ELSEVIER Theoretical Computer Science 218 (1999) 347–417

Theoretical Computer Science

Discrete parabolas and circles on 2D cellular automata

Marianne Delorme[a,1], Jacques Mazoyer[*,a], Laure Tougne[b,2]

[a] *Laboratoire de l'Informatique du Parallélisme, Ecole Normale Supérieure de Lyon, 46, Allée d'Italie, 69364 Lyon Cedex 07, France*

[b] *Laboratoire ERIC, Bât, 5 avenue Pierre Mendès – France, 69500 Bron, France*

Contents

1. Introduction

1.1. Cellular automata

A *cellular automaton* is a regular, homogeneous network of identical simple machines put on the nodes of the net, each of them communicating with some others,

* Corresponding author. E-mail: mazoyer@ens-lyon.fr.
[1] E-mail: marianne.delorme@ens-lyon.fr.
[2] E-mail: laure.tougne@dionysos.univ-lyon.fr.

PII: S0304-3975(98)00330-2

called its neighbors. The homogeneity of the network comes from its regularity, from the fact that only one finite automaton A with states set S modelizes each machine and from the neighborhood which is supposed to be uniform.

The notion of "regular" network is more delicate to formalize. Here we will limit ourselves considering a graph Γ, which usually will be $\mathbb{Z}$, $\mathbb{Z}^2$ or $\mathbb{Z}^3$, and possibly the hexagonal network. We call *cell* each pair $\{$vertex, automaton $A\}$.

Such a system can be considered as a dynamical one. At one time t, the global state of the system is given by an application $\Gamma \rightarrow S$, which assigns to each cell a state of A. And, starting from an initial configuration, the whole system synchronously evolves, at discrete times, from one configuration to the next one. This behavior is formalized through a sequence of configurations $(\Gamma_t)_{t\in\mathbb{N}}$.

In spite of their surface simplicity, cellular automata can be very powerful and present very complex, even chaotic, behaviors. As massively parallel discrete systems, possibly very sensible to the initial conditions, they are used in Physics as an alternative to differential equations.

Classical Physics modelizes real phenomena aid of differential equations, and the usual numerical methods are used to compute the system evolution. However, in case of some non-continuous e. d. p. for example, these numerical methods fail and the physicists may turn to cellular automata. Then the space ($\mathbb{R}^2$ or $\mathbb{R}^3$) is considered as tiled with square or cubic cells on which act some forces. The underlying grid of a cellular automaton can represent this space, in simple cases the actions on the cells can be modelized by a finite automaton, in more complex cases this modelization can require introducing real numbers or probabilities. In any case the cellular automaton (stricto sensu or generalized) simulates the physical phenomenon and its evolution allows to anticipate the phenomenon evolution.

1.2. Cellular automata, isotropy and discrete circles

Many physical phenomena are anisotrope, but some of them, as natural as the waves generated when throwing a stone into a calm water, are isotrope, which means, in Physics, that their space expansion is not direction-depending. In Mathematics, this property is expressed by a differential equation depending only on the time and the distance from the origin (and so, not depending on the polar angle). Then arises the question: how to simulate an isotrope phenomenon with a cellular automaton?

Many solutions have been proposed, some using hexagonal networks [3], others introducing probabilities [7, 13]. However, while actual interesting results are obtained with probabilistic automata, hexagonal networks lead to nothing.

Let us first observe that, as the hexagonal network is, in some sense, equivalent to the grid [12], only one question remains: how to simulate isotrope phenomena on a grid of finite automata? and an other one in line: is it possible to compute quickly a *discrete approximation* of the function $(x, y) \mapsto \sqrt{x^2+y^2}$, for example the function $(x, y) \mapsto \lfloor\sqrt{x^2+y^2}\rfloor$? This last function is not easy to handle with. As evidence, let us recall that Rózsa Peters [10] added this function to the basic recursive functions in order to get a definition of the recursive functions on one variable independent of many variables functions. Let us mention too that it is not known whether it is possible to send information in time $n+\sqrt{n}$ on a line of finite automata. But, for the moment, we have to precise what is a discrete approximation of the function $(x,y)\mapsto\sqrt{x^2+y^2}$, that means a discrete approximation of $\sqrt{\xi}$, where $\xi\in\mathbb{R}$.

Here, we will consider that such an approximation is determined by a denumerable partition of $\mathbb{R}$ in intervals $\{]a_z,b_z]/z\in\mathbb{Z}\}$ and a function ϕ from $\mathbb{Z}$ to $\mathbb{N}$ such that $\phi((a_{z_0},b_{z_0}))=\Phi(\xi)$ is taken as the discrete value of ξ if $\xi\in]a_{z_0},b_{z_0}]$. In order to give sense to that approximation, we have to give a meaning to "to compute". Here, to compute $\sqrt{\xi}$ will be, starting from an initial configuration C_0 in which the only marked cell is the origin, say $(0,0)$, arrive at a configuration in which the only marked points are the (x,y) such that $\sqrt{x^2+y^2}=\phi(\xi)$. This comes down to define a cellular automaton which build a *family of discrete circles*. So, now, we are interested in the problem, for itself, of the construction of discrete circles by means of cellular automata.

The notion of discrete circle is not new, and many definitions have been proposed, for example [1, 2, 5, 6, 9, 11]. We will keep the point of view sketched out in the case of the approximation of a real number through a natural one. So, let us precise that our approximation will be given by a partition of $\mathbb{R}^2$ by rectangles in $\mathbb{Z}\times\mathbb{Z}$: $\{R_{z,z'}=[sw_{z,z'},nw_{z,z'},se_{z,z'},ne_{z,z'}]/z,z'\in\mathbb{Z}\}$ and a function ϕ from $\mathbb{Z}^2$ to $\mathbb{N}$. Let us immediately remark that, as cellular automata have only a finite number of states, the rectangles $R_{z,z'}$ can take only a finite number of rational dimensions. So we can consider that they all are multiples of λ/μ with $\lambda,\mu\in\mathbb{N}$. Actually we will choose the family $\{R_{z,z'}=[(z,z'),(z,z'+1),(z+1,z'),(z+1,z'+1)]/z,z'\in\mathbb{Z}\}$ and show, due to a notion of grouping we will later introduce, that, making this choice, we do not lose any generality. Let us also notice that the function ϕ has to be cellular-automata-computable. To escape this difficulty and because z and z' play symmetrical roles, we will choose one of them, z for example. But it remains that the value of z will depend on the topology of $R_{z,z'}$ (to what mesh does the edges belong?). In any case if ξ is a point of a circle C, it belongs to a rectangle $R_{z,z'}$, and if $z(\xi)$ denotes its projection on the straight line $y=z'$, we could define $\Phi(\xi)$ as $\lfloor z(\xi)\rfloor$ or as $\lceil z(\xi)\rceil$ according to whether $(\lfloor z(\xi)\rfloor,z')$ belongs to $R_{\lfloor z(\xi)\rfloor,z'}$ or not. That amounts to put to open the square tiles up and right.

We not only want to build discrete circles with cellular automata, but also want the computation to be "as soon as possible". If the system starts from the initial configuration where only the origin is marked, the cell $(t,0)$ can, at best, be marked at time t. So, we may hope getting the circle $C((0,0),r_t)$ (centered at $(0,0)$, with radius r_t) if $r_t\in[t-1,t+1]$. Moreover, from the definition of a cellular automaton, r_t has to be rational and bounded. Actually, in the following, we choose to study the family $\odot$ of the circles centered at $(0,0)$ with the natural numbers as radii, and we prove it is sufficient because of some duality between circles radii and meshes sizes, more precisely, because studying the family $(C((0,0),t))_{t\in\mathbb{N}}$ on the net $\mathscr{R}_{\lambda/\mu}$ is equivalent to study the family $(C((0,0),t\lambda/\mu))_{t\in\mathbb{N}}$ on the net $\mathscr{R}$.

Let us explain a little more. If we think of an automata grid modelizing a physical phenomenon, we want the cell size as small as possible, actually infinitely small. So, that a cellular automaton modelize such a phenomenon has to be scale-resistant, that means not depending on the size of the underlying network meshes. More precisely, if $\mathscr{A}$ is a cellular automaton on the usual grid $\mathbb{Z}\times\mathbb{Z}$ and $\{(a+bx, c+dy)/x, y\in\mathbb{Z}\}$ a new grid, isomorph to the former one, it must exist some new cellular automaton $\mathscr{B}$ on this new grid, such that, knowing some configuration C of $\mathscr{A}$, we get a configuration C' of $\mathscr{B}$ in which the state-$\mathscr{B}$-machine at site $(a+bx_0, c+dy_0)$ contains the whole information of the $\mathscr{A}$-machines at sites $(a+bx_0+i, a+by_0+j)$, $i\in\{0,\dots,b-1\}$, $j\in\{0,\dots,d-1\}$ of C, and, moreover, that the global functions of $\mathscr{A}$ and $\mathscr{B}$ have the same physical interpretation. Formally, it is the notion of "grouping", which will be developed in Section 1.6, which renders this property. It is possible to give a geometrical interpretation of $\mathscr{B}$ that we denote now by $\mathscr{G}(\mathscr{A},a,c,b,d)$. For example, the transformation of $\mathscr{A}$ into $\mathscr{G}(\mathscr{A},a,c,b,d)$ corresponds to a quasi-affine-dilatation [4]. And if $\mathscr{A}$ marks the family $(C((0,0),t))_{t\in\mathbb{N}}$, $\mathscr{G}(\mathscr{A},0,0,k,k)$ is able to mark the family $(C((0,0),t))_{t\in\mathbb{N}}$ on $\mathscr{R}_{1/k}$. Finally, modifying $\mathscr{A}$ so that it marks the net $\mathscr{R}_{k'/k}$ we can get a cellular automaton marking $(C((0,0),t))_{t\in\mathbb{N}}$ on $\mathscr{R}_{k'/k}$ or $(C((0,0),tk'/k))_{t\in\mathbb{N}}$ on $\mathscr{R}$.

Two natural and basic digitizations of the family $\odot$ will be privileged here, following the fact that the rectangles of the grid will be open up-right or bottom-left, which lead to the notions of "floor circle" and "ceiling circle". But we will see that it is enough to study one of them to get the other one. And moreover that other discrete circles as the "Pitteway's circles" or the "arithmetical circles" can be obtained from the "floor circles".

Building these digitizations of the family $\odot$ will need a constant linking between structural study of this family in terms of discrete geometry and the constraints due to the "computing machinery".

1.3. Standard definitions

In this section, we recall some definitions concerning cellular automata, in limiting ourselves to the types of automata we need.

Definition 1.1. A *two dimensional cellular automaton* (or 2-CA), $\mathscr{A}$, is a 4-uplet $(2,S,B,\delta)$ such that

- S is a set the elements of which are the *states* of $\mathscr{A}$,

$$S=\{s_k/k\in\{0,\dots,|S|-1\}\},$$

- B is a finite subset of $\mathbb{Z}^2$, called the *neighborhood* of $\mathscr{A}$,

$$B=\{v_j=(x_j,y_j)/j\in\{0,\dots,|B|-1\}\},$$

- δ is a function from $S^{|B|}$ to S, called the *local transition function* of $\mathscr{A}$.

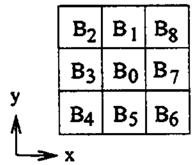


Fig. 1. Moore's neighborhood in dimension 2.

At each point of the grid ($\mathbb{Z}^2$) is attached the same finite automaton, and such a decorated point is called a cell. Each cell locally communicates with a finite number of neighbors. The neighborhood is fixed and geometrically uniform. In this paper we will only use the Moore's neighborhood, also called the 8-neighborhood, and a 2*D*-cellular automaton will, often, only be denoted by (S, δ).

Definition 1.2. Let $(x, y) \in \mathbb{Z}^2$. The *Moore's neighborhood* of the cell (x, y) is the set of cells, denoted by $B(x, y)$, such that

$$B(x,y) = \{(x+m, y+n) / \text{ with } m, n \in \{-1, 0, 1\}\}.$$

And we use the notations presented in Fig. 1:

$$B_0(x,y)=(x,y), \quad B_3(x,y)=(x-1,y), \quad B_6(x,y)=(x+1,y-1),$$

$$B_1(x,y)=(x,y+1), \quad B_4(x,y)=(x-1,y-1), \quad B_7(x,y)=(x+1,y),$$

$$B_2(x,y)=(x-1,y+1), \quad B_5(x,y)=(x,y-1), \quad B_8(x,y)=(x+1,y+1),$$

The local communications, which are deterministic and uniform, take place synchronously according to discrete times.

Definition 1.3. A configuration $C_{\mathscr{A}}$ of the cellular automaton $\mathscr{A}$ is an application from $\mathbb{Z}^2$ to S. For all t in $\mathbb{N}$, the configuration $C^t_{\mathscr{A}}$, at time t, becomes, at time $t+1$, the configuration $C^{t+1}_{\mathscr{A}}$ defined by $(x, y) \in \mathbb{Z}^2$ and

$$C^{t+1}_{\mathscr{A}}(x,y) = \delta(C^t_{\mathscr{A}}(x+x_1, y+y_1), \ldots, C^t_{\mathscr{A}}(x+x_{|B|}, y+y_{|B|})).$$

The function $F : S^{\mathbb{Z}^2} \to S^{\mathbb{Z}^2}$ which associates the configuration $C^{t+1}_{\mathscr{A}}$ to the configuration $C^t_{\mathscr{A}}$ is called the *global function* of $\mathscr{A}$.

A state q such that $\delta(q, \ldots, q) = q$ is called a *quiescent state*. In the following, we denote by C_0 the *initial configuration* such that all the cells of the grid are in a quiescent state except the cell $(0, 0)$.

The notion of *signal*, often used when working with cellular automata, is delicate enough and needs some clarification.

1.4. Signals

As we are in the way to design a cellular automaton with a given behavior on a well defined set of starting configurations, we will have to conceive the succession of configurations of this automaton, that means to find the efficient links which determine this succession. So it is very helpful and basic to get a convenient graphical representation of it, which is called a space-time diagram of the cellular automaton.

1.4.1. Space-time diagram

Let $\mathscr{A}$ be a two-dimensional cellular automaton (respectively a one-dimensional cellular automaton), with states set S. The configuration of $\mathscr{A}$, at time t, is a map which associates to each cell (i,j) (respectively i) a state s. We get a graphical representation of it in $\mathbb{Z}\times\mathbb{Z}\times\mathbb{N}$ (respectively $\mathbb{Z}\times\mathbb{N}$) in assigning to each point (i,j,t) (respectively (i,t)) a color (or a pattern) corresponding to s. And another one in $\mathbb{R}^3$ (respectively $\mathbb{R}^2$) in assigning to the elementary square of smaller vertex (i,j,t) (respectively the elementary cube of "smaller" vertex (i,t)), a color (or a pattern) corresponding to s.

So, each sequence $(C_t)_{t\geqslant t_0}$ of configurations can be seen as a colored part of $\mathbb{Z}\times\mathbb{Z}\times\mathbb{N}$ (respectively $\mathbb{Z}\times\mathbb{N}$), or $\mathbb{R}^3$ (respectively $\mathbb{R}^2$), according to the chosen representation, which is called a *space-time diagram* of $\mathscr{A}$. The points, elementary squares or elementary cubes belonging to such diagrams are sometimes called *sites*.

1.4.2. Signals

If we consider a state, or a finite set of states, of a cellular automaton as an individable particle of information, the line, surface or space possibly determined in the space-time diagram by this (or these) state(s) can be interpreted as the track of this information in the course of time. In this paper, such a track will be called a *signal* when it is a mono (finitely multi)-colored connected path of the space-time diagram. Let us precise the definition in case of $2D$-cellular automata.

Definition 1.4. A *one-dimensional signal*, or *signal*, *on a two-dimensional cellular automaton* is an application of a cofinal subset of $\mathbb{N}$ into $\mathbb{Z}\times\mathbb{Z}\times\mathbb{N}$. Its image is a set of sites $\{(x(t),y(t),t)/t\in\mathbb{N},\ t\geqslant t_0\}$, where x and y are applications from $\mathbb{N}$ to $\mathbb{N}$ which verify

$$(x(t+1),y(t+1),t+1)\in\{(x(t)+m(t),y(t)+n(t),t+1)/\ m(t),n(t)\in\{-1,0,1\}\}$$

Fig. 2 gives an example of such a signal.

1.4.3. Projected signals and space-time diagrams

If space-time diagrams of one-dimensional cellular automata can be very readable, it is usually no more the case for two-dimensional cellular automata since these diagrams are then three dimensional ones. It is why we will use a planar representation of signals. We project the signal onto the $(\vec{\mathcal{O}}x,\vec{\mathcal{O}}y)$ plane, and we mark each cell reached by the

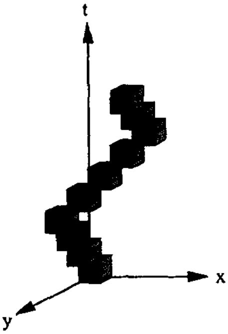


Fig. 2. Example of signal.

signal with the time it has reached it (see, e.g. Fig. 44). The obtained diagram will be called the projected-space-time diagram, even if the times are not written, which happens when there is no ambiguity.

As soon as we have exhibited the elementary bits of information, in finite number, necessary to solve our problem and understood the way they go through time, the signals give rise to a "geometrical diagram" (or a projected geometrical diagram), some sort of skeleton of a (or virtual) space-time diagram, and we have to prove that it can provide from the evolution of a cellular automaton which effectively installs such signals.

Knowing whether it is possible to get all (recursive) curves of $\mathbb{Z} \times \mathbb{Z} \times \mathbb{N}$ is an open question, while the answer, in case of dimension 1, is negative. Actually, the only signals easy to build are the linear ones, with rational slopes, which provide others through their possible interactions. In the following we will try to connect the significant points with straight lines segments.

1.4.4. Real-time signals

As we want to build as fast as possible the families of discrete circles, we want any significant information going as fast as possible. A signal representing such an information will be called a real time signal. Intuitively it does never stay on one cell, nor come back to an already reached one.

Definition 1.5. Let S be a signal. We denote by $S_T = \{(x(t), y(t), t)/t \in \mathbb{N},\ t_0 \leqslant t \leqslant T\}$ the sites visited by the signal S, created at time t_0, up to time T. We say that S is a *real-time signal* (or *S is propagated in real time*) if and only if its image is such that for all $T \geqslant t_0$,

$$S_T = S_{T-1} \cup \{(x(T-1) + m(T-1), y(T-1) + n(T-1), T)\}$$

and, there does not exist any time t, $t_0 \leqslant t \leqslant T-1$, such that

$$(x(T-1) + m(T-1), y(T-1) + n(T-1), t) \in S_{T-1}.$$

Let us notice that a real-time signal on the cell (x, y) at time t has, in case of the Moore neighborhood, seven possible moves. So it exist many real-time signals. Whereas there are only two real-time signals in dimension 1, being propagated at maximal speed on the left or on the right. Let us also remark that if a real-time signal is created at t_0 on (x_0, y_0), it reaches the cell (x, y) at time $t_0 + \|(x, y) - (x_0, y_0)\|_\infty$. In particular if $t_0 = x_0 = y_0 = 0$, it gets on (x, y) at time $\|(x, y)\|_\infty$. In this case we will not mark the times in the projected geometrical diagram (see Fig. 41).

1.5. Construction of figures

We will now precise what "building figures" means in this paper. A *figure* is a subset, here a finite one, of $\mathbb{Z} \times \mathbb{Z}$, and a *family of figures* an application from $\mathbb{N}$ into the set of finite subsets of $\mathbb{Z} \times \mathbb{Z}$, that we will denote $\mathscr{F} = (F_i)_{i \in \mathbb{N}}$. Constructing a figure F by a cellular automaton $\mathscr{A}$ is to select a subset S_F of $\mathscr{A}$-states such that the automaton starting from a convenient initial configuration reaches a configuration where the cell (x, y) is in a state belonging to S_F if and only if $(x, y) \in F$. This definition leads to the following one:

Definition 1.6. Let $\mathscr{A}$ be a 2-CA. We say that $\mathscr{A}$ *constructs the family of figures* $\mathscr{F} = (F_i)_{i \in \mathbb{N}}$ *of* $\mathbb{Z} \times \mathbb{Z}$, *according to the times* $(t_i)_{i \in \mathbb{N}}$ if and only if there exists a subset $S_{\mathscr{F}}$ and a sequence $(t_i)_{i \in \mathbb{N}}$ of times such that the automaton starting from an initial configuration C_0 at time $t = 0$ enters, at time $t = t_i$, a configuration where all the cells belonging to F_i are in a state of $S_{\mathscr{F}}$ and are the only ones to be in such states.

Notice that the initial configuration is C_0 because every configuration, such that all the cells are in a quiescent state except a finite number of them, is equivalent to it using the grouping method we will explain below.

We know, by means of mutual simulations between the evolution of a $2D$-cellular automaton on almost everywhere quiescent configurations and the one of a Turing machine tape, that the families of figures which can be constructed by $2D$-cellular automata are only the recursive ones. However the cellular automata can, as parallel systems, accelerate the construction. So, defining families of figures constructed "as soon as possible", that means in real time, is interesting. It is that way we plan to build the family of floor circles, for example.

Definition 1.7. Let $\mathscr{A}$ be a 2-CA. We say that $\mathscr{A}$ *constructs the family* $\mathscr{F}$ *in real time* if and only if $\mathscr{A}$ constructs $\mathscr{F}$ according to the times $(i)_{i \in \mathbb{N}}$.

Let us still observe that, as in the Turing machines case, we have time accelerations of a constant. More precisely, each family constructed according to the times $(i+c)_{i \in \mathcal{N}}$, where c is a given natural number, can be constructed in real time. A proof of this fact, using methods of [8], can be found in [14]. Such families will be called *constructed in quasi-real time*.

We have already announced that, starting from the construction of the floor circles family, we get, using the notion of grouping, the construction of other families of discrete circles (see Section 6). So we will consecrate the next section to this notion.

1.6. Grouped cellular automata

As pointed to in the introduction, we define applications from the $2D$-cellular automata set into itself that we call *groupings*.

Let $\mathscr{A}=(S,\delta)$ be a $2D$-cellular automaton, E a connected subset of a given $\mathscr{A}$-configuration. If the states of E are known, then it is possible to know the states of a subset of E, after one or several units of time. For example, if E is a known square of size $m\times m$, $m\geqslant 3$, we know the states of $\mathscr{A}$ in a square of size $m-2\times m-2$, inside the previous one, at the next time. That leads to define some functions $\tilde{\delta}_m$.

Definition 1.8. We denote by $\tilde{\delta}_m$ the application which associates to m^2 states of $\mathscr{A}$, $(m-2)^2$ states of $\mathscr{A}$, defined as follows:

- if $m=3, \tilde{\delta}_m=\delta$,
- if $m>3$ and, if we denote the m^2 states

$$\begin{pmatrix} q_{1,1} & \cdots & q_{1,m} \\ \vdots & & \vdots \\ q_{m,1} & \cdots & q_{m,m} \end{pmatrix},$$

then their $\tilde{\delta}_m$-image is

$$\begin{pmatrix} \delta\begin{pmatrix} q_{1,1} & q_{1,2} & q_{1,3} \\ \vdots & & \vdots \\ q_{3,1} & q_{3,2} & q_{3,3} \end{pmatrix} & \cdots & \delta\begin{pmatrix} q_{1,m-2} & q_{1,m-1} & q_{1,m} \\ \vdots & & \vdots \\ q_{3,m-2} & q_{3,m-1} & q_{3,m} \end{pmatrix} \\ \vdots & & \vdots \\ \delta\begin{pmatrix} q_{m-2,1} & q_{m-2,2} & q_{m-2,3} \\ \vdots & & \vdots \\ q_{m,1} & q_{m,2} & q_{m,3} \end{pmatrix} & \cdots & \delta\begin{pmatrix} q_{m-2,m-2} & q_{m-2,m-1} & q_{m-2,m} \\ \vdots & & \vdots \\ q_{m,m-2} & q_{m,m-1} & q_{m,m} \end{pmatrix} \end{pmatrix}$$

Let us observe that if we know nine $n\times n$ squares, in a configuration as the one of a cell with its Moore's neighborhood, we know the states of the central square after n successive applications of convenient functions $\tilde{\delta}_m$ with $m\leqslant 3n$. That determines a new function $\delta^{n,n}$ from $(S^{n^2})^9$ into S^{n^2}, and leads to the following definition:

Definition 1.9. Let $\mathscr{A}=(S,\delta)$ be a $2D$-cellular automaton. The $2D$-cellular automaton $(S^{n^2},\delta^{n,n})$ is called the n-grouped automaton of $\mathscr{A}$, and will be denoted $G_n(\mathscr{A})$.

This definition is purely syntactic. Writing the states of $G_n(\mathscr{A})$ as matrices is a convenient way to describe them, which allow to naturally simulate the evolutions of

$\mathscr{A}$ and $G_n(\mathscr{A})$. From a configuration C of $\mathscr{A}$ and a point (k,l), we get a tiling of C in $n \times n$ squares, each of them representing a $G_n(\mathscr{A})$-cell in a given state, then we get a configuration of $G_n(\mathscr{A})$. Actually, k and l being two given integers, there exists a bijection, we will denote $\Phi_{k,l,n}$, from the set of the $\mathscr{A}$-configurations onto the set of the $G_n(\mathscr{A})$-ones, defined by

$$\Phi_{k,l,n}((q_{(z,z')})_{z,z' \in \mathscr{Z}} = \begin{pmatrix} q_{nz-k,nz'-l} & \cdots & q_{nz-k,nz'-l+(n-1)} \\ \vdots & & \vdots \\ q_{nz-k+(n-1),nz'-l} & \cdots & q_{nz-k+(n-1),nz'-l+(n-1)} \end{pmatrix}_{z,z' \in \mathscr{Z}}.$$

And it holds:

Proposition 1.1. *Let be given integers $k,l,n,n>0$. Then, for each configuration C of $\mathscr{A}$, each natural integer t*

$$\Phi_{k,l,n}(C^{nt}) = (\Phi_{k,l,n}(C))^t.$$

There are many ways to interpret this last result. First, it is easy to understand that $G_n(\mathscr{A})$ brings a linear acceleration of factor n on the $\mathscr{A}$ evolution about. This implies, in particular, that if we admit that there exists a $2D$-cellular automaton $\mathscr{A}$ building the floor circles family in real time, then it exists one which constructs the discrete concentric circles with a radius multiple of n and centered in $(\lfloor n/2 \rfloor, \lfloor n/2 \rfloor)$: it is enough to consider $G_n(\mathscr{A})$ and $\Phi_{-\lfloor n/2 \rfloor, -\lfloor n/2 \rfloor, n}$.

