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A poset structure on quasifibonacci partitions

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ABSTRACT

In this paper, we study partitions of positive integers into distinct quasifibonacci numbers. A digraph and poset structure is constructed on the set of such partitions. Furthermore, we discuss the symmetric and recursive relations between these posets. Finally, we prove a strong generalization of Robbins' result on the coefficients of a quasifibonacci power series.

1. Introduction and statement of results

Let F_k denote the Fibonacci numbers (where we have shifted the usual initial condition); i.e. $(F_1, F_2, \dots) = (1, 2, 3, 5, 8, \dots)$.

Consider the formal power series

$$\begin{aligned} H(x) &= \prod_{k\geqslant 1}\left(1-x^{F_k}\right) \\ &= (1-x)\left(1-x^2\right)\left(1-x^3\right)\left(1-x^5\right)\left(1-x^8\right)\cdots \\ &= 1-x-x^2+x^4+x^7-x^8+x^{11}-x^{12}-x^{13}+x^{14}+\cdots. \end{aligned}$$

Let h_m be the coefficient of x^m in $H(x)$. It is clear that $h_m = h_m^+ - h_m^-$, where h_m^+ is the number of partitions of m into an even number of distinct Fibonacci numbers, and h_m^- is the number of partitions of m into an odd number of distinct Fibonacci numbers.

In [2], N. Robbins proved that $h_m \in \{-1, 0, 1\}$. In [1], F. Ardila gave a simpler proof for Robbins' result by giving a recursion on h_m.

In this paper, we consider *quasifibonacci numbers*, which serve as generalization of Fibonacci numbers.

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Definition 1.1. Given a positive integer $N \geqslant 2$. A sequence $A_1, A_2, \ldots$ of positive integers is called *quasifibonacci sequence of level N* if

- $A_{k+N} = A_{k+N-1} + \cdots + A_k$ for all $k \in \mathbb{Z}^+$, and
- $A_k > A_{k-1} + \cdots + A_1$ for all $1 \leqslant k \leqslant N$.

We also say that $A_1, A_2, \ldots$ are *quasifibonacci numbers.*

In particular, (shifted) Fibonacci numbers and Lucas numbers are quasifibonacci.

As we will see in this paper, Robbins' result can be generalized to *quasifibonacci numbers*. More precisely, we shall prove the following theorem:

Theorem 1.1. *Let $A_1, A_2, \ldots$ be a quasifibonacci sequence of even level. Consider the formal power series*

$$\begin{aligned} H(x) &= \prod_{k \geqslant 1} \left(1 - x^{A_k}\right) \\ &= \left(1 - x^{A_1}\right)\left(1 - x^{A_2}\right)\left(1 - x^{A_3}\right)\left(1 - x^{A_4}\right)\cdots \\ &= 1 + \sum_{m \geqslant 1} h_m x^m. \end{aligned}$$

Then $h_m \in \{-1, 0, 1\}$.

Similarly, we have $h_m = h_m^+ - h_m^-$ where h_m^+ (respectively h_m^-) is the number of partitions of m into an even (respectively odd) number of distinct quasifibonacci numbers.

In this paper, we study the structure of the set of such partitions. In fact, a digraph and poset structure on such sets will be constructed in Section 3. In Section 4, we will unveil intrinsic symmetry and recursive relations between these posets. Finally, as an application, we shall prove Theorem 1.1 in Section 5.

2. Notations

Definition 2.1. The following notations will be used throughout the paper.

- $\{0,1\}^\omega := \{(a_1, a_2, \ldots) \mid a_i \in \{0,1\},\ a_i = 0 \text{ for all but finitely many } i\text{'s}\}$.
- Given a quasifibonacci sequence $A_1, A_2, \ldots$, define

$$\begin{aligned} S_n &= S_n\left(\{A_k\}\right) \\ &:= \left\{ (a_1, a_2, \ldots) \in \{0,1\}^\omega \;\middle|\; \sum_{i=1}^{\infty} a_i A_i = n, a_i \in \{0,1\} \right\}. \end{aligned}$$

  S_n represents the set of partitions of n into distinct quasifibonacci numbers A_k.
- For $a = (a_1, a_2, \ldots) \in \{0,1\}^\omega$, define

$$s(a) := \sum_{i=1}^{\infty} a_i A_i.$$

  We also say that a is the *representation* of $s(a)$.
- For $k \geqslant N + 1$, define

$$A_{k,0} := A_k + \sum_{\substack{1 \leqslant i < k-N-1 \\ N \nmid i}} A_{k-N-1-i},$$

$$A_{k,1} := A_k + \sum_{\substack{1 \leqslant i < k-N \\ N \nmid i}} A_{k-N-i},$$

$$A_{k,2} := A_k + \sum_{\substack{1 \leqslant i < k-N+1 \\ N \nmid i}} A_{k-N+1-i}.$$

- For any $(a_1, a_2, \ldots) \in S_n$, define the *length* $l(a)$ to be the largest i such that $a_i = 1$. (Abusing the notation, we also identify a with $(a_1, a_2, \ldots, a_{l(a)})$, which is called the *reduced representation*.)

The next lemma gives some important arithmetic properties of the A_k's which will be used frequently throughout the paper.

Lemma 2.1. *Let $A_1, A_2, \ldots$ be a quasifibonacci sequence. Then*

(1) $A_{k+2} > A_k + A_{k-1} + \cdots + A_1$ *for any* $k \in \mathbb{Z}^+$,
(2) *for* $A_k \leqslant n < A_{k+1}$ *and any* $a \in S_n$, *we have* $l(a) \in \{k-1, k\}$,
(3) *for* $A_1 + A_2 + \cdots + A_{k-1} < n < A_{k+1}$ *and any* $a \in S_n$, *we have* $l(a) = k$,
(4) $$\sum_{\substack{1 \leqslant i \leqslant k-1 \\ N \nmid i}} A_{k-i} < A_k,$$
(5) $A_k < A_{k,0} < A_{k,1} < A_{k,2} < A_{k+1}$,
(6) $A_1 + A_2 + \cdots + A_k > 2A_{k,0}$.

