration associated with the discrete rotation \( [r_{\alpha}] \). We use the notation introduced above. The configuration \( C_\alpha \) is the coding of the orbit of \( 0 \) under the \( \mathbb{Z}^2 \)-action \( (\mathbb{T}^2, T_{k_\alpha}, T_{j_\alpha}) \) with respect to the labelling function \( l_{C_\alpha} \).
Let us assume that the group $G$ means that the position, in the discretization cell of a point $v \in \mathbb{Z}^2$, of the point $\{r_\alpha\}(v)$ of the lattice $\mathbb{Z}_x + \mathbb{Z}_y$ determines the directions of the images of the neighbors of $v$ under the action of the discrete rotation.

**Example:** The case $\alpha = \pi/4$

We detail here the case $\alpha = \pi/4$. In this case, the alphabet $Q_{\pi/4}$ has 9 elements. Consider the sequences in the lines of the two-dimensional word $C_{\pi/4}$. One has $m_{\pi/4} = m(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$, for $m \in \mathbb{Z}$. One can easily check that the one-dimensional words $(C_{\pi/4}(m, n_0))_{m \in \mathbb{Z}^2}$ are codings of the rotation $R_{1/\sqrt{2}}: \mathbb{R}/(\sqrt{2}\mathbb{Z}) \to \mathbb{R}/(\sqrt{2}\mathbb{Z})$, $x \mapsto x + \frac{1}{\sqrt{2}}$, with respect to the three intervals $[-1/2, -3/2 + \sqrt{2}], [-3/2 + \sqrt{2}, 1/2], [1/2, \sqrt{2} - 1/2]$. By renormalizing by $\frac{1}{\sqrt{2}}$, one obtains a coding of the rotation by $1/2$ over $T^1 = \mathbb{R}/\mathbb{Z}$ with respect to three intervals of respective lengths $1 - 1/\sqrt{2}$, $\sqrt{2} - 1$, and $1 - 1/\sqrt{2}$. One obtains a similar result for the sequences in the columns. Furthermore, the two-dimensional word $C_{\pi/4}$ presents some intriguing self-similarity properties, which have been studied in [12]. We plan to explore them by exploiting the self-similarity of the underlying dynamical system provided by Corollary 2, such as illustrated in Fig. 3, and by exhibiting a two-dimensional substitution generating the two-dimensional word $C_{\pi/4}$.

**4. Distribution of symbols in $C_\alpha$**

We can now deduce, from the $\mathbb{Z}^2$-action introduced in Section 3, combinatorial properties of the two-dimensional word $C_\alpha$, and in particular, results concerning the densities of symbols, by using classical tools from symbolic dynamics and ergodic theory.

Let $G_\alpha \subseteq T^2$ stand for the orbit of 0 under the $\mathbb{Z}^2$-action $(T^2, T_x, T_y)$ with respect to the labelling function $I_{C_\alpha}$: the configuration results from a coding of the orbit. The group $G_\alpha$ is the image, by the canonical projection $x \mapsto \{x\}$ onto $T^2$, of the lattice $L_\alpha := \mathbb{Z}_x + \mathbb{Z}_y$ of $\mathbb{R}^2$; $G_\alpha$ is invariant by rotation by $\pi/2$.

Let us recall that an angle $\alpha$ is said Pythagorean if $\cos \alpha$ and $\sin \alpha$ are both rational. The uniform density of $G_\alpha$ is a key ingredient of our combinatorial study. Let us distinguish two cases according to the fact that $\alpha$ is Pythagorean or not.

**Lemma 3.** The group $G_\alpha$ is dense in $T^2$ if and only if $\alpha$ is not Pythagorean. If $\alpha$ is not Pythagorean, then the two-dimensional sequence $(u_{m,n})_{m,n \in \mathbb{Z}^2}$ defined by $u_{m,n} := T_{T_x}^m \circ T_{T_y}^n(0)$ is equidistributed in $T^2$. If $\alpha$ is Pythagorean, then the configuration $C_\alpha$ is periodic, and its lattice of periods has dimension two.