We can also think of a spatial interpretation. Actually an illustration of it will take place in Section 6, where we will get realizations of quasi-affine-dilatations [4] through convenient $G_n(\mathscr{A}), \Phi_{0,1,n}$ and $\Phi_{0,-1,n}$.

We are now ready to start an analytic study of the floor circles, study which will help to conceive the automaton we are looking for.

2. Analytic study of circles on the grid, bundles of discrete parabolas

As we want to build discrete circles families by means of cellular automata and as a cell of the automaton will have to decide locally whether it belongs to a circle or not, we look for a definition of a discrete circle founded on local properties.

Let us consider the real circle $\mathscr{C}(O,R)$, where R is a positive integer, and the points of this circle with $\vec{O}x$-coordinate $x = R - k, k \in \mathbb{N}, 1 \leqslant k \leqslant R$. As the $\vec{O}y$-coordinate is $y = \sqrt{2kx + k^2}$, the points belonging to the intersection of $\mathscr{C}$ with the straight lines $x = R - k$ belong too to the parabolas $y = \sqrt{2kx + k^2}$ (see Fig. 3).

In the following, we consider the digitization of the parabolas $y = \sqrt{2kx + k^2}$, $k \geqslant 1$, which consists in taking the floor of y. We will see in Sections 5 and 6 how to obtain the ceiling- and the nearest-value-parabolas by cellular automata, starting from the floor parabolas. This digitization makes play a specific role to the grid, especially to the intersections of the real circles in $\odot$ and the lines of the grid. In fact, there exist

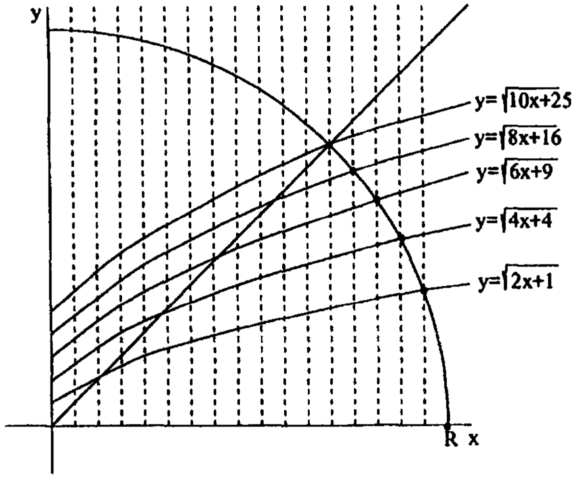


Fig. 3. Intersections between the real circle $\mathscr{C}$ and the family of lines of equation $x = R - k$.

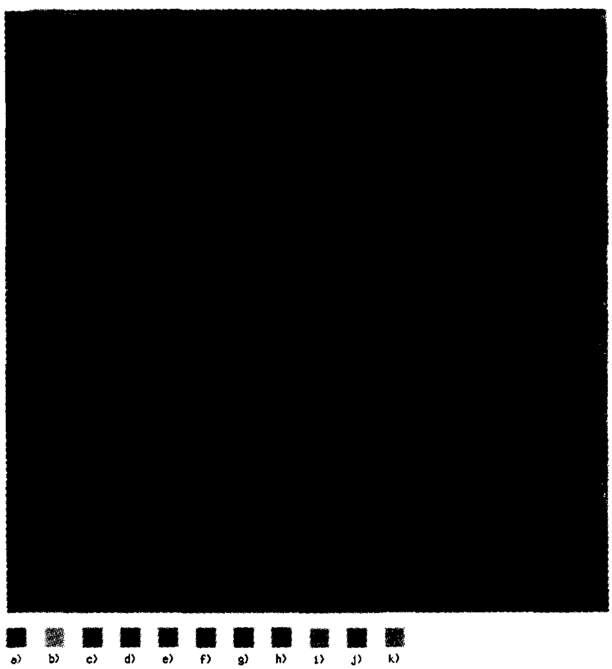


Fig. 4. We associate a color to each kind of intersection between ⊙ and the grid.

eleven sorts of intersections between these circles and the unit squares of the grid, and, if we give a color to each kind of intersection, we get Fig. 4 on which we find again the above mentioned parabolas, or more precisely, the floor-digitizations of them.

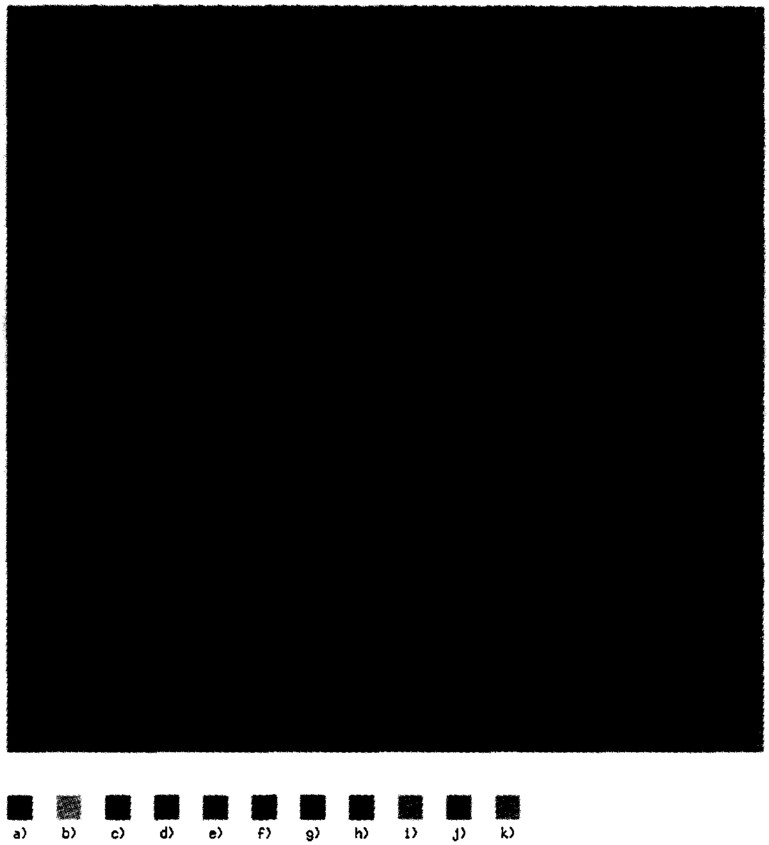


Fig. 5. Fig. 4 in a larger grid.

Finally, we observe that for a given $x = R-k$, the points that belong to the floor circle corresponding to $\mathscr{C}(O,R)$ are situated between the floor-digitizations of the parabolas $y = \sqrt{2kx + k^2}$ and $y = \sqrt{2(k+1)x + (k+1)^2}$. That means that the floor circles are made, in the first octant, of points situated on a succession of straight horizontal, vertical or diagonal segments, leaning on the bundle of floor-parabolas. And we have a feeling that building the bundles of floor-parabolas by means of cellular automata will easily lead to build the family $\lfloor \odot \rfloor$ of floor-circles we are interested in.

First, we describe and comment Fig. 5 which is the same as Fig. 4 but with more circles. Then we prove that the intersections between the circles and the grid are also the intersections between a bundle of parabolas (we define it) and the grid. Finally, we locally study this bundle of parabolas in order to construct it by cellular automaton.

2.1. Study and comments of Fig. 5

First, we can remark that there are no blank squares. As a matter of fact, all the squares of the grid are crossed by at least one circle.

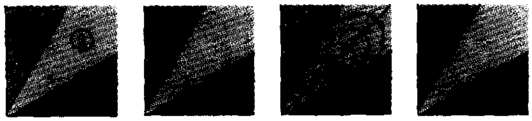


Fig. 6. The three parts of Fig. 5

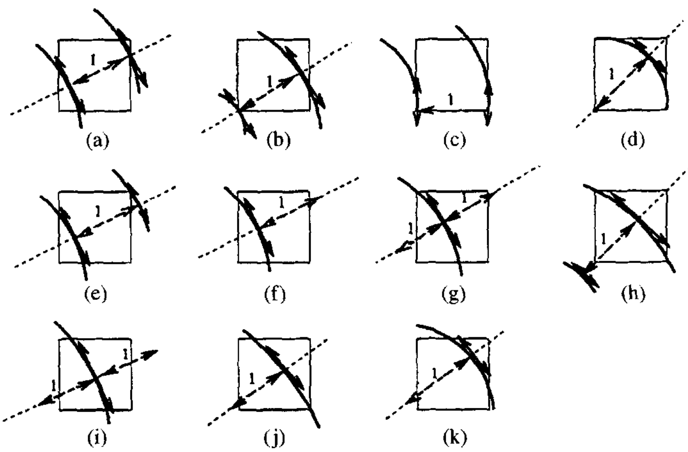


Fig. 7. The different kinds of intersection between the family $\odot$ and the grid (in the first octant).

When we look at this figure, we can distinguish three parts, parted by two lines, which demarcate a central part in the figure. Fig. 6(a) gives a number to each of these parts. In the parts 1 and 3, symmetric according to the first diagonal, we can see parabolic curves (Fig. 6(b)) on which are situated noticeable groups of two or four cells.

In the part 2, called the heart of the figure, we can distinguish some regularities, which are, in fact, regular curves centered on the first diagonal (Fig. 6(c)). If we attentively observe the figure, we can notice similar effects in the parts 1 and 3. More precisely, as the distance between two consecutive circles is minimal and equal to one on the radial direction, we can prove the following fact.

Fact 2.1. *In the first octant, there are eleven cases of intersection between the family of circles* $\odot$ *and the grid.*

Fig. 7 shows these different cases.

Notice that the groups of four or two cells we have mentioned before, correspond to the two cases that are represented in Fig. 8.

It is worth paying attention to the case a) in which a circle of radius R meets the center, called $P(x,y)$, of the 4-cells-group, because this means that there exists an

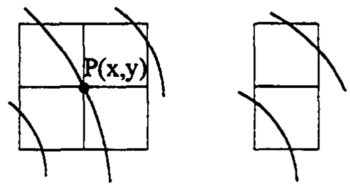


Fig. 8. The groups of 4 or 2 cells that we can distinguish in Fig. 5

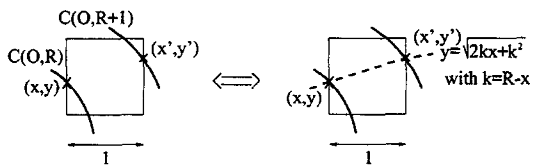


Fig. 9. Duality parabolas/circles.

integer k such that $y=\sqrt{2kx+k^2}$ is an integer number (with x an integer such that $x+k=R$).

We will see in the following subsections that the elements we have brought out in Fig. 5, namely the three parts, the parabolas, the different groups of cells, play a specific role for conceiving the wanted cellular automaton. But, first of all, we will study discrete parabolas.

2.2. *Floor parabolas*

We have seen that the intersections between the circle $\mathscr{C}(O,R)$ and the family of lines $x=R-k$ belong to the bundle of parabolas of equation $y=\sqrt{2kx+k^2}$. For all integers $x \geqslant 0, y \geqslant 0, k \geqslant 1$, we have

$$\begin{cases} x^2+y^2=R^2 \\ x=R-k \end{cases} \Leftrightarrow \begin{cases} y^2=2kx+k^2 \\ x=R-k \end{cases}$$

Consequently, studying the intersections between the family of circles $\odot$ and the grid is equivalent to study the intersections between the bundle of parabolas $y=\sqrt{2kx+k^2}, k \geqslant 1$, and the grid. Fig. 9 gives an example which shows this duality parabolas/circles.

Let h_k be the real parabola of equation $y=\sqrt{2kx+k^2}$. We will consider $\hat{h}_k=\{(x,y) \in \mathbb{Z} \times \mathbb{R}/y=\sqrt{2kx+k^2}\}$, and $\hat{H}_k=\{(x,y) \in \mathbb{Z}^2/y=\lfloor\sqrt{2kx+k^2}\rfloor\}$ which will be the wanted digitization of h_k. We, in fact, consider the broken lines which link the points of $\hat{H}_k$ as shown in Fig. 10. We denote them H_k, so $H_k=\{(x,y) \in [(i,j),(i+1,j')]; i,j,j' \in \mathbb{Z}, j=\lfloor\sqrt{2ki+k^2}\rfloor$ and $j'=\lfloor\sqrt{2k(i+1)+k^2}\rfloor\}$ and, from now on, we will say that H_k is the wanted *discrete parabola* or *floor-parabola*.

Then we obtain Fig. 11, on which we see the floor-parabolas corresponding to $y=\sqrt{2kx+k^2}$ for $1 \leqslant k \leqslant 300$, and their symmetric according to the first diagonal.

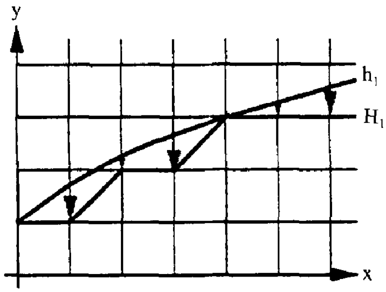


Fig. 10. The digitization we consider.

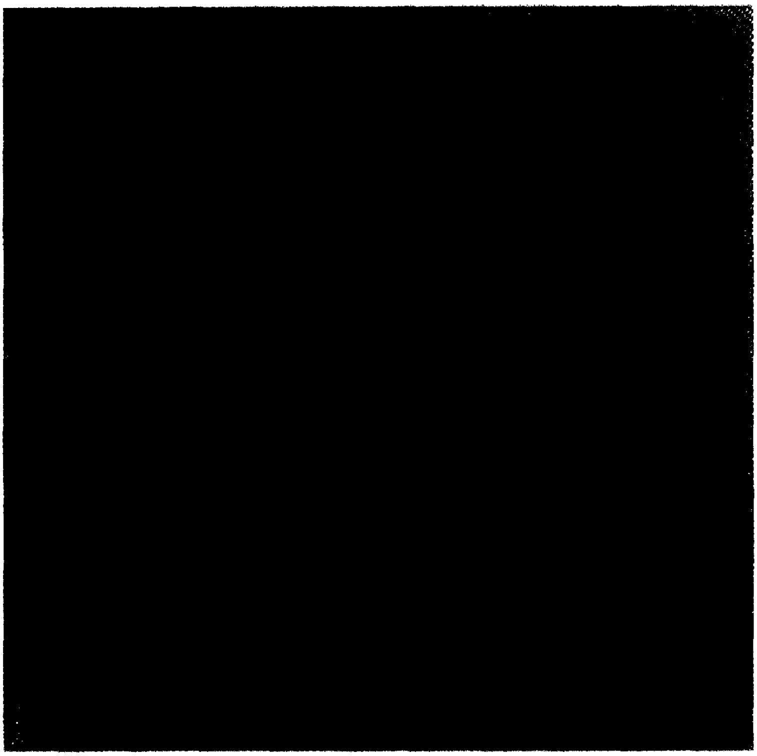


Fig. 11. Discrete parabolas $y = \lfloor\sqrt{2kx + k^2}\rfloor$ and $x = \lfloor\sqrt{2ky + k^2}\rfloor$ with $1 \leqslant k \leqslant 300$.

Before studying this figure, we have to precise some notations.

Notations 2.1 ($h_k, v_{k'}, H_k, V_{k'}, \mathscr{H}$ *and* $\mathscr{V}$). If k and k' are integers greater than or equal to 1, let

- h_k and $v_{k'}$ be the real parabolas of respective equations $y = \sqrt{2kx + k^2}$ and $x = \sqrt{2k'y + k'^2}$,

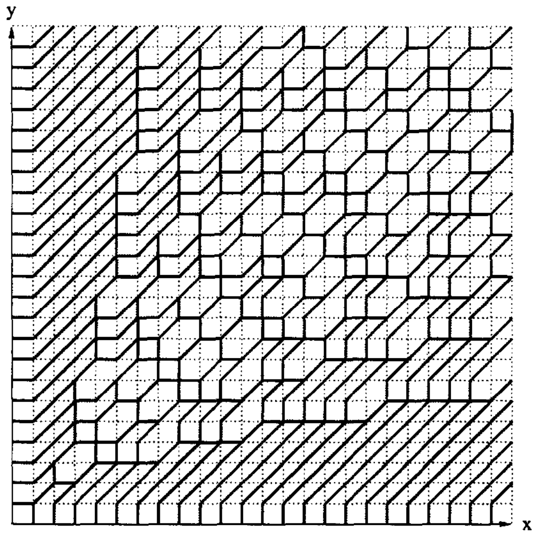


Fig. 12. Discrete parabolas $y = \lfloor\sqrt{2kx + k^2}\rfloor$ and $x = \lfloor\sqrt{2ky + k^2}\rfloor$ with $1 \leqslant k \leqslant 23$.

- H_k and $V_{k'}$ be the respective floor-digitizations of h_k and $v_{k'}$,
- $\mathscr{H}$ and $\mathscr{V}$ be respectively the sets of parabolas H_k and $V_{k'}$.

2.3. Description of Fig. 11

In order to study Fig. 11 more precisely, we look at Fig. 12 which, as less parabolas appear, is more readable.

First, we can see that the parabolas H_k are only composed of horizontal and diagonal segments and symmetrically, the parabolas $V_{k'}$ are only composed of vertical and diagonal segments. In fact, the cases (a)–(d) of Fig. 7 induce horizontal segment on the basis of the square. The cases (e) and (f) lead to diagonal segments. And the other cases correspond to the fact that there is no segment in the square or, there is an horizontal segment on the ceiling of the square. Fig. 13 shows the link between the horizontal and diagonal segments of the parabolas H_k and the intersections between the circles and the grid. If we attentively observe the central part of the figure which is shown in Fig. 14, we see that it is only composed of four patterns: a square, an hexagon and, a chevron and its symmetric (cf. Fig. 17), which are regularly arrange.

We will see in the following that these patterns are essential for the construction by cellular automata.

2.4. Local study of the bundles of parabolas

We first introduce some notations, then study the bundles of discrete parabolas and especially their intersections.

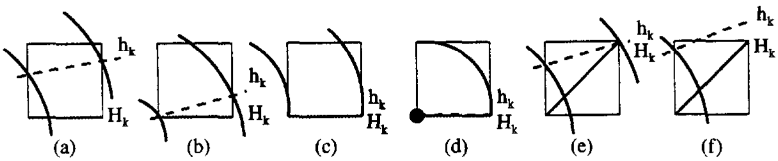


Fig. 13. Link between the discrete parabolas and the circles.

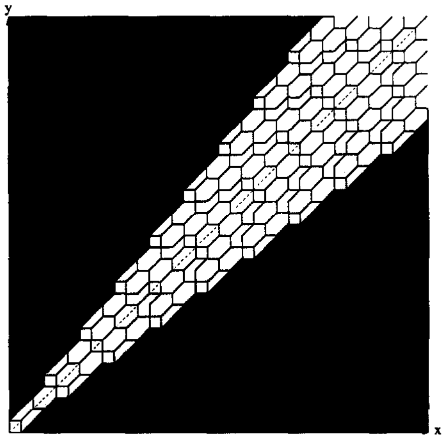


Fig. 14. Central part on Fig. 12.

2.4.1. Notations

For the following, we will give a name to the successions of horizontal and diagonal segments of a discrete parabola. Notice that we will suppose $k \geqslant 1$ because the case $k=0$ is trivial. Moreover, as the parabolas H_k and V_k are symmetric according to the first diagonal, all the definitions or proofs given for the ones are available for the others.

Definition 2.1. We say that, for $k \geqslant 1$,
- there is a *landing of length p at the point* (x, y) on H_k if and only if $(x+i, y) \in H_k$ for all $0 \leqslant i \leqslant p$, $(x-1, y) \notin H_k$ and $(x+p+1, y) \notin H_k$,
- there is a *slope of length m at the point* (x, y) on H_k if and only if $(x+i, y+i) \in H_k$ for all $0 \leqslant i \leqslant m$, $(x-1, y-1) \notin H_k$ and $(x+m+1, y+m+1) \notin H_k$.

We also need to distinguish some points.

Definition 2.2. We say that, for all $k \geqslant 1$,
- the point (x, y) is a *peak* if and only if $(x, y) \in H_k$, $(x-1, y-1) \in H_k$ and $(x+1, y) \in H_k$,
- the point (x, y) is an *hollow* if and only if $(x, y) \in H_k$, $(x-1, y) \in H_k$ and $(x+1, y+1) \in H_k$.

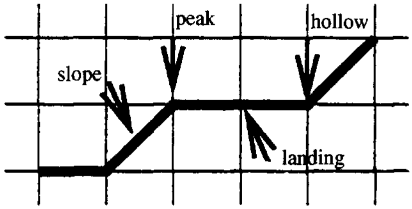


Fig. 15. Landing, slope, peak and hollow.

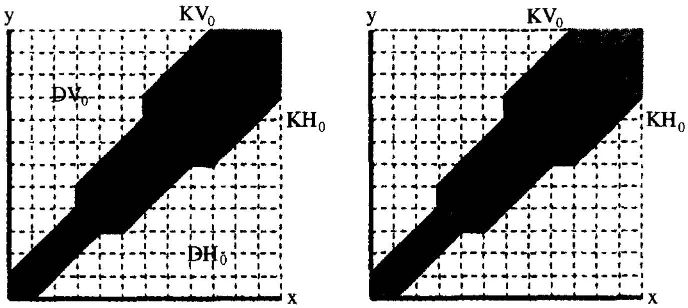


Fig. 16. KH_0, KV_0, K_0, DH_0 and DV_0.

Fig. 15 gives an illustration of those definitions.

And we give a name to the parts of the plane we observed previously.

Notations 2.2.

- We denote by KH_0 the discrete line of equation $y = \lfloor \frac{3}{4}x \rfloor$ and, by KV_0 the line which is symmetrical to KH_0 according to the first diagonal $(x = \lfloor \frac{3}{4}y \rfloor)$.
- We denote by K_{01} the set of points (x, y) such that $\lfloor \frac{3}{4}x \rfloor \leqslant y \leqslant x$. K_{02} is the set of points of the plane such that $\lfloor \frac{3}{4}y \rfloor \leqslant x \leqslant y$.

  We call *heart*, denoted by K_0, the set of points (x, y) of the plane which belong to $K_{01} \cup K_{02}$.
- We denote by DH_0 the set of points (x, y) such that $y < \lfloor \frac{3}{4}x \rfloor$ and by DV_0 the set of points such that $x < \lfloor \frac{3}{4}y \rfloor$.

Fig. 16 shows these parts of the plane and their notations.

We call *patterns* the white regions in Fig. 12 delimited by the elements of the two bundles $\mathscr{H}$ and $\mathscr{V}$ and their intersections. We are particularly interested in some of them which belong to the heart.

Definition 2.3. Let $(x, y) \in K_0$. We say that

- (x, y) is the *origin of a pattern A* if and only if there exist $k \geqslant 1$ and $k' \geqslant 1$ such that $(x, y) \in H_k \cap V_{k'}$, $(x + 1, y) \in H_k \cap V_{k'+1}$, $(x, y + 1) \in H_{k+1} \cap V_{k'}$, $(x + 1, y + 1) \in H_{k+1} \cap V_{k'+1}$.

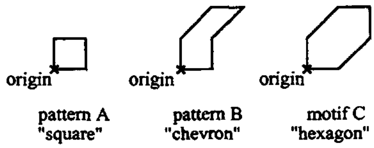


Fig. 17. Patterns A, B and C.

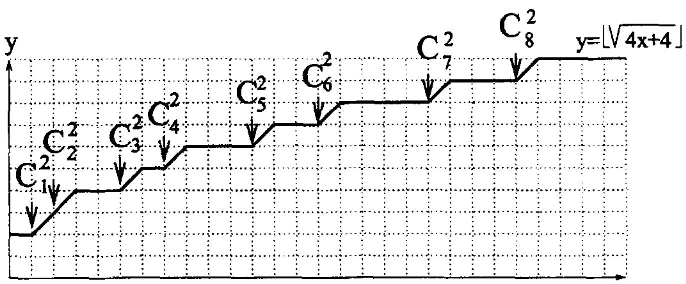


Fig. 18. Graphical representation of the points C_i^2 for $1 \leqslant i \leqslant 8$.

- (x,y) is the *origin of a pattern B* if and only if there exist $k \geqslant 1$ and $k' \geqslant 1$ such that $(x,y) \in H_k \cap V_{k'}$, $(x+1,y) \in H_k \cap V_{k'+1}$, $(x,y+1) \in V_{k'}$, $(x+1,y+1) \in V_{k'+1}$, $(x+1,y+2) \in H_{k+1} \cap V_{k'}$ et $(x+2,y+2) \in H_{k+1} \cap V_{k'+1}$.
- (x,y) is the *origin of a pattern C* if and only if there exist $k \geqslant 1$ and $k' \geqslant 1$ such that $(x,y) \in H_k \cap V_{k'}$, $(x+1,y) \in H_k \cap V_{k'+1}$, $(x,y+1) \in H_{k+1} \cap V_{k'}$, $(x+2,y+1) \in V_{k'+1}$, $(x+1,y+2) \in H_{k+1} \cap V_{k'}$, $(x+2,y+2) \in H_{k+1} \cap V_{k'+1}$.

See Fig. 17.

2.4.2. Study of the bundle $\mathscr{H}$

First, we study the bundle $\mathscr{H}$ in the quarter of the plan such that $x \geqslant 0$ and $y \geqslant 0$. Then, we pay special attention to the part K_{01}.

Notations 2.3. For all $k \in \mathbb{N}^*$, we denote by C_i^k $(i \geqslant 0)$ the points of coordinates (x_i^k, y_i^k) that belong to H_k and which satisfy:

$$\lfloor \sqrt{2k(x_i^k+1)+k^2} \rfloor = \lfloor \sqrt{2kx_i^k+k^2} \rfloor + 1,$$

and by $\mathscr{C}$ the family $(C_i^k)_{i \geqslant 1}$.