Proof. (1) By the definition of quasifibonacci numbers, it is clear that $A_{k+2} > A_{k+1} \geqslant A_k + \cdots + A_1$ when $1 \leqslant k \leqslant N$. The case for $k > N$ follows from induction. Actually, we use the following inequality:

$$A_{k+2} = A_{k+1} + \cdots + A_{k-N+2} \geqslant A_{k+1} + A_k$$

and then apply the induction hypothesis $A_{k+1} > A_{k-1} + \cdots + A_1$ to replace A_{k+1}.

(2) It is clear that $l(a) \leqslant k$. Suppose $l(a) \leqslant k-2$. By the previous lemma, $n \leqslant A_{k-2} + \cdots + A_1 < A_k$, a contradiction. Therefore, $l(a) \geqslant k-1$.

(3) This is straightforward.

(4) Write $k-1 = Nk_1 + r$ $(0 \leqslant r \leqslant N-1)$. Then

$$\begin{aligned}
\sum_{\substack{1 \leqslant i \leqslant k-1 \\ N \nmid i}} A_{k-i} &= \sum_{j=1}^{k_1} \sum_{i=1}^{N-1} A_{k-jN+i} + \sum_{i=1}^{r} A_i \\
&= \sum_{j=1}^{k_1} (A_{k-jN+N} - A_{k-jN}) + \sum_{i=1}^{r} A_i \\
&= A_k - A_{r+1} + \sum_{i=1}^{r} A_i \\
&< A_k.
\end{aligned}$$

(5) It is obvious that $A_k < A_{k,0} < A_{k,1} < A_{k,2}$. By (4),

$$\begin{aligned}
A_{k,2} &= A_k + \sum_{\substack{1 \leqslant i < k-N+1 \\ N \nmid i}} A_{k-N+1-i} \\
&< A_k + A_{k-N+1} \\
&\leqslant A_k + A_{k-1} \\
&\leqslant A_{k+1}.
\end{aligned}$$

(6) By (4), we have

$$\begin{aligned}
2A_{k,0} &= A_k + A_k + \sum_{\substack{1 \leqslant i < k-N-1 \\ N \nmid i}} A_{k-N-1-i} + \sum_{\substack{1 \leqslant i < k-N-1 \\ N \nmid i}} A_{k-N-1-i} \\
&< A_k + (A_{k-1} + \cdots + A_{k-N}) + \sum_{\substack{1 \leqslant i < k-N-1 \\ N \nmid i}} A_{k-N-1-i} + A_{k-N-1} \\
&\leqslant A_1 + A_2 + \cdots + A_k. \qquad \square
\end{aligned}$$

3. A digraph and poset structure on S_n

For each $n \geqslant 1$, we construct a digraph $G_n := G_n(\{A_k\})$ in the following way:

(1) Set $V(G_n) = S_n$. In particular, set $G_n = \emptyset$ if $S_n = \emptyset$.
(2) For $a = (a_1, a_2, \ldots)$, $b = (b_1, b_2, \ldots) \in S_n$, let $(a, b) \in E(G_n)$ if there exists $k \in \mathbb{Z}^+$ such that
- $a_{k+N} = 1$, $a_k = a_{k+1} = \cdots = a_{k+N-1} = 0$,
- $b_{k+N} = 0$, $b_k = b_{k+1} = \cdots = b_{k+N-1} = 1$,
- $a_t = b_t$ for all $t \notin \{k, k+1, \ldots, k+N\}$.

(Here (u, v) represents the directed edge $u \to v$.)

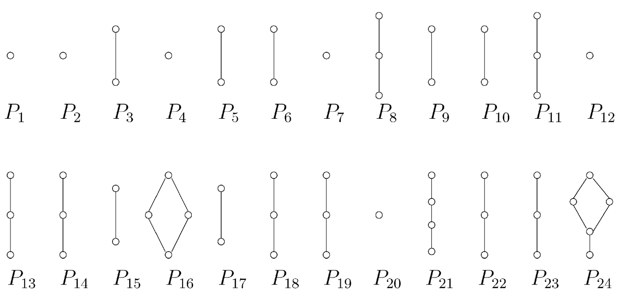
The digraph structure induces a partial order on S_n as follows:

For $a, b \in S_n$, set $a \geqslant b$ if there exists a path in G_n from a to b.

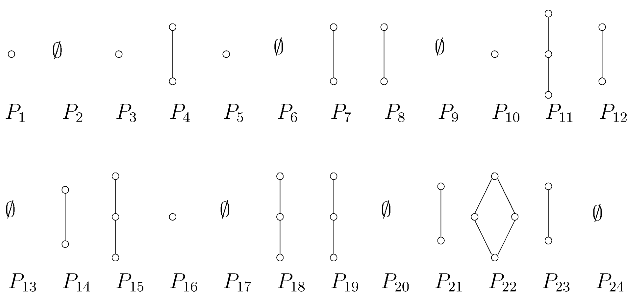
In other words, a covers b if and only if $(a, b) \in E(G_n)$. We call this partial order the *carry forward order* on S_n.

This makes S_n into a poset P_n. The following examples show the corresponding Hasse diagrams for $P_n(\{F_k\})$ and $P_n(\{L_k\})$, where $\{F_k\}$ and $\{L_k\}$ denote the Fibonacci numbers and Lucas numbers, respectively.

Example 3.1. First 24 Hasse diagrams for $P_n(\{F_k\})$:



Example 3.2. First 24 Hasse diagrams for $P_n(\{L_k\})$:



Comment: In general, we can construct digraph and poset structure on any finite subset of $\{0,1\}^\omega$. The corresponding partial order is still called the *carry forward order*.