**Proof.** Let us assume that $\alpha$ is not Pythagorean. We prove the equidistribution of the two-dimensional sequence $(u_{m,n})_{m,n \in \mathbb{Z}^2}$ in $T^2$ by using a classical argument on Weyl sums. Indeed, for $p, q \in \mathbb{Z}^2$, we set $f_{p,q} : \mathbb{R}^2 \to \mathbb{R}^2$, $(x, y) \mapsto e^{2\pi i (px + qy)}$. One first checks that $\int_{[0,1]^2} f_{p,q}(x, y) dx dy \neq 0$ if and only if $p = q = 0$. Furthermore, one has

$$f_{p,q}(u_{m,n}) = e^{2\pi i p(m \cos \alpha - n \sin \alpha)} \cdot e^{2\pi i q(m \sin \alpha + n \cos \alpha)} = e^{2\pi i pm(p \cos \alpha + q \sin \alpha)} \cdot e^{2\pi i qn(-p \sin \alpha + q \cos \alpha)}.$$

By hypothesis, one has either $\cos(\alpha)$ or $\sin(\alpha)$ irrational. Then one cannot have simultaneously $p \cos(\alpha) + q \sin(\alpha) \in \mathbb{Z}$ and $-p \sin(\alpha) + q \cos(\alpha) \in \mathbb{Z}$. One thus gets that for $(p, q) \in \mathbb{Z}^2$, $(p, q) \neq (0, 0)$, and then
By definition, one has

\[ \lim_{n \to +\infty} \frac{1}{(2N + 1)^2} \sum_{|m|,|n| \leq N} f_{p,q}(u_{m,n}) = 0, \]

which yields the equidistribution of \((u_{m,n})_{m,n} \in \mathbb{Z}^2\).

We assume now that \(\alpha\) is a Pythagorean angle. There exists a unique prime Pythagorean triple \((a, b, c) \in \mathbb{N}^3\) that satisfies \(1 \leq b \leq a \leq c\), \(\gcd(a, b, c) = 1\), \(\cos(\alpha) = \frac{a}{c}\), \(\sin(\alpha) = \frac{b}{c}\), and hence \(a^2 + b^2 = c^2\). Let \(u, v \in \mathbb{Z}^2\) such that \(ua - bv = \gcd(a, b)\). The vector \(u_\alpha + v_\alpha\) generates \(G_\alpha\), which hence is a finite cyclic group of order \(c\). Moreover, the vectors \(q_\alpha\) and \(q_\beta\) are period vectors for \(C_\alpha\), hence the lattice of periods of \(C_\alpha\) has dimension two. This ends the proof. \(\square\)

Let us note that more information on rotations with Pythagorean angles can be found in [14]. We can now Deduce, from Lemma 3, density results for \(C_\alpha\).

**Theorem 4.** Let \(C_\alpha\) be the configuration associated with the discrete rotation \([r_\alpha]\). For every symbol \(p \in Q_\alpha\), its density \(\eta_{C_\alpha}(p)\) in \(C_\alpha\) exists and is equal to

- the area of the frame \(I_p\) defined in Theorem 1, if \(\alpha\) is not Pythagorean,
- and to \(1/c \cdot \text{Card} (G_\alpha \cap I_p)\), if \(\alpha\) is Pythagorean, where \(c\) stands for the order of the group \(G_\alpha\).

**Proof.** By definition, one has

\[ \eta_{C_\alpha}(p) = \lim_{N \to +\infty} \frac{|r_\alpha|((-N, \ldots, N) \cap I_p)}{(2N + 1)^2}. \]

If \(\alpha\) is not Pythagorean, then the result follows directly from Lemma 3.

Let us assume now that \(\alpha\) is Pythagorean. One first checks that \(\eta_{C_\alpha}(p) = \lim_{N \to +\infty} \frac{|r_\alpha|((-c\lfloor N/c \rfloor, \ldots, c\lfloor N/c \rfloor) \cap I_p)}{(2N + 1)^2}\) exists and is equal to \(\text{Card} (G_\alpha \cap I_p)\). \(\square\)

We can deduce more generally the following combinatorial properties of the two-dimensional word \(C_\alpha\). Let us note that in the present paper, we focus on the statistical properties of repartition of the symbols in \(Q_\alpha\), because of their interest for the study of the discrete rotation \([r_\alpha]\).

**Theorem 5.** Let \(C_\alpha\) be the configuration associated with the discrete rotation \([r_\alpha]\). The density of rectangular patterns exists in \(C_\alpha\) for every pattern \(\chi\) that occurs in \(C_\alpha\). The two-dimensional word \(C_\alpha\) is uniformly recurrent, i.e., for every positive integer \(n\), there exists a positive integer \(N\) such that every square pattern of size \([N, N]\) of \(C_\alpha\) contains every square pattern of size \([n, n]\) of \(C_\alpha\). Furthermore, there exists a positive constant \(A\) such that the rectangular complexity function of \(C_\alpha\) satisfies

\[ \forall m, n, \ p_{C_\alpha}(m, n) \leq A \cdot mn. \]