Fig. 18 gives a graphical representation of the points C_i^2 for $1 \leqslant i \leqslant 8$.

We observe that a C_i^k is not always an hollow (for example C_2^2), however all those situated above the first diagonal are hollows. We have to point out the point of $\mathscr{C}$ the $\vec{O}x$-coordinate of which is the largest integer smaller or equal to $(4k-1)$. It is, in fact, the last hollow of H_k belonging to the heart.

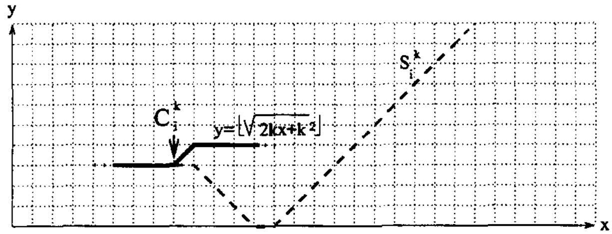


Fig. 19. Graphical representation of S_i^k.

Notations 2.4. We denote by $C_{n(k)}^k$ the point of coordinates $(x_{n(k)}^k, y_{n(k)}^k)$ such that $x_{n(k)}^k \leqslant 4k-1$, and $x_{n(k)+1}^k > 4k-1$.

We prove the following lemma.

Lemma 2.1. *For all* $k \geqslant 1$, *all* $i \geqslant 1$,
1. $y_i^k = k + i - 1$,
2. $x_{n(k)}^k = 4k - 1$,
3. $n(k) = 2k$.

Proof. 1. $y_i^k = k+i-1$. For all $k \geqslant 1$, we verify easily that the point C_1^k is the point $(1,k)$. By definition $y_{i+1}^k = y_i^k + 1$, so $y_i^k = y_1^k + i - 1$, and, as $y_1^k = k$, we obtain $y_i^k = k+i-1$.

2. $x_{n(k)}^k = 4k-1$. For $x = 4k-1$, $y = \lfloor\sqrt{2k(4k-1)+k^2}\rfloor = 3k-1 = \lfloor\sqrt{2k(4k)+k^2}\rfloor - 1$. Actually, $\lfloor\sqrt{9k^2-2k}\rfloor = 3k-1 \Leftrightarrow 3k-1 \leqslant \sqrt{9k^2-2k} < 3k$ and, to suppose $3k - 1 > \sqrt{9k^2-2k}$ or $3k \leqslant \sqrt{9k^2-2k}$ leads to contradictions with $k \geqslant 1$.
So, the point $(4k-1, 3k-1)$ is a point of $\mathscr{C}$ and consequently, it is the point $(x_{n(k)}^k, y_{n(k)}^k)$.

3. $n(k) = 2k$ We have seen that $y_{n(k)}^k = 3k-1$ and proved (point 1) that $y_i^k = k+i-1$. Hence, $y_{n(k)}^k = k + n(k) - 1 = 3k - 1$ gives $n(k) = 2k$. □

We denote by S_i^k the broken line born on C_i^k and defined as follows:

$$S_i^k(x,y) = \begin{cases} x_i^k \leqslant x \leqslant x_i^k + 1, & y = \lfloor\sqrt{2kx_i^k + k^2}\rfloor, \\ x_i^k + 1 \leqslant x \leqslant x_i^k + y_i^k + 1, & y = -x + (x_i^k + y_i^k + 1), \\ x_i^k + y_i^k + 1 \leqslant x \leqslant x_i^k + y_i^k + 2, & y = 0, \\ x \geqslant x_i^k + y_i^k + 2, & y = x - (x_i^k + y_i^k + 2). \end{cases}$$

Fig. 19 gives a graphical representation of S_i^k.

Lemma 2.2. *Let* C_i^k $(i \geqslant 1)$, *belong to* $\mathscr{C}$. *Let* S_i^k *be the broken line born on* C_i^k. *Then the point* C_{i+2k}^k *belongs to* S_i^k.

In fact, we want to prove that all the C_i^k can be obtain from the $2k$ first ones, which are in $(DV_0 \cup K_0)$, with the help of the broken lines S_i^k. Fig. 20 gives the example of the parabolas H_1 and H_2.

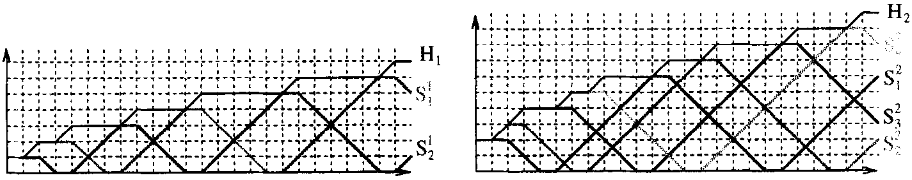


Fig. 20. Broken lines that join the hollows of the parabolas H_1 and H_2.

Proof. Two steps are necessary. The first one consists in finding the expression of the first coordinate of C^k_{i+2k} according to the first coordinate of C^k_i. And in the second part, we verify that the point C^k_{i+2k} belongs to the ascending part of a signal S^k_i.

Let (x, y) be on H_k, then $2kx + k^2 = y^2$. Moreover, $1 + 2 + 3 + \cdots + y = \frac{1}{2}(y^2 + y)$. Using these equalities for $(x^k_i + 1, y^k_i + 1)$, we get

$$2k(x^k_i + 1) + k^2 = 2 + 4 + 6 + \cdots + 2((y^k_i + 1) - 1) + (y^k_i + 1),$$

$$2k(x^k_{i+2k} + 1) + k^2 = 2 + 4 + 6 + \cdots + 2((y_{i+2k} + 1) - 1) + (y_{i+2k} + 1).$$

But, from Lemma 2.1 point 1, $y^k_{i+2k} + 1 = (y^k_i + 1) + 2k$, hence,

$$2k(x^k_{i+2k} + 1) - 2k(x^k_i + 1)$$
$$= (y^k_i + 1) + 2((y^k_i + 1) + 1) + \cdots + 2((y^k_i + 1) + 2k - 1) + (y^k_i + 1) + 2k,$$

$$2k(x^k_{i+2k} - x^k_i) = 2(y^k_i + 1) + 2((y^k_i + 1) + 1) + \cdots + 2((y^k_i + 1) + 2k - 1) + 2k,$$

$$2k(x^k_{i+2k} - x^k_i)$$
$$= 4k(y^k_i + 1) + 2 + 4 + \cdots + 2(2k + 1) + 2k = 4k(y^k_i + 1) + (2k)^2,$$

and, finally, $x^k_{i+2k} = x^k_i + 4k + 2i$. So, $x^k_{i+2k} - (x^k_i + y^k_i + 2) = x^k_i + 4k + 2i - x^k_i - k - i + 1 - 2 = 3k + i - 1 = y^k_i + 2k = y^k_{i+2k}$, which means $C^k_{i+2k} \in S^k_i$. □

2.4.3. *Properties of H_k in K_{01}*

Lemma 2.3. *For all $k \geqslant 1$,*

1. *$H_k \cap K_{01}$ is only composed of landings of length 1 or 2,*
2. *$H_k \cap (K_{01} \cup DH_0)$ is only composed of slopes of length 1.*

Proof. The following well known result will give the proof: let a, b be two reals ($a < b$), and f be a function such that $f : [a, b] \to \mathbb{R}$ which is continuous on $[a, b]$ and differentiable on $]a, b[$. Then

$$(b - a) \inf_{x \in]a,b[} f'(x) \leqslant f(b) - f(a) \leqslant (b - a) \sup_{x \in]a,b[} f'(x) \quad (*)$$

1. Landings of length 1 or 2. We suppose that $H_k \cap K_{01}$ has a landing of length n, $n \geqslant 1$, and we denote respectively a and b the first coordinates of the points which belong to the extremities of this landing. Fig. 21 represents this landing when $n = 3$.

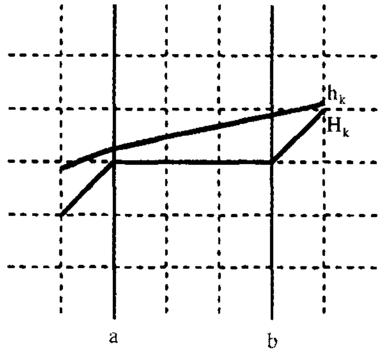


Fig. 21. Landing of length 3, a and b the first coordinates of the extremity points.

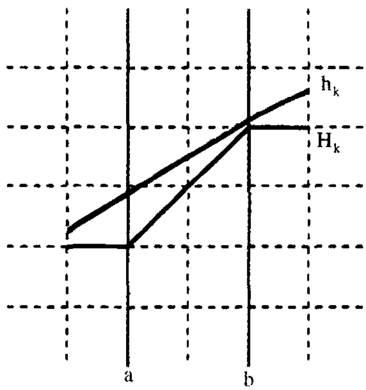


Fig. 22. Slope of length 2, a and b the first coordinates of the extremity points.

Using the inequality $(*)$ with the function h_k, $h_k : x \mapsto \sqrt{2kx+k^2}$, we get

$$(b-a) \inf_{x\in]a,b[} h'_k(x) \leqslant h_k(b) - h_k(a).$$

We have $(b-a)=n$ and $h_k(b)-h_k(a)<1$. But the function $x \mapsto h'_k(x) = k/\sqrt{2kx+k^2}$ is decreasing. So, we consider the point of maximal $\vec{O}x$-coordinate which belongs to $(H_k \cap K_{01})$. It is the point of $\vec{O}x$-coordinate $4k$, and we have $h'_k(4k) = \frac{1}{3}$.

Consequently, $(*)$ implies $n \times \frac{1}{3} \leqslant h_k(b) - h_k(a) < 1$, then $n<3$.

2. Slopes of length 1. We suppose that $H_k \cap (K_{01} \cup DH_0)$ has a slope of length m, $m \geqslant 1$, and we denote respectively by a and b the $\vec{O}x$-coordinates of the points which are at the extremities of this slope. Fig. 22 represents this slope when $m=2$.

   Using $(*)$ once more, we get $f(b)-f(a) \leqslant (b-a)\ sup_{x\in]a,b[} f'(x)$. We have $(b-a)=m$ and $h_k(b)-h_k(a)>m-1$. As the function $x \mapsto h'_k(x)$ is decreasing, we consider the point of minimal $\vec{O}x$-coordinate which belongs to $H_k \cap (K_{01} \cup DH_0)$. This point belongs to the first diagonal and then satisfies $x = \sqrt{2kx+k^2}$.

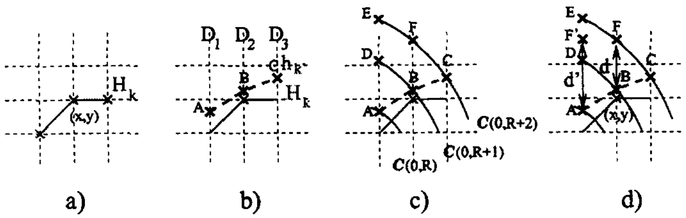


Fig. 23. Lemma 2.4, Fig. 1.

But, $x=\sqrt{2kx+k^2} \Leftrightarrow x^2=2kx+k^2 \Leftrightarrow (x-k)^2=2k^2$; and as $k\geqslant 1, x=\sqrt{2kx+k^2} \Leftrightarrow x-k=k\sqrt{2} \Leftrightarrow x=(1+\sqrt{2})k$.

Moreover,

$$h'_k((1+\sqrt{2})k)=\frac{k}{\sqrt{2k((1+\sqrt{2})k)+k^2}}=\frac{k}{\sqrt{(3+2\sqrt{2})k^2}}$$
$$=\frac{1}{\sqrt{3+2\sqrt{2}}}\simeq 0.41.$$

Consequently, $(*)$ implies $m-1<h_k(b)-h_k(a)\leqslant m\times 0.41$ and, as m is an integer, we get $m\leqslant 1$, hence $m=1$. □

Lemma 2.4. *For all points $(x,y)\in\mathbb{Z}^2$ such that $x\geqslant y\geqslant 1$, if $(x,y)\in H_k\cap K_{01}$ is a peak then $(x,y+1)\in H_{k+1}$.*

Proof. We suppose that (x,y) belongs to H_k and is a peak. By definition, this means that the points $(x-1,y-1)$ and $(x+1,y)$ belong to H_k. Fig. 23(a) localizes these points.

Consequently, the parabola h_k is as it is shown in Fig. 23(b). We respectively call D_1, D_2 and D_3 the straight vertical lines going through the points $(x-1,y-1)$, (x,y) and $(x+1,y)$. We respectively denote by A, B and C the points which are the intersections between the parabola h_k and the straight lines D_1, D_2 and D_3. These points are also the intersection points between three circles denoted by $C(O,R)$, $C(O,R+1)$ and $C(O,R+2)$, and the same lines. Let D, E and F be the following points: $D=C(O,R+1)\cap D_1$, $E=C(0,R+2)\cap D_1$ and $F=C(O,R+2)\cap D_2$. See Fig. 23(c).

Let d be the distance between the points B and F. As the minimal distance between two circles is equal to 1, we have $d\geqslant 1$. Let F' be the point which has the same second coordinate as F but which belongs to the line D_1 and, let d' be the distance between the points A and F'. Then to prove that the point $(x,y+1)$ belongs to the parabola H_{k+1} is equivalent to prove that the distance d' is less or equal to 2. This is our goal now.

We consider the line $\mathscr{D}$ which meets the points $O=(0,0)$ and A. Let P be the point which is the intersection between $\mathscr{D}$ and $C(O,R+2)$ (see Fig. 24(a)). Let $\mathscr{T}$ be the

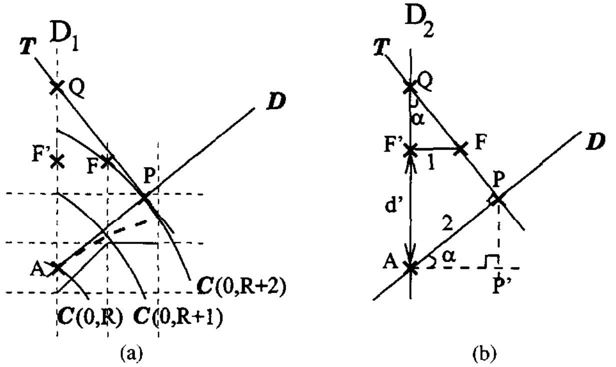


Fig. 24. Lemma 2.4, Fig. 2.

tangent line to the circle $C(O,R+2)$ at P. It cuts the line D_1 at Q. We remark that the distance between the points A and P is equal to 2.

Let P' be the projection of P on the horizontal line going through A. Let α be the angle $(\widehat{\vec{AP'},\vec{AP}})$.We also have: $\alpha=(\widehat{\vec{QA},\vec{QP}})$. See Fig. 24(b).

We remark that the more the angle α decreases, the more the distance between Q and A increases. Hence we consider the smallest angle α possible in K_{01}. This angle is obtained when $\mathscr{D}$ is the line of equation $y=\frac{3}{4}x$. So, we take $\tan\alpha=\frac{3}{4}$.

But, in the triangle APQ, $\sin\alpha=2/AQ$. Hence $AQ=2/\sin\tan^{-1}\frac{3}{4}=\frac{10}{3}$. Moreover, in the triangle $QF'F$, we have $\tan\alpha=1/QF'$. As $\tan\alpha=\frac{3}{4}$, $QF'=\frac{4}{3}$, we obtain $d'=AF'=\frac{10}{3}-\frac{4}{3}=2$. □

Lemma 2.5. *For all points $(x,y)\in\mathbb{Z}^2$ such that $x\geqslant y\geqslant 1$, if $(x,y)\in H_k\cap K_{01}$ is an hollow, then $(x,y+2)\in H_{k+1}$.*

Proof. We suppose that (x,y) belongs to $H_k\cap K_{01}$ and is an hollow. As $H_k\cap K_{01}$ is only composed of slopes of length 1 (Lemma 2.3), the point $(x+1,y+1)$ is a peak. And, as we previously proved it in Lemma 2.4, the point $(x+1,y+2)$ belongs to H_{k+1}. See Fig. 25(a).

As the parabolas H_k are only composed of horizontal and diagonal segments, either the point $(x,y+1)$ or the point $(x,y+2)$ belongs to the parabola H_{k+1}. Suppose that the point $(x,y+1)$ belongs to H_{k+1}. Then, for the same reasons as previously, the points $(x-1,y+1)$ and $(x+2,y+2)$ belong to H_{k+1}. See Fig. 25(b).

Notice that we want to construct $H_{k+1}\cap K_{01}$ from $H_k\cap K_{01}$. Consequently, we suppose that $x>y$ in order to have $(x,y+1)$ which belongs to K_{01}.

The situation of the previous figure implies the existence of two real numbers x_1 and x_2 such that:

$$(y+1)^2=2kx_1+k^2 \quad (1)$$

$$(y+2)^2=2(k+1)x_2+(k+1)^2 \quad (2)$$

with $-1<x_2-x_1<1$. See Fig. 25(c).

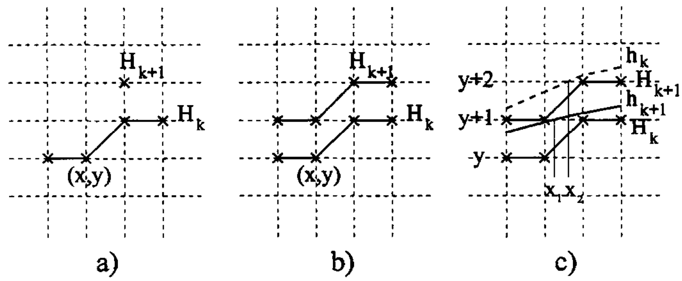


Fig. 25. Lemma 2.5, Fig. 1.

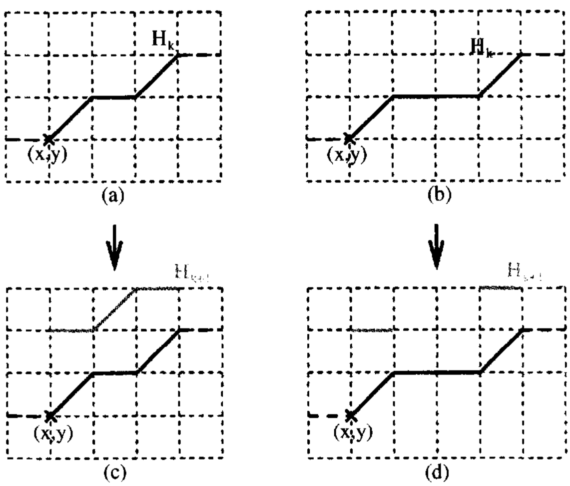
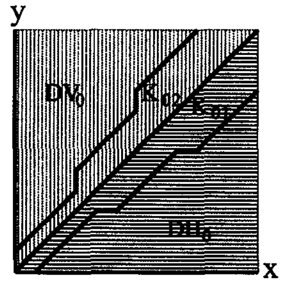


Fig. 26. The two patterns which are in $H_k \cap K_{01}$.

$(2)-(1) \Leftrightarrow y+1 = k(x_2-x_1+1)+x_2 \Leftrightarrow k(x_2-x_1+1)+x_2-y-1=0$ but, $-1<x_2-x_1<1$ hence, $k(x_2-x_1+1)>0$. Moreover, $x_2 \geqslant y+1$ because $x \geqslant y+1$ ((x,y) belongs á K_{01} but not to the first diagonal). Then we obtain a contradiction and, consequently, the point $(x, y+1)$ does not belong to $H_{k+1} \cap K_{01}$. □

As H_k is only composed of landings of length 1 or 2 and, of slopes of length 1 (Lemma 2.3), H_k is only composed of the two patterns that are shown in Fig. 26(a) and (b). We denote by (x,y) the "first" point of each of these patterns and we want to construct the parabola H_{k+1} from the parabola H_k (on the interval on which we suppose H_k to be known). We use Lemmas 2.4 and 2.5 which indicate that if a point (ξ,η) belongs to H_k and is a peak then $(\xi,\eta+1)$ belongs to H_{k+1} and if (ξ,η) is an hollow then $(\xi,\eta+2)$ belongs to H_{k+1}.

In case (a), the points (x,y) and $(x+2,y+1)$ are hollows and consequently, the points $(x,y+2)$ and $(x+2,y+3)$ belong to H_{k+1}. Moreover, the points $(x+1,y+1)$ and $(x+3,y+2)$ are peaks. Hence the points $(x+1,y+2)$ and $(x+3,y+3)$ belong to



$(K_{01} \cap DH_0)$ only have stairs of length 1.

$(K_{02} \cap DV_0)$ only have stairs of length 1.

Fig. 27. Lemma 2.6.

H_{k+1} (see Fig. (c)). In this case, we can construct H_{k+1} on all the interval on which we know H_k.

In the second case (Fig. (b)), the points (x, y) and $(x+3, y+1)$ are hollows. So, the points $(x, y+2)$ and $(x+3, y+3)$ belong to H_{k+1}. And the points $(x+1, y+1)$ and $(x+4, y+2)$ are peaks and, consequently, the points $(x+1, y+2)$ and $(x+4, y+3)$ belong to H_{k+1}. The parabolas H_k are only composed of horizontal and diagonal segments, so either the point $(x+2, y+2)$ or the point $(x+2, y+3)$ belongs to H_{k+1}. But, for the moment, we can not conclude for the position of the parabola H_{k+1} on $(x+2)$. For answering the question, we have to study the intersections of the bundles $\mathscr{H}$ and $\mathscr{V}$.

2.4.4. Intersections between the bundles

Lemma 2.6. *For all k and all k' integers greater or equal to* 1, *the parabolas H_k and V_k' meet on a segment of length less than or equal to $\sqrt{2}$.*

Proof. As the parabolas H_k and V_k' are only composed of horizontal or vertical and diagonal segments, their intersections are either points or diagonal segments.

But in $(K_{01} \cup DH_0)$, according to Lemma 2.3, the slopes in H_k are of length 1 and, symmetrically, in $(K_{02} \cup DV_0)$ the slopes in $V_{k'}$ also are of length 1. See Fig. 27.

Consequently, the intersections between two parabolas H_k and $V_{k'}$ are at most segments of length $\sqrt{2}$. □

Proposition 2.1. *For all points $(x, y) \in \mathbb{Z}^2$ such that $x \geq 1$, $y \geq 1$, we have: for all $k \geq 1$, all $k' \geq 1$,*

1. *if $(x, y) \in H_k \cap V_{k'}$ and $(x+1, y) \in H_k$ then $(x, y+1) \in V_{k'}$,*
2. *if $(x, y) \in H_k \cap V_{k'}$ and $(x+1, y+1) \in H_k$ then $(x+1, y+1) \in V_{k'}$,*
3. *if $(x, y) \in V_{k'}$ and (x, y) does not belong to any parabola H_k then $(x+1, y+1) \in V_{k'}$.*

Proof. 1. Let us suppose that $(x, y) \in H_k \cap V_{k'}$ and $(x+1, y) \in H_k$.

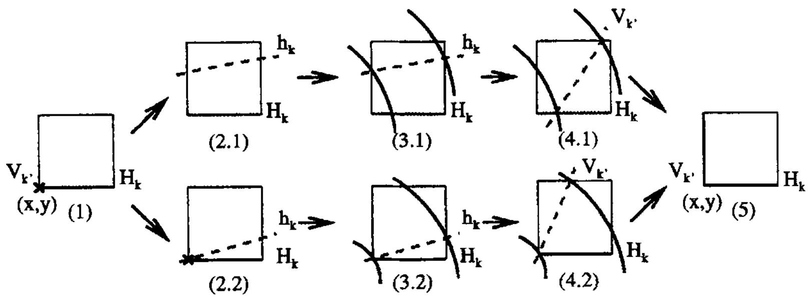


Fig. 28. Proposition 2.1, point 1.

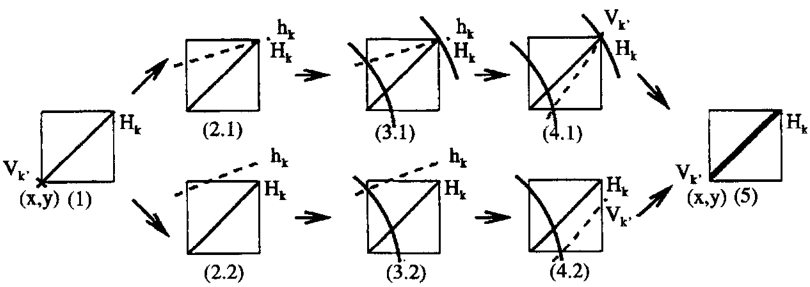


Fig. 29. Proposition 2.1, point 2.

From the definition, two cases are possible: either (x, y) does not belong to h_k (case 2.1 in Fig. 28), or (x, y) belongs to h_k (case 2.2). But, we have seen that the intersections between the parabolas h_k and the vertical lines are also the intersections between the circles and these same lines. Hence we obtain cases (a) and (b) of Fig. 7, presented in 3.1 and 3.2 in Fig. 28.

Symmetrically, the same link exists between the intersections between the parabolas $v_{k'}$ and the horizontal lines and the intersections between the circles and these same lines. So, we obtain Figs. 4.1 and 4.2. Finally when we digitize, we obtain Fig. (5) in all the cases.

2. Let us suppose that $(x, y) \in H_k \cap V_{k'}$ and $(x + 1, y + 1) \in H_k$.

From the definition we have, as previously, two possible cases. These ones are represented in 2.1 and 2.2 in Fig. 29. Then we deduce the position of the parabola $v_{k'}$ (Figs. 4.1 and 4.2). And finally, the respective digitizations lead to Fig. 5.