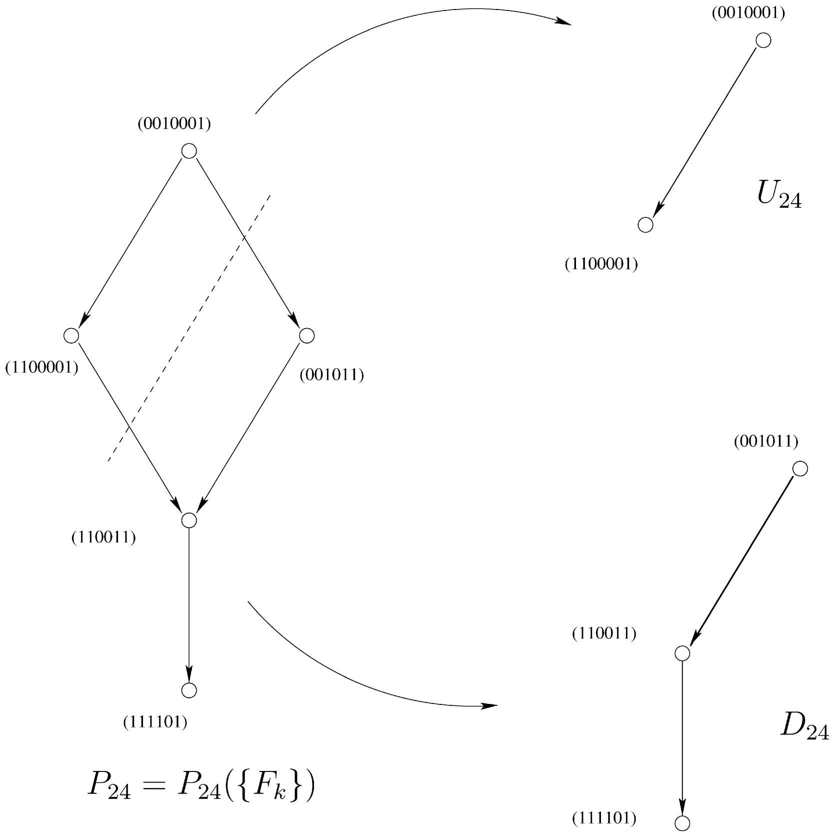
For $A_k \leqslant n < A_{k+1}$, define

$$T_n := \{a \in S_n \mid l(a) = k\},$$
$$R_n := \{a \in S_n \mid l(a) = k-1\}.$$

By Lemma 2.1(2), we have $S_n = T_n \cup R_n$.

Furthermore, let U_n, D_n denote the subposet of P_n restricted on the vertex set T_n and R_n, respectively. (Abusing the notation, U_n, D_n also denote the corresponding subdigraphs of G_n.)

Example 3.3. The figure below shows that P_{24} can be decomposed into U_{24} and D_{24}.



Now we study the structure of the posets P_n in detail.

Proposition 3.1. *Suppose $S_n \neq \emptyset$. Then there is a unique maximal element $\hat{1}$ in P_n. More precisely, $\hat{1}$ is the only element in S_n which does not contain N consecutive 1's.*

Furthermore, if $A_k \leqslant n < A_{k+1}$, then $l(\hat{1}) = k$.

Proof. Suppose $a \in S_n$ is a maximal element in P_n. It is clear that a does not contain N consecutive 1's. (Otherwise, assume that $a_k = a_{k+1} = \cdots = a_{k+N-1} = 1$ and $a_{k+N} = 0$. Set $b = (b_1, b_2, \ldots)$ where $b_k = b_{k+1} = \cdots = b_{k+N-1} = 0$, $b_{k+N} = 1$ and $b_t = a_t$ for $t \notin \{k, k+1, \ldots, k+N\}$. Then $b \in P_n$ but $b > a$.)

Now we show the uniqueness. Suppose both a and a' are maximal elements in P_n. Let k be the largest index such that $a_k \neq a'_k$. Without loss of generality, assume that $a_k = 0$, $a'_k = 1$ and $a_t = a'_t = 0$ for $t > k$. Since a is maximal, it does not contain N consecutive 1's. It follows that

$$s(a) \leqslant \sum_{\substack{1\leqslant i\leqslant k-1\\ N\nmid i}} A_{k-i}$$

which is the largest possible value of length $k-1$ with no N consecutive 1's.

However, Lemma 2.1(4) gives

$$s(a) \leqslant \sum_{\substack{1\leqslant i\leqslant k-1\\ N\nmid i}} A_{k-i} < A_k \leqslant s(a'),$$

a contradiction. Hence the maximal element is unique. Denote it by $\hat{1}$.

Now suppose $A_k \leqslant n < A_{k+1}$. By Lemma 2.1(2), $l(\hat{1}) \in \{k, k-1\}$. As proved above, any element of length $k-1$ with no N consecutive 1's is smaller than A_k. So we must have $l(\hat{1}) = k$. $\square$

Corollary 3.2. *Suppose*

$$n > \sum_{\substack{1\leqslant i\leqslant k-1\\ N\nmid i}} A_{k-i}$$

and $S_n \neq \emptyset$. Then $n \geqslant A_k$.

Proof. Assume that

$$\sum_{\substack{1\leqslant i\leqslant k-1\\ N\nmid i}} A_{k-i} < n < A_k.$$

Then we must have $l(\hat{1}) \leqslant k-1$. Thus

$$n = s(\hat{1}) < \sum_{\substack{1\leqslant i\leqslant k-1\\ N\nmid i}} A_{k-i},$$

a contradiction. $\square$

Similarly, we have the following dual result:

Proposition 3.3. *Suppose $S_n \neq \emptyset$. Then there is a unique minimal element $\hat{0}$ in P_n. More precisely, $\hat{0}$ is the only element in S_n which does not contain N consecutive* 0*'s in reduced representation.*

Although we have $l(\hat{1}) = k$ when $A_k \leqslant n < A_{k+1}$, we do not have $l(\hat{0}) = k-1$ in general.

