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On 0L-Languages

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INTRODUCTION

$0L$-, $1L$- and $2L$-languages are together called L-languages and were introduced by Lindenmayer (1968), originally as a tool for certain problems in theoretical biology. Herman and Van Dalen have discussed the strength of $1L$- and $2L$-systems (or, rather, their canonical extensions). Doucet did some work on deterministic $0L$-systems, and Rozenberg investigated some properties of L-languages and their generalizations. In this paper we are only concerned with $0L$-languages; we shall present some results characterizing this family, and then make a comparison with the "classical" Chomsky hierarchy. We shall assume the reader to be familiar with the main facts about the Chomsky-languages.

1. PRELIMINARIES

1.1 If A and B are two sets, then $A \subseteq B$ denotes inclusion of A in B. $A \subset B$ denotes strict inclusion, and $A \not\subseteq B$ denotes the negation of $A \subseteq B$; A and B are called *incomparable* if $A \not\subseteq B$ and $B \not\subseteq A$.

The number of elements of A is written as $\# A$. 2^A denotes the family of all subsets of A. $\varnothing$ is the empty set.

Let Σ be a finite set. Any sequence of elements of Σ is called a *word* over Σ. If x and y are two words over Σ, then their concatenation is written as xy.

Λ denotes the empty word. If $a \in \Sigma$, then a^2 means aa, a^3 means aaa, etc; $a^0 = \Lambda$.

If L_1, L_2 are sets of words over Σ, then $L_1 \cdot L_2 = \{xy : x \in L_1 \,\&\, y \in L_2\}$. Σ^* is the (Kleenean) closure of Σ under concatenation; it means that $\Sigma^* = \bigcup_{i=0}^{\infty} \Sigma^i$, where $\Sigma^0 = \{\Lambda\}$, and $\Sigma^i = \Sigma^{i-1} \cdot \Sigma$ for $i = 1, 2, 3, \ldots$. $\Sigma^+ = \Sigma^* - \{\Lambda\}$.

If x is a word over Σ, then the length of x is denoted by $|x|$. If Σ and Δ are two nonempty sets and h is a mapping, $h : \Sigma \to \Delta^*$, which is extended to Σ^* by defining $h(\Lambda) = \Lambda$ and $h(a_1 \cdots a_n) = h(a_1) \cdots h(a_n)$ for $a_1, \ldots, a_n \in \Sigma$, then h is called a *homomorphism*.

Let $X \subseteq \Sigma^*$, $Y \subseteq \Delta^*$, and $h : \Sigma \to \Delta^*$ be a homomorphism. Then $h(X) = \{y : (\exists x)_X\, (h(x) = y)\}$.

h^{-1}, defined by $h^{-1}(Y) = \{x : h(x) \in Y\}$ is called an *inverse homomorphism.*

1.2. For definitions of context-sensitive, context-free, and regular grammars, we refer the reader to an appropriate textbook, e.g., Ginsburg [4].

As regards the use of different kinds of letters, we shall usually denote terminal symbols by $a, b, c, \ldots$; terminal words by $\ldots w, x, y, z$; nonterminal symbols by $A, B, C, \ldots$; nonterminal words by $\ldots X, Y, Z$; and arbitrary words by $\alpha, \beta, \gamma, \ldots$.

In describing derivations, we shall use $\alpha \underset{G}{\to} \beta$ for a production rule, $\alpha \underset{G}{\Rightarrow} \beta$ for a derivation of length one, and $\alpha \overset{*}{\underset{G}{\Rightarrow}} \beta$ for any derivation of length zero or more. If there is no risk of confusion, the G is omitted.

For a fixed, finite alphabet Σ one can define the following three families of languages:

DEFINITION. $\mathscr{L}_\Sigma^{CS} = \{M : M \subseteq \Sigma^*$ and there exists a context-sensitive grammar such that $L(G) = M\}$.
$\mathscr{L}_\Sigma^{CF} = \{M : M \subseteq \Sigma^*$ and there exists a context-free grammar G such that $L(G) = M\}$.
$\mathscr{L}_\Sigma^{R} = \{M : M \subseteq \Sigma^*$ and there exists a regular grammar G such that $L(G) = M\}$.

For purposes of classification, we also define $\tilde{\mathscr{L}}_\Sigma^{CS} = \{M : M - \{\Lambda\} \in \mathscr{L}_\Sigma^{CS}\}$. A set $M \subseteq \Sigma^*$ is called a *context-sensitive* (*context-free, regular*) *language* if it is an element of $\tilde{\mathscr{L}}_\Sigma^{CS}(\mathscr{L}_\Sigma^{CF}, \mathscr{L}_\Sigma^{R})$.

It has been proved [cf. Ginsburg] that $\mathscr{L}_\Sigma^{R} \subset \mathscr{L}_\Sigma^{CF} \subset \mathscr{L}_\Sigma^{CS} \cup \{\Lambda\}$ for any alphabet Σ with $\#\Sigma > 1$. It has also been proved [cf. Ginsburg] that $\mathscr{L}_\Sigma^{R} = \mathscr{L}_\Sigma^{CF}$ if $\#\Sigma = 1$.

DEFINITION. A *linear bounded automaton* is a system $B = \langle K, \Sigma, \delta, q_0, F\rangle$, where K and Σ are disjoint, finite, nonempty sets, δ is a mapping of $K \times \Sigma$ into the subsets of $K \times \Sigma \times \{-1, 0, 1\}$, $q_0 \in K$, and $F \subseteq K - \{q_0\}$.
For arbitrary elements u and v of Σ^* and c in Σ write

(i) $ucpav \vdash uqcbv$ if $\delta(p, a)$ contains $(q, b, -1)$

(ii) $upav \vdash uqbv$ if $\delta(p, a)$ contains $(q, b, 0)$

(iii) $upav \vdash ubqv$ if $\delta(p, a)$ contains $(q, b, 1)$.

For each α in $\Sigma^*K\Sigma^*$ write $\alpha \overset{*}{\vdash} \alpha$. For α and β in $\Sigma^*K\Sigma^*$ write $\alpha \overset{*}{\vdash} \beta$ if there exist $\alpha = \alpha_0, \ldots, \alpha_k = \beta$ such that $\alpha_i \vdash \alpha_{i+1}$ for each $i < k$.

Let $T(B) = \{w \in \Sigma^* : q_0 w \overset{*}{\vdash} \beta$ for some $\beta \in \Sigma^*F\}$. A $w \in \Sigma^*$ is *accepted* by B if and only if $w \in T(B)$.

The class of languages accepted by linear bounded automata is identical with the class of context-sensitive languages [Kuroda].

1.3. We shall now define $0L$-systems and the languages generated by them.

DEFINITION. A *$0L$-system* is a system $G = \langle \Sigma, P, \sigma \rangle$, where Σ (the *alphabet*) is a finite, nonempty set, σ (the *axiom*) is an element of Σ^+, and P (the set of *productions*) is a finite subset of $\Sigma \times \Sigma^*$, such that

$$(\forall a)_\Sigma \, (\exists \alpha)_{\Sigma^*} \, (\langle a, \alpha \rangle \in P).$$

As in the grammars defined in 1.1, $a \to \alpha$ shall mean the same as $\langle a, \alpha \rangle \in P$.