3. Let us suppose that $(x, y) \in V_{k'}$ and (x, y) does not belong to any parabola H_k.

That (x, y) does not belong to any parabola H_k means that there does not exist any parabola h_k which cuts the segment $[(x, y), (x, y + 1)[$. In fact, there exists 5 possible configurations represented on the first line of Fig. 30. The second line shows the circles that correspond. Notice that we find again cases (g)–(l) of Fig. 7. So, we obtain in the line 3, the position of the parabola $v_{k'}$ in the different cases. And finally, the last line shows that the digitization which is obtained is the same in all the cases. □

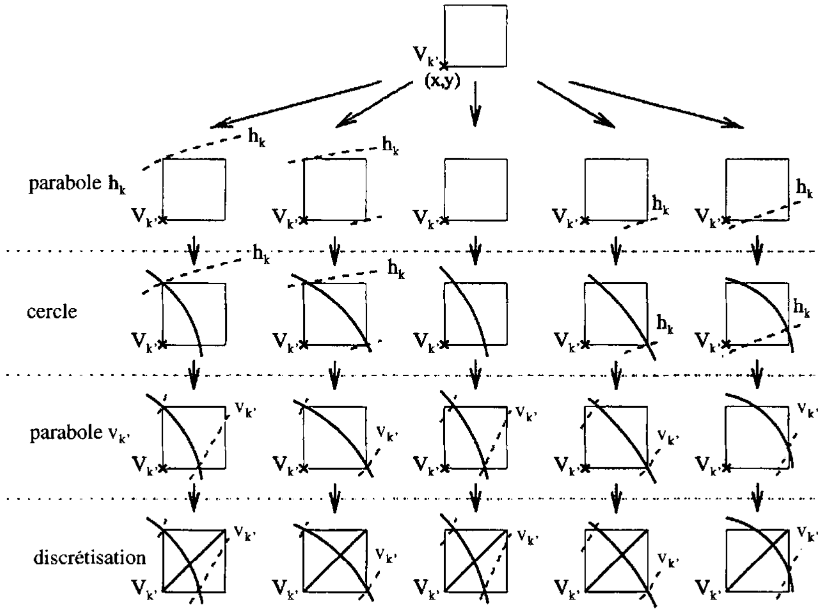


Fig. 30. Proposition 2.1, point 3.

The following paragraph is dedicated to the study of the heart K_0 of Fig. 11.

2.5. *Heart and patterns A*

We have seen that the heart is only composed of three patterns: a square, a chevron and a hexagon, denoted A, B and C, respectively. Moreover, we have remarked that these patterns are regularly displayed.

We are first going to give an analytic expression of the coordinates of the origin of the patterns A. Then, we will prove that if we know the origins of these patterns then we can construct the whole heart.

2.5.1. *Analytic expression of the origins of the patterns A*

Lemma 2.7. *For all points (x,y) such that $x \geqslant y \geqslant 0$, (x,y) is the origin of a pattern A if and only if there exists a positive integer R such that the three circles $C(O,R)$, $C(O,R+1)$ and $C(O,R+2)$ of $\odot$ cut the segment $[(x,y),(x+1,y+2)[$ and are the only ones of $\odot$ to cut it.*

Proof. First we can remark that for all point (x,y) in $\mathbb{Z}^2$, the segment $[(x,y),(x+1,y+2)[$ can be cut by at most three circles. Actually, the distance between two consecutive circles is minimal and equal to 1 in the radial direction and, the length of the segment $[(x,y),(x+1,y+2)]$ is $\sqrt{5}$.

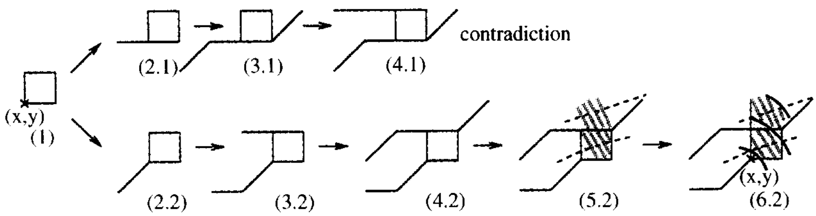


Fig. 31. Lemma 2.7, Fig. 1.

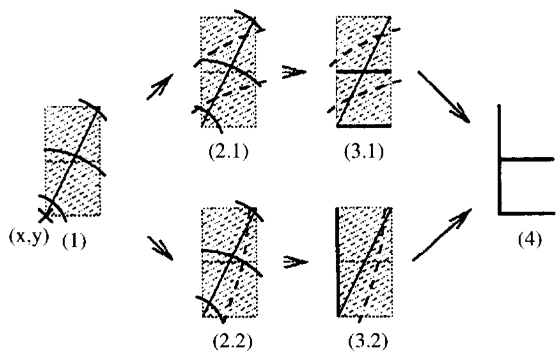


Fig. 32. Lemma 2.7, Fig. 2.

1. Let us suppose that (x, y) is the origin of a pattern A. By definition, saying that (x, y) belongs to H_k and is the origin of a pattern A means that $(x, y) \in H_k \cap V_{k'}$, $(x+1, y) \in H_k \cap V_{k'+1}$, $(x, y+1) \in H_{k+1} \cap V_{k'}$ and $(x+1, y+1) \in H_{k+1} \cap V_{k'+1}$. See Fig. 31(1). From this remark, as the parabolas H_k are only composed of horizontal and diagonal segments, two cases are possible: either $(x-1, y)$ belongs to H_k (Fig. 2.1), or $(x-1, y-1)$ belongs to H_k (Fig. 2.2). In the first case, as the length of the landings is at most 2, the point $(x-1, y)$ is a peak and $(x-2, y-1)$ is an hollow (Fig. 3.1). Then, from Lemmas 2.4 and 2.5, H_{k+1} has a landing of length 3; this is in contradiction with Lemma 2.3.
   In the second case, the point $(x-1, y-1)$ is an hollow because the length of the slopes is at most 1 and hence, the point $(x-1, y+1)$ belongs to H_{k+1} (Fig. 3.2). As the length of the landing is 2, the points $(x-2, y)$ and $(x+2, y+2)$ belong to H_{k+1} (Fig. 4.2). The parabolas h_k and h_{k+1} are as it is shown in Fig. 5.2. And, when we consider the circles, we obtain $C(O, R)$, $C(O, R+1)$ and $C(O, R+2)$ which cut the segment $[(x, y), (x+1, y+2)[$.
2. Let us suppose that the segment $[(x, y), (x+1, y+2)[$ is cut by the three circles $C(O, R)$, $C(O, R+1)$ and $C(O, R+2)$, and these circles are the only ones to cut it. Fig. 32(1) shows the three circles. As we have seen it in the introduction of this section, the intersections between these circles and the vertical lines are also the intersections of h_k with these same lines (Fig. 2.1), and symmetrically for the parabolas $v_{k'}$ (Fig. 2.2). Figs. 3.1 and 3.2 indicate the respective discrete

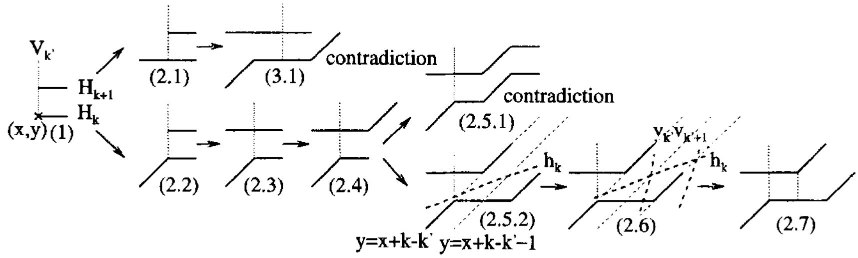


Fig. 33. Lemma 2.7, Fig. 3.

parabolas. Finally, we have: $(x,y) \in H_k \cap V_{k'}$, $(x+1,y) \in H_k$, $(x,y+1) \in H_{k+1} \cap V_{k'}$ and $(x,y+2) \in V_{k'}$ (Fig. 4).
Two cases are possible: either the point $(x-1,y)$ belongs to H_k, or the point $(x-1,y-1)$ belongs to H_k.
In the first case (Fig. 33, 2.1), as the length of the landings are at most 2 (Lemma 2.3), the point $(x-1,y)$ is a peak and $(x-2,y-1)$ is an hollow. Then, from Lemmas 2.4 and 2.5, the points $(x-1,y+1)$ and $(x,y+1)$ belong to H_{k+1} (Fig. 3.1). Consequently, the parabola H_{k+1} has a landing of length 3; this is in contradiction with Lemma 2.3.
In the second case, the point (x,y) is a peak and $(x-1,y-1)$ is an hollow (Fig. 2.2). So, from Lemmas 2.4 and 2.5, the point $(x-1,y+1)$ belongs to H_{k+1} (Fig. 2.3). As the length of the landings is at least 2, the point $(x+2,y+2)$ belongs to H_{k+1} (Fig. 2.4). In this situation, the point $(x+2,y+1)$ cannot belong to the parabola h_k (Fig. 2.5.1). We have proved it in the proof of Lemma 2.5. Hence, $(x+2,y)$ belongs to H_k. Moreover, we verify easily that the parabolas h_k and $v_{k'}$ meet on the points $(k' \pm \sqrt{2kk'}, k \pm \sqrt{2kk'})$. This means that the intersection points are in a line of equation $y=x+k-k'$. In our case, the point (x,y) belongs to the intersection between the parabolas H_k and $V_{k'}$. Hence the line of slope 1 going through (x,y) has $y=x+k-k'$ as equation. The parallel line of this last one going through $(x+1,y)$ has equation: $y=x+k-k'-1$ (Fig. 2.5.2). The intersection between h_k and $v_{k'+1}$ belongs to this last one. But, h_k cuts the line of equation $y=x+k-k'-1$ in the square of length side 1 and which has $(x+1,y)$ as point of minimal first and second coordinates (Fig. 2.6). The point is the intersection point between h_k and $v_{k'+1}$. Hence we obtain the form of $v_{k'+1}$. We digitize and we obtain that $(x+1,y)$ and $(x+1,y+1)$ belong to H_{k+1}; this completes the pattern A. □

Let us make the link between the different kinds of intersection which are represented in Fig. 7 and the patterns of Fig. 5. Two cases are possible: either the circle $C(O,R)$ goes through the point (x,y); then we have an intersection of type a) (Fig. 7) and the square which is higher is of type b). Or the circle $C(O,R)$ cuts the segments $](x,y),(x+1,y)[$ and $](x,y),(x,y+1)[$ and the intersection if of type b); and the square which is higher is always of type b). The first case is not visible on the la

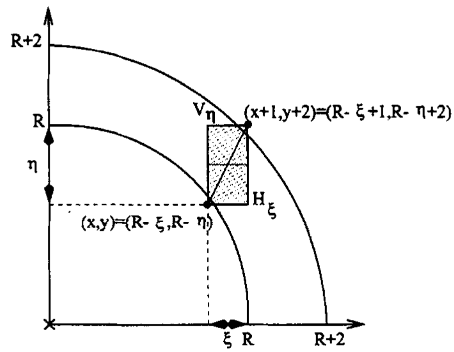


Fig. 34. Proposition 2.2, Fig. 1.

Fig. 5. On the contrary, the second one correspond to the superposition of two squares that have the same color and we can easily see them. Notice that they only appear in the central part of the figure.

The following proposition computes explicitly the coordinates of the origins of patterns A in the first octant. It also shows that they are in K_{01}.

Proposition 2.2. *For all (x,y) with $x \geqslant y \geqslant 0$, the point (x,y) is the origin of a pattern A if and only if there exists two integers k and i ($k \geqslant 1$, $i \geqslant 0$ and $i \geqslant \lfloor k(1+\sqrt{2}) \rfloor$) such that*

$$x = 2k + 4i + \lfloor \sqrt{2ki + k^2} \rfloor$$

$$y = 2k + 3i + 2\lfloor \sqrt{2ki + k^2} \rfloor$$

Proof. In a first part, we exhibit two functions f and g, from $\mathbb{R}$ into itself, such that a point (x,y) is the origin of a pattern A if and only if there exists an integer r such that $f(r) \leqslant 0$ and $g(r) > 0$. Then, we study these functions. Finally, we give a geometrical interpretation of them.

1. By definition, (x,y) is the origin of a pattern A if and only if there exists two integers η and ξ such that

   $(x,y) \in H_\xi \cap V_\eta$, $(x+1,y) \in H_\xi \cap V_{\eta+1}$, $(x,y+1) \in H_{\xi+1} \cap V_\eta$ and $(x+1,y+1) \in H_{\xi+1} \cap V_{\eta+1}$.

   Moreover, from Lemma 2.7, (x,y) is the origin of a pattern A if and only if there exists a positive integer R such that the three circles $C(O,R)$, $C(O,R+1)$ and $C(O,R+2)$ cut the segment $[(x,y),(x+1,y+2)[$ and are the only ones to cut it. Consequently, we have: $x = R - \xi$ and $y = R - \eta$. See Fig. 34.

   Then we have the following equivalence:

   (x,y) is an origin of pattern A if and only if there exists an integer R ($R \geqslant 1$) such that:

   $$(R-\xi)^2 + (R-\eta)^2 \leqslant R^2 \quad (3)$$

and

$$(R+2)^2 < (R-\xi+1)^2 + (R-\eta+2)^2 \quad (4)$$

We put $f(R)=(R-\xi)^2+(R-\eta)^2-R^2$ and $g(R)=(R-\xi+1)^2+(R-\eta+2)^2-(R+2)^2$.

2. Study of the functions f and g

First, we can write again the expressions of f and g as follows:
$f(R)=(R-\xi)^2+(R-\eta)^2-R^2=R^2-2(\xi+\eta)R+(\xi+\eta)^2-2\xi\eta$, that is to say, $f(R)=[R-(\xi+\eta)^2]^2-2\xi\eta$. And, by similar computations, $g(R)=[R-(\xi+\eta-1)^2]^2-2(\xi+1)\eta$.

The derivative of the function $f: R \mapsto [R-(\xi+\eta)^2]^2-2\xi\eta$ is $f'(R)=2R-2(\xi+\eta)$. This one is equal to zero for $R=\xi+\eta$. And, $f(\xi+\eta)=-2\xi\eta$.

Moreover, the points that belong to the intersection between the representative curve of f and the axis of first coordinates are the points $(-\sqrt{2\xi\eta}+\xi+\eta, 0)$ and $(\sqrt{2\xi\eta}+\xi+\eta, 0)$. Notice that $-\sqrt{2\xi\eta}+\xi+\eta \geqslant 0$ because $\xi \geqslant 1$ and $\eta \geqslant 1$.

Similarly, the derivative of the function $g: R \mapsto [R-(\xi+\eta-1)^2]^2-2(\xi+1)\eta$ is $g'(R)=2R-2(\xi+\eta-1)$. It is equal to zero for $x=\xi+\eta-1$ and $g(\xi+\eta-1)=-2(\xi+1)\eta$.

The points which are the intersections between the representative curve of g and the axis of first coordinates are the points $(-\sqrt{2(\xi+1)\eta}+\xi+\eta-1, 0)$ and $(\sqrt{2(\xi+1)\eta}+\xi+\eta-1, 0)$. As previously, $-\sqrt{2(\xi+1)\eta}+\xi+\eta-1 \geqslant 0$ and, $-\sqrt{2(\xi+1)\eta}+\xi+\eta-1 < -\sqrt{2\xi\eta}+\xi+\eta$.

From point (1), (x, y) is the origin of a pattern A if and only if there exists an integer R $(R \geqslant 1)$ such that $f(R) \leqslant 0$ and $g(R) > 0$.

Consequently, (x, y) is the origin of a pattern A if and only if there exists an integer R $(R \geqslant 1)$ such that $\sqrt{2(\xi+1)\eta}+\xi+\eta-1 < R \leqslant \sqrt{2\xi\eta}+\xi+\eta$.

Fig. 35 gives a graphical representation of the representative curves of f and g, and of the integer R.

We put $h(R)=[R-(\xi+\eta+1)]^2-2(\xi+1)\eta$.

As h is the symmetrical parabola to the parabola g according to the line of equation $x=\xi+\eta$, we have:

(x, y) is the origin of a pattern A if and only if there exists an integer R' $(R' \geqslant 1)$ such that $f(R') \leqslant 0$ and $h(R') > 0$.

Fig. 36 gives an illustration of this equivalence.

But,

$$f(R') \leqslant 0 \Leftrightarrow (R'-\xi)^2+(R'-\eta)^2-R'^2 \leqslant 0$$

$$f(R') \leqslant 0 \Leftrightarrow R'^2 \geqslant (R'-\xi)^2+(R'-\eta)^2$$

and,

$$g(R') > 0 \Leftrightarrow [(R'-(\xi+\eta+1)]^2-2(\xi+1)\eta > 0$$

$$g(R') > 0 \Leftrightarrow [R'-(\xi+1)]^2+(r'-\eta)^2-R'^2 > 0$$

$$g(R') > 0 \Leftrightarrow R'^2 < [R'-(\xi+1)]^2+(R'-\eta)^2$$

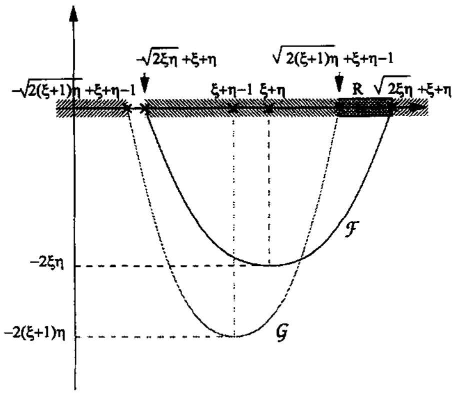


Fig. 35. Proposition 2.2, Fig. 2.

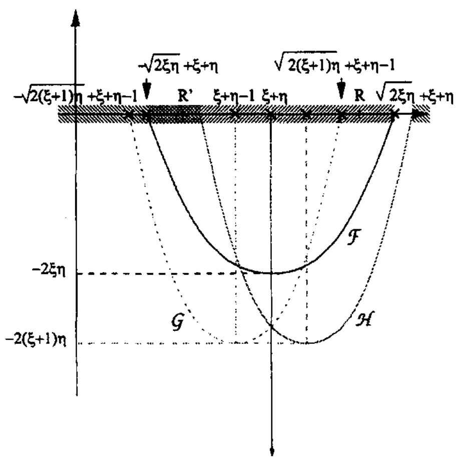


Fig. 36. Proposition 2.2, Fig. 3.

Consequently, we have the following equivalence:
(x,y) is the origin of a pattern A if and only if there exists an integer R' $(R' \geqslant 1)$ such that $(R'-\xi)^2+(R'-\eta)^2 \leqslant R'^2 < [R'-(\xi+1)]^2+(R'-\eta)^2$.

3. Geometrical interpretation

We consider the circle centered on (R',R') and of radius R'. The previous equivalence means that the distance between the center of the circle and the point of coordinates (ξ,η) is less or equal to the radius of the circle, which is itself strictly less that the distance between the center of the circle and the point $(\xi+1,\eta)$. So, we obtain Fig. 37(a).

We put $R'=k+i$ and $i=\eta-R'$; k and i are integers (see Fig. 37(b). From the Pythagore's theorem, the distance between the point (R',R') and the point which is the intersection between the circle and the segment $[(\xi,\eta),(\xi+1,\eta)]$ is equal to $\sqrt{2ki+k^2}$. Consequently, $\xi=k+i+\lfloor\sqrt{2ki+k^2}\rfloor$ and $\eta=k+2i$.

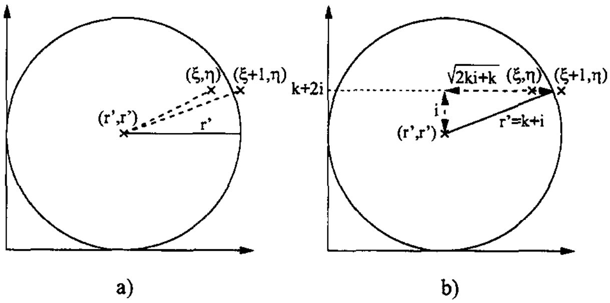


Fig. 37. Proposition 2.2, Fig. 4.

So, (x, y) is the origin of a pattern A if and only if there exists two integers k and i such that:

$$\xi = k + i + \lfloor\sqrt{2ki + k^2}\rfloor \quad \text{and} \quad \eta = k + 2i.$$

The point $(R', 0)$ has been previously defined as the symmetrical point of $(R, 0)$ according to the line of equation $x = \xi + \eta$. As $R' = k + i$, we have: $R = k + i + 2(\xi + \eta - k - i)$. That is to say: $R = 2\xi + 2\eta - k - i$.
If we replace ξ and η by their respective values, we obtain:

$$R = 2\xi + 2\eta - k - i$$
$$R = 2k + 2i + 2\lfloor\sqrt{2ki + k^2}\rfloor + 2k + 4i - k - i$$
$$R = 3k + 5i + 2\lfloor\sqrt{2ki + k^2}\rfloor.$$

But, $x = R - \xi$ and $y = R - \eta$. Hence,

$$x = 2k + 4i + \lfloor\sqrt{2ki + k^2}\rfloor \quad \text{and} \quad y = 2k + 3i + 2\lfloor\sqrt{2ki + k^2}\rfloor. \qquad \square$$

2.5.2. Reconstitution of the heart from the patterns A

In this paragraph, we explain how the heart of the figure can be constructed when the parabola H_1 in K_{01}, the points of the first diagonal belonging to H_k and the points origins of patterns A are known. Fig. 38 shows these elements.

We have seen that, using Lemmas 2.3–2.5, we can obtain the parabola H_{k+1}, except some points, from the parabola H_k. As a matter of fact, in the case where the point (x, y) is neither an hollow nor a peak, we have seen that we can not decide whether the point $(x, y + 1)$ or the point $(x, y + 2)$ belongs to H_{k+1}. But we remark that if we know the points which are origins of patterns A, we can solve this problem. Actually, the point $(x, y + 2)$ belongs to H_{k+1} (Fig. 40(a)) if and only if, from Proposition 2.1, the point $(x - 1, y)$ is the origin of a pattern B (Fig. c)). And, $(x, y + 1)$ belongs to H_{k+1} (Fig. (b)) if and only if the point $(x - 1, y)$ is the origin of a pattern A.

Let use consider the parabola H_2 for example. Let (x, y) be the point of H_2 on the first diagonal. As the point $(x, y - 1)$ is the origin of a pattern A, the point $(x + 1, y)$ belongs to H_2. As the length of the landings are at most 2, the point $(x+2, y+1)$ belongs

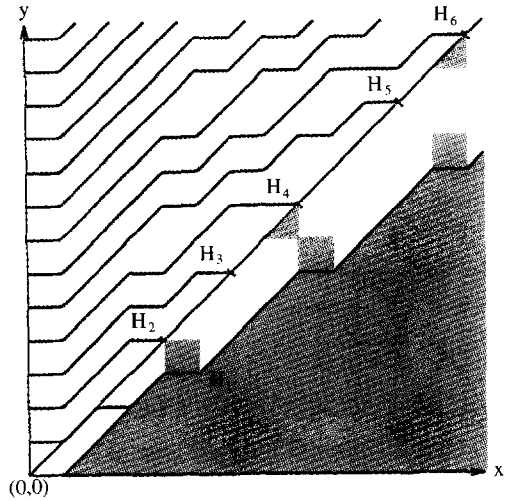


Fig. 38. Reconstitution of the heart from the patterns A, elements we suppose to know.

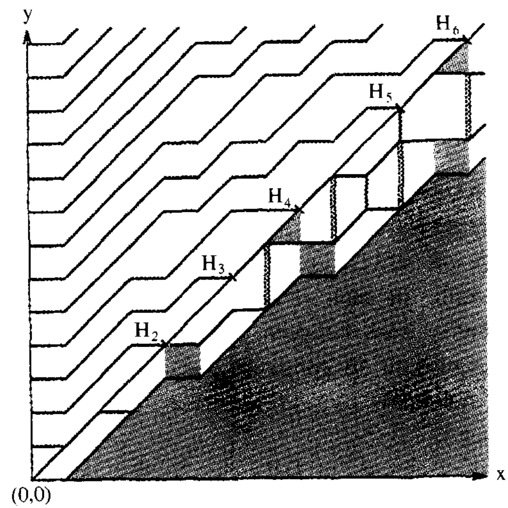


Fig. 39. Reconstitution of the heart from the patterns A.

to H_2. And, as the length of the slopes are at most 1, the point $(x+3, y+1)$ belongs to H_2 $(x+4, y+2)$ belongs to H_2 because this point is an origin of pattern A and consequently, $(x+5, y+2)$ also belongs to H_2. Fig. 39 shows the construction of the first parabolas.

We have proved some properties of the bundles $\mathscr{H}$ and $\mathscr{V}$. In the following section, we will prove that these properties are sufficient to define a cellular automaton which will construct these bundles. We will translate the proved lemmas in terms of cellular automata, and, in particular, explain how the broken lines introduced in Section 2.4.2 are transformed into signals allowing to build the wanted cellular automaton.

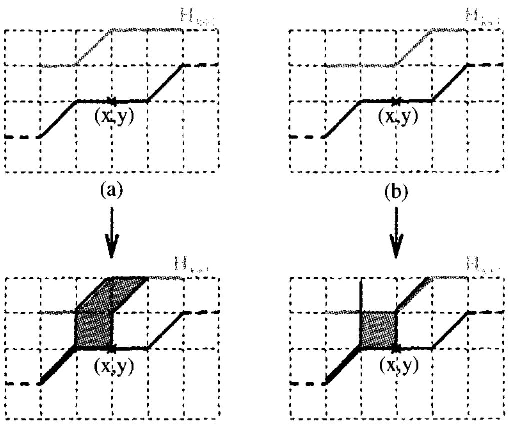


Fig. 40. How we obtain H_{k+1} from H_k.

3. Construction of the floor parabolas with a cellular automaton

In the first part we will explain the strategy of construction and bring to light the difficult points. In the second one, we will describe the automaton.

3.1. Strategy of construction

In the previous section, we have distinguished different parts in the bundles $\mathscr{H}$ and $\mathscr{V}$ (DV_0, DH_0 and the heart K_0 which is itself composed of K_{01} and K_{02}) and presented some of their respective properties. Let us say here and now that the construction of the bundles will be specific for each of these parts. We will also notice here that, as we strongly use the intersections between the bundles $\mathscr{H}$ and $\mathscr{V}$ to construct their elements, we do not know how to construct one of these bundles without the other one. Moreover, we do not know how to construct only one parabola H_k (except H_1 which is a particular case).

We have seen that if we know, in DH_0, the $2k$ first points of H_k, we can locate the $2k$ following ones (see Section 2.4.4). But, the $2k$ first points of H_k belong to $DV_0 \cup K_0$. So, the parabolas in DH_0 are constructed with the help of the ones in DV_0 and K_0. The difficulty here will be to make sure that the signals issued from the broken lines we previously put to light can be constructed by cellular automata.