**Proof.** We first deduce, from Corollary 2, that given two positive integers \(m, n\), there exists a finite partition of \(\mathbb{T}^2\) into finite unions of frames \(P_\alpha^{[m,n]} = \{ J_\chi, \ \chi \text{ pattern of size } [m, n] \text{ of } C_\alpha \} \) such that \(\chi\) occurs at position \(v\) in \(C_\alpha\) if and only if \([r_\alpha\](\(v\)) \in I_\chi\). Let us stress the fact that the sets \(J_\chi\) are not necessarily frames, nor even connected sets; indeed, they are obtained as finite intersections of frames \(I_p\) associated with symbols \(p \in Q_\alpha\). More precisely, \(I_\chi\) is obtained as follows:

\[ I_\chi = \cap_{0 \leq k \leq m-1} \cap_{0 \leq \ell \leq n-1} T_{k\ell} \circ T_k \circ I_\chi(k, \ell). \]

This allows us to deduce the existence of densities for all rectangular patterns of \(C_\alpha\). We thus obtain, analogously as for Theorem 4, that they are equal to the measure of \(I_\chi\) in the non-Pythagorean case, and to the cardinality of the intersection of \(G_\alpha\) with \(I_\chi\) in the Pythagorean case.

Let us now prove uniform recurrence. Let us assume that \(\alpha\) is non-Pythagorean. We assume w.l.o.g. that \(\cos(\alpha) \not\in \mathbb{Q}\). According to [18], given any interval \(I\) of \(\mathbb{T}^1\), there exists \(n_0\) such that among any finite sequence of points \([k \cos(\alpha)], [(k+1) \cos(\alpha)], \ldots, [(k+n_0) \cos(\alpha)]\), at least one of them belongs to \(I\). Let us fix a pattern \(\chi\) and a position \(v \in \mathbb{Z}^2\). We apply the previous result to the interval \(I_\chi \cap [-1/2, 1/2]\), and to the sequence \((T_k(v) \cap [-1/2, 1/2])_{k \in \mathbb{Z}} = (x_k + k \cos(\alpha))_{k \in \mathbb{Z}}\). Hence, given any \(v \in \mathbb{Z}^2\), the pattern \(\chi\) occurs at position \(v + k(1, 0)\), for some \(k\) with \(0 \leq k \leq n_0\), of the configuration \(C_\alpha\), which yields the uniform recurrence. If \(\alpha\) is Pythagorean, then the uniform recurrence follows from the fact that \(C_\alpha\) has a lattice of periods of rank 2.
We obtain an upper bound on the complexity function by counting the connected components of the sets obtained by taking intersections of the form $\cap_{0 \leq k \leq m-1, 0 \leq \ell \leq n-1} T_k \circ \mathcal{I}_k \circ \mathcal{I}_k$. We thus get $P_{C_\alpha}(m+1, n) - P_{C_\alpha}(m, n) \leq 5n$, for all $n \in \mathbb{N}$, which yields the desired result by a simple induction. \hfill \Box

**Remark 6.** Let us note that we deduce from Lemma 3 that the symbols which appear in $C_\alpha$ at indices of the form $2v$, for $v \in \mathbb{Z}^2$, are exactly the elements of $Q_\alpha$. Indeed, in the non-Pythagorean case, the sequence $(u_{2m, 2n})_{(m,n)\in \mathbb{Z}^2}$ is still dense. Otherwise, we use the fact that the Pythagorean triple $(a, b, c)$ introduced in the proof of Lemma 3 is assumed to be a prime triple, i.e., $\gcd(a, b, c) = 1$. We will use this remark in the next section.

5. Distribution of symbols in $C_\alpha'$

We now consider a second configuration $C_\alpha'$ studied, e.g., in [15]:

$$\forall v \in \mathbb{Z}^2, \ C_\alpha'(v) := \bigcup_{w} C_\alpha(w).$$

The configuration $C_\alpha'$ codes the action of $[r_\alpha]$ on the 4-neighbors of antecedents of points of $\mathbb{Z}^2$.

Let $Q'_\alpha$ stand for the set of values taken by $C_\alpha'$. We want to state a result analogous to Theorem 1 in order, first, to interpret the configuration $C_\alpha'$ as a coding of a symbolic dynamical system, and second, to compute the densities of the symbols in $C_\alpha'$. Let us note that Corollary 1 in [15] does not directly yield a dynamical interpretation of $C_\alpha'$.

Let us note that there exist elements $v \in \mathbb{Z}^2$ that have no antecedent by $[r_\alpha]$. Such an element is called a hole. An example of a hole is depicted in Fig. 5 below. According to [14], two holes can never be adjacent, i.e., if $v$ is a hole, then neither $v + i$ nor $v + j$ can be a hole. Our strategy for describing $C_\alpha'$ as a coding of a $\mathbb{Z}^2$-action is thus to create a “block configuration” by working with patterns of size $[2, 2]$ that occur in $C_\alpha'$. According to Remark 6, there is no restriction in working with even indices rather than with odd indices.