Furthermore, in Section 5, we will show that P_n is a modular lattice.

4. Symmetry and recursions

If we view $P_1, P_2, \ldots$ as a sequence, then there exist local symmetry relations between the posets. For instance, in Example 3.1, the posets are central symmetric from P_7 to P_{12}, and from P_{12} to P_{20}. In general, similar symmetry appears for all quasifibonacci sequences.

In order to describe this special symmetry relation, we recall the definition of *dual posets*.

Definition 4.1. Two posets P, Q are *dual posets* to each other if there exists an order-reversing bijection $\phi : P \to Q$ whose inverse is also order-reversing; that is

$$x \leqslant y \quad \text{in } P \quad \Leftrightarrow \quad \phi(y) \leqslant \phi(x) \quad \text{in } Q.$$

Proposition 4.1. *For $A_k \leqslant n < A_{k+1}$, let $n' = A_1 + A_2 + \cdots + A_k - n$. Then P_n is dual to $P_{n'}$.*

Proof. Define $\phi : P_n \to P_{n'}$ by setting $(a_1, a_2, \ldots, a_k) \mapsto (1 - a_1, 1 - a_2, \ldots, 1 - a_k)$. (Note that a_k is not necessarily nonzero.)

It is easy to check that $(a, b) \in E(G_n)$ if and only if $(\phi(b), \phi(a)) \in E(G_{n'})$. Hence P_n is dual to $P_{n'}$ via ϕ. □

Being more careful, we can derive similar symmetry on U_n and D_n.

Proposition 4.2. *For $A_k \leqslant n < A_k + A_{k-1}$, let $n' = A_1 + A_2 + \cdots + A_k - n$. Then U_n is dual to $D_{n'}$, and D_n is dual to $U_{n'}$.*

Proof. It suffices to show that $\phi(U_n) = D_{n'}$ and $\phi(D_n) = U_{n'}$.

Let $a = (a_1, a_2, \ldots, a_k) \in U_n$ ($a_k = 1$). Since $n - A_k < A_{k-1}$, $a_{k-1} = 0$. Thus $1 - a_k = 0$, $1 - a_{k-1} = 1$. Hence $l(\phi(a)) = k - 1$.

On the other hand, for any $b = (b_1, b_2, \ldots, b_k) \in D_n$, we must have $1 - b_k = 1$. So $l(\phi(b)) = k$.

Therefore, $\phi(U_n) = D_{n'}$ and $\phi(D_n) = U_{n'}$, as desired. □

Other than symmetry, there are intrinsic recursive relations in the poset sequence $\{P_n\}$. In order to describe the recursion clearly, we introduce the following notations.

Definition 4.2. Suppose $a = (a_1, a_2, \ldots)$, $b = (b_1, b_2, \ldots) \in \{0, 1\}^\omega$ satisfy $a_t = 0$ whenever $b_t = 1$. Define

$$a + b := (a_1 + b_1, a_2 + b_2, \ldots).$$

Similarly, suppose $a = (a_1, a_2, \ldots)$, $b = (b_1, b_2, \ldots) \in \{0, 1\}^\omega$ satisfy $a_t = 1$ whenever $b_t = 1$. Define

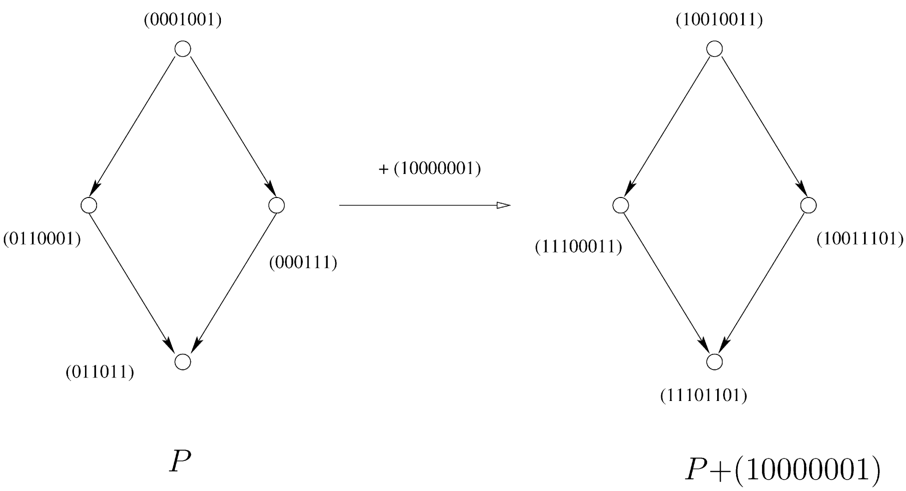
$$a - b := (a_1 - b_1, a_2 - b_2, \ldots).$$

Definition 4.3. Let P be a finite subset of $\{0, 1\}^\omega$. The poset structure on P is determined by the *carry forward order*. Let $b = (b_1, b_2, \ldots, b_k) \in \{0, 1\}^\omega$. Suppose that for any $a = (a_1, a_2, \ldots) \in P$, we have $a_t = 0$ whenever $b_t = 1$. Define

$$P + b = \{a + b \mid a \in P\},$$

regarded as a poset, with the induced partial order.

Example 4.1. The following figure gives an example for $P = P_{26}(\{F_k\})$ and $b = (10000001)$:



We can also define the addition between the posets:

Definition 4.4. Let P, Q be disjoint finite subsets of $\{0,1\}^\omega$ with digraph and poset structure defined in Section 3. Let P_1, Q_1 be subposets of P, Q, respectively. Suppose there is a bijection $\psi: P_1 \to Q_1$ such that a covers $\psi(a)$ for all $a \in P_1$.