The automaton we are looking for will construct simultaneously the parabolas H_k and V_k, time after time. At time t, only some of the parabolas $V_{k'}$ with $k' < k$ will be used to construct H_k. As the parabolas H_k and V_k meet on the first diagonal, the wanted algorithm will be available in $(DV_0 \cup K_{02})$ for the parabolas H_k and in $(DH_0 \cup K_{01})$ for the parabolas $V_{k'}$.

In K_{01}, the parabola H_k is constructed with the help of the parabola H_{k-1} and of the origins of patterns A (see Section 2.5.1). We will prove that H_{k+1} can be obtained by cellular automaton, with the help of signals deduced from the $H_k \cap (K_{01} \cup DH_0)$ part of the parabola H_k. So, the construction of H_k uses information in K_{01} and in DH_0. And we have symmetric statements for the parabolas $V_{k'}$.

Actually the construction will be achieved by recurrence. At time t, the $(t-k)$ first points of the parabolas H_k and V_k are known. We denote these known pieces of parabolas as follows:

$$H_{k,t} = \{(x,y) \in H_k; x \in \mathbb{N}, y \in \mathbb{N} \text{ and } x \leqslant t-k\},$$

$$V_{k,t} = \{(x,y) \in V_k; x \in \mathbb{N}, y \in \mathbb{N} \text{ and } x \leqslant t-k\}.$$

And we put

$$\mathscr{H}_t = \bigcup_{k\in\mathbb{N}} H_{k,t} \quad \text{and} \quad \mathscr{V}_t = \bigcup_{k\in\mathbb{N}} V_{k,t}$$

which represent the bundles $\mathscr{H}$ and $\mathscr{V}$ at time t.

As already seen in Section 2.5.2, constructing the heart needs some knowledge on the patterns A positions. During the cellular construction, this knowledge will be conveyed by lines (see Section 3.2.3), corresponding to the following sets, for $t \geqslant k+i$, $i \geqslant 1$,

$$DH_{i,k,t} = \{(x,y) \in \mathbb{N}^2;\ x = i + \theta \quad \text{and}$$

$$y = \lfloor \sqrt{2ki+k^2} \rfloor + \theta \text{ with } \theta = -(k+i)\},$$

$$DV_{i,k,t} = \{(x,y) \in \mathbb{N}^2;\ x = \lfloor \sqrt{2ki+k^2} \rfloor + \theta \quad \text{and}$$

$$y = i + \theta \text{ with } \theta = t-(k+i)\}.$$

These sets are initialized or extended at each time (see Section 3.2.3).

Clearly, the limits of the heart are also constructed at each time.

At the beginning of the recurrence step, $\mathscr{H}_t$, $\mathscr{V}_t$ and the set of auxiliary signals which are necessary on the ball $\mathscr{B}(0,t)$ are known. Then we proceed as follows:

1. from $\mathscr{H}_t \cap (K_0 \cup DV_0)$, we deduce $\mathscr{H}_{t+1} \cap DH_0$,
   from $\mathscr{V}_t \cap (K_0 \cup DH_0)$, we deduce $\mathscr{V}_{t+1} \cap DV_0$.
   See Section 3.2.1.
2. from $\mathscr{V}_t \cap (DV_0 \cup K_{02})$, we deduce $\mathscr{H}_{t+1} \cap (DV_0 \cup K_{02})$,
   from $\mathscr{H}_t \cap (DH_0 \cup K_{01})$, we deduce $\mathscr{V}_{t+1} \cap (DH_0 \cup K_{01})$.
   See Section 3.2.2.
3. from $\mathscr{H}_t \cap K_{01}$ we deduce $\mathscr{H}_{t+1} \cap K_{01}$,
   from $\mathscr{V}_t \cap K_{02}$ we deduce $\mathscr{V}_{t+1} \cap K_{02}$.
   See Section 3.2.4.
4. Then the auxiliary signals are known on $\mathscr{B}(0, t+1)$.

3.2. Construction of the automaton

We will first successively describe the various algorithms used to construct the parabolas in the different parts, then globally study the automaton.

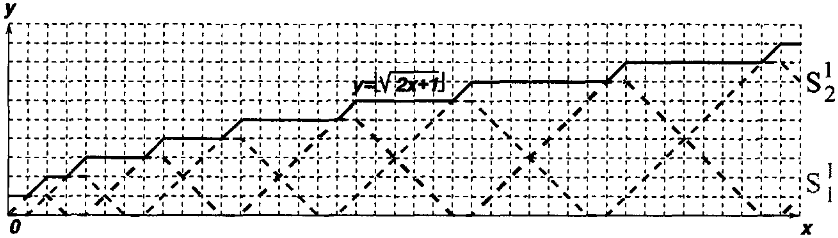


Fig. 41. Projected geometrical diagram without time of the automaton which constructs the parabola H_1.

3.2.1. The "long" landings

We call "long landings" the landings of the parabolas H_k which belong to DH_0. The first paragraph deals with the case of the parabola H_1. It is a particular case of the general construction, where the parabola H_k will be obtained from a part $H_k \cap (K_0 \cup DV_0)$ of itself. Finally, we explain how we can construct all the parabolas H_k from $\bigcup_{k\geqslant 0}(H_k \cap (DV_0 \cup K_0))$.

Lemma 3.1. *There exists a 2D-CA which constructs the parabola H_1 from the initial configuration C_0.*

Proof.
- From Section 2.4.2, two signals S (the traces of which are the broken lines S_1^1 and S_2^1) generated by the cell (0,0) and one signal T, the trace of which actually is H_1, are necessary to construct the parabola H_1.
- From a point C_i^1, each signal S are propagated one cell on the right, then in diagonal until it reaches $\vec{O}x$, then one cell on the right and finally, in diagonal until it crosses the signal T. From Lemma 2.2, the signals meet on a cell which is a point C_i^1. The corresponding cell then propagates adequately both signals. From a point C_i^1, the signal T is propagated in diagonal for one unit of time, then horizontally up to a new meeting with a signal S.

  The initial conditions are to be specified: at time 0, T is propagated from cell (0,0) one unit of time along $\vec{O}y$, at time 1, T is propagated horizontally from cell (0,1), while S_1^1 is propagated diagonally and S_2^1 horizontally from cell (0,0). □

Fig. 41 shows the projected geometrical diagram, without the times marked, of this automaton. Fig. 42 gives a graphical representation of the three-dimensional space-time diagram.

More generally, we have:

Lemma 3.2. *For all $k \geqslant 1$, there exists a 2D-CA which constructs H_k from $H_k \cap (K_0 \cup DV_0)$.*

Proof. From Lemma 2.2, we can construct the point C_{i+2k}^k from the point C_i^k with $i \geqslant 0$ with the help of signals S. As we suppose $H_k \cap (DV_0 \cup K_0)$ to be known, that is to say that the points C_i^k for $1 \leqslant i \leqslant 2k$ are given, we will be able to use this lemma.

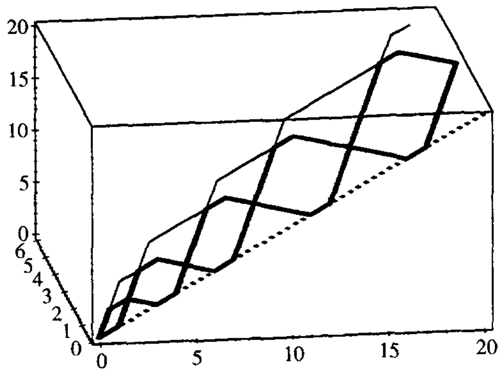


Fig. 42. Three-dimensional space-time diagram of the celllular automaton which constructs the parabola H_1.

Then the principle of this automaton will be the following one: [3]

Initially, all the cells are in a quiescent state (denoted by q) except: the cell $(0,0)$, which is in state q_1, and the cells which belong to $H_k \cap (DV_0 \cup K_0)$, which are in state q_2. We notice that a cell c which is in state q_2 "knows" whether it is a point of $\mathscr{C}$ or not. Actually, for such a cell, its neighbor cells $B_3(c)$ and $B_8(c)$ are also in the state q_2 while the $B_i(c)$, $i \notin \{3,8\}$, are in state q. For the other cells c', the only neighbor cells in state q_2 are the cells $B_3(c')$ and $B_7(c')$ while the other ones are in the quiescent state q.

- In a first time, the cell $(0,0)$ initializes a signal which is propagated in the direction $\vec{Oy}$. This signal reaches the cell $(0,k)$, which is the first cell it meets that belongs to the parabola H_k. This cell is marked because it is in the state q_2. Here the second phase begins. Notice that $\vec{Ox}$ is also marked by a signal.
- The state q_1 is not propagated any more. But, the cell $(0,k)$ initializes a signal which propagates a new signal denoted by S_H. At a given time, if this signal reaches a cell c then, at the next time, it will reach the cell $B_i(c)$ ($i=7$ or 8) such that $B_i(c)$ is in the state q_2. Moreover, if the cell $B_i(c)$ is a point C_i^k then the cell $B_i(c)$ propagates the signal but also initializes a signal S. The last one evolves as follows:
  - We suppose that the signal S is initialized at time t by the cell c,
  - Step 1: the signal S is propagated to the cell $B_7(c)$,
  - Step 2: the signal goes from the cell $B_7(c)$ to the cell $B_6(B_7(c))$, reaches the cell $B_6(B_6(B_7(c)))$, then the cell $B_6(B_6(B_6(B_7(c))))$ etc. until it reaches a cell, denoted by $(i,0)$, that belongs to $\vec{Ox}$ (itself marked by a signal initialized by $(0,0)$),
  - Step 3: then the signal S is propagated from the cell $(i,0)$ to the cell $(i+1,0)$,
  - Step 4: S goes from $(i+1,0)$ to $B_8(i+1,0)$, then to $B_8(B_8(i+1,0))$ etc.

When the signal S_H reaches the last cell which is in the state q_1, that is to say the last cell of $H_k \cap (DV_0 \cup K_0)$, the third phase begins.

---

[3] Moore's neighborhood:

B_2	B_1	B_8
B_3	B_0	B_7
B_4	B_5	B_6

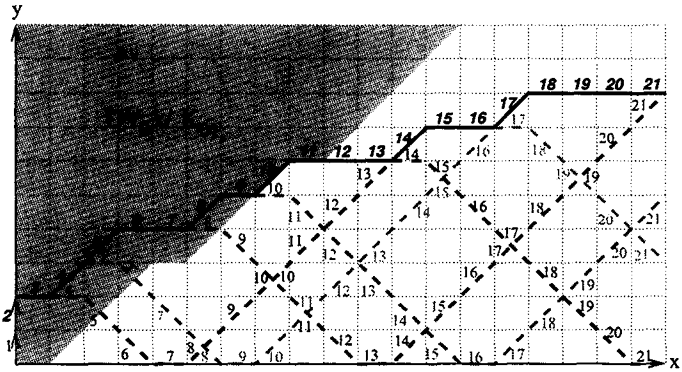


Fig. 43. Projected geometrical diagram of the celllular automaton which constructs H_2.

- If at a given time, only the signal S_H reaches a cell c then, at time $t+1$, the signal is propagated to the cell $B_7(c)$. On the contrary, if at time t a cell c is simultaneously reached by the signal S_H and by a signal S then, at time $t+1$, the signal S_H is propagated to the cell $B_8(c)$ and the cell c initializes a signal S.

We easily verify that the cells that are simultaneously reached by the two signals are the cells C_i^k because the traces of signals S are exactly the broken lines S_i^k of Lemma 2.2. Hence, the parabola H_k is the trace of the signal S_H we have constructed. □

Fig. 43 gives a graphical representation of the construction of the parabola H_2 from $H_2 \cap (K_0 \cup DV_0)$.

Proposition 3.1. *There exists a 2D-CA which constructs the parabolas H_k for $k \geqslant 1$ from $\bigcup_{k\geqslant 0}(H_k \cap (DV_0 \cup K_0))$.*

Proof. The principle of this automaton is almost the same as the one of the previous automaton. The difference is that each cell of $\vec{O}y$ reached by q_1, initializes a signal S_H but, at different times. Actually the cell $(0,k)$ initializes a signal S_H at time k, then all the signals S_H can be represented by the same state. Furthermore, we notice that all the signals are propagated in real time. So, the signals that control a given parabola do not mix with the signals that control an other one. In fact, when a signal S meets, at (x,y), the trace of a signal S_H it does not control, the signal S_H has already reached the cell of $\vec{O}x$-coordinate $(x+1)$. Hence the signal S "knows" that it is not the signal S_H it controls. Consequently, all the signals S can be constructed with the same states.

Hence, all these signals can be installed by means of a finite number of states (see Table 1). □

Fig. 44 shows the simultaneous construction of the two first parabolas. We remark that, as the interval of time is equal to 1, the number of states is finite.

Table 1
States which are used for the signals S

State	Meaning
e_1	initialization of the signals that control H_1
e_2	horizontal floor landing
e_3	ascendant phase
e_4	horizontal ceiling landing
e_5	descendant phase
e_6	meeting

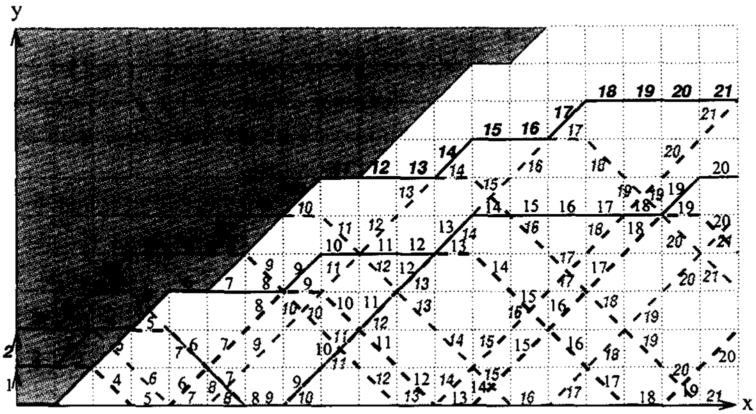


Fig. 44. Projected geometrical diagram of the cellular automaton which construct the parabolas H_1 and H_2.

Corollary 3.2. *For all $k \geqslant 1$, for all $n \geqslant 0$, the automaton which generates the parabolas H_k from $\bigcup_{k \geqslant 0}(H_k \cap (DV_0 \cup K_0))$, marks the point $(n, \lfloor\sqrt{2kn+k^2}\rfloor)$ at time $(n+k)$.*

Proof. From the previous proposition, the initialization of the parabola H_k is performed at time k. That is to say that the point $(0,k)$ is marked with a special state at time k. Then, the signal is propagated of one cell at each time. So, it reaches the cell $(n, \lfloor\sqrt{2kn+k^2}\rfloor)$ at time $k+n$. □

Corollary 3.3. *For all $t \geqslant 0$, the bundle $\mathscr{H}_{t+1} \cap DH_0$ can be constructed by cellular automaton from $\mathscr{H}_t \cap (DV_0 \cup K_0)$ or $\mathscr{H}_t \cap DH_0$ and the bundle $\mathscr{V}_{t+1} \cap DV_0$ from $\mathscr{V}_t \cap (DH_0 \cup K_0)$ or $\mathscr{V}_t \cap DV_0$.*

3.2.2. Construction of the orthogonal bundle

In Section 2.4.3, we proved that the respective positions of two parabolas H_k and $V_{k'}$ in a square on the grid are linked (cf. Proposition 2.1). From this result we deduce an automaton that constructs a parabola $V_{k'}$ aid of the bundle $\mathscr{H}$. Then, we generalize the algorithm to the construction of the bundle $\mathscr{V}$ from the bundle $\mathscr{H}$.

Lemma 3.3. *There exists a 2D-CA which constructs the parabolas $V_{k'}$ $(k' \geqslant 1)$ (respectively H_k $(k \geqslant 1)$) from the bundle of parabolas $\mathscr{H}$ (respectively $\mathscr{V}$).*

Proof. We suppose known the parabolas H_k for $k \geqslant 1$. That means that at initial time, all the cells of the quarter of the plane are in a quiescent state denoted by q, except the cell $(0,0)$ and the cells that belong to the parabolas H_k. These last ones are, for example, in state q_H.

Let S be the signal constructed as follows:

- we suppose that at time 0, the cell $(k',0)$ $(k' \geqslant 1)$ is in a special state denoted by $q_{S_{init}}$,
- The evolution is determined by the following rules: for all $t>0$, all $c \in \mathbb{Z}^2$,
  1. if $state(B_5(c),t) = q_{S_{init}}$ then $state(c,t+1) = q_S$.
  2. if $state(B_4(c),t) = q_S$ and $state(B_5(c),t) \neq q_H$ then $state(c,t+1) = q_S$.
  3. if $state(B_5(c),t) = q_S$ and $state(B_6(c),t) = q_H$ then $state(c,t+1) = q_S$.

In fact, S is always propagated in the diagonal direction (for $t>1$) except when the cell which is reached is such that its neighbor cell on the right (v_7) is in the state q_H. In this case, the signal is propagated one cell upward.

We can remark that for $t \geqslant 0$, the $\vec{O}y$-coordinate of the cell reached by the signal S is $y=t$. Then, for $t=0$ the trace of the signal S is the parabola $V_{k'}$. At time $t=1$, the cell which is reached is $(k',1)$ and we have already seen in Lemma 2.1 that for all $k' \geqslant 1$, $x = \lfloor \sqrt{2k' + k'^2} \rfloor = k'$. Moreover, for $t>1$, this automaton exactly translates Proposition 2.1. Hence, we are sure that the trace of the constructed signal is the parabola $V_{k'}$ for t, $t \geqslant 0$. □

Fig. 45 shows the construction of the parabola V_7 by this automaton for $0 \leqslant t \leqslant 19$.

Lemma 3.4. *There exists a 2D-CA which constructs the bundle of parabolas $\mathscr{V}$ (resp. $\mathscr{H}$) from the bundle of parabolas $\mathscr{H}$ (resp. $\mathscr{V}$).*

Proof. The initial configuration of this cellular automaton is different from the one of the previous one: we suppose the parabolas $V_{k'}$ for $k \geqslant 1$ to be known and the cell $(0,0)$ to be in a special state. This last cell initializes at time 0, a signal which goes along $\vec{O}x$ at speed 1. Each cell reached by this signal initializes itself a signal S which is propagated as it is described in the proof of Lemma 3.3. So we obtain all the parabolas $V_{k'}$ for $k' \geqslant 1$ at time intervals equal to 1. □

Fig. 46 shows the configuration of this last automaton at time 14.

Corollary 3.3. *For all $k \geqslant 1$, for all $n \geqslant 0$, the automaton that generates the bundle of parabolas $\mathscr{V}$ from the bundle $\mathscr{H}$, marks the point $(n, \lfloor \sqrt{2k'n + k'^2} \rfloor)$ of the parabola $V_{k'}$ at time $(n+k')$.*

Proof. A signal S is initialized at each time. Then, the signal that corresponds to the parabolas $V_{k'}$ is initialized at time k'. Moreover, at each time, each signal is propagated of one cell. Hence the cell $(n, \lfloor \sqrt{2k'n + k'^2} \rfloor)$ is marked at time $k'+n$. □

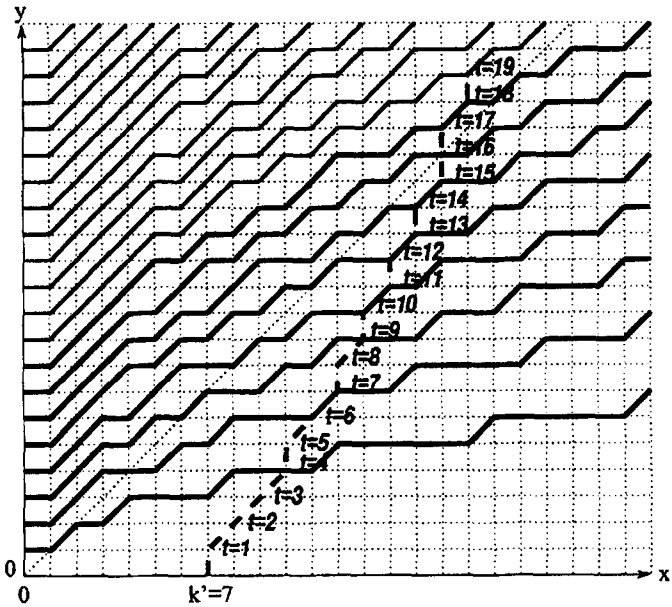


Fig. 45. Projected geometrical diagram of the cellular automaton which constructs the parabola V_7 from the bundle $\mathscr{H}$.

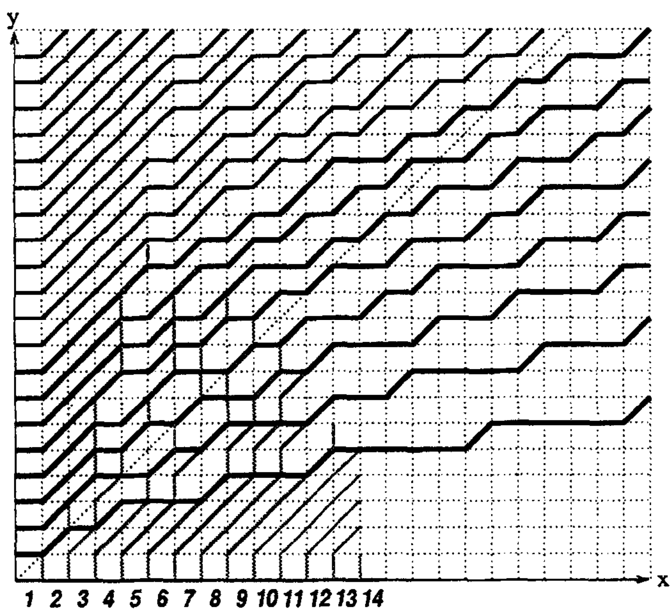


Fig. 46. Projected geometrical diagram of the cellular automaton which constructs the parabolas $V_{k'}$ from the parabolas H_k.

Corollary 3.4. *For all $t \geqslant 0$, the bundle $\mathscr{V}_{t+1} \cap (DH_0 \cup K_{01})$ is constructed by cellular automaton from the bundle $\mathscr{H}_t \cap (DH_0 \cup K_{01})$ and, the bundle $\mathscr{H}_{t+1} \cap (DV_0 \cup K_{02})$ is constructed by cellular automaton from the bundle $\mathscr{V}_t \cap (DV_0 \cup K_{02})$.*

3.2.3. The origins of patterns A

We will now define a cellular automaton which localizes the origins of the patterns A.

The coordinates (α_i, β_i) of the origins of the patterns A have the following expressions (see Section 2.5.1):

$$\alpha_i^k = 2k + 4i + \lfloor \sqrt{2ki + k^2} \rfloor,$$

$$\beta_i^k = 2k + 3i + 2\lfloor \sqrt{2ki + k^2} \rfloor$$

with $k \geqslant 1$ and $i \geqslant \lfloor k(1 + \sqrt{2}) \rfloor$.

We start studying the set of points (α_i^k, β_i^k) with $i \geqslant 0$ and $k \geqslant 1$. Let us remark that $\lfloor \sqrt{2ki + k^2} \rfloor - \lfloor \sqrt{2k(i-1) + k^2} \rfloor$ takes only two values: 0 or 1. So, only two cases have to be considered to compute the coordinates of (α_i^k, β_i^k) from the coordinates of $(\alpha_{i-1}^k, \beta_{i-1}^k)$:

- $\lfloor \sqrt{2ki + k^2} \rfloor - \lfloor \sqrt{2k(i-1) + k^2} \rfloor = 1$ and then $\alpha_i^k = x_{i-1}^k + 5$ and $\beta_i^k = y_{i-1}^k + 5$
- $\lfloor \sqrt{2ki + k^2} \rfloor - \lfloor \sqrt{2k(i-1) + k^2} \rfloor = 0$ and then $\alpha_i^k = x_{i-1}^k + 4$ and $\beta_i^k = y_{i-1}^k + 3$

with, $\alpha_0^k = 3k$ and $\beta_0^k = 4k$. This means that if the point $(i, \lfloor \sqrt{2ki + k^2} \rfloor)$ is the origin of a slope (case where $\lfloor \sqrt{2ki + k^2} \rfloor - \lfloor \sqrt{2k(i-1) + k^2} \rfloor = 1$) then the point (α_i^k, β_i^k) is obtained from the point $(\alpha_{i-1}^k, \beta_{i-1}^k)$ adding 5 to its coordinates. On the contrary, if the point $(i, \lfloor \sqrt{2ki + k^2} \rfloor)$ is not the origin of a slope (case where $\lfloor \sqrt{2ki + k^2} \rfloor - \lfloor \sqrt{2k(i-1) + k^2} \rfloor = 0$) then the point (α_i^k, β_i^k) is obtained from the point $(\alpha_{i-1}^k, \beta_{i-1}^k)$ adding 4 to the first coordinate and 3 to the second one.

In fact, we construct by cellular automaton the points (α_i^k, β_i^k) with $i \geqslant 0$ and $k \geqslant 1$ and we only consider the ones that belong to K_{01}. Then we obtain all the origins of patterns A from Proposition 2.2.

Lemma 3.5. *There exists a 2D-CA which, the parabola H_k $(k \geqslant 1)$ being given, marks the points (α_i^k, β_i^k), $i \geqslant 0$.*

Proof. Let k be an integer $(k \geqslant 1)$. We consider the parabola H_k and we want to localize the points of coordinates (α_i^k, β_i^k) for $i \geqslant 0$ from this one. Fig. 47 gives a graphical representation in the case $k = 1$. The crosses localize the points we want to construct.

The principle of the construction is the following one: each cell of the parabola H_k, the coordinates of which are $(i, \lfloor \sqrt{2ki + k^2} \rfloor)$, emits a signal which helps to construct the point (α_i^k, β_i^k). We have $\alpha_i^k - i = \beta_i^k - \lfloor \sqrt{2ki + k^2} \rfloor$. This implies that a signal born on the cell $(i, \lfloor \sqrt{2ki + k^2} \rfloor)$ and which is propagated in diagonal, that is to say which goes at each time from a cell c to the cell $B_8(c)$, will meet the point (α_i^k, β_i^k). We have to define how the signal knows that it arrives at this point.