We then introduce a particular domain of $\mathbb{R}^2$ that is a fundamental domain for the lattice $\mathbb{Z}l_\alpha + \mathbb{Z}j_\alpha$, such that if we know the projection of a point $p \in \mathbb{Z}l_\alpha + \mathbb{Z}j_\alpha$ in that domain, then we can recover the symbols that appear in the block configuration; therefore, we find out what the symbols that appear in $C_\alpha'$ are. We thus deduce a symbolic dynamical system for the block configuration. Finally, we use this dynamical system, in order to get the density of the symbols both in the block configuration and in $C_\alpha'$.

5.1. Dynamical system for $C_{B_\alpha}$

We denote by $(Q_\alpha')^{[2,2]}$ the set of patterns of size $[2, 2]$ that occur in $C_\alpha'$. Let $(C_\alpha')^{[2,2]}$ be the configuration with values in the finite alphabet $(Q_\alpha')^{[2,2]}$ that maps $v$ to the pattern of size $[2, 2]$ that occurs at position $2v$ in $C_\alpha'$. Since $(C_\alpha')^{[2,2]}(v)$ is an application that returns patterns of size $[2, 2]$, then $C_\alpha'(v)$ is obtained by taking the value at position $(x_v \mod 2, y_v \mod 2)$ in the $[2, 2]$ pattern $(C_\alpha')^{[2,2]}([x_v/2], [y_v/2])$.

For any $v \in \mathbb{Z}^2$, one sets

$$F_B(v) = [x_v - \frac{1}{2}, x_v + \frac{3}{2}] \times [y_v - \frac{1}{2}, y_v + \frac{3}{2}].$$

Let

$$F_{D_\alpha} := \left(\left[-\frac{1}{2}, \cos \alpha - \frac{1}{2}\right] \cup \left[\cos \alpha - \frac{1}{2}, \cos \alpha + \sin \alpha - \frac{1}{2}\right] \cup \left[-\frac{1}{2}, \sin \alpha - \frac{1}{2}\right] \cup \left[-\frac{1}{2}, \sin \alpha - \frac{1}{2}\right]\right).$$

The set $F_{D_\alpha}$ is a fundamental domain for the lattice $L_\alpha = \mathbb{Z}l_\alpha + \mathbb{Z}j_\alpha$ (see Fig. 4), i.e., $\cup_{\gamma \in L_\alpha} F_{D_\alpha} + \gamma$ is a partition of $\mathbb{R}^2$. We thus set $\mathbb{T}_\alpha^2 := \mathbb{R}^2/(\mathbb{Z}l_\alpha + \mathbb{Z}j_\alpha)$. Furthermore, we denote by $v \mapsto \{v\}_\alpha$ the canonical projection on $\mathbb{T}_\alpha^2$, $\mathbb{T}_\alpha^2$ that is in one-to-one correspondence with $F_{D_\alpha}$.

**Theorem 7.** Let $\alpha \in [0, \pi/4]$. Let $C_\alpha'$ be the configuration associated with the discrete rotation $[r_\alpha]$. There exists a partition $P_\alpha' = \{J_{\rho'}\}, \rho' \in Q_\alpha'$ of $F_{D_\alpha}$ into a finite number of frames such that

$$\forall v \in \mathbb{Z}^2, \ (C_\alpha')^{[2,2]}(v) = p' \iff \{2v\}_\alpha \in J_{\rho'}.$$
Fig. 4. An exchange of pieces between $F_{D\alpha}$ and the canonical representation of $\mathbb{R}^2/L\alpha$, obtained by performing translations in $L\alpha$.

Fig. 5. From a point $p_0 \in \mathbb{Z}k\alpha + \mathbb{Z}j\alpha$ contained in the domain $F_{D\alpha}(2v)$ (in dark gray), we can recover all the symbols that contribute to the block of size $[2, 2]$ at position $2v$ in $C'_\alpha$. $F_B(2v)$ is depicted in light gray.

Fig. 6. A partition of the domain $F_{D\alpha}$, for $\alpha \approx 0.464705 \text{ rad}$. This partition gives the pattern of size $[2, 2]$ that appears in $(C'_\alpha)^{[2,2]}(v)$, according to the position of $-\{2v\}_\alpha$ inside that domain. On the coordinate axis, the positions are labeled by expressions of the form $kc + k's + k''$, meaning that the corresponding line is located at $k \cos(\alpha) + k' \sin(\alpha) + k'' - \frac{1}{2}$ in $F_{D\alpha}$. For readability reasons, the scale is monotone but not linear.