The relation $\Rightarrow$ in $0L$-systems differs from the corresponding relation in the grammars of 1.1:

DEFINITION. Let $G = \langle \Sigma, P, \sigma \rangle$ be a $0L$-system; let $x \in \Sigma^+$, $x = a_1 \cdots a_m$ with $m \geqslant 1$ and $a_j \in \Sigma$ for $j = 1,..., m$; let $y \in \Sigma^*$. Then $x \underset{G}{\Rightarrow} y$ if and only if

$$(\exists p_1 ,..., p_m)_P \, (\forall j)_{1,...,m} \, (p_j = \langle a_j , \alpha_j \rangle \, \& \, y = \alpha_1 \cdots \alpha_m).$$

In the usual way, $\overset{*}{\underset{G}{\Rightarrow}}$ is defined as the transitive and reflexive closure of $\underset{G}{\Rightarrow}$.

DEFINITION. Let $G = \langle \Sigma, P, \sigma \rangle$ be a $0L$-system. The *language* generated by G is defined as $L(G) = \{x : \sigma \overset{*}{\underset{G}{\Rightarrow}} x\}$.

DEFINITION. Let Σ be a fixed alphabet.
$\mathcal{O}_\Sigma = \{M : M \subseteq \Sigma^* \, \& \text{ there exists a } 0L\text{-system } G \text{ such that } L(G) = M\}$.

A set $M \subseteq \Sigma^*$ is called a *$0L$-language* if it is an element of $\mathcal{O}_\Sigma$. A special class of $0L$-languages is defined by $\mathcal{O}_\Sigma^\Lambda = \{M : M \in \mathcal{O}_\Sigma \, \& \, \Lambda \in M\}$.

The difference in definition of $0L$-languages and context-free languages, for example, can be summed up in three points:

(i) There is no terminal alphabet; every string derived in the system (sentential form) is an element of its language.

(ii) The axiom of a $0L$-system is a word of length one or more.

(iii) Productions are always applied simultaneously; in other words, if a word derives another word, productions are applied to all the letters in it.

2. Closure Properties

$0L$-languages are remarkable by their nearly complete lack of closure properties under the usually considered operations. We shall show this.

THEOREM 2.1. *$0L$-languages are not closed with respect to*

(i) *Union*

(ii) *Complement*

(iii) *Intersection*

(iv) *The star operator* (*or Kleenean closure*)

(v) *The* $+$ *operator*

(vi) *Homomorphisms*

(vii) *Inverse homomorphisms*

(viii) *Intersection with regular sets.*

Proof. We shall make use of the following $0L$-languages to provide counterexamples for the different sections:

$K_1 = \{a^2, a^4, a^6, a^8, a^{10}, ...\}$,
generated by $\langle\{a\}, \{a \to a, a \to a^3\}, a^2\rangle$.

$K_2 = \{a^3, a^4, a^5, a^6, a^7, ...\}$,
generated by $\langle\{a\}, \{a \to a, a \to a^2\}, a^3\rangle$.

$K_3 = \{\Lambda, a, a^2, a^3, a^4\}$,
generated by $\langle\{a\}, \{a \to \Lambda, a \to a\}, a^4\rangle$.

$K_4 = \{a^3, a^6, a^{12}, a^{24}, a^{48}, ...\}$,
generated by $\langle\{a\}, \{a \to a^2\}, a^3\rangle$.

$K_5 = \{a, a^3, a^4, a^9, a^{10}, a^{11}, ..., a^{16}, a^{27}, a^{28}, ...\}$,
generated by $\langle\{a\}, \{a \to a^3, a \to a^4\}, a\rangle$.

$K_6 = \{ba, ba^2, ba^3, ba^4, ...\}$,
generated by $\langle\{a, b\}, \{a \to a, b \to b, b \to ba\}, ba\rangle$.

$K_7 = \{b, a^5, a^{15}\}$,
generated by $\langle\{a, b\}, \{a \to a, b \to a^5, b \to a^{15}\}, b\rangle$.

$K_8 = \{aa\} \cup \{b^{2^k} : k \geqslant 2\}$,
generated by $\langle\{a, b\}, \{a \to bb, b \to bb\}, aa\rangle$.

We shall also make use of the fact that two-element subsets of $\{a\}^+$, such as $\{a^3, a^5\}$, are not $0L$-languages (in 3.1 (iii) we shall prove this for any finite subset M of $\{a\}^+$ for which $\#M \geqslant 2$).

(i) A trivial counterexample is $\{a\} \cup \{a^2\}$; the component sets are in $\mathcal{O}_{\{a\}}$, but their union is not. Less trivial is $K_1 \cup K_4 = \{a^2, a^3, a^4, a^6, a^8, \ldots\}$. There is no $0L$-system $H = \langle\{a\}, P, \sigma\rangle$ such that $L(H) = K_1 \cup K_4$; for it is clear that $a \to \Lambda$ cannot be in P, so $\sigma = a^2$; also $a^2 \Rightarrow a^3$, which means that both $a \to a$ and $a \to a^2$ must be in P; with these rules, however, $a^4 \Rightarrow a^5$, and $a^5 \notin K_1 \cup K_4$.

(ii) Trivial non-$0L$-languages are the complements of Σ^* (the empty set) and of Σ^+ (the set containing only Λ). The complement of K_1, $\{a\}^* - K_1 = \{\Lambda, a, a^3, a^5, a^7, \ldots\}$, is also not in $\mathcal{O}_{\{a\}}$. For, if $H = \langle\{a\}, P, \sigma\rangle$ is such that $L(H) = \{a\}^* - K_1$, then both $a \to \Lambda$ and some $a \to a^m$ must be in P. This, however, enables one to produce words of even length (take any word of $\{a\}^* - K_1$, apply the second rule to two of its letters, and the first rule to the others), which do not belong in $\{a\}^* - K_1$. The statement still holds for complements with respect to Σ^+ instead of $\Sigma^* : \{a\}^+ - K_2 = \{a, a^2\} \notin \mathcal{O}_{\{a\}}$.

(iii) Again, all those intersections containing either no element or only Λ can serve as counterexamples. Also, $K_1 \cap K_3 = \{a^2, a^4\}$ is not in $\mathcal{O}_{\{a\}}$. Less trivial is $K_2 \cap K_5 = \{a^3, a^4, a^9, a^{10}, a^{11}, \ldots\}$. By an argument similar to that in (i) one can easily show that this set is not in $\mathcal{O}_{\{a\}}$.