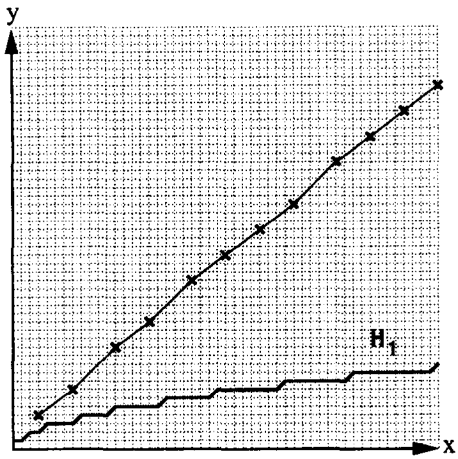


Fig. 47. Graphical representation of the points (α_i^1, β_i^1) and of the parabola H_1.

Table 2
States which are used for the signals which localizes the origins of patterns A

State	Meaning
q_{init}	initialization of the signals born on the cells $(0,k)$
q_{op}	the signal is born on the origin of a landing
q_p	the signal is not born on the origin of a landing
e_1, e_2	to count 3
e_3, e_3, e_5, e_6	to count 5
e_c	origin of pattern A

It is sufficient to remember that we obtain y_{i+1}^k from y_i^k adding 5 if the point of the parabola H_k is the origin of a landing, and three if not. So, the cells $(i, \lfloor\sqrt{2ki+k^2}\rfloor)$ for $i \geqslant 0$ will emit two kinds of signals according to the fact that they are origins of landing or not. Then, if at a given time t a cell which is reached by such a signal, sees in its neighborhood (B_3) the point marked by the previous signal, it will count 3 or 5 according to its type to mark a new point. The signal initialized by the cell $(0,k)$ cannot proceed like that: there is no previous signal. But, we have, for all $k \geqslant 1$, $\alpha_0^k = 3k$ and $\beta_0^k = 4k$. So, we just have to add another signal that marks these cells. The necessary states to realize these signals are sum up in Table 2.

Fig. 48 represents the trace of the signals used to construct the points (α_i^1, β_i^1) from H_1.

We have to look after the fact that the signals which are generated by the cells (x,y) of a parabola H_k such that $x - y = constant$ (the cells of H_k belonging to slopes of length m, $m \geqslant 1$) do not intermingle. Then, when a cell of H_k generates a signal, it enters a special state q_e which means "I have generated a signal". And this state is propagated on the parabola H_k. A cell of H_k only generates a signal if its neighbor on the parabola is in the state q_e and if the signal generated by this neighbor does not go through itself. We remark that for a fixed $\vec{O}x$-coordinate x the number of stairs before

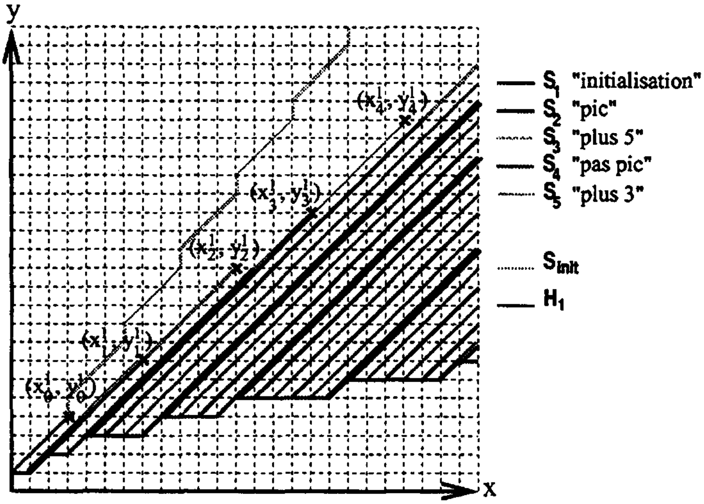


Fig. 48. Trace of the signals used to construct the points (α_i^1, β_i^1) from H_1.

x on H_{k+1} is strictly greater than the number of stairs on H_k. So, the signals on H_{k+1} will be more delayed than on H_k and hence, they will not mix each other. □

Corollary 3.5. *For all* $k \geqslant 1$, *for all* $i \geqslant 0$, $2k + 3i + \lfloor\sqrt{2ki + k^2}\rfloor$ *times are necessary to mark the point* (α_i^k, β_i^k), *origin of a pattern* A, *from the point* $(i, \lfloor\sqrt{2ki + k^2}\rfloor)$.

Proof. The point $(2k + 4i + \lfloor\sqrt{2ki + k^2}\rfloor, 2k + 3i + 2\lfloor\sqrt{2ki + k^2}\rfloor)$ is localized from the point $(i, \lfloor\sqrt{2ki + k^2}\rfloor)$ with the help of a diagonal signal which spreads of one cell at each time. So, the time necessary to reach this point is equal to the difference between the first coordinates (or the second) of these two points. Fig. 49 gives an illustration of this corollary. □

3.2.4. The heart

Lemma 3.6. *There exists a 2D-CA which constructs the parabola* H_{k+1} $(k \geqslant 1)$ *in* K_{01} *from the parabola* H_k *in* K_{01} *and the origins of patterns* A.

Proof. This cellular automaton is very simple because, as we have remarked it before, problems occur for the second points of the landings of length 2. As a matter of fact, if H_k has a landing of length 1 at a point (x, y) then we know, from Lemmas 2.4 and 2.5, that the points $(x, y + 2)$, $(x + 1, y + 2)$, $(x + 2, y + 3)$ and $(x + 3, y + 3)$ belong H_{k+1}. This corresponds to two different signals which are emitted by the hollows and the peaks. The first one is propagated two cells upwards and the second one one cell upwards. In the case where (x, y) is the origin of a landing of length 2, the cell $(x + 2, y + 1)$ is neither a peak nor an hollow. But, it will evolve according to its neighbor $(x + 1, y + 1)$. If this last cell is the origin of a pattern A (it is marked by a

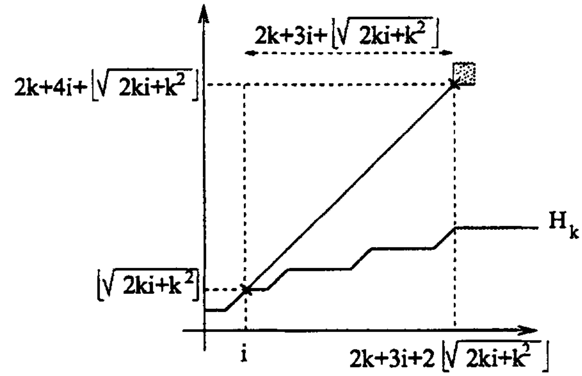


Fig. 49. Corollary 3.5.

special state) then it initializes a signal which is propagated one cell upwards. On the other case, it initializes a signal which is propagated two cells upwards. □

3.2.5. An automaton that constructs the parabolas from C_0

Proposition 3.2. *There exists a 2-CA which generates the parabolas H_k and $V_{k'}$ for $k \geqslant 1$ and $k' \geqslant 1$ from the initial configuration C_0.*

Proof. Notice some signals that can be easily constructed:

- the signal, denoted by S_a, which marks $\vec{O}x$,
- the signal, denoted by S_b, which marks $\vec{O}y$,
- the signal, denoted by S_d, which marks the first diagonal,
- the signal, denoted by S_{KH_0}, which localizes the cells (x,y) such that $y = \lfloor \frac{3}{4}x \rfloor$ and,
- the signal, denoted by S_{KV_0}, which localizes the cells (x,y) such that $x = \lfloor \frac{3}{4}y \rfloor$.

Fig. 50 shows the traces of these signals.

At initial time, these signals are initialized by the cell $(0,0)$. Then, each cell which is reached by the signal S_b initializes a signal S_H and, each cell reached by the signal S_a initializes a signal S_V. The signals S_H and S_V which are respectively initialized by the cells $(0,1)$ and $(1,0)$ are particular cases and their evolution has been described in Section 3.2.2. The evolution of other signals S_H and S_V, initialized by the cells $(0,k)$ and $(k,0)$ with $k>1$, is performed in three phases.

In the first one, the signals S_H (respectively S_V) use their intersection with the signals S_V initialized by the cells $(l,0)$ with $l<k$ (respectively S_H initialized by the cells $(0,l)$ with $l<k$). The algorithm which allows to obtain the orthogonal bundle has been described in Section 3.2.2. This phase comes to an end when the signals S_H and S_V meet the signal S_d. Notice that during this phase, the cells which are reached by the signals S_H and S_V also initialize new signals. These last ones will localize the patterns A and they evolve as it is described in Section 3.2.3.

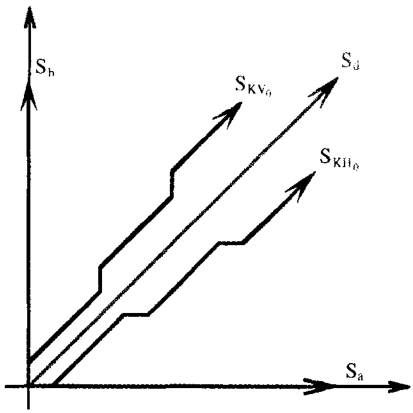


Fig. 50. Proposition 3.2.

In the second phase, the evolution of the signals S_H and S_V follows the algorithm which has been described in Section 3.2.3: signals which are initialized by the hollows (which go two cells upwards) and by the peaks (which go one cell upwards) are used. This phase ends when the signals S_H and S_V, respectively, meet the signals S_{KH_0} and S_{KV_0}. Notice that we have to verify that the origins of the patterns A which help to the construction of a given signal S_H, are marked before we have to use them. From Corollary 3.5, $2k+3i+\lfloor\sqrt{2ki+k^2}\rfloor$ steps are necessary to localize the point $(2k+4i+\lfloor\sqrt{2ki+k^2}\rfloor, 2k+3i+2\lfloor\sqrt{2ki+k^2}\rfloor)$ from the point (α_i^k, β_i^k) and $\lfloor\sqrt{2ki+k^2}\rfloor - k$ units are necessary as amount of shift. But the point (α_i^k, β_i^k) is itself constructed at time $(k+i)$ from Corollary 3.2. Hence the point $(2k+4i+\lfloor\sqrt{2ki+k^2}\rfloor, 2k+3i+2\lfloor\sqrt{2ki+k^2}\rfloor)$ is known at time $t=k+3i+2\lfloor\sqrt{2ki+k^2}\rfloor$. Now we suppose that the point $(2k+4i+\lfloor\sqrt{2ki+k^2}\rfloor, 2k+3i+2\lfloor\sqrt{2ki+k^2}\rfloor)$ helps to the construction of the parabola H_K $(K>k)$. The point of first coordinate $2k+4i+\lfloor\sqrt{2ki+k^2}\rfloor$ of the parabola H_K is reached, from Corollary 3.2, at time $t'=2k+4i+\lfloor\sqrt{2ki+k^2}\rfloor+K$. But, from Proposition 2.2, we have

$$K=R-x=k+i+\lfloor\sqrt{2ki+k^2}\rfloor>\lfloor\sqrt{2ki+k^2}\rfloor$$

because $k\geqslant 1$. Hence we have $t'>t$.

The last phase is the construction of the “long landings”, that is explained in Section 3.2.2.

Fig. 51 shows the evolution of the automaton from the time $t=10$ to the time $t=11$. □

Corollary 3.6. *For all $k\geqslant 1$, for all $n\geqslant 0$, the automaton which generates the parabolas H_k from the initial configuration C_0 constructs the point $(n, \lfloor\sqrt{2kn+k^2}\rfloor)$ at time $(n+k)$.*

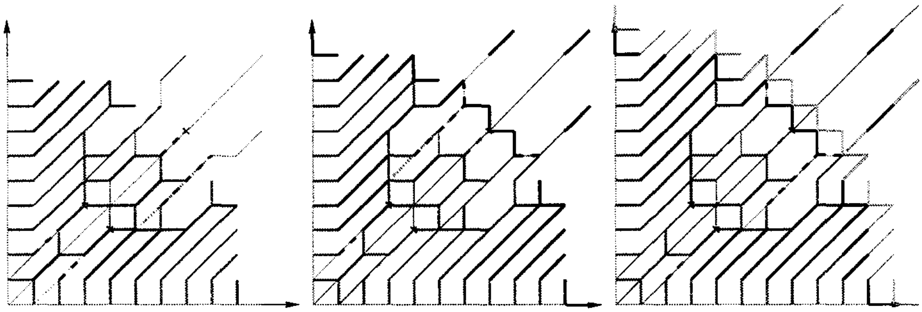


Fig. 51. Evolution of the automaton which constructs the parabolas from the time $t=10$ to the time $t=12$.

In this section, we have described the automaton that constructs the parabolas $\mathscr{H}$ and $\mathscr{V}$. This construction is not easy because first it is founded on a recurrence and second, the algorithms vary according to different parts of the plane.

Moreover, notice that, as all the signals which are used, are real time signals, the parabolas are constructed "as soon as" possible.

We will use this construction in the following section in order to construct the floor circles.

4. Construction of the floor circles

We will show here how the knowledge of the bundles of parabolas previously studied lead to cellular automata that, simultaneously, build these bundles of parabolas and the whole family of floor circles $\lfloor\odot\rfloor$ corresponding to $\odot$. But first of all we have to precise the definition of a floor circle.

Definition 4.1. For every positive integer R, the *floor circle* centered on $(0,0)$ of radius R, denoted by $C_{floor}(0,R)$, is defined in the first octant by

$$C_{floor,1^{st}octant}(0,R)=\bigcup_{k=0}^{R}(\{(x,y)\in\mathbb{N}^2/x=R-k,\lfloor\sqrt{2kx+k^2}\rfloor\leqslant y< \lfloor\sqrt{2(k+1)x+(k+1)^2}\rfloor \text{ and } x\geqslant y\geqslant 0\}).$$

Floor circles consist in points belonging to vertical segments spreading from one floor parabola (included) up to the next one (excluded). We have to notice that on a parallel to $\vec{O}x$, we can have more than one point. Let us study how one of these vertical segments follows another.

Let us call $D_{R,x}$ the intersection of $C_{floor}(0,R)$ and the vertical line of $\vec{O}x$-coordinate x and $E_{R,x}$ its bottom point. $E_{R,x}$ belongs to the parabola H_{R-x}. Two cases appear:

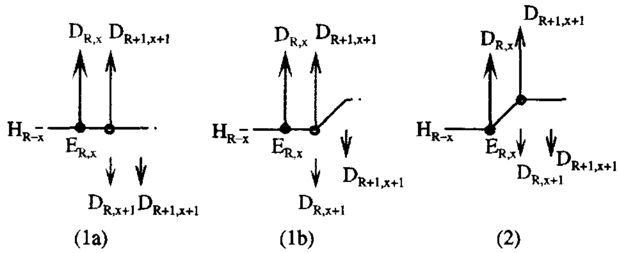


Fig. 52. How to obtain a segment of the floor circle $\mathscr{C}_{floor}(0,R)$ from a segment of $\mathscr{C}_{floor}(0,R-1)$.

1. If $(x+1,y)\in H_{R-x}$, then $D_{R+1,x+1}$ begins on $E_{R+1,x+1}=(x+1,y)$. The top point of $D_{R,x+1}$ is $(x+1,y-1)$ (immediately under $E_{R+1,x+1}$ and the top point of $D_{R+1,x+2}$ is not known: it may be $(x+2,y-1)$ or $(x+2,y)$ (see Fig. 52(1a) or (1b)).
2. If $(x+1,y)\notin H_{R-x}$, then $(x+1,y+1)\in H_{R-x}$. Thus $E_{R+1,x+1}$ is $(x+1,y+1)$. The top point of $D_{R,x+1}$ is $(x+1,y)$ (immediately under $E_{R+1,x+1}$ and the top point of $D_{R+1,x+2}$ is $(x+2,y)$ because a slope in $DH_0\cup K_{01}$ has length 1 (see Fig. 52(2)).

In consequence, if $(x,y)\in C_{floor}(0,R)$, then $(x+1,y)\in C_{floor}(0,R+1)$. And $(x+2,y)\in C_{floor}(0,R+1)$ if and only if $(x+1,y)$ is an hollow of H_{R-x}.

4.0.6. Construction of the family of the floor circles in real time

Lemma 4.1. *There exists a 2-CA which, from the initial configuration C_{HV}, constructs the family of floor circles in real time.*

Proof. We suppose that the circle of radius $R-1$ and the bundle of parabolas $\mathscr{H}$ are already constructed and we aim to get the circle of radius R (Fig. 53). The study following Definition 4.1, shows that a cell always belongs to $C_{floor}(0,R)$ if its left (B_3) neighbor belongs to $C_{floor}(0,R-1)$.[4] This previous study also shows that the only cells of $C_{floor}(0,R)$ not obtained by this right shift are those:

- the B_3-neighbor of which belongs to $C_{floor}(0,R)$.
- the B_3-B_3-neighbor of which belongs to $C_{floor}(0,R-1)$.
- the B_3-neighbor of which is an hollow.

We summarize these three previous conditions by the two following ones:

- the B_4-neighbor of which belongs to $C_{floor}(0,R-1)$.
- the B_3-neighbor of which is an hollow.

Consequently, to construct $C_{floor}(0,R)$ from $C_{floor}(0,R-1)$ it is sufficient to associate to each (x,y) in $C_{floor}(0,R-1)$, its B_7-neighbor and, if its B_1-neighbor is an hollow, its B_8-neighbor (Fig. 54).

This defines a local rule we translate in cellular automata terms.

---

[4] Moore's neighborhood:

B_2	B_1	B_8
B_3	B_0	B_7
B_4	B_5	B_6

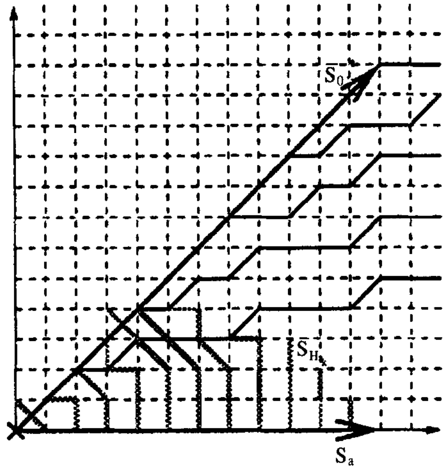


Fig. 53. Configuration at time $t = 12$ of the cellular automaton which constructs the family of floor circles.

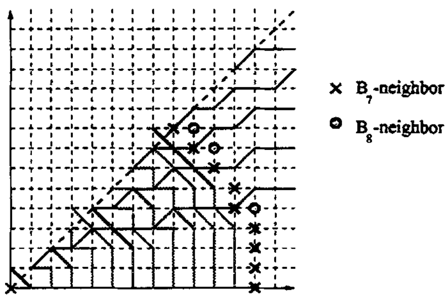


Fig. 54. Construction in real time of the floor circles.

Proof. The cells of the circle of radius R are in the Moore's neighborhood of the cells which belong to the circle of radius $(R-1)$. So, they can be constructed in one unit of time using the parabolas. That is to say that a new circle is constructed at each time and, as the circle of radius 1 appears at time 1, the construction is made in real time. □

The previous Lemma 4.1 suppose that the parabolas are already constructed. But, it is possible to construct simultaneously the circles and the parabolas, as is shown by the following proposition.

Proposition 4.1. *There exists a 2-CA which constructs, in real time, the family of floor circles from the initial configuration* C_0.

Proof. At initial time, the only cell in a non quiescent state is $(0,0)$. It is the origin of all the signals which allow to construct parabolas and circles.

In Section 3, we have described a cellular automaton which constructs the bundles of parabolas $\mathscr{H}$ and $\mathscr{V}$ from the initial configuration C_0. Now we compound the states of this automaton with the states of the automaton which constructs in real time the circles from the parabolas we have seen previously. Let us remark that this new automaton does not construct the family of circles in real time but in quasi-real time. As a matter of fact, the circle of radius R uses the points of first coordinates $x=R-k$ of the parabolas H_k. But, these points are constructed with the help of the automaton described in Section 3.2.4, at time $t=k+x=k+R-k=R$. Thus at time R, the intersections between the circle of radius R and the bundle of parabolas $\mathscr{H}$ are known. But, two times more are necessary to obtain the circle between the parabolas. As a matter of fact, we consider the cell (x,y) which belongs to the parabola H_k. The circle cut H_k at a point which will be different according to the fact that (x,y) is an hollow or not. But, the point of first coordinate $x+1$ of H_k is constructed at time $t'=k+x+1$. Consequently, the circle R is constructed next time; that is to say at time $R+2$.

But, from the result recalled in Section 1.5, we can conclude that there exists a 2-CA which simultaneously constructs the floor parabolas and circles in real time. □

Following the same idea (constructing adequate bundles of parabolas CA-constructible), we will show how to build, in real time on cellular automata, other discrete circles namely ceiling circles, Pitteway's circles and arithmetical circles. The construction of the parabolas will still be the main difficulty.

Let us start with the ceiling circles.

5. Ceiling parabolas and circles

Let us denote $\bar{\mathscr{H}}$ and $\bar{\mathscr{V}}$, $k \geqslant 1$, the bundles of parabolas obtained by putting $\lceil\ \rceil$ in place of $\lfloor\ \rfloor$ in the definition of floor-parabolas. Their elements, the ceiling-parabolas, are denoted $\bar{H}_k$ and $\bar{V}_k$. We could get $\bar{\mathscr{H}}$ and $\bar{\mathscr{V}}$ as we did for $\mathscr{H}$ and $\mathscr{V}$, but we are going to exhibit an other method which allows to get these new bundles from the previous ones, and which is founded on the following remark.

Remark 5.1. For all x real, we have:

$$\begin{cases} \lceil x\rceil = \lfloor x\rfloor & \text{if } x \text{ is an integer,}\\ \lceil x\rceil = \lfloor x\rfloor + 1 & \text{else.}\end{cases}$$

Using these equalities for our problem, we obtain:

$$\begin{cases} \bar{H}_k(x)=H_k(x) & \text{if } h_k(x) \text{ is an integer} \quad (5),\\ \bar{H}_k(x)=H_k(x)+1 & \text{else} \quad (6).\end{cases}$$

So, let (x,y) belong to the parabola H_k. If the parabola h_k does not go through (x,y) then the point $(x,y+1)$will belong to the parabola $\bar{H}_k$. If not, (x,y) belongs to the parabola $\bar{H}_k$ too. It follows that the parabola $\bar{H}_k$ is always one cell higher than the

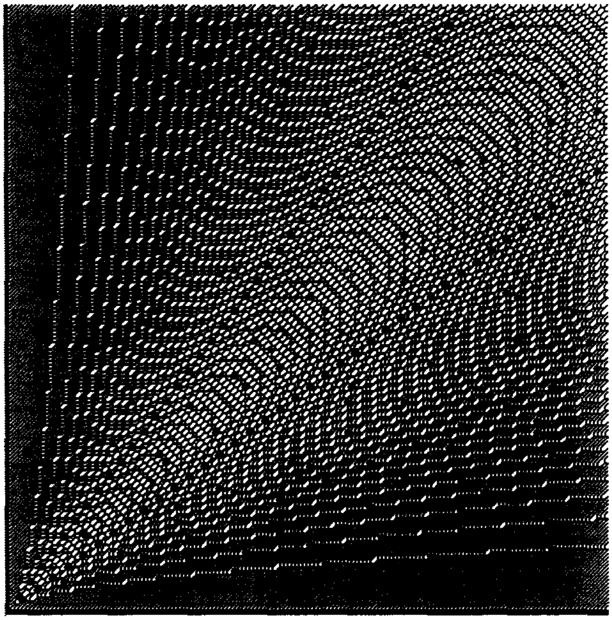


Fig. 55. Bundles $\mathscr{H}$ and $\mathscr{V}$, and the points (x, y) such that $y^2 = 2kx + k^2$.

parabola H_k except when $2kx + k^2$ is a perfect square. Consequently, if there exists a cellular automaton that marks the set of points:

$$\mathscr{P} = \{(x, y) \in \mathbb{Z}^2;\ y^2 = 2kx + k^2 \text{ with } k \text{ integer}\}$$

then, we can construct a 2-CA that gives the bundles $\bar{\mathscr{H}}$ and $\bar{\mathscr{V}}$ from the bundles $\mathscr{H}$ and $\mathscr{V}$.

We will prove that such automata exist.

5.0.7. A cellular automaton that marks the points (x, y) from $\mathbb{Z}^2$ such that $y^2 = 2kx + k^2$

Fig. 55 represents the bundles $\mathscr{H}$ and $\mathscr{V}$ where the points (x, y) from $\mathbb{N}^2$ such that $y^2 = 2kx + k^2$ are marked by black squares.

Let (x'^k, y'^k) be on the parabola H_k. Let $S'^k_{x',y'}$ be the broken line born on (x'^k, y'^k) and defined as follows:

$$S'^k_{x',y'}(x, y) = \begin{cases} x'^k \leqslant x \leqslant x'^k + y'^k, & y = -x + (x'^k + y'^k), \\ x \geqslant x'^k + y'^k, & y = x - (x'^k + y'^k). \end{cases}$$

It is simply the broken line that, starting from a point of the parabola, goes diagonally down to the $\vec{O}x$ axis, then diagonally up to the parabola again.

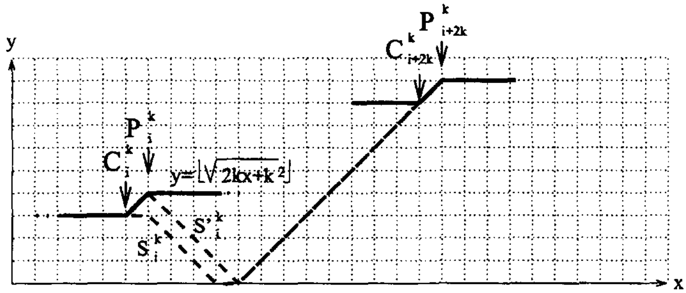


Fig. 56. Broken line that joint the peaks P_i^k and P_{i+2k}^k.