Define

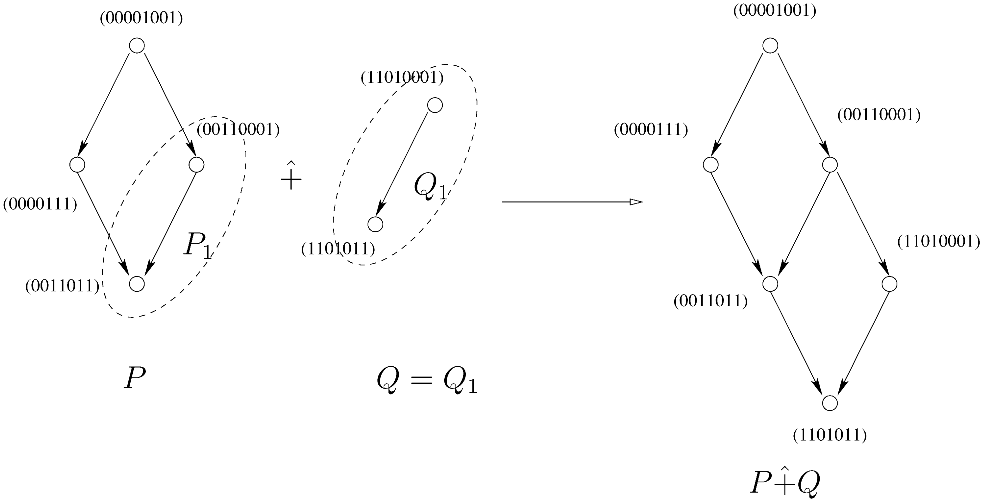
$$P \underset{(P_1,Q_1)}{\hat{+}} Q = P \cup Q$$

regarded as a digraph with

$$V(P \underset{(P_1,Q_1)}{\hat{+}} Q) = V(P) \cup V(Q),$$
$$E(P \underset{(P_1,Q_1)}{\hat{+}} Q) = E(P) \cup E(Q) \cup \{(a, \psi(a)) \mid a \in P_1\}.$$

We will drop the label "(P_1, Q_1)" if this causes no confusion. Moreover, it ψ is not hard to see that ψ is a *strictly order-preserving map* (i.e., $\psi(p_1) < \psi(p_1')$ whenever $p_1 < p_1'$). So we can view $P \hat{+} Q$ as a poset induced by the digraph.

In particular, $P_n = U_n \hat{+} D_n$.



From now on, let τ_k denote the only element in $\{0,1\}^\omega$ with the kth entry being 1 and the others being 0; i.e., $\tau_k = (00\ldots01) \in S_{A_k}$.

Let η_k denote the only element $(a_1, a_2, \ldots, a_{k-1})$ in $\{0,1\}^\omega$ with $a_{k-1} = a_{k-2} = \cdots = a_{k-N} = 1$ and $a_{k-N-1} = \cdots = a_1 = 0$; i.e., $\eta_k = (00\ldots0011\ldots11) \in S_{A_k}$.

Now we describe the recursion explicitly. In particular, we show that each P_n with $A_k \leqslant n < A_{k+1}$ can be expressed in terms of $P_1, P_2, \ldots, P_{A_k - 1}$.

Proposition 4.3. *If $A_{k,2} < n < A_{k+1}$, then P_n is isomorphic to P_{n-A_k}. More precisely, we have $P_n = P_{n-A_k} + \tau_k$, $U_n = U_{n-A_k} + \tau_k$ and $D_n = D_{n-A_k} + \tau_k$.*

Proposition 4.4. *If $A_{k,1} < n \leqslant A_{k,2}$, then*

$$\begin{aligned} P_n &= (P_{n-A_k} + \tau_k) \hat{+} (D_{n-A_k} + \eta_k) \\ &= (P_{n-A_k} + \tau_k) \underset{(D_{n-A_k}+\tau_k, D_{n-A_k}+\eta_k)}{\hat{+}} (D_{n-A_k} + \eta_k), \end{aligned}$$

$U_n = P_{n-A_k} + \tau_k$ and $D_n = D_{n-A_k} + \eta_k$.

Proposition 4.5. *If* $A_{k,0} < n \leqslant A_{k,1}$, *then*

$$\begin{aligned} P_n &= (P_{n-A_k} + \tau_k) \hat{+} (P_{n-A_k} + \eta_k) \\ &= (P_{n-A_k} + \tau_k) \underset{(P_{n-A_k}+\tau_k, P_{n-A_k}+\eta_k)}{\hat{+}} (P_{n-A_k} + \eta_k), \end{aligned}$$

$U_n = P_{n-A_k} + \tau_k$ *and* $D_n = P_{n-A_k} + \eta_k$.

The only remaining case is $A_k \leqslant n \leqslant A_{k,0}$. By Proposition 4.1, P_n is dual to $P_{n'}$ where $n' = A_1 + A_2 + \cdots + A_k - n$.

By Lemma 2.1(1) and (6), we have

$$\begin{aligned} n' &\leqslant A_1 + A_2 + \cdots + A_k - A_k \\ &= A_1 + A_2 + \cdots + A_{k-1} \\ &< A_{k+1}, \\ n' &\geqslant A_1 + A_2 + \cdots + A_k - A_{k,0} \\ &> A_{k,0}. \end{aligned}$$

Hence $P_{n'}$ can be determined by the propositions above. Moreover, by Proposition 4.2, U_n and D_n are dual to $D_{n'}$ and $U_{n'}$, respectively. So they can also be determined by recursions.