(iv) Consider K_6^*, and assume that there exists a $H = \langle\{a, b\}, P, \sigma\rangle$ such that $L(H) = K_6^*$. From the two facts that $\Lambda \in K_6^*$ and that all other words contain both a's and b's, it follows that both $a \to \Lambda$ and $b \to \Lambda$ must be in P. But this implies that $a \to a^m \notin P$ for every $m \geqslant 1$ (otherwise a^m would be in $L(H)$). So, for any rule $a \to \alpha$ in P, either $\alpha = \Lambda$ or $\alpha = \alpha_1 b \alpha_2$ for some α_1 and α_2 in $\{a, b\}^*$. But this means that H, which should produce words of the form ba^n for arbitrary large n, is not able to do so.

(v) Obviously, $K_8^+ = \{s_1 \cdots s_k : k \geqslant 1 \,\&\, (s_i = a^2 \vee s_i = b^4)\}$. Assume that there exists a $H = \langle\{a, b\}, P, \sigma\rangle$ such that $L(H) = K_8^+$. Then $a \to \Lambda$ and $b \to \Lambda$ are not in P, since $\Lambda \notin K_8^+$. So $\sigma = aa$, and, as $a^2 \Rightarrow a^4$, $a \to a^2$ is in P ($a \to a$ in P and $a \to a^3$ in P lead to obvious contradictions). As b^4 is in $L(H)$, either $a^2 \Rightarrow b^4$ or $a^4 \Rightarrow b^4$. In both cases, the existence of some P-rule $a \to b^k$ ($k = 1, 2$, or 3) is necessary. But with such a rule, $a^2 \Rightarrow a^2 b^k$ with $k = 1, 2$ or 3, and such a word is not in K_8^+. So the required H does not exist, and K_8^+ is no $0L$-language.

(vi) A trivial counterexample is the erasing homomorphism defined by $h(a) = \Lambda$, applied to any $0L$-language over $\{a\}$. A more interesting case is the homomorphism $h_1 : \{a\}^* \to \{a\}^*$ defined by $h_1(a) = a^5$; $h_1(K_3) = \{\Lambda, a^5, a^{10}, a^{15}, a^{20}\}$, which is not in $\mathcal{O}_{\{a\}}$, as the reader will easily see.

(vii) Again, $\{\Lambda\}$ and $\varnothing$ can be used as a proof: if $h_2 : \{a\}^* \to \{a\}^*$ maps a into a^{10}, then $h_2^{-1}(K_3) = \{\Lambda\}$; and $h_1^{-1}(K_4) = \varnothing$. Less trivial is $h_1^{-1}(K_5) = \{a^2, a^3, a^6,...\}$. By a similar argument as used in (i), one can readily show that this set is not in $\mathcal{O}_{\{a\}}$. Still another counterexample is provided by $h_1^{-1}(K_7) = \{a, a^3\} \notin \mathcal{O}_{\{a,b\}}$.

(viii) Let R denote the regular language $\{a^3, a^4, a^5, a^6,...\}$; then $K_3 \cap R = \{a^3, a^4\} \notin \mathcal{O}_{\{a\}}$. Both $\varnothing$ and $\{\Lambda\}$ are regular sets; so intersecting them with an appropriate $0L$-language again provides two counterexamples.

COROLLARY 2.2. *$0L$-languages are not closed with respect to either* gsm-*mappings or inverse* gsm-*mappings.*[1]

3. $0L$-LANGUAGES OVER A ONE-LETTER ALPHABET

Among the $0L$-languages a special place is taken by those over a one-letter alphabet. We shall first say something about their closure properties. Looking at the counterexamples used in the proof of Theorem 2.1, one can see that it holds also for this special class, with one exception.

3.1. THEOREM *The $0L$-languages over a one-letter alphabet are closed with respect to the Kleenean closure.*

Proof. Let $G = \langle\{a\}, P, a^r\rangle$ be an arbitrary $0L$-system over $\{a\}$. First, we shall exclude the special case of $L(G) = \{a\}$. It is clear that a^* is a $0L$-language. Second, if $L(G)$ is a finite set (different from $\{a\}$), then the proof follows from Remark 3.2. So we assume that $L(G)$ is infinite. Given $i < r$, if there are any words in $L(G)$ of the form $a^{k \cdot r+i}$ for some $k \geqslant 0$, then we shall denote the shortest among them by w_i. Together with a^r, the different w_i form the set T. The set of immediate successors of words in T together with a^r is denoted by $\hat{T}$; more formally, $\hat{T} = \{y \in \{a\}^* : (\exists x)_T(x \underset{G}{\Rightarrow} y)\} \cup \{a^r\}$. Of course $\hat{T}$ is finite. Now take the $0L$-system $H = \langle\{a\}, Q, a^r\rangle$ with $Q = \{a \to \Lambda\} \cup \{a \to t : t \in \hat{T}\}$. We shall show that $L(H) = (L(G))^*$.

(i) Let $x \in L(H)$. Either $x = \Lambda$, or $x = a^r$, or x is something else.

[1] A *generalized sequential machine* (gsm) is a sextuple $S = \langle K, \Sigma, \Delta, \delta, \lambda, q_1\rangle$, where K is a finite nonempty set (of states), Σ and Δ are (input and output) alphabets, δ is a mapping of $K \times \Sigma$ into K, λ is a mapping of $K \times \Sigma$ into Δ^*, and q_1 is an element of K. δ and λ are extended to $K \times \Sigma^*$ in the usual way. If $X \subseteq \Sigma^*$, then the gsm-*mapping* of X by S is defined by $S(X) = \{\lambda(q_1, x) : x \in X\}$. If $Y \subseteq \Delta^*$, then the *inverse* gsm-*mapping* of Y by S is defined by $S^{-1}(Y) = \{x \in X : \lambda(q_1, x) \in Y\}$.

In the first two cases, $x \in (L(G))^*$. In the third case x must have an immediate predecessor, y, in $L(H)$ (meaning that $y \underset{H}{\Rightarrow} x$), of the form a^m. This means that x itself is of the form $x_1 \cdots x_m$ with all x_i taken from $\{\Lambda\} \cup \hat{T}$ and, consequently, words from $L(G)$ (if they are not equal to Λ). So x is a concatenation of words from $L(G)$, and $x \in (L(G))^*$.

(ii) Now it must be proved that $y \in (L(G))^*$ implies $y \in L(H)$. It is, however, sufficient to prove that $x \in L(G)$ implies $x \in L(H)$. For let $y = x_1 \cdots x_m$ with all x_i in $L(G)$. $a^r \overset{*}{\underset{H}{\Rightarrow}} a^{kr}$ for every k, in particular, $a^r \overset{*}{\underset{H}{\Rightarrow}} a^{mr}$. If $a^r \underset{G}{\Rightarrow} x_i$ implies $a^r \underset{H}{\Rightarrow} x_i$ for all i, then also $a^r \overset{*}{\underset{H}{\Rightarrow}} a^{mr} \underset{H}{\Rightarrow} x_1 \cdots x_m = y$. The argument fails only if $y = \Lambda$, but in that case y is in $L(H)$ anyhow.