Lemma 5.1. *For all $k \geqslant 1$ and all points (x, y) in H_k such that $x \geqslant 4k$, (x, y) belongs to $\mathscr{P}$ if and only if there exists a point (x', y') in $\mathscr{P}$ such that the broken line $S'^k_{x',y'}$ born on (x', y') meets the parabola H_k at (x, y).*

Proof. Let us suppose that (x, y) on H_k is in $\mathscr{P}$. From Remark 5.1, (x, y) simultaneously belongs to H_k and $\bar{H}_k$. As $x \geqslant 4k$, (x, y) is a peak of the parabola H_k, and $(x-1, y-1)$ is a point of the family $\mathscr{C}$ which can be denoted by C_{i+2k}^k with $i \geqslant 1$ (See Section 2.4.2). So, the point C_i^k of $\mathscr{C}$, with coordinates x_i^k and y_i^k, exists on H_k and we consider the point $(x_i^k + 1, y_i^k + 1)$. We know that $y_i^k = k + i - 1$, this implies that $(x_i^k + 1, y_i^k + 1)$ belong to $\mathbb{Z}^2$. Moreover, if we put $x - 1 = x_{i+2k}^k$ and $y - 1 = y_{i+2k}^k$, we know that $x_{i+2k}^k - x_i^k = 4k + 2i$ and we get $2k(x_i^k + 1) + k^2 = 2k(x_{i+2k}^k - 4k - 2i + 1) + k^2 = 2k(x_{i+2k}^k + 1) + k^2 - 8k^2 - 4ki = (y_{i+2k}^k + 1)^2 - 8k^2 - 4ki = (3k + i)^2 - 8k^2 - 4ki = (k + i)^2 = (y_i^k + 1)^2$, which proves that $(x_i^k + 1, y_i^k + 1)$ is on H_k. Finally, it is easy to verify that the broken line born on $(x_i^k + 1, y_i^k + 1)$, $S'^k_{x_i^k+1, y_i^k+1}$ goes through (x, y). So $(x_i^k + 1, y_i^k + 1)$ is the wanted point. □

The converse assertion is trivial.

So, all the points (x, y) which belong to $\mathscr{P} \cap DH_0$ can be obtained from the points that belong to $\mathscr{P} \cap (DV_0 \cup K_0)$ (cf. the analytic study, Section 2). Moreover, as in the construction of the bundles $\mathscr{H}$ and $\mathscr{V}$, the points that belong to $\mathscr{P} \cap DV_0$ are obtained using broken lines (Fig. 56) that are symmetric to the broken lines $S'^k_{x,y}$ according to the first diagonal and which lean on the bundle $\mathscr{V}$. Consequently, we just have to consider the points in $\mathscr{P} \cap K_0$ and, more precisely only the points in $\mathscr{P} \cap K_{01}$ (because the points in $\mathscr{P} \cap K_{02}$ are obtained symmetrically). We go on in the study of $\mathscr{P}$ in proving three lemmas which lead to the fact that the points of $\mathscr{P}$ can be marked by a cellular automaton.

Lemma 5.2. *For all $k \geqslant 1$ and every point (x_0, y_0) in $H_k \cap K_{01}$, if (x_0, y_0) is in $\mathscr{P}$ then (x_0, y_0) is the origin of a pattern A.*

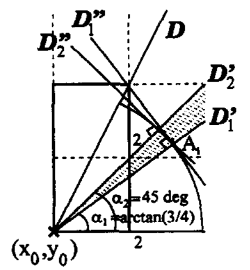


Fig. 57. Lines $\mathscr{D}$, $\mathscr{D}_1'$ and $\mathscr{D}_2'$.

Proof. If (x_0, y_0) belongs to $\mathscr{P}$, $(x_0, y_0) \in \mathbb{Z}^2$ and $y_0^2 = 2kx_0 + k^2$. So, there exists an integer $R = x_0 + k$ such that $x_0^2 + y_0^2 = R^2$.

Moreover, from Lemma 2.7, it is equivalent to prove that (x_0, y_0) is the origin of a square and to prove that there exists an integer R such that the circles $C(O,R)$, $C(O,R+1)$ and $(O,R+2)$ meet the segment $[(x,y),(x+1,y+2)[$ and are the only ones that cut it. As $x_0^2 + y_0^2 = R^2$, the circle $C(O,R)$ meets the segment $[(x,y),(x+1,y+2)[$ at (x_0,y_0). We are going to prove that the circle $C(O,R+2)$ is the circle of greatest radius that meets $[(x,y),(x+1,y+2)[$.

We consider the straight line $\mathscr{D}$ of slope $\frac{1}{2}$ which goes through (x_0,y_0). Its equation is $y = 2x + (y_0 - 2x_0)$. Let $\mathscr{D}'$ be the straight line which goes through (x_0,y_0) and $(0,0)$. In K_{01}, $\mathscr{D}'$ makes an angle α with the horizontal line which goes through (x_0,y_0) such that $\arctan\frac{3}{4} \leqslant \alpha \leqslant 45°$. Fig. 57 gives a graphical representation of the lines $\mathscr{D}_1'$ and $\mathscr{D}_2'$ which are the limits of variation of $\mathscr{D}'$. Notice that in the direction of the line $\mathscr{D}'$ (the radial direction), the distance between two consecutive circles is equal to 1. So, the circle of radius $(R+2)$ cuts the line $\mathscr{D}'$ at the point A such that $x_0 + \sqrt{2} \leqslant x_A \leqslant x_0 + 1.6$ and $y_0 + 1.2 \leqslant y_A \leqslant y_0 + \sqrt{2}$.

We consider the line $\mathscr{D}''$ which is orthogonal to $\mathscr{D}'$ through A. Fig. 57 gives a graphical representation of the lines $\mathscr{D}_1''$ and $\mathscr{D}_2''$ which are the limits of variation of $\mathscr{D}''$. Notice that if $\mathscr{D}_1''$ cuts the segment $[(x_0,y_0),(x_0+1,y_0+2)]$ then the line $\mathscr{D}_2''$ also cuts it. Consequently, we only consider $\mathscr{D}_1''$ and the angle $\alpha' = \arctan\frac{3}{4}$. The equation of $\mathscr{D}''$ is $y = \frac{-4}{3}x + (\frac{4}{3}x_0 + y_0 + \frac{10}{3})$.

As the intersection, (x,y), between $\mathscr{D}''$ and $\mathscr{D}$ satisfies the equations $y = \frac{-4}{3}x + (\frac{4}{3}x_0 + y_0 + \frac{10}{3})$ and $y = 2x + (y_0 - 2x_0)$, we get $x = x_0 + 1$.

Consequently, the circle $C(O,R+2)$ cuts the segment $[(x_0,y_0),(x_0+1,y_0+2)[$ and is the last one to cut it. Hence, the point (x_0,y_0) is an origin of pattern A. □

Lemma 5.3. *For all $k \geqslant 1$ and all points (x_0,y_0) which belong to $H_k \cap K_{01}$, if (x_0,y_0) is in $\mathscr{P}$ then there exist two integers l and i such that*:

$$x_0 = 2l + 4i + \lfloor\sqrt{2li + l^2}\rfloor,$$
$$y_0 = 2l + 3i + 2\lfloor\sqrt{2li + l^2}\rfloor,$$

and $2li + l^2 = m^2$ where m is an integer.

Proof. From the hypothesis, (x_0, y_0) is the origin of a pattern A. So, by Proposition 2.2, there exist two integers l and i such that

$$x_0 = 2l + 4i + \lfloor\sqrt{2li + l^2}\rfloor,$$
$$y_0 = 2l + 3i + 2\lfloor\sqrt{2li + l^2}\rfloor$$

or there exists an integer R such that the circles $C(O,R)$, $C(O,R+1)$ and $C(O,R+2)$ cut the segment $[(x_0, y_0),(x_0+1, y_0+2)[$ and, as (x_0, y_0) also is in $\mathscr{P}$, such that $C(O,R)$ goes through (x_0, y_0). Then, still with the notations of Proposition 2.2, there exists an integer R, $R \geqslant 1$, such that

$$(R-\xi)^2 + (R-\eta)^2 = R^2 \quad (7)$$

and

$$(R+2)^2 < (R-\xi+1)^2 + (R-\eta+2)^2 \quad (4).$$

Now, if we consider the functions f and g defined in the proof of Proposition 2.2, we can say that there exists an integer R ($R \geqslant 1$) such that $f(R) = 0$ and $g(R) > 0$ or such that $R = \xi + \eta + \sqrt{2\xi\eta}$ (with $\xi = R - x_0$ et $\eta = R - y_0$). But, we have showed (in the proof of Proposition 2.2) that R can be expressed as follows: $R = 3l+5i+2\lfloor\sqrt{2li + l^2}\rfloor$. Consequently, as $\xi = l + i + \lfloor\sqrt{2li + l^2}\rfloor$ and $\eta = l + 2i$, we have

$$\sqrt{2(l+i+\lfloor\sqrt{2li + l^2}\rfloor)(l+2i)} + 2l + 3i + \lfloor\sqrt{2li + l^2}\rfloor$$
$$= 3l + 5i + 2\lfloor\sqrt{2li + l^2}\rfloor.$$

This is also equivalent to:

$$\sqrt{(l+2i)^2 + 2(l+2i)\lfloor\sqrt{2li + l^2}\rfloor + (\sqrt{2li + l^2})^2} = l + 2i + \lfloor\sqrt{2li + l^2}\rfloor.$$

We suppose that $\sqrt{2li + l^2} = \lfloor\sqrt{2li + l^2}\rfloor + \varepsilon$ with $\varepsilon \geqslant 0$.

We obtain:

$$\sqrt{(l+2i+\lfloor\sqrt{2li + l^2}\rfloor)^2 + 2\varepsilon\lfloor\sqrt{2li + l^2}\rfloor + \varepsilon^2} = l + 2i + \lfloor\sqrt{2li + l^2}\rfloor.$$

This implies that $\varepsilon = 0$, and, hence, that $\sqrt{2li + l^2} = \lfloor\sqrt{2li + l^2}\rfloor$. □

Lemma 5.4. *For all $k \geqslant 1$ and every point (x, y) in $H_k \cap \mathscr{P}$, the point $(x', y') = (2k + 4x + \lfloor\sqrt{2kx + k^2}\rfloor, 2k + 3x + 2\lfloor\sqrt{2kx + k^2}\rfloor)$ is in $\mathscr{P}$.*

Proof. Let (x, y) be in $H_k \cap \mathscr{P}$. We have $\lfloor\sqrt{2kx + k^2}\rfloor = \sqrt{2kx + k^2}$.

If we put $A_x^k = (2k + 4x + \lfloor\sqrt{2kx + k^2}\rfloor)^2 + (2k + 3x + 2\lfloor\sqrt{2kx + k^2}\rfloor)^2$.

From the hypothesis, we get

$$A_x^k = (2k + 4x + \sqrt{2kx + k^2})^2 + (2k + 3x + 2\sqrt{2kx + k^2})^2,$$

$$A_x^k = 13k^2 + 38kx + 25x^2 + 4(3k + 5x)\sqrt{2kx + k^2},$$

$$A_x^k = (3k + 5x)^2 + 2(3k + 5x)(2\sqrt{2kx + k^2}) + (2\sqrt{2kx + k^2})^2,$$

$$A_x^k = (3k + 5x + 2\sqrt{2kx + k^2})^2. \quad \square$$

So, to each point (x, y) in $H_k \cap \mathscr{P}$ corresponds a point $(x', y') = (2k + 4x + \lfloor \sqrt{2kx + k^2} \rfloor, 2k + 3x + 2\lfloor \sqrt{2kx + k^2} \rfloor)$ which is also in $\mathscr{P}$. Consequently, we use the same recurrence as in the construction of the floor parabolas (see Section 3.1). We just add some signals in order to also mark the points of the set $\mathscr{P}$.

Lemma 5.5. *There exists a 2-CA which, from C_0, constructs the bundles $\mathscr{H}$ and $\mathscr{V}$ and, marks the points of the set $\mathscr{P}$.*

Proof. We have proved in the previous lemmas that, if (x, y) is in $\mathscr{P}$,

- either (x, y) belongs to DH_0 (and symmetrically to DV_0). In this case, from Lemma 5.1, there exists a point (x', y') in $\mathscr{P}$ such that the broken line $S'^k_{x',y'}$ born on (x', y') cuts the parabola H_k at the point (x, y). So, the point (x, y) is constructed by cellular automaton from the point (x', y') with the help of signals the traces of which are the broken lines S'^k (as it has been done in Section 3.2.1 for the signals S). Symmetrically, we mark the points (x, y) in $DV_0 \cap \mathscr{P}$,
- or, (x, y) belongs to K_{01} (and symmetrically to K_{02}). In this case, we have proved in Lemma 5.2, that the point (x, y) is the origin of a pattern A. Consequently, from Proposition 2.2, there exists two integers l and i such that:

  $$x = 2l + 4i + \lfloor \sqrt{2li + l^2} \rfloor,$$

  $$y = 2l + 3i + 2\lfloor \sqrt{2li + l^2} \rfloor$$

  and, from Lemma 5.3, we have $\lfloor \sqrt{2li + l^2} \rfloor = \sqrt{2li + l^2}$. So, from Lemma 3.5, the point (x, y) is obtained from point $(i, \lfloor \sqrt{2ki + l^2} \rfloor)$ with the help of a diagonal signal (cf. Section 3.2.3) born on $(i, \lfloor \sqrt{2ki + l^2} \rfloor)$ which belongs to $\mathscr{P}$ and obtained by the first point.

Finally, we simultaneously mark the set $\mathscr{P}$ and construct the bundles $\mathscr{H}$ and $\mathscr{V}$. We just have to add signals the traces of which are the broken lines S'^k and to specify by 2 new states the diagonal signals that are born on points $(i, \lfloor \sqrt{2ki + l^2} \rfloor)$ such that $\lfloor \sqrt{2li + l^2} \rfloor = \sqrt{2li + l^2}$. $\square$

5.0.8. Cellular automaton that constructs the bundle $\bar{\mathscr{H}}$

To get this bundle, we just have to modify a little the cellular automaton that generates the parabolas $\mathscr{H}$ and $\mathscr{V}$. Actually, we have seen that the parabola $y = \lceil \sqrt{2kx + k^2} \rceil$ is always one cell higher than the parabola $y = \lfloor \sqrt{2kx + k^2} \rfloor$ except on the points (x, y)

belonging to $\mathbb{Z}^2$. Consequently, a signal, denoted $S_{\bar{H}}$, initialized by the cell $(0,k)$, is propagated one cell higher than the signal S_H (that has been described in the proof of Proposition 3.2) and one time later. It meets the signal S_H on the points (x,y) in $\mathbb{Z}^2$, and corresponds to the parabola $\lceil\sqrt{2kx+k^2}\rceil$. And we have:

Lemma 5.6. *There exists a 2-CA which, from C_0, constructs the bundles $\mathscr{H}$, $\mathscr{V}$, $\bar{\mathscr{H}}$ and $\bar{\mathscr{V}}$.*

Corollary 5.1. *For all $k \geqslant 1$, all $n \geqslant 0$, the automaton that generates the parabolas H_k from the initial configuration C_0 constructs the point $(n, \lceil\sqrt{2kn+k^2}\rceil)$ at time $(n+k+1)$.*

5.0.9. Construction of the ceiling circles

The definition of the ceiling circles is analogous to the one of the floor circles, replacing the $\lfloor\,\rfloor$ by $\lceil\,\rceil$.

Definition 5.1. The *ceiling circle* centered on $(0,0)$ or radius R, denoted by $C_{ceil}(0,R)$, is defined in the first octant by

$$C_{ceil,1^{st}octant}(0,R) = \bigcup_{k=0}^{R} (\{(x,y) \in \mathbb{N}^2 / x = R-k, \lceil\sqrt{2kx+k^2}\rceil \leqslant y < \lceil\sqrt{2(k+1)x+(k+1)^2}\rceil \text{ and } x \geqslant y \geqslant 0\}).$$

The ceiling circles are obtained from the parabolas $\bar{H}_k$ and $\bar{V}_{k'}$ with cellular automata that are identical to the cellular automata described in the proof of lemma. The study following Definition 4.1 remains available. Concerning the generation of the ceiling circles from the initial configuration C_0, we prove the following lemma:

Lemma 5.7. *There exists a 2-CA which constructs the family of ceiling circles in real time from the initial configuration C_0.*

Proof. The construction of the parabola $\bar{H}_k$ is performed one time later according to the parabolas H_k. Consequently, at time $R+1$, the intersections between the ceiling circle of radius R and the bundle $\bar{\mathscr{H}}$ are known. Moreover, we have seen, in the proof of Lemma 4.1, that two units of times are necessary to construct the circles leaning on the parabolas. Then, the ceiling circle of radius R is constructed at time $(R+3)$. Hence, from Section 1.5, there exists a 2-CA that constructs the family of ceiling circles in real time. □

In the following we still apply the same ideas to construct the Pitteway's circles the arithmetical ones. The main idea is still to build convenient bundles of parabolas starting from the knowledge of the bundles $\mathscr{H}$ and $\mathscr{V}$. But we will use a method sophisticated enough, the method of grouping for cellular automata, which will appear as very efficient.

6. Pitteway's and arithmetical circles

6.1. *Pitteway's circle*

Let us recall the definition of this discrete circle.

Definition 6.1. The *Pitteway's circle* centered on $(0,0)$ of radius R, denoted by $C_{Pitteway}$ $(0,R)$, is defined in the first octant by

$$C_{Pitteway,1^{st}octant}(0,R)=\bigcup_{k=0}^{R}(\{(x,y)\in\mathbb{N}^2/x=R-k,[\sqrt{2kx+k^2}]\leqslant y<$$
$$[\sqrt{2(k+1)x+(k+1)^2}] \text{ and } x\geqslant y\geqslant 0\}.$$

6.1.1. *Construction of the bundles $\tilde{\mathscr{H}}$ and $\tilde{\mathscr{V}}$*

Let us recall that every real number x can be written $x=\lfloor x\rfloor+\{x\}$, where $\lfloor x\rfloor$ is the integer part of x and $\{x\}$ its fractional part, $0\leqslant\{x\}<1$, and that the nearest integer to x, denoted $[x]$, is defined as follows:

$$[x]=\begin{cases}\lfloor x\rfloor & \text{if } \{x\}<\frac{1}{2},\\ \lceil x\rceil & \text{if } \{x\}\geqslant\frac{1}{2}.\end{cases}$$

It is easy to verify that

$$[x]=\begin{cases}\dfrac{\lfloor 2x\rfloor}{2} & \text{if } 0\leqslant\{x\}<\frac{1}{2},\\ \dfrac{\lfloor 2x\rfloor+1}{2} & \text{if } \frac{1}{2}\leqslant\{x\}<1.\end{cases}$$

So we obtain:

$$[\sqrt{2kx+k^2}]=\begin{cases}\dfrac{\lfloor 2\sqrt{2kx+k^2}\rfloor}{2} & \text{if } 0\leqslant\{\sqrt{2kx+k^2}\}<\frac{1}{2},\\ \dfrac{\lfloor 2\sqrt{2kx+k^2}\rfloor+1}{2} & \text{if } \frac{1}{2}\leqslant\{\sqrt{2kx+k^2}\}<1,\end{cases}$$

which can be written, with $K=2k$ and $X=2x$,

$$[\sqrt{2kx+k^2}]=\begin{cases}\dfrac{\lfloor\sqrt{2KX+K^2}\rfloor}{2} & \text{if } 0\leqslant\{\sqrt{2kx+k^2}\}<\frac{1}{2},\\ \dfrac{\lfloor\sqrt{2KX+K^2}\rfloor+1}{2} & \text{if } \frac{1}{2}\leqslant\{\sqrt{2kx+k^2}\}<1,\end{cases}$$

and be rewritten

$$y=[\sqrt{2kx+k^2}]\Leftrightarrow$$
$$Y=\lfloor\sqrt{2KX+K^2}\rfloor \text{ with } K=2k, X=2x \quad\text{and}\quad Y=2y \text{ or } Y=2y-1.$$

This means that we can construct the parabola $y=[\sqrt{2kx+k^2}]$ from the parabola $Y=\lfloor\sqrt{2KX+K^2}\rfloor$ with $K=2k$, $X=2x$ and $Y=2y$ or $Y=2y-1$ grouping the cells

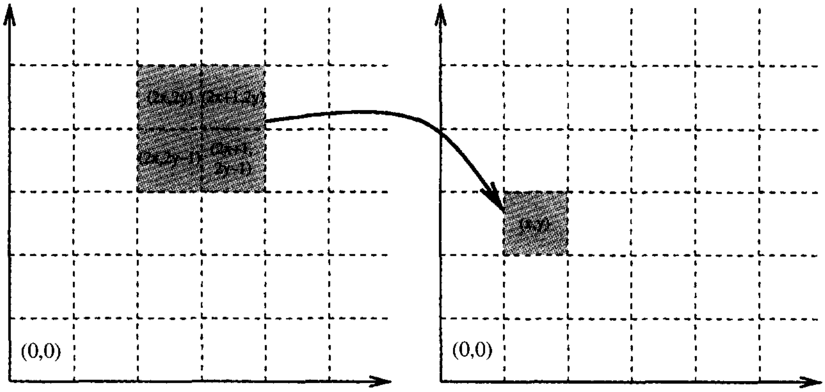


Fig. 58. How we group the cells.

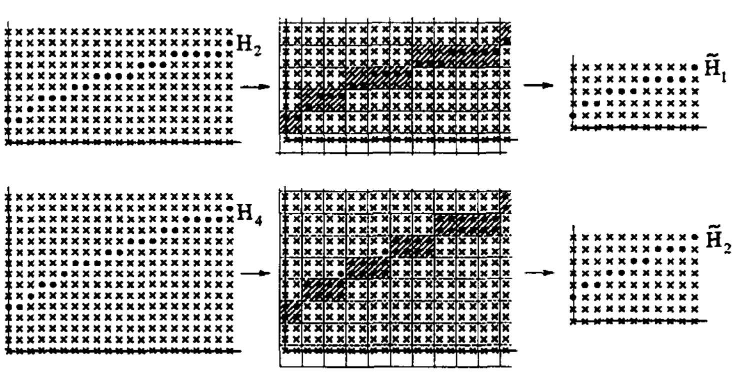


Fig. 59. How to obtain the parabolas $\tilde{H}_1$ and $\tilde{H}_2$.

as we have indicated it in Fig. 58. In fact, the cell (x, y) belongs to the parabola $\tilde{H}_k$ if and only if the cell $(2x, 2y)$ or the cell $(2x, 2y-1)$ belongs to the parabola H_{2k}.

Hence we have the following lemma:

Lemma 6.1. *There exists a 2-CA which, for all k, $k \geqslant 1$, constructs the parabola $y = [\sqrt{2kx + k^2}]$ from the parabola $y = \lfloor \sqrt{2(2k)x + (2k)^2} \rfloor$.*

Fig. 59 shows how to obtain the parabola $y = [\sqrt{2x+1}]$ from $y = \lfloor \sqrt{4x+4} \rfloor$ and the parabola $y = [\sqrt{2x+1}]$ from $y = \lfloor \sqrt{4x+4} \rfloor$, grouping the cells 4 by 4 and using the function $\Phi_{0,1,2}$ (cf. Section 1.6). The distinguished states for the grouped cellular automaton are the states two components of which are distinguished states for the automaton of Lemma 4.1.

6.1.2. Construction of the Pitteway's circles

Lemma 6.2. *The Pitteway's circle of radius R is obtained from the floor circle of radius 2R in "grouping the cells 4 by 4".*

Proof. We have proved in the previous paragraph that the bundle $\tilde{\mathscr{H}}$ is obtained from the bundle $\mathscr{H}$ using cells grouping. So, the points which are the intersection between the Pitteway's circle of radius R and the bundle $\tilde{\mathscr{H}}$ are known. We just have to examine the points which belong to the circle and which are between the parabolas. For this, we remark that we have the following equivalence: for all x and all y reals,

$$(x,y)\in C(0,R) \Leftrightarrow (2x,2y)\in C(O,2R).$$

Consequently, the points which are between the parabolas are obtained with the help of the same method of grouping. □

So we can directly use all the results we have proved for the floor circles to construct the Pitteway's circles.

Proposition 6.1. *There exists a 2-CA which constructs, from the initial configuration C_0, the Pitteway's circle of radius R ($R \geqslant 0$).*

Proof. From Lemma 6.2, we just have to construct the floor circle of radius $2R$ and to group the cell 4 by 4. □

We can also construct the family of Pitteway's circles.

Lemma 6.3. *There exists a 2-CA which constructs in real time, from the initial configuration C_{HV}, the family of Pitteway's circles.*

Proof. From Lemma 4.1, there exists a 2-CA which constructs, from the configuration C_{HV}, the floor circle of radius $2R$ at time $2R$. The grouped cellular automaton which is obtained from this automaton, simulates, at each time, two times of the former one. So, we obtain using cells grouping the Pitteway's circle of radius R, at time R from the initial configuration C_{HV}. □

And finally, we have:

Proposition 6.2. *There exists a 2-CA which, from the initial configuration C_0, constructs in real time the family of Pitteway's circles.*

Proof. From Lemma 4.1, the family of floor circle is generated by cellular automata in real time. Consequently, the floor circle of radius $2R$ is constructed at time $2R$. The grouping cellular automaton constructed from this automaton, simulates at each time two times of this one. So the necessary time to obtain the Pitteway's circle of radius R is R. □

6.2. Arithmetical circle

E. Andrès, in his master thesis [1], defines the *arithmetical circle* $C_{arith}(0,R)$ centered on $(0,0)$ and with radius R, as the solution of the following system of diophantian

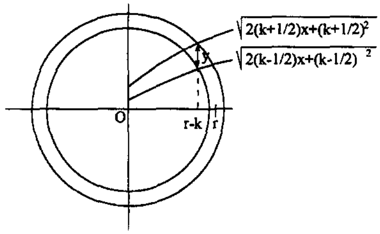


Fig. 60. Lemma 6.4.

equations:

$$(x,y)\in C_{arith}(O,R) \Leftrightarrow x^2+y^2\in[(R-\tfrac{1}{2})^2,(R+\tfrac{1}{2})^2[,$$

where $x,y\in\mathbb{Z}$ and $R\in\mathbb{N}$.