Proof of Proposition 4.3. Assume $S_n \neq \emptyset$. (Otherwise the proposition is trivially true.) Let $\hat{1}$ be the maximal element in P_n. Then $l(\hat{1}) = k$ (Proposition 3.1). So $\hat{1} - \tau_k \in S_{n-A_k}$, which implies $S_{n-A_k} \neq \emptyset$. Note that

$$n - A_k > A_{k,2} - A_k = \sum_{\substack{1 \leqslant i < k-N+1 \\ N \nmid i}} A_{k-N+1-i}.$$

By Corollary 3.2, we have $n - A_k \geqslant A_{k-N+1}$.

For any $a \in S_n$,

$$\begin{aligned} s(a) = n &\geqslant A_k + A_{k-N+1} \\ &> (A_{k-1} + \cdots + A_{k-N}) + (A_1 + A_2 + \cdots + A_{k-N-1}) \quad \text{(Lemma 2.1(1))} \\ &= A_1 + A_2 + \cdots + A_{k-1}. \end{aligned}$$

Hence $l(a) = k$ for all $a \in S_n$. In particular, it is valid to do the subtraction $a - \tau_k$.

Therefore, the map

$$\begin{aligned} \phi : P_n &\to P_{n-A_k}, \\ a &\mapsto a - \tau_k \end{aligned}$$

gives an isomorphism from P_n to P_{n-A_k} and $P_n = P_{n-A_k} + \tau_k$. $\square$

Proof of Proposition 4.4. Assume $S_n \neq \emptyset$. In this case, $S_n = T_n \cup R_n$.

Applying a similar argument, we obtain $n \geqslant A_k + A_{k-N}$. Furthermore, by Proposition 3.1, $l(\hat{1} - \tau_k) = k - N$.

Define

$$\begin{aligned} \phi_1 : U_n &\to P_{n-A_k}, \\ a &\mapsto a - \tau_k. \end{aligned}$$

Then ϕ_1 gives an isomorphism from U_n to P_{n-A_k} and $U_n = P_{n-A_k} + \tau_k$.

On the other hand, for any $a = (a_1, a_2, \dots, a_{k-1}) \in D_n$, we claim that $a_{k-N} = a_{k-N+1} = \cdots = a_{k-1}$. Assume the contrary. Then

$$\begin{aligned} s(a) &\leqslant A_1 + \cdots + A_{k-N-1} + A_{k-N+1} + \cdots + A_{k-1} \\ &< (A_{k-N} + A_{k-N-1}) + A_{k-N+1} + \cdots + A_{k-1} \\ &= A_{k-N-1} + A_k \\ &< A_{k-N} + A_k \\ &\leqslant n, \end{aligned}$$

a contradiction.

Hence, the subtraction $a - \eta_k$ is valid.

Note that $l(a - \eta_k) \leqslant k - N - 1 = l(\hat{1} - \tau_k) - 1$. Thus $a - \eta_k \in D_{n-A_k}$ for all $a \in D_n$. So the map

$$\begin{aligned} \phi_2 : D_n &\to D_{n-A_k}, \\ a &\mapsto a - \eta_k \end{aligned}$$

gives an isomorphism from D_n to D_{n-A_k} and $D_n = D_{n-A_k} + \eta_k$.

Therefore,

$$P_n = (P_{n-A_k} + \tau_k) \hat{+} (D_{n-A_k} + \eta_k)$$

via the natural map $\psi : D_{n-A_k} + \tau_k \to D_{n-A_k} + \eta_k$: $a + \tau_k \mapsto a + \eta_k$. □

Proof of Proposition 4.5. The proof is almost the same. Assume $S_n \neq \emptyset$.

In this case, we have $n \geqslant A_k + A_{k-N-1}$ and $l(\hat{1} - \tau_k) = k - N - 1$.

For $a \in D_n$, we still have $a_{k-N} = a_{k-N+1} = \cdots = A_{k-1} = 1$. So the map

$$\begin{aligned} \phi_2 : D_n &\to P_{n-A_k}, \\ a &\mapsto a - \eta_k \end{aligned}$$

gives an isomorphism from D_n to P_{n-A_k}. This completes the proof. □

5. Applications

5.1. Modularity of the lattices P_n

Theorem 5.1. *P_n, U_n, D_n are modular lattices.*

Proof. We prove the statement by induction on n.

Base case: When $n < A_{N+1}$, P_n is either $\emptyset$ or a single-element set. The statement is trivially true.

Inductive step: Consider $A_k \leqslant n < A_{k+1}$ $(k \geqslant N + 1)$. It is clear that the dual poset of a modular lattice is also a modular lattice. When $A_k \leqslant n \leqslant A_{k,0}$, we have $A_{k,0} < A_1 + A_2 + \cdots + A_k - n < A_{k+1}$ (Lemma 2.1(1) and (6)). So, by symmetry (Proposition 4.1), it suffices to consider $A_{k,0} < n < A_{k+1}$.

(1) If $A_{k,2} < n < A_{k+1}$, then, by Proposition 4.3, P_n, U_n, D_n are isomorphic to P_{n-A_k}, U_{n-A_k} and D_{n-A_k}, respectively. So they are modular lattices by the induction hypothesis.

(2) If $A_{k,1} < n \leqslant A_{k,2}$, then, by Proposition 4.4, U_n, D_n are isomorphic to P_{n-A_k} and D_{n-A_k}, respectively. So they are lattices by induction.

To show that P_n is a lattice, we need to show that for any $x, y \in P_n$, $x \vee y$ and $x \wedge y$ exist. Indeed, if $x, y \in U_n$ or $x, y \in D_n$, then $x \vee y$ and $x \wedge y$ exist by the induction hypothesis. If $x \in U_n$ and $y \in D_n$, we claim that $x \vee y$ coincides with $x \vee \psi^{-1}(y)$, which exists by induction. Clearly, $x \vee \psi^{-1}(y)$ is an upper bound of x and y. On the other hand, if $z \in U_n$ is an upper bound of y, it must be an upper bound of $\psi^{-1}(y)$. Therefore, $x \vee y = x \vee \psi^{-1}(y)$, as desired. Hence P_n is a lattice.