So let x be in $L(G)$. Either $x = a^r$ (and then $x \in L(H)$), or x has a direct predecessor in $L(G) : v \underset{G}{\Rightarrow} x$. Then, v is of the form a^u, with $u \geqslant 1$. According to the construction of the set T, a^u (being an element of $L(G)$) can always be divided into elements of T; to be more precise, if $a^u = a^{k \cdot r+i}$ (with $1 \leqslant i \leqslant r$) and $w_i = a^{l \cdot r+i}$, then $v = a^u = a^r \cdots a^r w_i$ (with a^r repeated $k - l$ times, and $k - l$ never negative). As v is now written as a concatenation of $k - l + 1$ elements of $T : v = v_1 \cdots v_{k-l+1}$, the corresponding division of x, $x = x_1 \cdots x_{k-l+1}$ (determined by $v_i \underset{G}{\Rightarrow} x_i$ for each i) has the property that all x_i are in $\hat{T}$.

The remainder of the proof is now easy; $v_1 \underset{H}{\Rightarrow} x_1$, for Q contains both $a \to x_1$ and $a \to \Lambda$ (to be applied once and $| v_1 | - 1$ times, respectively). Similarly, $v_2 \underset{H}{\Rightarrow} x_2, \ldots, v_{k-l+1} \underset{H}{\Rightarrow} x_{k-l+1}$. So $v \underset{H}{\Rightarrow} x$.

By induction, $v \overset{*}{\underset{G}{\Rightarrow}} x$ implies $v \overset{*}{\underset{H}{\Rightarrow}} x$; so $x \in L(G)$ implies $x \in L(H)$. As mentioned before, it follows that $(L(G))^* \subseteq L(H)$.

3.2. *Remark* One can also prove that, for any nonempty finite set $S \subseteq \{a\}^*$, with $S^* \neq \{\Lambda\}$, S^* is a $0L$-language; for, if $S = \{x_1, \ldots, x_k\}$, then S^* is generated by the $0L$-system $H = \langle \{a\}, \{a \to \Lambda, a \to x_1, \ldots, a \to x_k\}, x_k \rangle$.

3.3. Theorem 3.1 suggests an "arithmetical" characterization of certain $0L$-languages.

Let $\mathbf{N}$ denote the set of natural numbers. Given a one-letter alphabet $\Sigma = \{a\}$, we can represent the natural numbers in terms of Σ^* in an obvious manner, a^n representing n for $n = 0, 1, 2, \ldots$.

Given a set $M \subseteq \mathbf{N}$, we shall denote by Add(M) the smallest subset of $\mathbf{N}$ that contains M and is closed under addition. Then Theorem 3.1 and Remark 3.2 state that, if M is representable by a $0L$-language (or finite), then Add(M) is representable by a $0L$-language. We shall show that, using the same representation, one can characterize, for any one-letter alphabet Σ, the family $\mathcal{O}_\Sigma^\Lambda$.

For a finite $F = \{m_1, ..., m_s\} \subset \mathbf{N} - \{0\}$ and a natural number n ($n \neq 0$), we define $\mathscr{S}(F, n)$ by

$$\mathscr{S}(F, n) = \{p \in \mathbf{N} : (\exists k_1, ..., k_s)_{\mathbf{N}}(p = k_1 m_1 + k_2 m_2 + \cdots + k_s m_s)\} \cup \{n\}.$$

3.4. LEMMA *Let $\Sigma = \{a\}$, and $K \subseteq \Sigma^*$, K infinite. If $K \in \mathcal{O}_\Sigma^\Lambda$, then there exist a natural number $n > 0$ and a nonempty finite set $F \subset \mathbf{N} - \{0\}$ such that $K = \{a^j : j \in \mathscr{S}(F, n)\}$.*

Proof. Let $G = \langle \{a\}, P, \sigma \rangle$ be a $0L$-system such that $L(G) = K$. Let $\sigma = a^{m_0}$, and $P = \{a \to a^k : k \in \bar{B}\}$ for some $\bar{B} \subset \mathbf{N}$. We claim that $L(G) = \{a^j : j \in \mathscr{S}(B, m_0)\}$, where $B = \bar{B} - \{0\}$.

(i) Let $x \in L(G)$. If $x = \sigma$, then $x = a^{m_0}$ and $m_0 \in \mathscr{S}(B, m_0)$. If $x \neq \sigma$, then there exists a $y \in L(G)$ with $y \underset{G}{\Rightarrow} x$. Let $y = a^r$. Then obviously there exist $k_1, ..., k_s \in \mathbf{N}$ (with $k_1 + \cdots + k_s = r$) and $m_1, ..., m_s \in B$ such that $x = a^q$ with $q = k_1 m_1 + k_2 m_2 + \cdots + k_s m_s$ (intuitively speaking, $a \to a^{m_1}$ is applied to k_1 occurrences of a in y, $a \to a^{m_2}$ to k_2 occurrences, and so on, thus obtaining x from y). So $q \in \mathscr{S}(B, m_0)$. Hence, if $x \in L(G)$, then $x \in \{a^j : j \in \mathscr{S}(B, m_0)\}$.

(ii) Let $j \in \mathscr{S}(B, m_0)$. If $j = m_0$, then, of course, $a^j = \sigma \in L(G)$. If $j \neq m_0$, then there exist $k_1, ..., k_n \in \mathbf{N}$ such that $j = k_1 m_1 + k_2 m_2 \cdots k_s m_s$ (where $B = \{m_1, ..., m_s\}$). Note that, because $L(G)$ is infinite, there exists a word $a^g \in L(G)$ such that $g > k_1 + \cdots + k_s$. Now, if $j = 0$, then obviously $a^j \in L(G)$, and if $j \neq 0$, then $a^g \underset{G}{\Rightarrow} a^j$ (apply $a \to a^{m_1}$ to k_1 occurrences of a in a^g, $a \to a^{m_2}$ to k_2 occurrences, and so on; to the remaining $g - (k_1 + \cdots + k_s)$ occurrences, apply $a \to \Lambda$). So $j \in \mathscr{S}(B, m_0)$ implies $a^j \in L(G)$.

Combining (i) and (ii), one obtains

$$K = L(G) = \{a^j : j \in \mathscr{S}(B, m_0)\}.$$

3.5. LEMMA *Let $n \in \mathbf{N} - \{0\}$ and F be a nonempty finite subset of $\mathbf{N} - \{0\}$. Then there exists a $0L$-language $K \subseteq \{a\}^*$ such that $K = \{a^j : j \in \mathscr{S}(F, n)\}$.*

Proof. Take $G = \langle \{a\}, \{a \to a^f : f \in F \cup \{0\}\}, a^n \rangle$. In a similar way to the previous proof, one can show that $L(G) = \{a^j : j \in \mathscr{S}(F, n)\}$.