The points that belong to the arithmetical circle $C_{arith}(0,R)$ are the integer points which are between the real circle $C(O,R-\frac{1}{2})$ and the real circle $C(O,R+\frac{1}{2})$.

As we want to construct the arithmetical circle with a cellular automaton, we must transform this definition to put in light "local" properties which can be used by the automaton.

6.2.1. Link with the parabolas

We have the following lemma:

Lemma 6.4. *For every point* $(x,y)\in\mathbb{R}^2$ *such that* $x\geqslant y\geqslant 0$, *every* k *integer* $(k\geqslant 1)$ *we have*:

$$(r-\tfrac{1}{2})^2\leqslant x^2+y^2<(r+\tfrac{1}{2})^2 \Leftrightarrow$$
$$2(k-\tfrac{1}{2})x+(k-\tfrac{1}{2})^2\leqslant y^2<2(k+\tfrac{1}{2})x+(k+\tfrac{1}{2})^2 \quad \textit{and} \quad x=r-k.$$

Fig. 60 gives an illustration of this lemma. So, for a fixed first coordinate x, we are interested in the points which belong to the vertical segment of length 1 which is between the parabolas $y=\sqrt{2(k-\frac{1}{2})x+(k-\frac{1}{2})^2}$ and $y=\sqrt{2(k+\frac{1}{2})x+(k+\frac{1}{2})^2}$; we denote them by $h_{k-\frac{1}{2}}$ and $h_{k+\frac{1}{2}}$. Let $H_{k-\frac{1}{2}}$ and $H_{k+\frac{1}{2}}$ be the respective digitizations of the parabolas $h_{k-\frac{1}{2}}$ and $h_{k+\frac{1}{2}}$.

As the integer points which belong to the real circle of equation $x^2+y^2=(R+\frac{1}{2})^2$ do not belong to the arithmetical circle centered on $(0,0)$ and of radius R, we get: for every point (x,y) such that $x\geqslant y\geqslant 0$ (x and y are integers), every integer k $(k\geqslant 1)$,

$$(r-\tfrac{1}{2})^2\leqslant x^2+y^2<(r+\tfrac{1}{2})^2 \Leftrightarrow$$
$$\begin{cases} \bar{H}_{k-\frac{1}{2}}(x)\leqslant y\leqslant H_{k+\frac{1}{2}}(x) & \text{if } h_{k+\frac{1}{2}}(x) \text{ is not an integer,}\\ \bar{H}_{k-\frac{1}{2}}(x)\leqslant y<H_{k+\frac{1}{2}}(x) & \text{else.}\end{cases}$$

6.2.2. Parabolas $H_{k+\frac{1}{2}}$ and $\bar{H}_{k-\frac{1}{2}}$ for $k \geqslant 1$

In this part, we prove that the parabolas $H_{k-\frac{1}{2}}$ and $\bar{H}_{k-\frac{1}{2}}$ for $k \geqslant 1$, can be respectively obtained from the parabolas H_α and $\bar{H}_\alpha$ with $\alpha = 2k-1$ grouping the cells 4 by 4. So, to construct the arithmetical circle we will just have to consider the integer points which are between these parabolas.

We put $\alpha = 2k-1$. As the first part of this paragraph is available for $H_{k-\frac{1}{2}}$ and $\bar{H}_{k-\frac{1}{2}}$, we use the notation $I(x)$ which means $\lfloor x \rfloor$ or $\lceil x \rceil$. Then we have

$$y = I\left(\sqrt{2\left(k-\frac{1}{2}\right)x + \left(k-\frac{1}{2}\right)^2}\right) \Leftrightarrow y = I\left(\sqrt{\alpha x + \frac{\alpha^2}{4}}\right)$$

which can also be written as

$$y = I(\sqrt{2(k-\tfrac{1}{2})x + (k-\tfrac{1}{2})^2}) \Leftrightarrow y = I(\tfrac{1}{2}\sqrt{2\alpha(2x) + \alpha^2})$$

and, if we put $X = 2x$,

$$y = I(\sqrt{2(k-\tfrac{1}{2})x + (k-\tfrac{1}{2})^2}) \Leftrightarrow y = I(\tfrac{1}{2}\sqrt{2\alpha X + \alpha^2}). \quad (8)$$

But, for every positive real x, we have

$$\left\lfloor \frac{1}{2}x \right\rfloor = \begin{cases} \dfrac{\lfloor x \rfloor}{2} & \text{if } \lfloor x \rfloor \text{ even,} \\ \dfrac{\lfloor x \rfloor - 1}{2} & \text{else.} \end{cases}$$

As a matter of fact, $\lfloor \frac{1}{2}\{x\} \rfloor = 0$ and consequently, $\lfloor \frac{1}{2}x \rfloor = \lfloor \frac{1}{2}\lfloor x \rfloor \rfloor$.

Just as for the ceiling,

$$\left\lceil \frac{1}{2}x \right\rceil = \begin{cases} \dfrac{\lceil x \rceil}{2} & \text{if } \lceil x \rceil \text{ even,} \\ \dfrac{\lceil x \rceil + 1}{2} & \text{else} \end{cases}$$

because $\lceil \frac{1}{2}x \rceil = \lceil \frac{1}{2}\lceil x \rceil + \frac{1}{2}(\{x\}-1) \rceil$ and $\lceil \frac{1}{2}(\{x\}-1) \rceil = 0$.

If we apply this to (8), we obtain:

$$y = \left\lfloor \sqrt{2\left(k-\frac{1}{2}\right)x + \left(k-\frac{1}{2}\right)^2} \right\rfloor$$
$$= \begin{cases} Y = \lfloor \sqrt{2\alpha X + \alpha^2} \rfloor \text{ with } X = 2x, Y = 2y & \text{if } \lfloor \sqrt{2\alpha X + \alpha^2} \rfloor \text{ even,} \\ Y = \lfloor \sqrt{2\alpha X + \alpha^2} \rfloor \text{ with } X = 2x, Y = 2y + 1 & \text{else,} \end{cases}$$

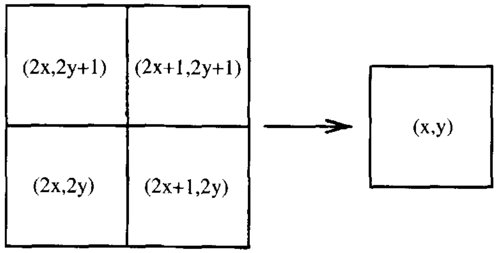


Fig. 61. How to obtain $\bar{H}_{k-\frac{1}{2}}$ from $\bar{H}_k$.

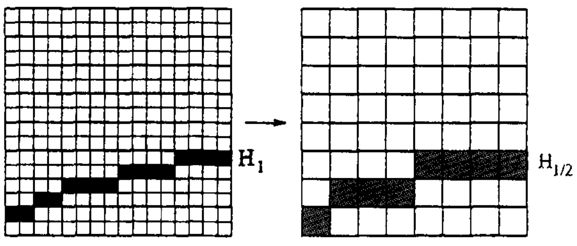


Fig. 62. How to obtain the parabola $H_{\frac{1}{2}}$ from the parabola H_1.

and

$$y = \left\lceil \sqrt{2\left(k-\frac{1}{2}\right)x + \left(k-\frac{1}{2}\right)^2} \right\rceil$$
$$= \begin{cases} Y = \lceil\sqrt{2\alpha X + \alpha^2}\rceil \text{ with } X = 2x, Y = 2y & \text{if } \lceil\sqrt{2\alpha X + \alpha^2}\rceil \text{ even,} \\ Y = \lceil\sqrt{2\alpha X + \alpha^2}\rceil \text{ with } X = 2x, Y = 2y-1 & \text{else.} \end{cases}$$

So, the parabolas $H_{k-\frac{1}{2}}$ with k integer ($k \geqslant 1$) are obtained from the parabolas H_α with $\alpha = 2k-1$, by grouping the cells 4 by 4 as indicated in Fig. 61.

The parabolas $\bar{H}_{k-\frac{1}{2}}$, k integer ($k \geqslant 1$), are also obtained using cells grouping from the parabolas $\bar{H}_\alpha$ with $\alpha = 2k-1$. But, it is not the same way of grouping; we proceed as it is indicated in Fig. 58.

Fig. 62 gives the example of the parabola $H_{\frac{1}{2}}$.

6.2.3. Construction of the arithmetical circles by cellular automata

Proposition 6.3. *There exists a 2-CA which constructs the family of arithmetical circle centered on* $(0,0)$ *from the initial configuration* C_0.

Proof. As we have proved it we can see the arithmetical circle (in the first octant) as a set of vertical, diagonal and horizontal segments based on the parabolas $H_{k-\frac{1}{2}}$ and

$\bar{H}_{k-\frac{1}{2}}$, and these last ones are constructible from H_α and $\bar{H}_\alpha$. So, the cellular automaton which constructs the arithmetical circle must generates:

- the parabolas H_k for $k \geqslant 1$, the algorithm is described in 3,
- the parabolas $\bar{H}_k$ for $k \geqslant 1$, see Section 5.0.8,
- the parabolas $H_{k-\frac{1}{2}}$ and $\bar{H}_{k-\frac{1}{2}}$. We only consider the parabolas H_k and $\bar{H}_k$ such that k is odd (marked by new states) and we group the cells 4 by 4 as it is indicated in Section 6.2.2, that is to say by 4 *via* $\Phi_{0,0,2}$ and $\Phi_{0,-1,2}$ and the set of the distinguished states of the grouping cellular automaton are the ones the component $(0,0)$ or $(0,1)$ of which is distinguished for the cellular automaton of Lemma 4.1.
- and, finally, the signals that describe the circles (we are going to precise them in the following).

Fig. 63 sums up the different constructions which allow to obtain the arithmetical circles. In the first octant, the signal that describe the arithmetical circle of radius R is initialized by the cell $(R,0)$. It evolves in real time in the direction of the increasing ordinates until it reaches the parabola $H_{\frac{1}{2}}$ except if $h_{\frac{1}{2}(r)}$ is an integer. In this case, the parabolas $H_{\frac{1}{2}}$ and $\bar{H}_{\frac{1}{2}}$ meet. Then, it goes one cell up on the left: then it is on the parabola $\bar{H}_{\frac{1}{2}}$. And so on until it reaches the first diagonal. Fig. 63 shows this signal for various length of circles. □

One the initial and basic stage of our work was the observation of Fig. 5. To finish, we will show that building of the bundles H_k and $\bar{H}_k$ by cellular automata lead to the possibility to get Fig. 5 by means of a cellular automaton, that means that it is possible to mark the different type of intersections between the family $\mathcal{ON}$ of real circles and the grid by a cellular atomaton.

6.3. *Construction of Fig. 5*

6.3.1. *Intersections between a real circle of fixed radius and the grid*

We will only consider the intersections between the real circle of radius R and the grid in the first octant. So, we are interested in the squares the vertices of which are integer points (x,y) such that $x \geqslant 0$ and $y \leqslant x+1$. We will call A_0, A_1, A_2 and A_3 the edges of a square of the grid, P_0, P_1, P_2 and P_3 its vertices, as indicated in Fig. 64.

First, we can remark that for a fixed first coordinate x $(x \geqslant R)$, the circle crosses an edge or goes through a vertex of the grid. Fig. 65 shows these intersections.

1. *Case of a vertex*: Let $P(x_P, y_P)$ be a vertex of a square C. If P belongs to the circle $\mathscr{C}(O,R)$, $x_P^2 + y_P^2 = R^2$ (9) where x_P and y_P are two integers. But, (9) $\Leftrightarrow y_P^2 = 2kR - k^2$ if we put $x_P = R-k$, and (9) $\Leftrightarrow y_P^2 = 2kx_P + k^2$ if we substitute $x_P + k$ to R. Consequently, for a fixed first coordinate x_P, the circle goes through the point P if and only if $y_P = \sqrt{2kx_P + k^2}$ with $k \geqslant 1$, x_P and y_P integers. This kind of points has already been studied in Section 5.0.7, where it has been proved that there exists a 2-CA that mark them. So, there exists a 2-CA that localizes the places where the circle goes through a point of the grid.

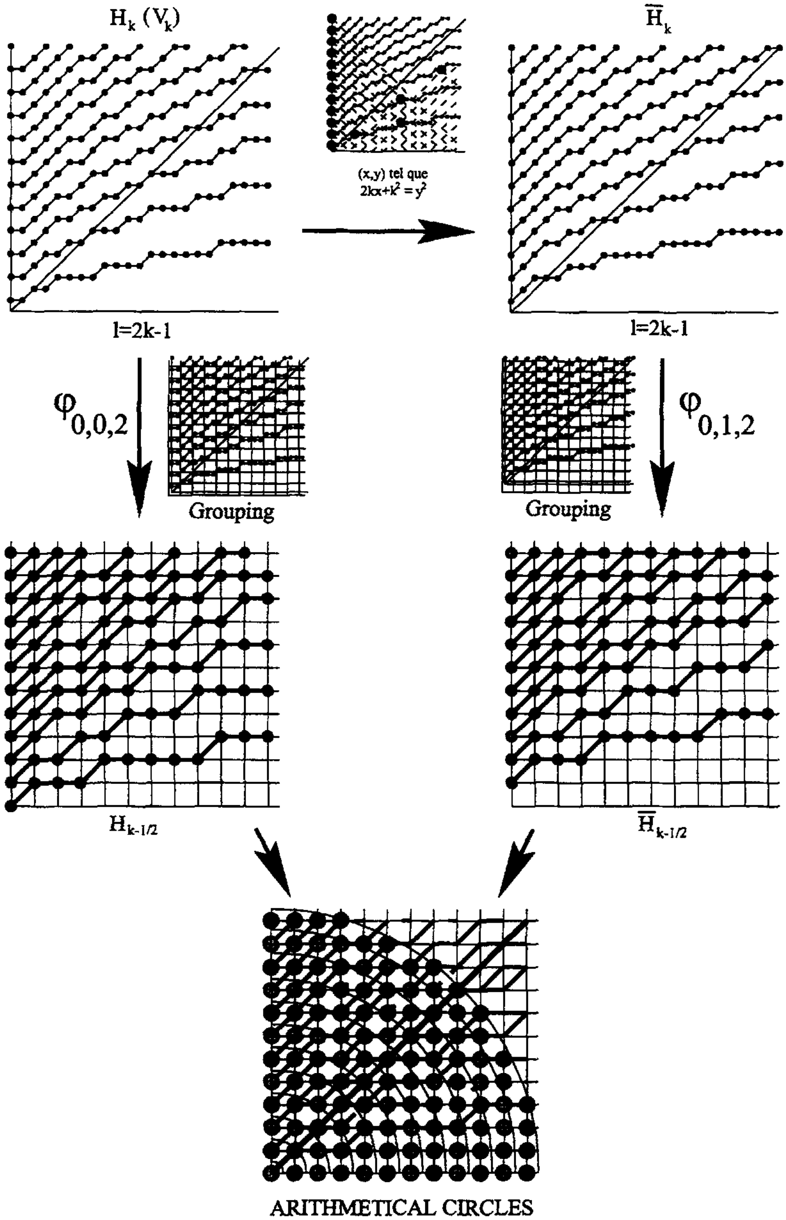


Fig. 63. How to obtain the arithmetical circles from the parabolas H_k for $k \geqslant 1$.

2. We denote by (α, β) the point which is the intersection between the circle and the straight line $x = \alpha$ in the part of the plane we consider. We suppose that β is not an integer and we denote C the square of the grid such that (α, β) belongs to the edge $[P_1, P_2]$ of C.

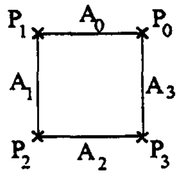


Fig. 64. A square of the grid.

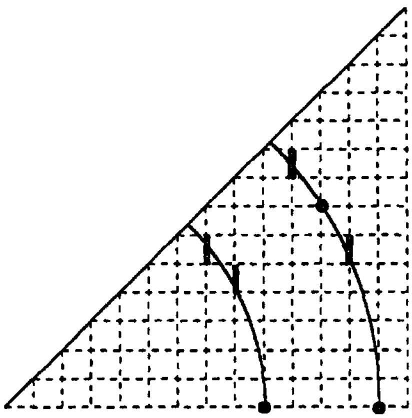


Fig. 65. Intersections with the grid.

So $\alpha^2+\beta^2=R^2$ (10) and (10) $\Leftrightarrow \beta^2=2k\alpha+k^2$ with $k=R-\alpha$.

As the points P_1 and P_2 are the extremities of the edge, $y_{P_1}=\lceil\sqrt{2k\alpha+k^2}\rceil$ and $y_{P_2}=\lfloor\sqrt{2k\alpha+k^2}\rfloor$.

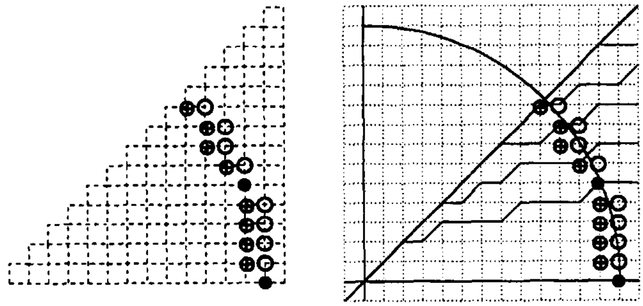
As the parabolas H_k and $\bar{H}_k$ are constructible with cellular automata, we can also distinguish by cellular automata the cases where the circle cuts a vertical edge.

Finally, for a fixed first coordinate x, we know the square of the grid which is crossed and, moreover, we can say whether it is crossed on an edge or on a vertex.

So we get:

Lemma 6.5. *There exists a 2-CA which, from the initial configuration C_1 such that all the cells are in a quiescent state except two: the cell $(0,0)$ and the cell $(R,0)$, indicates the edge and the vertices of the squares of the grid that are crossed by the real circle of radius R.*

Proof. The configuration we want to get at the end is represented in Fig. 66(a). An edge is crossed by the circle if and only if its two extremities are not in a quiescent state on the figure. The black points corresponds to the vertices of the squares that belong to the circle. Fig. 66(b) shows the position of the circle of radius 13 according to the marked points. Let us consider the evolution of the two following signals between two parabolas: the first one is composed of the white points and the second one of the



(a) (b)

Fig. 66. Final configuration for the circle of radius 13 (a) and the real circle (b).

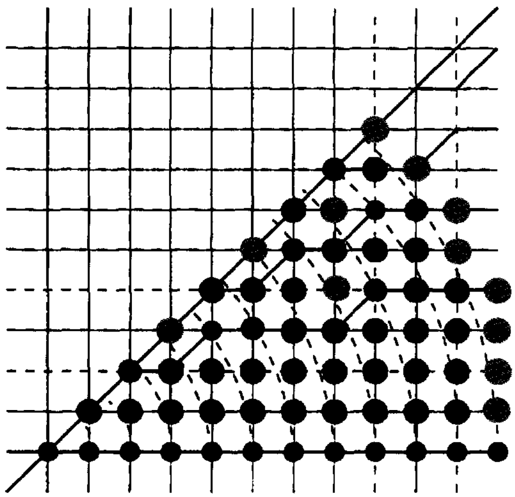


Fig. 67. Intersections between the grid and the family of concentric circles obtained by cellular automaton.

points with a cross (the black points belong to the two signals). If at the first coordinate x $(x<R)$, the real circle cuts a vertical edge, we have seen that the vertices P_1 and P_2 of the corresponding square respectively belong to the parabolas $\bar{H}_k$ and H_k with $k=R-x$. And, if the circle goes through a vertex, this one is such that $H_k(x)=\bar{H}_k(x)$. It is the same for the first coordinate $x-1$. Consequently, between the vertical edge or the vertex which is crossed at the first coordinate x and, the edge or the vertex which is crossed at the first coordinate $(x-1)$, the real circle only cuts horizontal edges. The initialization of these signals is realized by the cell $(R,0)$ such that $H_0(R)=\bar{H}_0(R)$. □

6.3.2. Fig. 5

Coming back to Fig. 5, we get:

Theorem 6.1. *There exists a 2-CA which, from the initial configuration C_0, says for each square of the grid how it is cut by the family of circles* $\odot$.

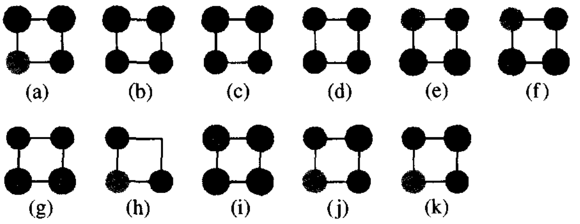


Fig. 68. The different kinds of intersection constructed by cellular automaton.

Proof. We just have to apply the algorithm described in the proof of Lemma 6.5 for all the circles of the family. We obtain Fig. 67.

Fig. 68 indicates how to obtain Fig. 5 from Fig. 67. □

Corollary 6.1. *The characterization of the intersections by cellular automaton is realized in real time.*

Proof. The principle of this automaton is the same as the principle of the one that constructs the floor circles in real time (cf. Lemma 4.1). At each time, the automaton defines the intersections with a new circles with the help of the parabolas. □

7. Conclusion

In this paper, we essentially succeed in conceiving a cellular that builds in real time Fig. 5. This is based on some noticeable facts.

The first one is that the family $\odot$ is stable by dilatation and that the grouping method allows to realize on cellular automata such transformations. One of the efficient consequences is that instead of studying the intersection of $\odot$ with the grid, it is enough to study the intersection of one circle of $\odot$ with one mesh.

The choice of the grid as underlying partition of $\mathbb{R}^2$ allows to consider only the sides of the meshes, more precisely two adjacent sides, but which cannot be reduced to one of them by symmetry. This implies that we come down to the well known digitizations through $\lfloor\,\rfloor$ and $\lceil\,\rceil$, (thus even to one of them), generating all the possible topologies on this grid.

We observe that we managed to build in real time all the well-known discrete circles but the Bresenham's one. Knowing the parabolas bundles, it is not difficult starting from a point of $\vec{Ox}-coordinates$, to construct one Bresenham's circle (see [14]). This is not really interesting in cellular automata point of view. A real time construction depends on the existence of a local definition of Bresenham's circle.

One issue arises: is it possible to build, in an analogous way, other families of curves? That set the question of the supporting curves (the parabolas in the present work) and the one of the dilatation stability of the family.

Due to the universality of 2-CA, to construct recursive families of curves with 2-CA is not a problem. The point is to get them quickly, even as soon as possible. We know that it is possible for some non monotonic curves or for curves other than conics. It would be interesting to know whether some families could be constructed aid of quadratic supporting curves (for example the family of hyperbolas centered at the origin, with $\vec{Ox}$ as symmetry axis and going through $(0,n)$), or the family of ellipsis centered at the origin and going through $(0,n)$), or aid of polynomial supporting curves of higher degree (for example some families of strophoïds).

On another hand, from a cellular automata point of view, it could be interesting to compare 2D-CA and 3D-CA, in studying real-time generation of spheres centered at the origin and with natural integer radius.

All the previous remarks show close links with geometry. But, we have still to notice that an interesting fact in this work is the connection which arises with arithmetics, and which should be deeper studied.

The Moore's neighborhood has been very efficient to achieve our constructions. But, in 2-CA , the most usual neighborhood is the Von Neumann's one with 4 neighbors. Classical cellular simulations between Moore and von Neumann neighborhoods allow to conclude that the family $\odot$ may be set up with von Neumann's neighborhood following times $2i$, $i \in \mathbb{N}$. Can this family be constructed using Von Neumann's neighborhood as soon as possible?

Looking to Fig. 11 and to the family $\odot$ drawn on a computer screen with an incrementation step of two pixels, we observe some "moirés" effects. It would be worthy of studying these effects. In fact, moirés are well known as interaction between two families of curves. To describe analytically their positions may allow to give digitizations with another moirés curves. These moirés require more attention. This point is connected with human vision. Is it possible to see moirés founded on non-quadratic equations?

The notion of grouping has been revealed efficient for, and well adapted to, the family $\odot$. Among other results, it allows to generalize the construction using von Neumann's neighborhood to constructions using every neighborhood induced by archimedean tilings.

The cellular automata that construct the floor parabolas and the floor circles have been tested on computer. The states of these cellular automata were defined as uplets: one component per kind of signal (13 components) and the number of states by component never exceeded 5. Thus, the number of states is less than 5^{13} but we have not tried to optimize them.

Acknowledgements

We would like to thank Eric Remila who pointed out the present proof of Proposition 2.2, which is a shorter version of the first one.

References

[1] E. Andrès, Cercles discrets et rotations discrètes, PhD thesis, Université Louis Pasteur, Strasbourg, 1994.

[2] J.E. Bresenham, Algorithm for computer control of a digital plotter, IBM Systems J. 4 (1) (1965) 25–30.

[3] D. D'Humières, Numerical experiments on lattice gases: mixtures and Galilean invariances, Complex Systems 1 (1987) 633–647.

[4] M.A. Jacob, E. Andrès, On discrete rotations, Discrete Geometry for Computer Imagery, 1995, pp. 161–174.

[5] C.E. Kim, Digital disks, IEEE Trans. Pattern Anal. machine Intell. PAMI-6(3) (1984).

[6] Z. Kulpa, On the properties of discrete circles, rings, and disks, Comput. Graphics Image Process. 10 (1979) 348–365.

[7] M. Marcus, B. Hess, Isotropic cellular automaton for modelling excitable media, Nature 347 (6288) (September 1990) 56–58.

[8] J. Mazoyer, Entrées et sorties sur lignes d'automates, in: Algorithmique Parallèle, 1992.

[9] A. Nakamura, K. Aizawa, Digital circles, Computer Vision 26 (1984) 242–255.

[10] R. Peters, Recursive Functionen, Academaiai Kiadó, Budapest, 1951.

[11] M.L.V. Pitteway, Integer circles, etc.- some further thoughts, Comput. Graphics Image Process. 3 (1974) 262–265.

[12] Zs. Róka, Automates cellulaires sur graphes de Cayley, PhD thesis, Ecole Normale Supérieure de Lyon, 1994.

[13] H.E. Schepers, M. Marcus, Two types of performance of an isotropic cellular automaton: stationary (turing) patterns and spiral waves, Physica A 188 (1992) 347–343.

[14] L. Tougne, Cercles discrets sur automates cellulaires, PhD thesis, Ecole Normale Supérieure de Lyon, January 1997.