Now we check the modularity. We need to show that x, y both cover $x \wedge y$ if and only if x, y are both covered by $x \vee y$.

Indeed, if $x, y \in U_n$ and $x, y \in D_n$, the statement follows by the induction hypothesis.

Suppose $x \in U_n$ and $y \in D_n$. Assume that $x \vee y$ covers both x, y. Obviously, $x \vee y \in U_n$. So $x \vee y = \psi^{-1}(y)$ and $\psi^{-1}(y)$ covers x. Hence $\psi(x)$ is covered by both x and y, as desired. Similarly, if $x \wedge y$ is covered by both x and y, then $\psi^{-1}(y)$ covers both x and y.

(3) If $A_{k,0} < n \leqslant A_{k,1}$, then, by Proposition 4.5, U_n, D_n are both isomorphic to P_{n-A_k}. The rest of proof is similar to case (2). □

5.2. Quasifibonacci sequences of even level

In this section we will prove Theorem 1.1. As mentioned in the introduction section, h_m is the difference of the number of partitions into an even number of A_k's and the number of partitions into an odd number of A_k's.

To distinguish these two kinds of partitions, we define the *sign function* $\sigma : \{0,1\}^\omega \to \pm 1$ by setting $\sigma(a) = 1$ if a contains an even number of 1's and $\sigma(a) = -1$ otherwise. In general, for any finite subset P of $\{0,1\}^\omega$ with natural partial ordering, define

$$\sigma(P) := \sum_{a \in P} \sigma(a).$$

It is clear that $h_n = \sigma(P_n)$. We also define $f_n = \sigma(U_n)$ and $g_n = \sigma(D_n)$. Then, obviously, $h_n = f_n + g_n$.

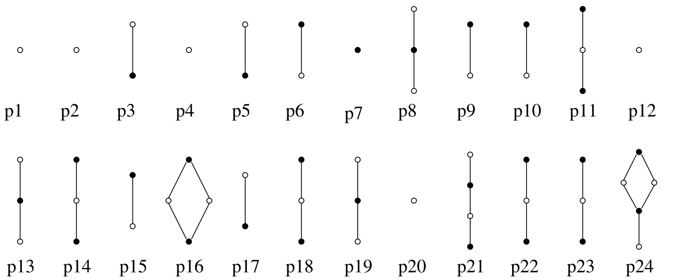
The following lemma will be useful in the proof below.

Lemma 5.2.

- *Let $a, b \in \{0,1\}^\omega$. Then $\sigma(a \pm b) = \sigma(a)\sigma(b)$.*
- *Let $a \in \{0,1\}^\omega$ and P a finite subset of $\{0,1\}^\omega$. Then $\sigma(P + a) = \sigma(P)\sigma(a)$.*
- *Let P, Q be finite subsets of $\{0,1\}^\omega$. Then $\sigma(P \hat{+} Q) = \sigma(P) + \sigma(Q)$.*

Proof. Straightforward. □

To visualize the relation of odd and even partitions, we color the digraph with two colors. In the corresponding Hasse diagram, a vertex $a \in P_n$ is colored black if $\sigma(a) = 1$ and colored white if $\sigma(a) = -1$. The figure below shows the first 24 colored Hasse diagrams for $P_n(\{F_k\})$:



Suppose a covers b in P_n. By the definition, b has $N - 1$ more 1's than a. Hence adjacent vertices in the corresponding digraph have different colors if N is even.

Now we prove Theorem 1.1 by proving the following stronger result:

Proposition 5.3. *For any $n \in \mathbb{Z}^+$, $f_n, g_n, h_n \in \{-1, 0, 1\}$.*

Proof. We perform induction on n.

Base case: When $n < A_{N+1}$, P_n is either $\emptyset$ or a single-element set. The statement is true.

Inductive step: Consider $A_k \leqslant n < A_{k+1}$ $(k \geqslant N+1)$.

(1) If $A_{k,2} < n < A_{k+1}$, then $U_n = U_{n-A_k} + \tau_k$, $D_n = D_{n-A_k} + \tau_k$, $P_n = P_{n-A_k} + \tau_k$. Hence we have

$$f_n = \sigma(\tau_k) f_{n-A_k} = -f_{n-A_k} \in \{-1, 0, 1\}.$$

Similarly, we have $g_n = -g_{n-A_k} \in \{-1, 0, 1\}$ and $h_n = -h_{n-A_k} \in \{-1, 0, 1\}$.

(2) If $A_{k,1} < n \leqslant A_{k,2}$, then $U_n = P_{n-A_k} + \tau_k$ and $D_n = P_{n-A_k} + \eta_k$ and $P_n = (P_{n-A_k} + \tau_k) \hat{+} (D_{n-A_k} + \eta_k)$. Hence we have

$$f_n = \sigma(\tau_k) h_{n-A_k} = -h_{n-A_k} \in \{-1, 0, 1\},$$
$$g_n = \sigma(\eta_k) g_{n-A_k} = g_{n-A_k} \in \{-1, 0, 1\}$$

and

$$\begin{aligned} h_n &= \sigma(P_{n-A_k} + \tau_k) + \sigma(D_{n-A_k} + \eta_k) \\ &= \sigma(P_{n-A_k})\sigma(\tau_k) + \sigma(D_{n-A_k})\sigma(\eta_k) \\ &= -h_{n-A_k} + g_{n-A_k} \\ &= -f_{n-A_k} \in \{-1, 0, 1\}. \end{aligned}$$