3.6. We can now give a characterization of $0L$-languages containing Λ over a one-letter alphabet.

THEOREM 3.6. *Let $\Sigma = \{a\}$, and $K \in \mathcal{O}_\Sigma^\Lambda$.*

(i) *K is finite if and only if either $K = \{\Lambda, a^m\}$ for some $m > 0$ or K is prefix-closed (meaning that $K = \{\Lambda, a, a^2, ..., a^m\}$ for some $m > 0$).*

(ii) *K is infinite if and only if there exist an* $n \in \mathbf{N}$, $n > 0$, *and a nonempty finite set* $F \subset \mathbf{N} - \{0\}$ *such that* $K = \{a^j : j \in \mathscr{S}(F, n)\}$.

Proof. (i) Let $G = \langle \{a\}, P, a^m \rangle$ be a $0L$-system such that $L(G)$ is finite and $L(G) \in \mathcal{O}_\Sigma^\Lambda$. Then obviously $a \to \Lambda$ is in P. If P contains nothing else, $L(G) = \{\Lambda, a^m\}$. If, on the other hand, P contains some $a \to a^r$, then $r = 1$ (otherwise $L(G)$ would be infinite); but then $L(G)$ is prefix-closed. Conversely, for an arbitrary $m > 0$, both $\{\Lambda, a^m\}$ and $\{\Lambda, a, a^2, ..., a^m\}$ are obviously elements of $\mathcal{O}_\Sigma^\Lambda$.

(ii) For infinite K, the theorem follows directly from lemmas 3.4 and 3.5.

3.7. Corollary *If K is an infinite set in* $\mathcal{O}_{\{a\}}$, *then there exist a finite set F and positive integers M and d such that* $K = F \cup \{M + kd : k \geqslant 0\}$.

Proof. The statement follows from Lemma 3.4 if one takes for d the greatest common divisor of $m_1, ..., m_s$ (in the same notation of Lemma 3.4). The reader will be familiar with the fact that the set $\{k_1 m_1 + k_2 m_2 + \cdots + k_s m_s : k_1, ..., k_s \geqslant 0\}$ can be written as $F \cup \{\overline{M} + kd : k \geqslant 0\}$ for some integer $\overline{M}$ and some finite set F.

Now if k_0 is the smallest integer for which $\overline{M} + k_0 d \geqslant n$, then put $M = \overline{M} + k_0 d$. This completes the proof.

4. 0L-Languages and Chomsky's Hierarchy

In this section we shall discuss the connections between context-free and context-sensitive languages and the $0L$-languages.

4.1. Theorem *For any alphabet* Σ, $\mathcal{O}_\Sigma$ *and* $\mathscr{L}_\Sigma^R$ *are incomparable, but not disjoint.*

Proof. (i) The $0L$-languages K_1, K_2, K_3, K_6, K_7, as used in the proof of 2.1, are all regular sets. So $\mathcal{O}_\Sigma \cap \mathscr{L}_\Sigma^R \neq \varnothing$.

(ii) The $0L$-language K_4 is not regular, since it is not ultimately periodic [4]. So $\mathcal{O}_\Sigma \not\subseteq \mathscr{L}_\Sigma^R$.

(iii) All finite sets are regular, but many of them are not in $\mathcal{O}_\Sigma$. In fact, every finite subset M of $\{a\}^+$ which has two or more elements is not a $0L$-language. To see this, assume the contrary. Then there exists a $0L$-system $G = \langle \{a\}, P, a^k \rangle$ such that $L(G) = M$. Now all possible assumptions on P lead to contradictions: if P contains any $a \to a^m$ with $m > 1$, then M is infinite; if P contains $a \to \Lambda$, then $\Lambda \in M$; and in the only remaining case (namely, P consisting of nothing but $a \to a$), $\# M = 1$. Hence $M \notin \mathcal{O}_\Sigma$.

Trivial examples of sets in $\mathcal{L}_\Sigma^R$ but not in $\mathcal{O}_\Sigma$ are $\{\Lambda\}$ and $\varnothing$.

Remark. The following example shows that the regular sets outside $\mathcal{O}_\Sigma$ are not all finite. Let $M_1 = \{a^{2n} : n \geqslant 1\} \cup \{a^3\}$. Then $M_1 \notin \mathcal{O}_\Sigma$, for any Σ. To see this, try to construct a $0L$-system $G = \langle \{a\}, P, \sigma \rangle$ such that $L(G) = M$. First, note that $a \to \Lambda$ is not in P (or else Λ would be in M_1). Consequently, the shortest word of M_1, a^2, must be the axiom. Hence $a^2 \overset{*}{\Rightarrow} a^3$, and even $a^2 \Rightarrow a^3$. The latter statement implies that $a \to a$ and $a \to a^2$ are both in P. But then $a^3 \Rightarrow a^5$, which is contradictory, since $a^5 \notin M_1$.

4.2. THEOREM[2] *If $G = \langle V, \Sigma, P, S \rangle$ is a context-free grammar, then there exists a $0L$-system H such that $L(H) \cap \Sigma^* = L(G)$.*

Proof. Define $Q = P \cup \{a \to a : a \in V\}$, and then $H = \langle V, Q, S \rangle$.

(i) Suppose that $S \underset{G}{\overset{*}{\Rightarrow}} x$. We shall prove by induction on the number of derivation steps that $S \underset{H}{\overset{*}{\Rightarrow}} x$. If $S \underset{G}{\Rightarrow} x$, then obviously $S \underset{H}{\Rightarrow} x$. Now suppose that the statement is valid for all derivations of k steps. Let $S \underset{G}{\Rightarrow} w_1 \underset{G}{\Rightarrow} w_2 \cdots \underset{G}{\Rightarrow} w_k \underset{G}{\Rightarrow} x$. Then $w_k = w'Aw''$, and $x = w'\alpha w''$, and $A \to \alpha$ is in P. Since $a \to a$ is in Q for every $a \in V$, $w' \underset{H}{\Rightarrow} w'$ and $w'' \underset{H}{\Rightarrow} w''$.[3] Since $P \subseteq Q$, $A \underset{H}{\Rightarrow} \alpha$. So $w_k \underset{H}{\Rightarrow} x$. From the induction hypothesis it follows that $S \underset{G}{\overset{*}{\Rightarrow}} w_k$ implies $S \underset{H}{\overset{*}{\Rightarrow}} w_k$; hence $S \underset{H}{\overset{*}{\Rightarrow}} w_k \underset{H}{\Rightarrow} x$.

(ii) For the proof that $L(H) \cap \Sigma^* \subseteq L(G)$, assume that $x \in V^+$, $y \in V^*$ and $x \underset{H}{\Rightarrow} y$. Let $x = a_1 \cdots a_m$ with each a_i in Σ. There exist productions $a_1 \to \alpha_1, \ldots, a_m \to \alpha_m$ in P such that $\alpha_1 \cdots \alpha_m = y$. Now[4]

$$a_1 \cdots a_m \underset{G}{\overset{\prime}{\Rightarrow}} \alpha_1 a_2 \cdots a_m \underset{G}{\overset{\prime}{\Rightarrow}} \alpha_1 \alpha_2 a_3 \cdots a_m \underset{G}{\overset{\prime}{\Rightarrow}} \cdots \underset{G}{\overset{\prime}{\Rightarrow}} \alpha_1 \alpha_2 \cdots \alpha_m$$

holds in G. So, if $x \underset{H}{\Rightarrow} y$, then $x \underset{G}{\overset{*}{\Rightarrow}} y$.