We define by $l_{(C'_\alpha)^{[2,2]}}: T^2_\alpha \rightarrow (Q'_\alpha)^{[2,2]}$ the labelling function that associates, with elements of the frame $J_{p'} \in P'_\alpha$ of $F_{D\alpha}$, the corresponding pattern $p'$ of size $[2, 2]$, i.e.,

$$\forall v \in \mathbb{Z}^2, (C'_\alpha)^{[2,2]}(v) = l_{(C'_\alpha)^{[2,2]}}(-\{2v\}_\alpha).$$

The configuration $(C'_\alpha)^{[2,2]}$ is thus a coding of the orbit of 0 under the $\mathbb{Z}^2$-action ($T^2_\alpha, v \mapsto v + (\{ -2i\}_\alpha, v \mapsto v + \{ -2j\}_\alpha$) with respect to the labelling function $l_{(C'_\alpha)^{[2,2]}}$. 
The proof is based on the following idea: for any 
\[ n \in \mathbb{Z} \]
Lemma 7. Table describing \( \eta_{\alpha}^n(p) \) for each symbol \( p \) that appears in \( C_\alpha \), with respect to the value of \( \alpha \).

**Proof.** The proof is based on the following idea: for any \( v \in \mathbb{Z}^2 \), there exists a unique \( \gamma \in L_\alpha = r_\alpha(\mathbb{Z}) \) such that \( v \in \gamma + F_D \alpha \). We apply this property to \( -2v \). Hence, for any \( v \in \mathbb{Z}^2 \), there exists a unique \( w \in \mathbb{Z}^2 \) such that \( r_\alpha(w) = 2v + F_D \alpha \). One thus has \( r_\alpha(-w) = 2v = \{ -2v \} \). Let us prove that it is possible to deduce the value of \( (C_\alpha')^{(2,1)}(v) \) from the location of \( \{ -2v \} \) in \( F_D \alpha \).

For that purpose, we first check that for all points \( w \) of \( \mathbb{Z}^2 \) that have their image by \( r_\alpha \) in \( F_B(2v) \), we can compute \( C_\alpha(w) \) according to Theorem 1 and Remark 6. Indeed, let \( w \) be the unique element in \( \mathbb{Z}^2 \) such that \( r_\alpha(w) \in 2v + F_D \alpha \); if \( x_{r_\alpha(w)} - 2v < \frac{1}{2} \), \( \{ r_\alpha(w) - 2v \} = 0 \), else \( \{ r_\alpha(w) - 2v \} = 1 \); we thus deduce the value of \( C_\alpha(w) \) according to Theorem 1. Consequently, we get a first partition of \( F_D \alpha \) into a finite number of frames yielding the value of \( C_\alpha(w) \).

The same argument applies for all points \( w' = r_\alpha(w) \) of \( \mathbb{Z}_k + \mathbb{Z}_j \), that are inside \( F_B(2v) \). We thus refine our first partition by intersecting it by translating by vectors of \( L_\alpha \), which ends the proof. \( \Box \)

### 5.2. Application

We can perform the same combinatorial study as in Section 4. In particular, Lemma 3 extends in a natural way. We do not detail here the corresponding results, but we focus on the following application to the density of the symbols. We assume in particular that \( \alpha \) is not a Pythagorean angle. As in the study of \( C_\alpha \), the orbit of \( 0 \) under the \( \mathbb{Z}^2 \)-action is dense and uniformly distributed in \( \mathbb{Z}^2 \). We thus deduce that

\[
\forall p \in Q_\alpha', \quad \eta_{C_\alpha'}(p) = \sum_{p' \in (Q_\alpha')^{(2,2)}} n(p', p) \mu(f_{p'}),
\]

where \( n(p', p) \) is the function that returns the number of occurrences of \( p \) in the pattern \( p' \) of size \([2, 2]\), and \( \mu(f_{p'}) \) stands for the area of the frame \( f_{p'} \) associated with the symbol \( p' \) according to Theorem 7.

However, practically, the computations for these symbolic maps are quite tedious. For each symbol \( p \), there exist 40 patterns \( p' \) of size \([2, 2]\) to compute. This leads to approximately 360 inequations... and there are approximately 25 symbols \( p \) to consider! The results describing the densities of the symbols in \( C_\alpha' \) have been summarized in Fig. 7. In the Pythagorean case, the study is also similar to the one developed for \( C_\alpha \).

### References