(3) If $A_{k,0} < n \leqslant A_{k,1}$, then $U_n = P_{n-A_k} + \tau_k$, $D_n = P_{n-A_k} + \eta_k$ and $P_n = (U_{n-A_k} + \tau_k) \hat{+} (D_{n-A_k} + \eta_k)$. Hence we have

$$f_n = \sigma(\tau_k)\sigma(P_{n-A_k}) = -h_{n-A_k} \in \{-1, 0, 1\},$$
$$g_n = \sigma(\eta_k)\sigma(P_{n-A_k}) = h_{n-A_k} \in \{-1, 0, 1\}$$

and

$$\begin{aligned} h_n &= \sigma(P_{n-A_k} + \tau_k) + \sigma(P_{n-A_k} + \eta_k) \\ &= \sigma(P_{n-A_k})\sigma(\tau_k) + \sigma(P_{n-A_k})\sigma(\eta_k) \\ &= -h_{n-A_k} + h_{n-A_k} \\ &= 0. \end{aligned}$$

(4) If $A_k \leqslant n \leqslant A_{k,0}$, then U_n, D_n, P_n are dual to $D_{n'}$, $U_{n'}$ and $P_{n'}$, respectively, where $n' = A_1 + A_2 + \cdots + A_k - n$ lies between $A_{k,0}$ and A_{k+1}. Hence we have

$$f_n = (-1)^k \sigma(D_{n'}) = (-1)^k g_{n'} \in \{-1, 0, 1\}^{\omega},$$
$$g_n = (-1)^k \sigma(U_{n'}) = (-1)^k f_{n'} \in \{-1, 0, 1\}^{\omega}$$

and

$$h_n = (-1)^k \sigma(P_{n'}) = (-1)^k h_{n'} \in \{-1, 0, 1\}^{\omega}.$$

This completes the proof. □

To write down the recursion explicitly, we have

$$(f_n, g_n, h_n) = \begin{cases} (-1)^{k+1} g_{\gamma_k - n - A_k} \quad (-1)^{k+1} f_{\gamma_k - n - A_k} \quad (-1)^{k+1} h_{\gamma_k - n - A_k} & \text{if } A_k \leqslant n < \gamma_k - A_{k,2}; \\ (-1)^{k} g_{\gamma_k - n - A_k} \quad (-1)^{k+1} h_{\gamma_k - n - A_k} \quad (-1)^{k+1} f_{\gamma_k - n - A_k} & \text{if } \gamma_k - A_{k,2} \leqslant n < \gamma_k - A_{k,1}; \\ (-1)^{k} h_{\gamma_k - n - A_k} \quad (-1)^{k+1} h_{\gamma_k - n - A_k} \quad 0 & \text{if } \gamma_k - A_{k,1} \leqslant n \leqslant A_{k,0}; \\ -h_{n-A_k} \quad h_{n-A_k} \quad 0 & \text{if } A_{k,0} < n \leqslant A_{k,1}; \\ -h_{n-A_k} \quad g_{n-A_k} \quad -f_{n-A_k} & \text{if } A_{k,1} < n \leqslant A_{k,2}; \\ -f_{n-A_k} \quad -g_{n-A_k} \quad -h_{n-A_k} & \text{if } A_{k,2} < n < A_{k+1} \end{cases}$$

where $\gamma_k = A_1 + A_2 + \cdots + A_k$.

Comment: If we take $A_k = F_k$, then the recursion above is precisely the one in [1].

5.3. *Quasifibonacci sequences of odd level*

In this case, N is an odd number. Hence for each pair of adjacent vertices (a, b) in G_n, the parity of the number of 1's in a and b must be the same. In other words, every adjacent pair of vertices have same color. Since P_n are lattices, G_n are connected graphs. Therefore all vertices in P_n have same color. It is easily seen that h_n is not bounded in this case. Instead we have the following estimate.

Proposition 5.4. *Let $k \geqslant N$ be an integer. For any $A_k \leqslant n < A_{k+1}$, $|h_n| \leqslant 2^{k-N}$.*

Proof. Base case: When $n < A_{N+1}$, P_n is either $\emptyset$ or a single-element set. So $h_n \in \{0, -1\}$. The statement is true.

Inductive step: Consider $A_k \leqslant n < A_{k+1}$ $(k \geqslant N + 1)$. With a similar argument, we can derive the following recursion:

$$h_n = \begin{cases} (-1)^{k+1} h_{\gamma_k - n - A_k} & \text{if } A_k \leqslant n < \gamma_k - A_{k,2}; \\ (-1)^{k+1}(g_{\gamma_k - n - A_k} + h_{\gamma_k - n - A_k}) & \text{if } \gamma_k - A_{k,2} \leqslant n < \gamma_k - A_{k,1}; \\ (-1)^{k+1} 2h_{\gamma_k - n - A_k} & \text{if } \gamma_k - A_{k,1} \leqslant n \leqslant A_{k,0}; \\ -2h_{n-A_k} & \text{if } A_{k,0} < n \leqslant A_{k,1}; \\ -g_{n-A_k} - h_{n-A_k} & \text{if } A_{k,1} < n \leqslant A_{k,2}; \\ -h_{n-A_k} & \text{if } A_{k,2} < n < A_{k+1} \end{cases}$$

where $\gamma_k = A_1 + A_2 + \cdots + A_k$.

Note that $|g_m| \leqslant |h_m|$ $(\forall m \in \mathbb{Z}^+)$. Hence, in any of the six cases, $|h_n| \leqslant 2 \cdot 2^{k-N-1} = 2^{k-N}$. This completes the proof. $\square$

Comment: This upper bound is the best possible because there exists $A_k \leqslant n < A_{k+1}$ satisfying $|h_n| = 2^{k-N}$ for all $k \geqslant N$. However, it is possible to improve the result by splitting the intervals into more pieces and refining the estimate.

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References

[1] F. Ardila, The coefficients of a Fibonacci power series, Fibonacci Quart. 42 (3) (2004) 202–204.
[2] N. Robbins, Fibonacci partitions, Fibonacci Quart. 34 (4) (1996) 306–313.