Again using induction, it follows that $S \underset{H}{\overset{*}{\Rightarrow}} y$ implies $S \underset{G}{\overset{*}{\Rightarrow}} y$. By (i) and (ii) we have now proved that $S \underset{G}{\overset{*}{\Rightarrow}} x$ if and only if $S \underset{H}{\overset{*}{\Rightarrow}} x$, for every $x \in V^*$. Hence $L(G) = \Sigma^* \cap L(H)$. In fact, $L(H) = \{x : S \underset{G}{\overset{*}{\Rightarrow}} x\}$.

4.3. THEOREM[2] *Let $H = \langle \Sigma, P, \sigma \rangle$ be a $0L$-system with the property that $a \to a$ is in P for every $a \in \Sigma$.*

[2] Theorems 4.2 and 4.3 have also been found, independently, by A. Lindenmayer [10]. Theorem 4.3 is also stated in [Van Dalen].

[3] If $x = a_1 \cdots a_m$ and $a_i \underset{G}{\Rightarrow} w_i$ for each a_i, then clearly $x \underset{G}{\Rightarrow} w_1 \cdots w_m$.

[4] Define $x \underset{G}{\overset{\prime}{\Rightarrow}} y$ if and only if $x \underset{G}{\Rightarrow} y$ or $x = y$.

Then $L(H)$ is a context-free language.

The proof is quite similar to the proof of Theorem 4.2 and can be omitted.

4.4. COROLLARY *If $H = \langle \Sigma, P, \sigma \rangle$ is a 0L-system, then any subsystem of H containing the identity production for each of its letters generates a context-free language. More formally: if $\tilde{\Sigma} \subseteq \{a : a \in \Sigma \;\&\; \langle a, a \rangle \in P\}$, $\tilde{P} = \{\langle a, \alpha \rangle \in P : a \in \tilde{\Sigma}\}$, $\tilde{\sigma}$ is any axiom over $\tilde{\Sigma}$, and $\tilde{H} = \langle \tilde{\Sigma}, \tilde{P}, \tilde{\sigma} \rangle$, then $L(\tilde{H})$ is context-free.*

4.5. COROLLARY *A 0L-language $M \subseteq \Sigma^*$ is context-free if and only if there exists a 0L-system $G = \langle V, P, \sigma \rangle$ with the properties*

(i) $(\forall a)_V(\langle a, \alpha \rangle \in P)$,

(ii) $L(G) \cap \Sigma^* = M$.

The "if" follows from Theorem 4.3, the "only if" from Theorem 4.2.

4.6. Theorem 4.3 also holds for a larger class of languages, obtained by extending the notion of 0L-languages.

A system $G = \langle \Sigma, P, \Theta \rangle$ will be called a *0L-system with context-free root* if Σ and P are defined as in 0L-systems, and Θ is a context-free language over Σ, serving as a set of axioms. The relations $\underset{G}{\Rightarrow}$ and $\overset{*}{\underset{G}{\Rightarrow}}$ are defined as in 0L-systems, and the only difference lies in the definition of the language generated by $G : L(G) = \{x \in \Sigma^* : (\exists \sigma)_\Theta(\sigma \overset{*}{\underset{G}{\Rightarrow}} x)\}$.

4.6. THEOREM *Let $G = \langle \Sigma, P, \Theta \rangle$ be a 0L-system with context-free root. If $a \to a \in P$ for every $a \in \Sigma$, then $L(G)$ is context-free.*

Proof. Let $\Theta = L(C)$ for some context-free grammar $C = \langle U, \Gamma, Q, S \rangle$. Obviously, $(U - \Gamma) \cap \Sigma = \varnothing$. Now take the context-free grammar $H = \langle V, \Sigma, R, S \rangle$, with $V = \Sigma \cup (U - \Gamma) \cup \bar{\Sigma}$ (where $\bar{\Sigma} = \{\bar{a} : a \in \Sigma\}$) and $R = \{\langle a, \bar{\alpha} \rangle : \langle a, \alpha \rangle \in Q\} \cup \{(\bar{a}, \bar{\alpha} \rangle : \langle a, \alpha \rangle \in P\} \cup \{\langle \bar{a}, a \rangle : a \in \Sigma\}$. In much the same way as in 4.2 and 4.4 one can prove that $L(G) = L(H)$.

4.7. THEOREM *For every alphabet Σ, $\mathcal{O}_\Sigma$ and $\mathcal{L}_\Sigma^{CF}$ are incomparable, but not disjoint.*

Proof. (i) Theorem 4.3 shows that $\mathcal{O}_\Sigma \cap \mathcal{L}_\Sigma^{CF} \neq \varnothing$.

(ii) $\mathcal{O}_\Sigma \not\subseteq \mathcal{L}_\Sigma^{CF}$, as is shown by the 0L-system

$$G = \langle \{a, b, c\}, \{\langle a, aa \rangle, \langle b, bb \rangle, \langle c, cc \rangle\}, abc \rangle.$$

Clearly, $L(G) = \{a^{2^n}b^{2^n}c^{2^n} : n \geqslant 0\}$. That $L(G)$ is not context-free can be easily shown analogously to the proof (by Bar-Hillel, Perles and Shamir, and by others) that $\{a^nb^nc^n : n \geqslant 1\}$ is not context-free.

The statement still holds for the special case that $\#\Sigma = 1$; the example mentioned in 4.1 (ii) showed that $\mathcal{O}_{\{a\}} \not\subseteq \mathcal{L}^R_{\{a\}}$, and, as mentioned in 1.1, $\mathcal{L}^R_{\{a\}} = \mathcal{L}^{CF}_{\{a\}}$.

(iii) $\mathcal{L}^{CF}_\Sigma \not\subseteq \mathcal{O}_\Sigma$ follows from $\mathcal{L}^R_\Sigma \not\subseteq \mathcal{O}_\Sigma$, which was proved in Theorem 4.1. This says nothing, of course, about the not-regular context-free sets. That these are not generally in $\mathcal{O}_\Sigma$ is shown, e.g., by the set $\{a^nb^n : n \geqslant 2\}$, which is context-free, not regular, and not $0L$. The last of these three statements is quite easily proved along the lines of 2.1 (i).

4.8. After Theorems 4.1 and 4.7 which established connections between $0L$-languages and regular and context-free languages, we shall now proceed to prove the strict inclusion of the class of $0L$-languages in the class of context-sensitive languages. The following lemma (which we need for Theorem 4.9) can be proved in a purely formal way, but this would necessitate the use of a rather complex formalism to describe a derivation, resulting in an obscure and tedious proof. Instead, we prefer to give a semiformal proof which is both reasonably clear and readily translatable into a proof of greater rigour.

LEMMA 4.8. *Let G be a $0L$-system. Then there exists a number C_G such that for every word w in $L(G)$ there exists a derivation such that $|u| \leqslant C_G \cdot |w|$ for every word u in that derivation.*

Proof. (i) First we shall define the notions *productive* and *improductive* element, *ancestor*, and *age*.

We assume a derivation $D : x \overset{*}{\Rightarrow} y$ to consist of a sequence of words beginning with x and ending with y, together with the precise set of productions used in each step. We shall only consider finite derivations. Within a one-step derivation $D : x \Rightarrow y$, one can in an obvious way[5] for any substring $\bar{x}$ of x specify the substring $\bar{y}$ of y derived (in D) from $\bar{x}$. This notion can also be extended to longer derivations $x \overset{*}{\Rightarrow} y$. Of course, such an $\bar{x}$ may consist of only one letter. So, given a $D : x = a \cdots a_j \cdots a_m \overset{*}{\Rightarrow} y$, the substring $\bar{y}(a_j)$ of y derived from a_j is precisely and uniquely defined. We shall call a_j (D-) *improductive* if and only if this $\bar{y}(a_j)$ is empty. (This simply means that a_j does not contribute to the last word of D). All letters in words of D (or letters in D, as we shall somewhat loosely call them) which are not D-improductive we shall call (D-) *productive*.

[5] A similar notion was introduced by Parikh for context-free grammars.

If, in a certain $D : x \stackrel{*}{\Rightarrow} y$, $\bar{y}$ is a substring of y, $\bar{y}$ is derived from some letter a_j (so $\bar{y} = \bar{y}(a_j)$ for some a_j in D), and $\bar{\bar{y}}$ is any substring of $\bar{y}(a_j)$, then a_j is called a *(D-)ancestor* of $\bar{\bar{y}}$. Furthermore, in the derivation $x \stackrel{*}{\Rightarrow} y_k \Rightarrow y_{k-1} \Rightarrow \cdots \Rightarrow y_1 \Rightarrow y$, a is the *k-th ancestor* of some $\bar{\bar{y}}$ in y if a is an ancesotr of $\bar{\bar{y}}$ and a is in y_k.
It is not difficult to see that, in a given derivation,

(1) except in the first word of the derivation, each letter has one and only one first (= direct) ancestor;

(2) all ancestors of a productive letter are also productive;

(3) all strings derived from an improductive letter consist wholly of improductive letters.

The *(D-)age* of a D-improductive letter is defined as the number of its D-improductive ancestors + 1. Thus, if an improductive letter has age 1, this means that all its ancestors are productive; if it has age 4, it means that its first, second and third ancestors are improductive, all further ancestors (if they exist) being productive (this follows from (2)).

(ii) With the aid of the newly defined concepts, we can now prove our lemma.
Let $G = \langle \Sigma, P, \sigma \rangle$ be a $0L$-system, with $\# \Sigma = n$, and

$$\max\{| \alpha | : \langle a, \alpha \rangle \in P\} = K.$$

Let $w \in L(G)$, and let $D : \sigma = w_0 \Rightarrow w_1 \Rightarrow \cdots \Rightarrow w_{m-1} \Rightarrow w_m = w$ be some derivation of w.
Consider an arbitrary word w_q in $D(1 \leqslant q \leqslant m - 1)$. w_q can be partitioned into substrings, as follows : $\bar{w}_q$ contains all D-productive letters of w_q, x_1 contains all D-improductive letters of D-age 1,..., x_p contains all D-improductive letters of D-age p. Note that

(4) each of these substrings may lie scattered over w_q (but this will not affect the argument).

(5) w_q may contain letters of any age p, except that $p \leqslant q$.

(6) $| \bar{w}_0 | \leqslant | \bar{w}_1 | \leqslant \cdots \leqslant | \bar{w} |$.

(7) $| \bar{w}_j | \leqslant | w_j |$ for all j, $0 \leqslant j \leqslant m - 1$.

Now, still in w_q, consider a letter in some x_j for which $j \geqslant n + 1$. It has more than n D-improductive ancestors, which implies that, among these, some letters occur at least twice; in other words, the line of ancestors contains a loop. This loop can be removed without any effect on w (all letters in the loop being improductive). By removing this loop and all similar loops throughout the derivation, one obtains a new and simplified derivation D', with the properties

(a) $\sigma \overset{*}{\underset{D'}{\Rightarrow}} w$
(b) no D'-improductive letter has a D'-age $> n$.

Denote the words in D' by adding $'$ to the older names, and the result for w_q is $w_q' = \bar{w}_q x_1' x_2' \cdots x_r'$ (note that statement (4) applies to this notation), where $r \leqslant n$ and $r \leqslant q$. It is now possible to estimate the length of w_q':

(8) $|w_q'| = |\bar{w}_q| + |x_1'| + |x_2'| + \cdots + |x_r'|$.

(9) The first D'-ancestor of each letter of x_1' is in $\bar{w}_{q-1}$; so

$$|x_1'| \leqslant K \cdot |\bar{w}_{q-1}|.$$

By the same argument, $|x_2'| \leqslant K^2 \cdot |\bar{w}_{q-2}|, \ldots, |x_r'| \leqslant K^r \cdot |\bar{w}_{q-r}|$. Now apply (6), (7), and (9) to (8):

$$\begin{aligned}(10)\quad |w_q'| &\leqslant |\bar{w}_q| + K \cdot |\bar{w}_{q-1}| + K^2 \cdot |\bar{w}_{q-2}| + \cdots + K^r \cdot |\bar{w}_{q-r}| \\ &\leqslant (1 + K + \cdots + K^r)\, |\bar{w}_q| \\ &\leqslant (1 + K + \cdots + K^r)\, |w| \\ &\leqslant (1 + K + \cdots + K^n)\, |w| = \frac{K^{n+1} - 1}{K - 1} |w|.\end{aligned}$$

By putting $C_G = (K^{n+1} - 1)/(K - 1)$, we obtain $|w_q'| \leqslant C_G \cdot |w|$, which is of the required form.

4.9. THEOREM *The 0L-languages are context-sensitive.*

Proof. As it is known [Kuroda] that the class of context-sensitive languages coincides with the class of languages accepted by linear bounded automata, it is sufficient to show that, for every $0L$-language, there is an *lba* accepting it.

Although it is by no means difficult to describe the actual machine accepting a given $0L$-language, the construction is at the same time straightforward and uninteresting, and we think an informal description will be enough. The "track" technique we use follows that of Hopcroft and Ullman.

Let $G = \langle \Sigma, P, \sigma \rangle$ be a $0L$-system. As shown in 4.8, there is a C such that every $w \in L(G)$ possesses a derivation $\sigma \overset{*}{\underset{G}{\Rightarrow}} w$ with the property that no word in that derivation is longer than $C \cdot |w|$. Of course, we are free to take $2C + 1$ instead of C.

Figure 1 illustrates an *lba* accepting $L(G)$. It uses a tape containing $2C + 1$ tracks (which can be easily coded into the required one-track tape) as follows:

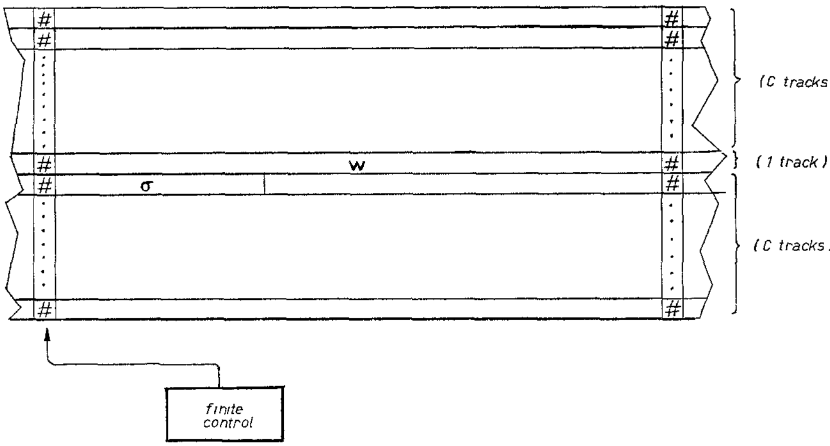

FIG. 1.

The word in question w is written on the central track, and, in the initial configuration (illustrated), the axiom σ is written in the lower half of the tape; the rest of the tape is empty (σ may, of course, be longer than w; but the whole lower half of the tape is available, and σ is never longer than $C \cdot | w |$).

The machine first compares σ and w. If $\sigma = w$, then of course $w \in L(G)$, and the machine stops. If $\sigma \neq w$, then the machine derives a word from σ (nondeterministically, if G is nondeterministic) and writes it in the upper half of the tape, meanwhile erasing σ from the lower half. This word is again compared with w and, if the result is negative, the machine produces a new word from it, writing the new one in the lower half and erasing the old one.

This procedure is repeated until either some comparison yields a positive result or the length of a word exceeds the available space. Note that, if a word runs off the tape, this does not mean that $w \notin L(G)$; w may have other derivations which stay within the limits.

For economy of operation, the machine could be instructed to compare w with a newly produced word only if this word has the correct length (i.e., occupies precisely one track of the available tape).

In previous proofs in this paper, all examples of non-$0L$-languages were context-sensitive. Together with the statement we just proved, this means that, for any Σ, $\mathcal{O}_\Sigma \subset \mathcal{L}_\Sigma^{CS}$.

4.10. We can now roughly locate the $0L$-languages within Chomsky's hierarchy. Let Σ be a fixed alphabet.

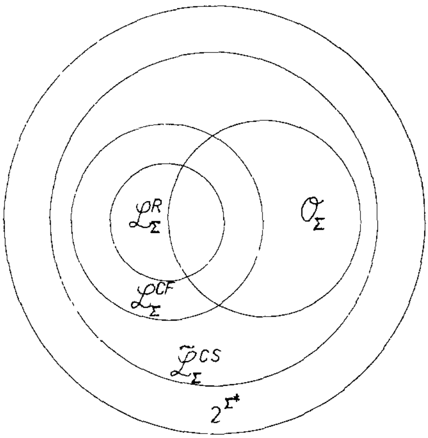

FIG. 2.

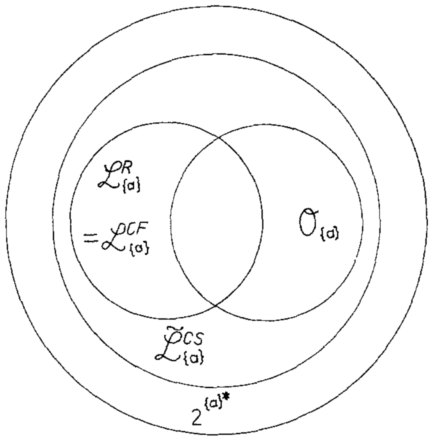

FIG. 3.

Figure 2 combines the contents of Theorems 4.1, 4.7, and 4.9. Figure 3 does not contain much extra information but merely illustrates the situation for a one-letter alphabet $\{a\}$.

CONCLUDING REMARKS

The results so far obtained raise some new problems. Of course, the classification of $0L$-languages with respect to Chomsky's hierarchy is open to refinement. There is also the arithmetical characterization of $0L$-languages. We found one for $\mathcal{C}^A_\Sigma$ if $\#\Sigma = 1$, but this is a rather restricted class; it should be extended to $\mathcal{O}_\Sigma$ with $\#\Sigma = 1$, and perhaps even to all $0L$-languages. The latter problem seems to be difficult.

As Theorem 2.1 shows, $0L$-languages display an extraordinary resistance to the usual Boolean and related operations. Whether this has to do with their biological origin is perhaps difficult to say; but there certainly is a need to devise operations better suited to these languages and/or their biological applications.

SUMMARY

In $0L$-languages, words are produced from each other by the simultaneous transition of all letters according to a set of production rules; the context is ignored.

(i) $0L$-languages are not closed under the operations usually considered.

(ii) $0L$-languages over a one-letter alphabet are discussed separately; a characterization is given of a subclass.

(iii) $0L$-languages are incomparable with regular sets, incomparable with context-free languages, and strictly included in context-sensitive languages.

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REFERENCES

1. Y. BAR-HILLEL, M. PERLES, AND E. SHAMIR, On formal properties of simple phrasestructure grammars, *Z. Phonetik, Sprachwiss. Kommunikat.* **14** (1961), 143–172.
2. D. VAN DALEN, A note on some systems of Lindenmayer, *Math. Systems Theory* **5** (1971, in press).
3. P. G. DOUCET, On the membership question in some Lindenmayer systems, *Indag. Math.* (in press).
4. S. GINSBURG, "The Mathematical Theory of Context-Free Languages," McGraw-Hill, New York, 1966.
5. G. T. HERMAN, The computing ability of a developmental model for filamentous organisms, *J. Theoret. Biol.* **25** (1969), 421–435.
6. G. T. HERMAN, The role of environment in developmental models, *J. Theoret. Biol.* **29** (1970), 329–341.
7. J. E. HOPCROFT AND J. D. ULLMAN, "Formal Languages and Their Relation to Automata," Addison-Wesley, Reading, Mass., 1969.
8. S. Y. KURODA, Classes of languages and linear bounded automata, *Information and Control* **7** (1964), 207–223.
9. A. LINDENMAYER, Mathematical models for cellular interaction in development I, II, *J. Theoret. Biol.* **18** (1968), 280–315.
10. A. LINDENMAYER, Developmental systems without interactions, their languages and grammars. *J. Theoret. Biol.* **30** (1971), 455–484.
11. R. J. PARIKH, On context-free languages, *J. Assoc. Comput. Mach.* **13** (1966), 570–581.
12. G. ROZENBERG, On some generalizations of Lindenmayer systems, to be published.